Chapter 10. Isosceles Triangles

Exercise 10(A)

Solution 1:

In $\triangle ABC$,

$\angle BAC + \angle ACB + \angle ABC = 180^0$

$48^0 + \angle ACB + \angle ABC = 180^0$

But $\angle ACB = \angle ABC$ [AB = AC]

$2 \angle ABC = 180^0 - 48^0$

$2 \angle ABC = 132^0$

$\angle ABC = 66^0 = \angle ACB \ldots \text{(i)}$

$\angle ACB = 66^0$

$\angle ACD + \angle DCB = 66^0$

$18^0 + \angle DCB = 66^0$

$\angle DCB = 48^0 \ldots \text{(ii)}$

Now, In $\triangle DCB$,

$\angle DBC = 66^0$ [From (i), Since $\angle ABC = \angle DBC$]

$\angle DCB = 48^0$ [From (ii)]

$\angle BDC = 180^0 - 48^0 - 66^0$

$\angle BDC = 66^0$

Since $\angle BDC = \angle DBC$

Therefore, BC = CD

Equal angles have equal sides opposite to them.
Solution 2:

Given: \( \angle ACE = 130^\circ; AD = BD = CD \)

Proof:

(i)

\[ \angle ACD + \angle ACE = 180^\circ \quad [DCE \text{ is a st. line}] \]
\[ \Rightarrow \angle ACD = 180^\circ - 130^\circ \]
\[ \Rightarrow \angle ACD = 50^\circ \]

Now, \( CD = AD \)

\[ \Rightarrow \angle ACD = \angle DAC = 50^\circ \ldots \{i\} \]

[Since angles opposite to equal sides are equal]

In \( \triangle ADC \),

\[ \angle ACD = \angle DAC = 50^\circ \]
\[ \angle ACD + \angle DAC + \angle ADC = 180^\circ \]
\[ 50^\circ + 50^\circ + \angle ADC = 180^\circ \]
\[ \angle ADC = 180^\circ - 100^\circ \]
\[ \angle ADC = 80^\circ \]

(ii)

\[ \angle ADC = \angle ABD + \angle DAB \quad [\text{Exterior angle is equal to sum of opp. interior angles}] \]

But \( AD = BD \)

\[ \therefore \angle DAB = \angle ABD \]
\[ \Rightarrow 80^\circ = \angle ABD + \angle ABD \]
\[ \Rightarrow 2\angle BD = 80^\circ \]
\[ \Rightarrow \angle ABD = 40^\circ = \angle DAB \ldots \{ii\} \]

(iii)

\[ \angle BAC = \angle DAB + \angle DAC \]

substituting the values from (i) and (ii)

\[ \angle BAC = 40^\circ + 50^\circ \]
\[ \Rightarrow \angle BAC = 90^\circ \]
Solution 3:

\[ \angle FAB = 128^\circ \quad [\text{Given}] \]
\[ \angle BAC + \angle FAB = 180^\circ \quad [\text{FAC is a st. line}] \]
\[ \Rightarrow \angle BAC = 180^\circ - 128^\circ \]
\[ \Rightarrow \angle BAC = 52^\circ \]

In \( \triangle ABC \),
\[ \angle A = 52^\circ \]
\[ \angle B = \angle C \quad [\text{Given } AB = AC \text{ and angles opposite to equal sides are equal}] \]
\[ \angle A + \angle B + \angle C = 180^\circ \]
\[ \Rightarrow \angle A + \angle B + \angle B = 180^\circ \]
\[ \Rightarrow 52^\circ + 2\angle B = 180^\circ \]
\[ \Rightarrow 2\angle B = 128^\circ \]
\[ \Rightarrow \angle B = 64^\circ = \angle C \quad \ldots \ldots \ (i) \]
\[ \angle B = \angle ADE \quad [\text{Given } DE \parallel BC] \]

(i)

Now,
\[ \angle ADE + \angle CDE + \angle B = 180^\circ \quad [\text{ADB is a st. line}] \]
\[ \Rightarrow 64^\circ + \angle CDE + 64^\circ = 180^\circ \]
\[ \Rightarrow \angle CDE = 180^\circ - 128^\circ \]
\[ \Rightarrow \angle CDE = 52^\circ \]

(ii)

Given \( DE \parallel BC \) and \( DC \) is the transversal.
\[ \Rightarrow \angle CDE = \angle DCB = 52^\circ \quad \ldots \ldots \ (ii) \]

Also, \[ \angle ECB = 64^\circ \quad \ldots \ldots \ [\text{From (i)}] \]

But,
\[ \angle ECB = \angle DCE + \angle DCB \]
\[ \Rightarrow 64^\circ = \angle DCE + 52^\circ \]
\[ \Rightarrow \angle DCE = 64^\circ - 52^\circ \]
\[ \Rightarrow \angle DCE = 12^\circ \]
Solution 4:

(i) Let the triangle be \(ABC\) and the altitude be \(AD\).

\[
\begin{align*}
\angle DBA &= \angle DAB = 37^\circ & [\text{Given } BD = AD \text{ and angles opposite to equal sides are equal}] \\
\text{Now,} & \\
\angle CDA &= \angle DBA + \angle DAB & [\text{Exterior angle is equal to the sum of opp. interior angles}] \\
&= 37^\circ + 37^\circ \\
&= 74^\circ \\
\Rightarrow & \\
\angle CDA &= 74^\circ \\
\text{Now in } \triangle ADC, & \\
\angle CDA &= \angle CAD = 74^\circ & [\text{Given } CD = AC \text{ and angles opposite to equal sides are equal}] \\
\text{Now,} & \\
\angle CAD + \angle CDA + \angle ACD &= 180^\circ \\
&= 74^\circ + 74^\circ + \chi = 180^\circ \\
&= \chi = 180^\circ - 148^\circ \\
&= \chi = 32^\circ
\end{align*}
\]
(ii) Let triangle be ABC and altitude be AD.

In $\triangle ABD$,

$\angle DBA = \angle DAB = 50^\circ$  \hspace{1cm} [Given $BD = AD$ and angles opposite to equal sides are equal]

Now,

$\angle CDA = \angle DBA + \angle DAB$  \hspace{1cm} [Exterior angle is equal to the sum of opp. interior angles]

$\therefore \angle CDA = 50^\circ + 50^\circ$

$\Rightarrow \angle CDA = 100^\circ$

In $\triangle ADC$,

$\angle DAC = \angle DCA = x$  \hspace{1cm} [Given $AD = DC$ and angles opposite to equal sides are equal]

$\therefore \angle DAC + \angle DCA + \angle ADC = 180^\circ$

$\Rightarrow x + x + 100^\circ = 180^\circ$

$\Rightarrow 2x = 80^\circ$

$\Rightarrow x = 40^\circ$
Solution 5:

Let \( \angle ABO = \angle OBC = x \) and \( \angle ACO = \angle OCB = y \)

In \( \triangle ABC \),
\[ \angle BAC = 180^\circ - 2x - 2y \] \((i)\)

Since \( \angle B = \angle C \) \[ AB = AC \]
\[ \frac{1}{2} B = \frac{1}{2} C \]
\[ \Rightarrow x = y \]

Now,
\[ \angle ACD = 2x + \angle BAC \] \[ \text{[Exterior angle is equal to sum of opp. interior angles]} \]
\[ = 2x + 180^\circ - 2x - 2y \] \[ \text{[From (i)]} \]
\[ \angle ACD = 180^\circ - 2y \] \((ii)\)

In \( \triangle OBC \),
\[ \angle BOC = 180^\circ - x - y \]
\[ \Rightarrow \angle BOC = 180^\circ - y - y \] \[ \text{[Already proved]} \]
\[ \Rightarrow \angle BOC = 180^\circ - 2y \] \((iii)\)

From (i) and (ii)
\[ \angle BOC = \angle ACD \]
Solution 6:

Given: \( \angle PLN = 110^\circ \)

(i) We know that the sum of the measure of all the angles of a quadrilateral is \(360^\circ\).

In quad. PQNL,

\[
\angle QPL + \angle PLN + \angle LNQ + \angle NQP = 360^\circ
\]
\[\Rightarrow 90^\circ + 110^\circ + \angle LNQ + 90^\circ = 360^\circ\]
\[\Rightarrow \angle LNQ = 360^\circ - 290^\circ\]
\[\Rightarrow \angle LNQ = 70^\circ\]
\[\Rightarrow \angle LNM = 70^\circ\] \((i)\)

In \(\triangle LMN\),

\[\text{LM} = \text{LN} \quad \text{[Given]}\]
\[\Rightarrow \angle LNM = \angle LMN \quad \text{[angles opp. to equal sides are equal]}\]
\[\Rightarrow \angle LMN = 70^\circ \] \((ii)\) \[\text{[From (i)]}\]

(ii)

In \(\triangle LMN\),

\[\angle LMN + \angle LNM + \angle MLN = 180^\circ\]

But, \(\angle LNM = \angle LMN = 70^\circ\) \[\text{[From (i) and (ii)]}\]
\[\Rightarrow 70^\circ + 70^\circ + \angle MLN = 180^\circ\]
\[\Rightarrow \angle MLN = 180^\circ - 140^\circ\]
\[\Rightarrow \angle MLN = 40^\circ\]
Solution 7:

In $\triangle ABC$,

$AC = BC$ \hspace{1cm} \text{[Given]}

$\therefore \angle CAB = \angle CBD$ \hspace{1cm} \text{[angles opp. to equal sides are equal]}

$\Rightarrow \angle CBD = 55^\circ$

In $\triangle ABC$,

$\angle CBA + \angle CAB + \angle ACB = 180^\circ$

but, $\angle CAB = \angle CBA = 55^\circ$

$\Rightarrow 55^\circ + 55^\circ + \angle ACB = 180^\circ$

$\Rightarrow \angle ACB = 180^\circ - 110^\circ$

$\Rightarrow \angle ACB = 70^\circ$

Now,

In $\triangle ACD$ and $\triangle BCD$,

$AC = BC$ \hspace{1cm} \text{[Given]}

$CD = CD$ \hspace{1cm} \text{[Common]}

$AD = BD$ \hspace{1cm} \text{[Given: CD bisects AB]}

$\therefore \triangle ACD \cong \triangle BCD$

$\Rightarrow \angle DCA = \angle DCB$

$\Rightarrow \angle DCB = \frac{\angle ACB}{2} = \frac{70^\circ}{2}$

$\Rightarrow \angle DCB = 35^\circ$
Solution 8:
Let us name the figure as following:

In \( \triangle ABC \),
AD = AC \quad \text{[Given]}
\therefore \angle ADC = \angle ACD \quad \text{[angles opp. to equal sides are equal]}
\Rightarrow \angle ADC = 42^\circ

Now,
\angle ADC = \angle DAB + \angle DBA \quad \text{[Exterior angle is equal to the sum of opp. interior angles]}

But,
\angle DAB = \angle DBA \quad \text{[Given : BD = DA]}
\therefore \angle ADC = 2\angle DBA
\Rightarrow 2\angle DBA = 42^\circ
\Rightarrow \angle DBA = 21^\circ

For x:
\[ x = \angle CBA + \angle BCA \quad \text{[Exterior angle is equal to the sum of opp. interior angles]}

We know that,
\angle CBA = 21^\circ
\angle BCA = 42^\circ
\therefore x = 21^\circ + 42^\circ
\Rightarrow x = 63^\circ
Solution 9:

In ΔABD and ΔDBC,

\[ BD = BD \quad \text{[Common]} \]
\[ \angle BDA = \angle BDC \quad \text{[each equal to } 90^\circ\text{]} \]
\[ \angle ABD = \angle DBC \quad \text{[BD bisects } \angle ABC\text{]} \]

\[ \therefore \triangle ABD \cong \triangle DBC \quad \text{[ASA criterion]} \]

Therefore,

\[ AD = DC \]

\[ x + 1 = y + 2 \]
\[ \Rightarrow x = y + 1 \ldots (i) \]

and \[ AB = BC \]

\[ 3x + 1 = 5y - 2 \]

Substituting the value of \( x \) from \((i)\)

\[ 3(y + 1) + 1 = 5y - 2 \]
\[ \Rightarrow 3y + 3 + 1 = 5y - 2 \]
\[ \Rightarrow 3y + 4 = 5y - 2 \]
\[ \Rightarrow 2y = 6 \]
\[ \Rightarrow y = 3 \]

Putting \( y = 3 \) in \((i)\)

\[ x = 3 + 1 \]
\[ \therefore x = 4 \]
Solution 10:
Let P and Q be the points as shown below:

Given: \( \angle PDQ = 58^\circ \)

\[ \angle PDQ = \angle EDC = 58^\circ \quad [\text{Vertically opp. angles}] \]

\[ \angle EDC = \angle ACB = 58^\circ \quad [\text{Corresponding angles: } AC \parallel ED] \]

In \( \triangle ABC \),
\( AB = AC \quad [\text{Given}] \)
\[ \therefore \angle ACB = \angle ABC = 58^\circ \quad [\text{angles opp. to equal sides are equal}] \]

Now,
\[ \angle ACB + \angle ABC + \angle BAC = 180^\circ \]
\[ \Rightarrow 58^\circ + 58^\circ + a = 180^\circ \]
\[ \Rightarrow a = 180^\circ - 116^\circ \]
\[ \Rightarrow a = 64^\circ \]

Since \( AE \parallel BD \) and \( AC \) is the transversal
\[ \angle ABC = b \quad [\text{Corresponding angles}] \]
\[ \therefore b = 58^\circ \]

Also since \( AE \parallel BD \) and \( ED \) is the transversal
\[ \angle EDC = c \quad [\text{Corresponding angles}] \]
\[ \therefore c = 58^\circ \]
Solution 11:

In $\triangle ACD$,

$AC = CD \quad \text{[Given]}$

$\therefore \angle CAD = \angle CDA$

$\angle ACD = 58^\circ \quad \text{[Given]}$

$\angle ACD + \angle CDA + \angle CAD = 180^\circ$

$\Rightarrow 58^\circ + 2\angle CAD = 180^\circ$

$\Rightarrow 2\angle CAD = 122^\circ$

$\Rightarrow \angle CAD = \angle CDA = 61^\circ \ldots \ldots \text{(i)}$

Now,

$\angle CDA = \angle DAB + \angle DBA \quad \text{[Ext. angle is equal to sum of opp. int. angles]}$

But,

$\angle DAB = \angle DBA \quad \text{[Given : AD = DB]}$

$\therefore \angle DAB + \angle DAB = \angle CDA$

$\Rightarrow 2\angle DAB = 61^\circ$

$\Rightarrow \angle DAB = 30.5^\circ \ldots \ldots \text{(ii)}$

In $\triangle ABC$,

$\angle CAB = \angle CAD + \angle DAB$

$\therefore \angle CAB = 61^\circ + 30.5^\circ$

$\Rightarrow \angle AB = 91.5^\circ$

Solution 12:

In $\triangle ACD$,

$AC = AD = CD \quad \text{[Given]}$

Hence, $ACD$ is an equilateral triangle

$\therefore \angle ACD = \angle CDA = \angle CAD = 60^\circ$

$\angle CDA = \angle DAB + \angle ABD \quad \text{[Ext. angle is equal to sum of opp. int. angles]}$

But,

$\angle DAB = \angle ABD \quad \text{[Given : AD = DB]}$

$\therefore \angle ABD + \angle ABD = \angle CDA$

$\Rightarrow 2\angle ABD = 60^\circ$

$\Rightarrow \angle ABD = \angle ABC = 30^\circ$
Solution 13:

Let $\angle A = 8x$ and $\angle B = 5x$

Given: $AB = AC$

$\Rightarrow \angle B = \angle C = 5x$ [Angles opp. to equal sides are equal]

Now,

$\angle A + \angle B + \angle C = 180^\circ$
$\Rightarrow 8x + 5x + 5x = 180^\circ$
$\Rightarrow 18x = 180^\circ$
$\Rightarrow x = 10^\circ$

Given that:

$\angle A = 8x$

$\Rightarrow \angle A = 8 \times 10^\circ$
$\Rightarrow \angle A = 80^\circ$
Solution 14:

In \( \triangle ABC \),
\( \angle A = 60^\circ \)
\( \angle C = 40^\circ \)
\[ \therefore \angle B = 180^\circ - 60^\circ - 40^\circ \]
\[ \Rightarrow \angle B = 80^\circ \]

Now,
BP is the bisector of \( \angle ABC \)

\[ \therefore \angle PBC = \frac{\angle ABC}{2} \]
\[ \Rightarrow \angle PBC = 40^\circ \]

In \( \triangle PBC \)
\( \angle PBC = \angle PCB = 40^\circ \)
\[ \therefore BP = CP \quad [\text{Sides opp. to equal angles are equal}] \]
Solution 15:

Let $\angle PBC = \angle PCB = x$

In the right angled triangle $ABC$,

$\angle ABC = 90^\circ$
$\angle ACB = \angle PBC = x$
$\Rightarrow \angle BAC = 180^\circ - (90^\circ + x)$
$\Rightarrow \angle BAC = (90^\circ - x). \quad \text{(i)}$

And

$\angle ABP = \angle ABC - \angle PBC$
$\Rightarrow \angle ABP = 90^\circ - x. \quad \text{(ii)}$

Therefore in the triangle $ABP$;

$\angle BAP = \angle ABP$

Hence,

$PA = PB \text{ [sides opp. to equal angles are equal]}$
Solution 16:

\( \triangle ABC \) is an equilateral triangle

\[ \Rightarrow \text{Side } AB = \text{Side } AC \]

\[ \Rightarrow \angle ABC = \angle ACB \quad \text{[If two sides of a triangle are equal, then angles opposite to them are equal]} \]

Similarly, Side AC = Side BC

\[ \Rightarrow \angle CAB = \angle ABC \quad \text{[If two sides of a triangle are equal, then angles opposite to them are equal]} \]

Hence, \( \angle ABC = \angle CAB = \angle ACB = y \) (say)

As the sum of all the angles of the triangle is \( 180^\circ \)

\[ \angle ABC + \angle CAB + \angle ACB = 180^\circ \]

\[ \Rightarrow 3y = 180^\circ \]

\[ \Rightarrow y = 60^\circ \]

\[ \angle ABC = \angle CAB = \angle ACB = 60^\circ \]

Sum of two non-adjacent interior angles of a triangle is equal to the exterior angle.

\[ \Rightarrow \angle CAB + \angle CBA = \angle ACE \]

\[ \Rightarrow 60^\circ + 60^\circ = \angle ACE \]

\[ \Rightarrow \angle ACE = 120^\circ \]

Now \( \triangle ACE \) is an isosceles triangle with \( AC = CF \)

\[ \Rightarrow \angle EAC = \angle AEC \]

Sum of all the angles of a triangle is \( 180^\circ \)

\[ \angle EAC + \angle AEC + \angle ACE = 180^\circ \]

\[ \Rightarrow 2\angle AEC + 120^\circ = 180^\circ \]

\[ \Rightarrow 2\angle AEC = 180^\circ - 120^\circ \]

\[ \Rightarrow \angle AEC = 30^\circ \]
Solution 17:

\( \triangle DBC \) is an isosceles triangle 
As \( \text{Side } CD = \text{Side } DB \)
\[ \Rightarrow \angle DBC = \angle DCB \quad [\text{If two sides of a triangle are equal, then angles opposite to them are equal}] \]

And \( \angle B = \angle DBC = \angle DCB = 28^\circ \)
As the sum of all the angles of the triangle is 180°
\[ \angle DCB + \angle DBC + \angle BCD = 180^\circ \]
\[ \Rightarrow 28^\circ + 28^\circ + \angle BCD = 180^\circ \]
\[ \Rightarrow \angle BCD = 180^\circ - 56^\circ \]
\[ \Rightarrow \angle BCD = 124^\circ \]

Sum of two non-adjacent interior angles of a triangle is equal to the exterior angle.
\[ \Rightarrow \angle DBC + \angle DCB = \angle DAC \]
\[ \Rightarrow 28^\circ + 28^\circ = 56^\circ \]
\[ \Rightarrow \angle DAC = 56^\circ \]

Now \( \triangle ACD \) is an isosceles triangle with \( AC = DC \)
\[ \Rightarrow \angle ADC = \angle DAC = 56^\circ \]

Sum of all the angles of a triangle is 180°
\[ \angle ADC + \angle DAC + \angle DCA = 180^\circ \]
\[ \Rightarrow 56^\circ + 56^\circ + \angle DCA = 180^\circ \]
\[ \Rightarrow \angle DCA = 180^\circ - 112^\circ \]
\[ \Rightarrow \angle DCA = 64^\circ = \angle ACD \]
Solution 18:

We can see that the \( \triangle ABC \) is an isosceles triangle with Side \( AB = AC \).
\[ \Rightarrow \angle ACB = \angle ABC \]

As \( \angle ACB = 65^\circ \)

hence \( \angle ABC = 65^\circ \)

Sum of all the angles of a triangle is \( 180^\circ \)

\[ \angle ACB + \angle CAB + \angle ABC = 180^\circ \]

\[ 65^\circ + 65^\circ + \angle CAB = 180^\circ \]

\[ \angle CAB = 180^\circ - 130^\circ \]

\[ \angle CAB = 50^\circ \]

As \( BD \) is parallel to \( CA \)

Therefore, \( \angle CAB = \angle DBA \) since they are alternate angles.

\[ \angle CAB = \angle DBA = 50^\circ \]

We see that \( \triangle ADB \) is an isosceles triangle with Side \( AD = AB \).
\[ \Rightarrow \angle ADB = \angle DBA = 50^\circ \]

Sum of all the angles of a triangle is \( 180^\circ \)

\[ \angle ADB + \angle DAB + \angle DBA = 180^\circ \]

\[ 50^\circ + \angle DAB + 50^\circ = 180^\circ \]

\[ \angle DAB = 180^\circ - 100^\circ = 80^\circ \]

\[ \angle DAB = 80^\circ \]

The angle \( DAC \) is sum of angle \( DAB \) and \( CAB \).

\[ \angle DAC = \angle CAB + \angle DAB \]

\[ \angle DAC = 50^\circ + 80^\circ \]

\[ \angle DAC = 130^\circ \]

Exercise 10(B)
Solution 1:

Const: AB is produced to D and AC is produced to E so that exterior angles $\angle DBC$ and $\angle ECB$ is formed.

In $\triangle ABC$,

\[ AB = AC \quad \text{[Given]} \]

\[ \therefore \angle C = \angle B \ldots \text{(i)} \quad \text{[angles opp. to equal sides are equal]} \]

Since angle B and angle C are acute they cannot be right angles or obtuse angles.

\[ \angle ABC + \angle DBC = 180^\circ \quad \text{[ABD is a st. line]} \]

\[ \Rightarrow \angle DBC = 180^\circ - \angle ABC \]

\[ \Rightarrow \angle DBC = 180^\circ - \angle B \ldots \text{(ii)} \]

Similarly,

\[ \angle ACB + \angle ECB = 180^\circ \quad \text{[ABD is a st. line]} \]

\[ \Rightarrow \angle ECB = 180^\circ - \angle ACB \]

\[ \Rightarrow \angle ECB = 180^\circ - \angle C \ldots \text{(iii)} \]

\[ \Rightarrow \angle ECB = 180^\circ - \angle B \ldots \text{(iv)} \quad \text{[from (i) and (iii)]} \]

\[ \Rightarrow \angle DBC = \angle ECB \quad \text{[from (ii) and (iv)]} \]
Now,
\[ \angle DBC = 180^\circ - \angle B \]
But \( \angle B = \text{Acute angle} \)
\[ :. \angle DBC = 180^\circ - \text{Acute angle} = \text{obtuse angle} \]

Similarly,
\[ \angle ECB = 180^\circ - \angle C. \]
But \( \angle C = \text{Acute angle} \)
\[ :. \angle ECB = 180^\circ - \text{Acute angle} = \text{obtuse angle} \]

Therefore, exterior angles formed are obtuse and equal.
Solution 2:

Const: Join AD.

In ΔABC,
AB = AC  \[\text{[Given]}\]
∴ ∠C = ∠B, \ldots.(i)  \[\text{[angles opp. to equal sides are equal]}\]

(i)

In ΔBPD and ΔCQD,
∠BPD = ∠CQD  \[\text{[Each = 90°]}\]
∠B = ∠C  \[\text{[proved]}\]
BD = DC  \[\text{[Given]}\]
∴ ΔBPD ≅ ΔCQD  \[\text{[AAS criterion]}\]
∴ DP = DQ  \[\text{[cpct]}\]

(ii) We have already proved that ΔBPD ≅ ΔCQD

Therefore, BP = CQ [cpct]

Now,
AB = AC [Given]
⇒ AB - BP = AC - CQ
⇒ AP = AQ
(iii)

In $\triangle APD$ and $\triangle AQD$,

$DP = DQ$ [proved]

$AD = AD$ [common]

$AP = AQ$ [Proved]

$\therefore \triangle APD \cong \triangle AQD$ [SSS]

$\Rightarrow \angle PAD = \angle QAD$ [cpct]

Hence, $AD$ bisects angle $A$.

**Solution 3:**

(i)

In $\triangle AEB$ and $\triangle AFC$,

$\angle A = \angle A$ [Common]

$\angle AEB = \angle AFC = 90^\circ$ [Given: $BE \perp AC$]

$\therefore \angle AEB \cong \angle AFC$ [AAS]

$\Rightarrow \triangle AEB \cong \triangle AFC$ [AAS]

$\therefore BE = CF$ [cpct]

(ii) Since $\triangle AEB \cong \triangle AFC$

$\angle ABE = \angle AFC$

$\therefore AF = AE$ [congruent angles of congruent triangles]
Solution 4:

Const: Join CD.

In \( \triangle ABC \),
\[
AB = AC \quad \text{[Given]}
\]
\[
\therefore \angle C = \angle B \quad \text{(i)} \quad \text{[angles opp. to equal sides are equal]}
\]

In \( \triangle ACD \),
\[
AC = AD \quad \text{[Given]}
\]
\[
\therefore \angle ADC = \angle ACD \quad \text{(ii)}
\]

Adding (i) and (ii)
\[
\angle B + \angle ADC = \angle C + \angle ACD
\]
\[
\angle B + \angle ADC = \angle BCD \quad \text{(iii)}
\]

In \( \triangle BCD \),
\[
\angle B + \angle ADC + \angle BCD = 180^\circ
\]
\[
\angle BCD + \angle BCD = 180^\circ \quad \text{[From (iii)]}
\]
\[
2\angle BCD = 180^\circ
\]
\[
\angle BCD = 90^\circ
\]
Solution 5:

\[ \angle A = 36^\circ \]
\[ \angle B = \angle C = \frac{180^\circ - 36^\circ}{2} = 72^\circ \]
\[ \angle ACD = \angle BCD = 36^\circ \text{ [\: CD is the angle bisector of } \angle C\text{]} \]
\[ \triangle ADC \text{ is an isosceles triangle since } \angle DAC = \angle DCA = 36^\circ \]
\[ \therefore AD = CD \text{......(i)} \]

In \( \triangle DCB \),
\[ \angle CDB = 180^\circ - (\angle DCB + \angle DBC) \]
\[ = 180^\circ - (36^\circ + 72^\circ) \]
\[ = 180^\circ - 108^\circ \]
\[ = 72^\circ \]
\[ \triangle DCB \text{ is an isosceles triangle since } \angle CDB = \angle CBD = 72^\circ \]
\[ \therefore DC = BC \text{......(ii)} \]

From (i) and (ii), we get
\[ AD = BC \]
Hence proved
Solution 6:

In $\triangle ABC$,

$AB = AC$ \hspace{1cm} [\text{Given}]

$\therefore \angle C = \angle B$ \hspace{1cm} [\text{angles opp. to equal sides are equal}]

$\Rightarrow \frac{1}{2} \angle C = \frac{1}{2} \angle B$

$\Rightarrow \angle BCF = \angle CBE$ \hspace{1cm} (ii)

In $\triangle BCE$ and $\triangle CBF$,

$\angle C = \angle B$ \hspace{1cm} [\text{From (i)}]

$\angle BCF = \angle CBE$ \hspace{1cm} [\text{From (ii)}]

$BC = BC$ \hspace{1cm} [\text{Common}]

$\therefore \triangle BCE \cong \triangle CBF$ \hspace{1cm} [\text{AAS}]

$\Rightarrow BE = CF$ \hspace{1cm} [\text{cpct}]
Solution 7:

In ΔABC,
AB = AC      \[\text{[Given]}\]
\Rightarrow \angle ACB = \angle ABC \quad \text{[angles opp. to equal sides are equal]}
\Rightarrow \angle ABC = \angle ACB \quad \text{[i]}

\angle DBC = \angle ECB = 90^\circ \text{[Given]}
\Rightarrow \angle DBC = \angle ECB \quad \text{[ii]}

Subtracting (i) from (ii)
\angle DCB - \angle ABC = \angle ECB - \angle ACB
\Rightarrow \angle DBA = \angle ECA \quad \text{[iii]}

In ΔDBA and ΔECA,
\angle DBA = \angle ECA \quad \text{[From (iii)]}
\angle DAB = \angle EAC \quad \text{[Vertically opposite angles]}
AB = AC \quad \text{[Given]}
\Rightarrow \angle DBA \cong \angle ECA \quad \text{[ASA]}
\Rightarrow BD = CE \quad \text{[cpct]}

Also,
AD = AE \quad \text{[cpct]}
Solution 8:

![Diagram of triangle ABC with additional line segments]

DA is produced to meet BC in L.

In $\triangle ABC$,

$AB = AC$ \quad \text{[Given]}

$\therefore \angle ACB = \angle ABC$ \quad \text{(i)} \quad \text{[angles opposite to equal sides are equal]}

In $\triangle DBC$,

$DB = DC$ \quad \text{[Given]}

$\therefore \angle DCB = \angle DBC$ \quad \text{(ii)} \quad \text{[angles opposite to equal sides are equal]}

Subtracting (i) from (ii)

$\angle DCB - \angle ACB = \angle DBC - \angle ABC$

$\Rightarrow \angle DCA = \angle DBA$ \quad \text{(iii)}

In $\triangle DBA$ and $\triangle DCA$,

$DB = DC$ \quad \text{[Given]}

$\angle DBA = \angle DCA$ \quad \text{[From (iii)]}

$AB = AC$ \quad \text{[Given]}

$\therefore \triangle DBA \cong \triangle DCA$ \quad \text{[SAS]}

$\Rightarrow \angle BDA = \angle CDA$ \quad \text{(iv)} \quad \text{[cpct]}
In $\triangle DBA$,
$\angle BAL = \angle DBA + \angle BDA.........(v)$

[Ext. angle = sum of opp. int. angles]

From (iii), (iv) and (v)
$\angle BAL = \angle DCA + \angle CDA......(vi)$

In $\triangle DCA$,
$\angle CAL = \angle DCA + \angle CDA.........(vii)$

[Ext. angle = sum of opp. int. angles]

From (vi) and (vii)
$\angle BAL = \angle CAL........(viii)$

In $\triangle BAL$ and $\triangle CAL$,
$\angle BAL = \angle CAL$ [From (viii)]
$\angle ABL = \angle ACL$ [From (i)]
$AB = AC$ [Given]
\therefore $\triangle BAL \cong \triangle CAL$ [ASA]
$\Rightarrow \angle ALB = \angle ALC$ [cpct]
and $BL = LC.........(ix)$ [cpct]

Now,
$\angle ALB + \angle ALC = 180^\circ$
$\Rightarrow \angle ALB + \angle ALB = 180^\circ$
$\Rightarrow 2\angle ALB = 180^\circ$
$\Rightarrow \angle ALB = 90^\circ$
\therefore $AL \perp BC$

or $DL \perp BC$ and $BL = LC$
\therefore $DA$ produced bisects BC at right angle.
Solution 9:

In $\triangle ABC$, we have $AB = AC$

$\Rightarrow \angle B = \angle C$ [angles opposite to equal sides are equal]

$\Rightarrow \frac{1}{2} \angle B = \frac{1}{2} \angle C$

$\Rightarrow \angle OBC = \angle OCB.$ ...........(i)

$\Rightarrow OB = OC.$ ............(ii) [angles opposite to equal sides are equal]

Now,

In $\triangle ABO$ and $\triangle ACO$,

$AB = AC$ [Given]

$\angle OBC = \angle OCB$ [From (i)]

$OB = OC$ [From (ii)]

$\triangle ABO \cong \triangle ACO$ [SAS criterion]

$\Rightarrow \angle BAO = \angle CAO$ [cpct]

Therefore, AO bisects $\angle BAC$. 
Solution 10:

In $\triangle ABC$,
\[ AB = AC \quad \text{[Given]} \]
\[ \therefore \angle C = \angle B \quad \text{[angles opp. to equal sides are equal]} \]

\[ \Rightarrow \frac{1}{2} AB = \frac{1}{2} AC \]
\[ \Rightarrow BF = CE \quad \text{[ii]} \]

\[ \Rightarrow \frac{1}{2} AB = \frac{1}{2} AC \]
\[ \Rightarrow BF = CE \quad \text{[ii]} \]

In $\triangle BCE$ and $\triangle CBF$,
\[ \angle C = \angle B \quad \text{[From (i)]} \]
\[ BF = CE \quad \text{[From (ii)]} \]
\[ BC = BC \quad \text{[Common]} \]
\[ \therefore \triangle BCE \cong \triangle CBF \quad \text{[SAS]} \]
\[ \Rightarrow BE = CF \quad \text{[cpct]} \]
Solution 11:

In $\triangle APQ$,

$AP = AQ$ \quad [\text{Given}]

$\therefore \angle APQ = \angle AQP$ \quad (i)

[angles opposite to equal sides are equal]

In $\triangle ABP$,

$\angle APQ = \angle BAP + \angle ABP$ \quad (ii)

[Ext. angle is equal to sum of opp. int. angles]

In $\triangle AQC$,

$\angle AQP = \angle CAQ + \angle ACQ$ \quad (iii)

[Ext. angle is equal to sum of opp. int. angles]

From (i), (ii) and (iii)

$\angle BAP + \angle ABP = \angle CAQ + \angle ACQ$

But, $\angle BAP = \angle CAQ$ \quad [\text{Given}]

$\Rightarrow \angle CAQ + \angle ABP = \angle CAQ + \angle ACQ$

$\Rightarrow \angle ABP = \angle ACQ$

$\Rightarrow B = C$ \quad (iv)

In $\triangle ABC$,

$\angle B = \angle C$

$\Rightarrow AB = AC$ \quad [\text{Sides opposite to equal angles are equal}]

Solution 12:

Since $AE \parallel BC$ and $DAB$ is the transversal

$\therefore \angle DAE = \angle ABC = \angle B$ \quad [\text{Corresponding angles}]

Since $AE \parallel BC$ and $AC$ is the transversal

$\angle CAE = \angle ACB = \angle C$ \quad [\text{Alternate Angles}]

But $AE$ bisects $\angle CAD$

$\therefore \angle DAE = \angle CAE$

$\Rightarrow \angle B = \angle C$

$\Rightarrow AB = AC$ [Sides opposite to equal angles are equal]
Solution 13:

\[ \triangle ABC \text{ is equilateral} \quad \text{(i) [Given]} \]

\[ AP = BQ = CR \quad \text{(ii) [Given]} \]

Subtracting (ii) from (i)

\[ AB - AP = BC - BQ = CA - CR \]

\[ BP = CQ = AR \quad \text{(iii)} \]

\[ \therefore \angle A = \angle B = \angle C \quad \text{(iv) [angles opp. to equal sides are equal]} \]

In \( \triangle BPQ \) and \( \triangle CQR \),

\[ BP = CQ \quad \text{[From (iii)]} \]

\[ \angle B = \angle C \quad \text{[From (iv)]} \]

\[ BQ = CR \quad \text{[Given]} \]

\[ \therefore \triangle BPQ \cong \triangle CQR \quad \text{[SAS criterion]} \]

\[ \Rightarrow PQ = QR \quad \text{(v)} \]

In \( \triangle CQR \) and \( \triangle APR \),

\[ CQ = AR \quad \text{[From (iii)]} \]

\[ \angle C = \angle A \quad \text{[From (iv)]} \]

\[ CR = AP \quad \text{[Given]} \]

\[ \therefore \triangle CQR \cong \triangle APR \quad \text{[SAS criterion]} \]

\[ \Rightarrow QR = PR \quad \text{(vi)} \]

From (v) and (vi)

\[ PQ = QR = PR \]

Therefore, \( \triangle PQR \) is an equilateral triangle.
Solution 14:

In \(\triangle ABE\) and \(\triangle ACF\),

\(\angle A = \angle A\) [Common]

\(\angle AEB = \angle AFC = 90^\circ\) [Given: BE \(\perp\) AC; CF \(\perp\) AB]

BE = CF [Given]

\(\therefore \triangle ABE \cong \triangle ACF\) [AAS criterion]

\(\Rightarrow AB = AC\)

Therefore, \(\triangle ABC\) is an isosceles triangle.
Solution 15:

AL is bisector of angle A. Let D is any point on AL. From D, a straight line DE is drawn parallel to AC.

DE || AC [Given]

\[ \angle ADE = \angle DAC \ldots (i) \] [Alternate angles]

\[ \angle DAC = \angle DAE \ldots (ii) \] [AL is bisector of \( \angle A \)]

From (i) and (ii)

\[ \angle ADE = \angle DAE \]

\[ \therefore AE = ED \] [Sides opposite to equal angles are equal]

Therefore, AED is an isosceles triangle.
Solution 16:

(i)

In \( \triangle ABC \),

\[ AB = AC \]

\[ \Rightarrow \frac{1}{2} AB = \frac{1}{2} AC \]

\[ \Rightarrow AP = AQ \ldots (i) \] [Since P and Q are mid-points]

In \( \triangle BCA \),

\[ PR = \frac{1}{2} AC \] [PR is line joining the mid-points of AB and BC]

\[ \Rightarrow PR = AQ \ldots (ii) \]

In \( \triangle CAB \),

\[ QR = \frac{1}{2} AB \] [QR is line joining the mid-points of AC and BC]

\[ \Rightarrow QR = AP \ldots (iii) \]

From (i), (ii) and (iii)

\[ PR = QR \]
AB = AC

⇒ ∠B = ∠C

Also,

\[ \frac{1}{2} AB = \frac{1}{2} AC \]

⇒ BP = CQ  \[\text{[P and Q are mid-points of AB and AC]}\]

In \( \triangle BPC \) and \( \triangle CQB \),

BP = CQ

∠B = ∠C

BC = BC

Therefore, \( \triangle BPC \cong \triangle CQB \) [SAS]

BP = CP
Solution 17:

(i) In $\triangle ACB$,

$AC = AC$ [Given]

$\therefore \angle ABC = \angle ACB$ .... (i) [angles opposite to equal sides are equal]

$\angle ACD + \angle ACB = 180^\circ$ .... (ii) [DCB is a straight line]

$\angle ABC + \angle CBE = 180^\circ$ .... (iii) [ABE is a straight line]

Equating (ii) and (iii)

$\angle ACD + \angle ACB = \angle ABC + \angle CBE$

$\Rightarrow \angle ACD + \angle ACB = \angle ACB + \angle CBE$ [From (i)]

$\Rightarrow \angle ACD = \angle CBE$

(ii)

In $\triangle ACD$ and $\triangle CBE$,

$DC = CB$ [Given]

$AC = BE$ [Given]

$\angle ACD = \angle CBE$ [Proved Earlier]

$\therefore \triangle ACD \cong \triangle CBE$ [SAS criterion]

$\Rightarrow AD = CE$ [CPCT]

Solution 18:

AB is produced to E and AC is produced to F. BD is bisector of angle CBE and CD is

bisector of angle BCF. BD and CD meet at D.
In $\triangle ABC$,

$AB = AC$ [Given]

$\therefore \angle C = \angle B$ [angles opposite to equal sides are equal]

$\angle CBE = 180^\circ - \angle B$ [ABE is a straight line]

$\Rightarrow \angle CBD = \frac{180^\circ - \angle B}{2}$ [BD is bisector of $\angle CBE$]

$\Rightarrow \angle CBD = 90^\circ - \frac{\angle B}{2}$ .............(i)

Similarly,

$\angle BCF = 180^\circ - \angle C$ [ACF is a straight line]

$\Rightarrow \angle BCD = \frac{180^\circ - \angle C}{2}$ [CD is bisector of $\angle BCF$]

$\Rightarrow \angle BCD = 90^\circ - \frac{\angle C}{2}$ .............(ii)

Now,

$\Rightarrow \angle CBD = 90^\circ - \frac{\angle C}{2}$ [\therefore \angle B = \angle C]

$\Rightarrow \angle CBD = \angle BCD$

In $\triangle BCD$,

$\angle CBD = \angle BCD$

$\therefore BD = CD$

In $\triangle ABD$ and $\triangle ACD$,

$AB = AC$ [Given]

$AD = AD$ [Common]

$BD = CD$ [Proved]

$\therefore \triangle ABD \cong \triangle ACD$ [SSS criterion]

$\Rightarrow \angle BAD = \angle CAD$ [cpct]

Therefore, AD bisects $\angle A$. 
Solution 19:

In $\triangle ABC$,

CX is the angle bisector of $\angle C$

$\Rightarrow \angle ACY = \angle BCX$ ...... (i)

In $\triangle AXY$,

$AX = AY$ [Given]

$\angle AXY = \angle AYX$ ...... (ii) [angles opposite to equal sides are equal]

Now $\angle XYC = \angle AXB = 180^\circ$ [straight line]

$\Rightarrow \angle AYX + \angle AYC = \angle AXY + \angle BXY$

$\Rightarrow \angle AYC = \angle BXY$ ....... (iii) [From (ii)]

In $\triangle AYC$ and $\triangle BXC$

$\angle AYC + \angle ACY + \angle CAY = \angle BXC + \angle BCX + \angle XBC = 180^\circ$

$\Rightarrow \angle CAY = \angle XBC$ [From (i) and (iii)]

$\Rightarrow \angle CAY = \angle ABC$
Solution 20:

Since IA \parallel CP and CA is a transversal

\[ \therefore \angle CAI = \angle PCA \text{ [Alternate angles]} \]

Also, IA \parallel CP and AP is a transversal

\[ \therefore \angle IAB = \angle APC \text{ [Corresponding angles]} \]

But \[ \therefore \angle CAI = \angle IAB \text{ [Given]} \]

\[ \therefore \angle PCA = \angle APC \]

\[ \Rightarrow AC = AP \]

Similarly,

BC = BQ

Now,

PQ = AP + AB + BQ

= AC + AB + BC

= Perimeter of \( \triangle ABC \)
Solution 21:

In $\triangle ABD$,

$\angle BAE = \angle 3 + \angle ADB$

$\Rightarrow 108^0 = \angle 3 + \angle ADB$

But $AB = AC$

$\Rightarrow \angle 3 = \angle 2$

$\Rightarrow 108^0 = \angle 2 + \angle ADB$. $(i)$

Now,

In $\triangle ACD$,

$\angle 2 = \angle 1 + \angle ADB$

But $AC = CD$

$\Rightarrow \angle 1 = \angle ADB$

$\Rightarrow \angle 2 = \angle ADB + \angle ADB$

$\Rightarrow \angle 2 = 2\angle ADB$

Putting this value in $(i)$

$\Rightarrow 108^0 = 2\angle ADB + \angle ADB$

$\Rightarrow 3\angle ADB = 108^0$

$\Rightarrow \angle ADB = 36^0$
Solution 22:

ABC is an equilateral triangle.
Therefore, \( AB = BC = AC = 15 \text{ cm} \)
\( \angle A = \angle B = \angle C = 60^\circ \)
In \( \triangle ADE \), \( DE \parallel BC \) [Given]
\( \angle AED = 60^\circ \) [\( \because \angle ACB = 60^\circ \)]
\( \angle ADE = 60^\circ \) [\( \because \angle ABC = 60^\circ \)]
\( \angle DAE = 180^\circ - (60^\circ + 60^\circ) = 60^\circ \)
Similarly, \( \triangle BDF \) & \( \triangle GEC \) are equilateral triangles.
\( = 60^\circ \) [\( \because \angle C = 60^\circ \)]
Let \( AD = x \), \( AE = x \), \( DE = x \) [\( \because \triangle ADE \) is an equilateral triangle]
Let \( BD = y \), \( FD = y \), \( FB = y \) [\( \because \triangle BDF \) is an equilateral triangle]
Let \( EC = z \), \( GC = z \), \( GE = z \) [\( \because \triangle GEC \) is an equilateral triangle]
Now, \( AD + DB = 15 \Rightarrow x + y = 15 \ldots (i) \)
\( AE + EC = 15 \Rightarrow x + z = 15 \ldots (ii) \)
Given, \( DE + DF + EG = 20 \)
\( \Rightarrow x + y + z = 20 \)
\( \Rightarrow 15 + z = 20 \) [from (i)]
\( \Rightarrow z = 5 \)
From (ii), we get \( x = 10 \)
\( \therefore y = 5 \)
Also, \( BC = 15 \)
\( BF + FG + GC = 15 \)
\( \Rightarrow y + FG + z = 15 \)
\( \Rightarrow 5 + FG + 5 = 15 \)
\( \Rightarrow FG = 5 \)
Solution 23:

In right $\triangle BEC$ and $\triangle BFC$,

$BE = CF$ [Given]

$BC = BC$ [Common]

$\angle BEC = \angle BFC$ [each $= 90^\circ$]

$\therefore \triangle BEC \cong \triangle BFC$ [RHS]

$\Rightarrow \angle B = \angle C$

Similarly,

$\angle A = \angle B$

Hence, $\angle A = \angle B = \angle C$

$\Rightarrow AB = BC = AC$

Therefore, $ABC$ is an equilateral triangle.
Solution 24:

DA $\parallel$ CE [Given]

$\Rightarrow \angle 1 = \angle 4$ \text{........(i)} [Corresponding angles]

$\angle 2 = \angle 3$ \text{........(ii)} [Alternate angles]

But $\angle 1 = \angle 2$ \text{........(iii)} [AD is the bisector of $\angle A$]

From (i), (ii) and (iii)

$\angle 3 = \angle 4$

$\Rightarrow AC = AE$

$\Rightarrow \triangle ACE$ is an isosceles triangle.
Solution 25:

Produce AD upto E such that AD = DE.

In \( \triangle ABD \) and \( \triangle EDC \),

\[
AD = DE \quad \text{[by construction]}
\]

BD = CD \quad \text{[Given]}

\( \angle 1 = \angle 2 \) \quad \text{[vertically opposite angles]}

\[\because \triangle ABD \cong \triangle EDC \quad \text{[SAS]}\]

\[\Rightarrow AB = CE \quad \text{(i)}\]

and \( \angle BAD = \angle CED \)

But, \( \angle BAD = \angle CAD \) \quad \text{[AD is bisector of } \angle BAC]\)

\[\therefore \angle CED = \angle CAD\]

\[\Rightarrow AC = CE \quad \text{(ii)}\]

From (i) and (ii)

\[AB = AC\]

Hence, \( \triangle ABC \) is an isosceles triangle.
Solution 26:

Since AB = AD = BD

\[ \triangle ABD \] is an equilateral triangle.

\[ \therefore \angle ADB = 60^\circ \]

\[ \Rightarrow \angle ADC = 180^\circ - \angle ADB \]

\[ = 180^\circ - 60^\circ \]

\[ = 120^\circ \]

Again in \( \triangle ADC \),

\[ AD = DC \]

\[ \therefore \angle 1 = \angle 2 \]

But,

\[ \angle 1 + \angle 2 + \angle ADC = 180^\circ \]

\[ \Rightarrow 2\angle 1 + 120^\circ = 180^\circ \]

\[ \Rightarrow 2\angle 1 = 60^\circ \]

\[ \Rightarrow \angle 1 = 30^\circ \]

\[ \Rightarrow \angle C = 30^\circ \]

\[ \therefore \angle ADC : \angle C = 120^\circ : 30^\circ \]

\[ \Rightarrow \angle ADC : \angle C = 4 : 1 \]

Solution 27:

(i)

In \( \triangle CAE \), \( \angle CAE = \angle AEC = \frac{180^\circ - 68^\circ}{2} = 56^\circ \) \( \therefore CE = AC \)

In \( \angle BEA \), \( a = 180^\circ - 56^\circ = 124^\circ \)

In \( \triangle ABE \), \( \angle ABE = 180^\circ - (a + \angle BAE) \)

\[ = 180^\circ - (124^\circ + 14^\circ) \]

\[ = 180^\circ - 138^\circ = 42^\circ \]
In $\triangle AEB \ & \triangle CAD$,

$\angle EAB = \angle CAD$ [Given]

$\angle ADC = \angle AEB$ [∴ $\angle ADE = \angle AED$ \{AE = AD\}]

$180^\circ - \angle ADE = 180^\circ - \angle AED$

$\angle ADC = \angle AEB$ [Given]

∴ $\triangle AEB \cong \triangle CAD$ [ASA]

AC = AB [By C.P.C.T.]

$2a + 2 = 7b - 1$

$\Rightarrow 2a - 7b = -3 .... (i)$

CD = EB

$\Rightarrow a = 3b .... (ii)$

Solving (i) & (ii), we get

$a = 9, b = 3$