Chapter 2

Complex Numbers

Ex 2.1

Simplify the following:

Question 1.

i1947 + i1950

Solution:

$$i^{1947} + i^{1950} = i^{4(486)+3} + i^{4(487)+2} = i^3 + i^2 = -i - 1$$

Question 2.

i1948 – i-1869

Solution:

$$i^{1948} - i^{-1869} = i^{4(487)} - i^{-(4(467)+1)} = 1 - i^{-4(487)} \cdot i^{-1} = 1 - (i^4)^{-(487)} \cdot \frac{1}{i}$$

$$= 1 - \frac{1 \times +i}{i \times +i} = 1 - \frac{i}{-1} = 1 + i$$

Question3.

∑12n=1in

Solution:

$$\sum_{n=1}^{12} i^n = (i^1 + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + (i^9 + i^{10} + i^{11} + i^{12}) = 0$$

Question 4.

i59+1i59

Solution:

$$i^{59} + \frac{1}{i^{59}} = i^{4(14)+3} + \frac{i}{i^{4(14)+3}} = i^3 + \frac{1}{i^3} = -i + \frac{1 \times i}{-i \times i} = -i + i = 0$$

Question 5.

i i2 i3 …….. i2000

Solution:

$$i\, i^2\, i^3 \ldots i^{2000} = i^{(1+2+3+\ldots+2000)} = i^{\frac{2000(2001)}{2}} = \left(i^{1000}\right)^{2001} = 1$$

Question 6.

∑10n=1in+50

Solution:

$$\sum_{n=1}^{10} i^{n+50} = \sum_{n=1}^{10} i^n \cdot i^{50} = \sum_{n=1}^{10} i^n \cdot i^{48} \cdot i^2 = -1\left[\sum_{n=1}^{10} i^n\right]$$

$$= -1[(i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + i^9 + i^{10}]$$

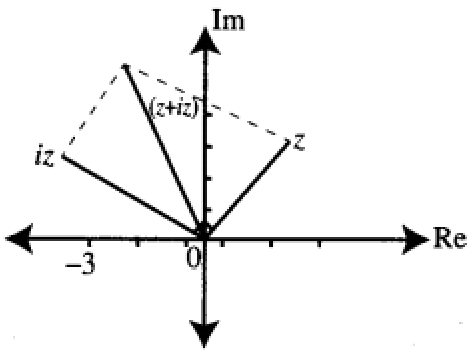
$$= -1\,[0 + 0 + i + i^2] = -1(i - 1) = 1 - i$$

Ex 2.2

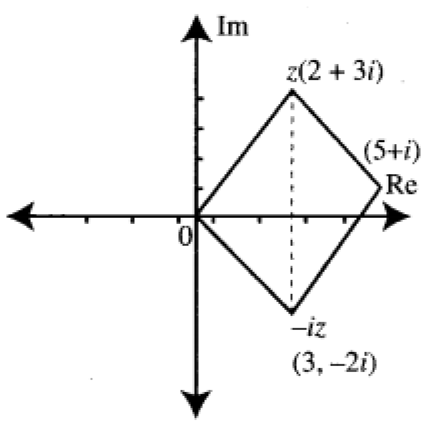
Question 1.
Evaluate the following if z = 5 – 2i and w = -1 + 3i
(i) z + w
(ii) z – iw
(iii) 2z + 3w
(iv) zw
(v) z2 + 2zw+ w2
(vi) (z + w)2
Solution:
(i) z + w = (5 – 2i) + (-1 + 3i)
= 4 + i
(ii) z – iw= (5 – 2i) -i (-1 + 3i)
= 5 – 2i + i + $3i^2$ (∵ i^2 = -1)
= 5 – 2i + i – 3(-1)
= 5 – 2i + i + 3
= 8 – i
(iii) 2z + 3w = 2 (5 – 2i) + 3 (- 1 +3i)
= 10 – 4i – 3 + 9i
= 7 + 5i
(iv) zw = (5 – 2i) (- 1 + 3i)
= -5 + 2i + 15i – $6i^2$
= -5 + 17i + 6
= 1 + 17i
(v) z2 + 2zw + w2
= $(z + w)^2$
= $(4 + i)^2$
= (4) + 2(4)(i) + $(i)^2$
= 16 + 8i + i^2
= 15 + 8i
(vi) (z + w)2 = z^2 + 2zw + w^2
= $(z + w)^2$ = $(4 + i)^2$
= 16 + 8i + i^2
= 15 + 8i

Question 2.
Given the complex number z = 2 + 3i, represent the complex numbers in the Argand diagram.
(i) z, iz, and z + iz
(ii) z, -iz, and z – iz
Solution:

(i) z, iz and z + iz.
z = 2 + 3i
iz = i(2 + 3i) = -3 + 2i
z + iz = 2 + 3i – 3 + 2i = -1 + 5i



(ii) z = 2 + 3i
-iz = -i(2 + 3i)
= -2i – 3i2
= (3 – 2i)
z – iz = (2 + 3i) + (3 – 2i)
= 5 + i



Question 3.
Find the values of the real numbers x and y, if the complex numbers.
(3 – i) x – (2 – i) y + 2i + 5 and 2x + (-1 + 2i) y + 3 + 2i are equal
Solution:
Given that the complex numbers are equal
(3 – i) x -(2 – i) y + 3 + 2i + 5
= 2x + (-1 + 2i)y + 3 + 2i
3x – ix – 2y + iy + 2i +5
= 2x – y + 2iy + 3 + 2i
(3x – 2y + 5) + i(y – x + 2)
= (2x – y + 3) + i(2y + 2)
Equating real and imaginary parts separately

3x – 2y + 5 = 2x – y + 3
x – y = -2 ………. (1)
y – x + 2 = 2y +
-x – y = 0 ………. (2)
solving 1 and 2

$$\begin{array}{rcll} x - y & = & -2 & (1) \\ -x - y & = & 0 & (2) \\ \hline -2y & = & -2 & \end{array}$$

y = 1
Substituting y = 1 in (1)
x – 1 = -2
x = -2 + 1 = -1
values of x and y are -1, 1

Ex 2.3

Question 1.
If $z_1 = 1 - 3i$, $z_2 = -4i$, and $z_3 = 5$, show that
(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
Solution:
(i) Given $z_1 = 1 - 3i$, $z_2 = -4i$, $z_3 = 5$
$z_1 + z_2 = (1 - 3i) + (-4i) = 1 - 7i$
$(z_1 + z_2) + z_3 = 1 - 7i + 5 = 6 - 7i$ ....(1)
$z_2 + z_3 = (-4i) + (5) = 5 - 4i$
$z_1 + (z_2 + z_3) = (1 - 3i) + (5 - 4i)$
$= 6 - 7i$ .... (2)
From (1) and (2)
$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
Hence proved.

(ii) $z_1 z_2 = (1 - 3i)(-4i)$
$= -4i - 12i^2$
$= -12 - 4i$
$(z_1 z_2)z_3 = (-12 - 4i)(5)$
$= -60 - 20i$ .... (1)
$z_2 z_3 = (-4i)(5) = -20i$
$z_1(z_2 z_3) = (1 - 3i)(-20i) = -20i + 60i^2$
$= -60 - 20i$ .... (2)
∴ from 1 and 2
$(z_1 z_2)z_3 = z_1(z_2 z_3)$
Question 2.
If $z_1 = 3$, $z_2 = -7i$, and $z_3 = 5 + 4i$, show that
(i) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
(ii) $(z_1 z_2) z_3 = z_1 z_3 + z_2 z_3$
Solution:
(i) $z_1 = 3$, $z_2 = -7i$, $z_3 = 5 + 4i$
$z_1 (z_2 + z_3) = 3(-7i + 5 + 4i)$
$= 3(5 - 3i)$
$= 15 - 9i$ ...... (1)
$z_1 z_2 + z_1 z_3 = 3(-7i) + 3(5 + 4i)$
$= -21i + 15 + 12i$
$= 15 - 9i$ ...... (2)
from (1) & (2), we get
∴ $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
(ii) $(z_1 + z_2) z_3 = (3 - 7i)(5 + 4i)$
$= 15 + 12i - 35i - 28i^2$
$= 15 - 23i + 28$
$= 43 - 23i$ .... (1)

$z_1 z_3 + z_2 z_3 = 3(5 + 4i) – 7i(5 + 4i)$
$= 15 + 12i – 35i – 28 i^2$
$= 15 – 23i + 28$
$= 43 – 23i$ ….. (2)
from (1) & (2), we get
$\therefore (z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$
Question 3.
If $z_1 = 2 + 5i$, $z_2 = -3 – 4i$, and $z_3 = 1 + i$, find the additive and multiplicative inverse of z_1, z_2, and z_3.
Solution:
$z_1 = 2 + 5i$
(a) Additive inverse of $z_1 = -z_1 = -(2 + 5i) = -2 – 5i$
(b) Multiplicative inverse of

$$z_1 = \frac{1}{z_1} = \frac{1}{2+5i} \times \frac{2-5i}{2-5i} = \frac{(2-5i)}{4+25} = \frac{2-5i}{29}$$

$z_2 = -3 – 4i$
(a) Additive inverse of $z_2 = -z_2 = -(-3 – 4i) = 3 + 4i$
(b) Multiplicative inverse of

$$z_2 = \frac{1}{z_2} \frac{1}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{-3+4i}{9+16} = \frac{-3+4i}{25}$$

$z_3 = 1 + i$
Additive inverse z_3 is $-z_3$
$\Rightarrow -(1 + i) = -1 – i$
Multiplicative inverse z_3 is $(z_3)^{-1}$
We know
$z_3 z_3^{-1} = 1$
$\Rightarrow (1 + i) (u + iv) = 1$ [$\because z_3^{-1} = u + iv$]
$u + iv + iu – v = 1$
$(u – v) + i (u + v) = 1 + i\,0$
Equating real and imaginary parts
$u – v = 1$
$u + v = 0$
Solving them, we get $u = \frac{1}{2}$ and $v = -\frac{1}{2}$
$\because z_3^{-1} = \frac{1}{2}(1 – i)$

Ex 2.4

Question 1.
Write the following in the rectangular form:

(i) $\overline{(5+9i)+(2-4i)}$

(ii) $\frac{10-5i}{6+2i}$

(iii) $\overline{3i}+\frac{1}{2-i}$

Solution:

(i) $z = \overline{(5+9i)+(2-4i)} = \left(\overline{5+9i}\right)+\left(\overline{2-4i}\right) = 5-9i+2+4i = 7-5i$

(ii) $z = \frac{10-5i}{6+2i} \times \frac{6-2i}{6-2i} = \frac{60-20i-30i+10i^2}{36+4} = \frac{60-50i-10}{40}$

$= \frac{50-50i}{40} = \frac{50(1-i)}{40} = \frac{5}{4}(1-i)$

(iii) $z = \overline{3i} + \frac{1}{2-i} = -3i + \frac{1}{2-i} \times \frac{2+i}{2+i} = -3i + \frac{2+i}{4+1}$

$= \frac{-15i+2+i}{5} = \frac{2-14i}{5} = \frac{2}{5} - \frac{14}{5}i$

Question 2.
If z = x + iy, find the following in rectangular form.

(i) $\text{Re}(\frac{1}{z})$

(ii) $\text{Re}(i\bar{z})$

(iii) $\text{Im}(3z + 4\bar{z} - 4i)$

Solution:

(i) $\text{Re}(\frac{1}{z}) = \text{Re}(\frac{1}{x+iy} \times \frac{x-iy}{x-iy})$

$= \text{Re}(\frac{x-iy}{x^2+y^2})$

$= \frac{x}{x^2+y^2}$

(ii) Re(iz¯¯¯) = Re i(x – iy) = ix + i^2y = ix + y
= y + ix
= y

(iii) Im $(3z + 4\bar{z} - 4i)$

$3z + 4\bar{z} - 4i = 3(x + iy) + 4(x - iy) - 4i$

$= 3x + 3iy + 4x - 4iy - 4i$

$= 7x - iy - 4i$

$= 7x + i(-y - 4)$

$= -y - 4$

Question 3.

If $z_1 = 2 - i$ and $z_2 = -4 + 3i$, find the inverse of $z_1 z_2$ and $\frac{z_1}{z_2}$

Solution:

$z_1 = 2 - i$, $z_2 = -4 + 3i$

(i) $z_1 z_2 = (2 - i)(-4 + 3i)$

$= (-8 + 6i + 4i - 3i^2)$

$= (-8 + 10i + 3)$

$= (-5 + 10i)$

$$(z_1z_2)^{-1} = \frac{1}{(z_1z_2)} = \frac{1}{-5+10i} \times \frac{-5-10i}{-5-10i} = \frac{-5-10i}{25+100} = \frac{5(-1-2\ )}{125} = \frac{-1-2i}{25}$$

(ii) $$\frac{z_1}{z_2} = \frac{2-i}{-4+3i} \times \frac{-4-3i}{-4-3i} = \frac{-8-6i+4i+3i^2}{16+9} = \frac{-8-2i-3}{25} = \frac{-11-2i}{25}$$

$$\left(\frac{z_1}{z_2}\right)^{-1} = \frac{25}{-11-2i} = \frac{25}{-11-2i} \times \frac{-11+2i}{-11+2i} = \frac{25(-11+2i)}{121+4} = \frac{25(-11+2i)}{125}$$

$$= \frac{1}{5}(-11+2i)$$

Question 4.

The complex numbers u, v, and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$. If $v = 3 - 4i$ and $w = 4 + 3i$, find u in rectangular form.

Solution:

$v = 3 - 4i$, $w = 4 + 3i = i(3 - 4i)$

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w} = \frac{1}{3-4i} + \frac{1}{i(3-4i)} = \frac{1}{3-4i}\left[1 + \frac{1 \times -i}{i \times -i}\right]$$

$$= \frac{1}{3-4i}[1-i] = \frac{1-i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i-3i+4}{9+16} \quad \Rightarrow \frac{1}{u} = \frac{7+i}{25}$$

$$u = \frac{25}{7+i} \times \frac{7-i}{7-i} = \frac{25(7-i)}{49+1} = \frac{25(7-i)}{50} = \frac{1(7-i)}{2} = \frac{7}{2} - \frac{i}{2}$$

Question 5.
Prove the following properties:
(i) z is real if and only if $z = \bar{z}$

(ii) $Re(z) = \frac{z+\bar{z}}{2}$ and $Im(z) = \frac{z-\bar{z}}{2i}$

Solution:
(i) Let z = x + iy
∴ $\overline{z}$ = x – iy
given that z = $\overline{z}$
x + iy = x – iy
Equating real and imaginary parts
x = x
y = -y
2y = 0 ⇒ y = 0
∴z = x $\bar{z}$ implies z is real.

(ii) $\frac{z+\bar{z}}{2i} = \frac{x+iy+x-iy}{2} = \frac{2x}{2} = x$

Real part of z = x

(iii) $\frac{z-\bar{z}}{2i} = \frac{(x+iy)-(x-iy)}{2i} = \frac{x+iy-x+iy}{2i} = \frac{2iy}{2i} = y$

Img part of z = y.

Question 6.
Find the least value of the positive integer n for which (√3 + i)n
(i) real
(ii) purely imaginary
Solution:
$(\sqrt{3} + i)^n$
$(\sqrt{3} + i)^2$
$= 3 - 1 + 2\sqrt{3}\, i$
$= (2 + 2\sqrt{3}\, i)$
$(\sqrt{3} + i)^3 = (\sqrt{3} + i)^2 (\sqrt{3} + i)$
$= (2 + 2\sqrt{3}\, i) (\sqrt{3} + i)$
$= 2\sqrt{3} + 2i + 6i - 2\sqrt{3}$
$(\sqrt{3} + i) = 8i$ ⇒ purely Imaginary when n = 3
$(\sqrt{3} + i)^4 = (\sqrt{3} + i)^3 (\sqrt{3} + i)$
$= 8i (\sqrt{3} + i)$
$= (-8 + 8\sqrt{3}\, i)$
$(\sqrt{3} + i)^5 = (\sqrt{3} + i)^4 (\sqrt{3} + i)$
$= (-8 + 8\sqrt{3}\, i) (\sqrt{3} + i)$
$= -8\sqrt{3} - 8i + 24i - 8\sqrt{3}$

$= -16\sqrt{3} + 16i$
$(\sqrt{3} + i)^6 = (\sqrt{3} + i)^5 (\sqrt{3} + i)$
$= (\sqrt{3} + i)(-16\sqrt{3} + 16i)$
$= 16(\sqrt{3} + i)(-\sqrt{3} + i)$
$= 16(-3 + i\sqrt{3} - i\sqrt{3} - 1)$
$= -64$ purely real when $n = 6$
Another Method:

$(\sqrt{3} + i) = r(\cos\theta + i\sin\theta)$, $\quad r = \sqrt{x^2 + y^2} = \sqrt{4} = 2.$

$(\sqrt{3} + i) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ $\quad \alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$

$$(\sqrt{3} + i)^n = 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n = 2^n\left[\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right]$$

when $n = 3$, $(\sqrt{3} + i)^3 = 2^3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = +8(0 + i) = 8i$ purely imaginary

when $n = 6$, $(\sqrt{3} + i)^6 = 2^6[\cos\pi + i\sin\pi] = -2^6 = -64$ purely real.

Question 7.
Show that
(i) $(2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10}$ is purely imaginary
(ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

Solution:
(i) $(2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10}$

$$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$$

$$= \frac{140+120i-35i+30}{49+36} = \frac{170+85i}{85} = \frac{85(2+i)}{85} = (2+i)$$

$$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12} = (2-i)^{12} + (2+i)^{12}$$

$$\bar{z} = \overline{(2-i)^{12} + (2+i)^{12}} = \overline{(2-i)^{12}} + \overline{(2+i)^{12}} = (2+i)^{12} + (2-i)^{12}$$

$\bar{z} = z \quad \Rightarrow z$ is purely real.

Let $z = \left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$

$$\overline{z} = \overline{\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}} = \overline{\left(2+i\sqrt{3}\right)^{10}} - \overline{\left(2-i\sqrt{3}\right)^{10}}$$

$$= \left(2-i\sqrt{3}\right)^{10} - \left(2+i\sqrt{3}\right)^{10} = \left[\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}\right]$$

$\overline{z} = -z$ $\Rightarrow$ z is purely imaginary.

(*ii*) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

$$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i} = \frac{171-19i-63i-7}{81+1}$$

$$= \frac{164-82i}{82} = \frac{82(2-i)}{82} = (2-i)$$

Ex 2.5

Question 1.

(i) $\frac{2i}{3+4i}$

(ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$

(iii) $(1 - i)^{10}$

(iv) 2i(3 – 4i) (4 – 3i)

Solution:

(i) $$\left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|} = \frac{2}{\sqrt{9+16}} = \frac{2}{5}$$

(ii) $$\left|\frac{2-i}{1+i} + \frac{1-2i}{1-i}\right| = \left|\frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}\right| = \left|\frac{2-2i-i-1+1+i-2i+2}{1+1}\right|$$

$$= \left|\frac{4-4i}{2}\right| = |2-2i| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

(iii) $(1 - i)^{10}$

Solution:

$|z| = |\sqrt{1^2 + 1^2}|^{10}$ [$\because |z^n| = |z|^n$]

$= (\sqrt{2})^{10} = 2^5 = 32$

(iv) 2i(3 – 4i)(4 – 3i)

Solution:

= |2i||3 – 4i||4 – 3i|

= 2i(12 – 9i – 16i – 12)

= 2i(-25i) = 50

∴ |z| = 50

Question 2.

For any two complex numbers z_1 and z_2, such that $|z_1| = |z_2| = 1$ and $z_1 z_2 \neq -1$, then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Solution:

$|z_1|^2 = 1$

$\Rightarrow z_1\bar{z}_1 = 1$

$\Rightarrow z_1 = \frac{1}{\bar{z}_1}$

Similarly $z_2 = \frac{1}{\bar{z}_2}$

$$\frac{z_1+z_2}{1+z_1z_2} = \frac{\frac{1}{\overline{z}_1}+\frac{1}{\overline{z}_2}}{1+\frac{1}{\overline{z}_1}\frac{1}{\overline{z}_2}} = \frac{\frac{\overline{z}_2+\overline{z}_1}{\overline{z}_1\,\overline{z}_2}}{\frac{\overline{z}_1\,\overline{z}_2+1}{\overline{z}_1\,\overline{z}_2}} = \frac{\overline{z}_1+\overline{z}_2}{\overline{z}_1\,\overline{z}_2+\overline{1}}$$

$\Rightarrow$ We have if $z = \overline{z}$ only when z is real.

$$= \overline{\frac{z_1+z_2}{1+z_1z_2}}$$

$$\left(\frac{z_1+z_2}{1+z_1z_2}\right) = \overline{\left(\frac{z_1+z_2}{1+z_1z_2}\right)} \qquad \therefore \frac{z_1+z_2}{1+z_1z_2} \text{ is real.}$$

Question 3.
Which one of the points 10 – 8i, 11 + 6i is closest to 1 + i.
Solution:

Let z_1 = 1 + i

z_2 = 10 – 8i

z_3 = 11 + 6i

$|z_1 - z_2|$ = |1 + i – 10 + 8i|

= |-9 + 9i|

= $\sqrt{(-9^2)+(9^2)}$

= $\sqrt{162}$ = 9√2

$|z_1 - z_3|$ = |1 + i – 11 – 6i|

= |-10 – 5i|= $\sqrt{(-10^2)+(-5^2)}$

= $\sqrt{100+25}$

= $\sqrt{125}$

= $\sqrt{5\times 25}$ = 5√5

∴ 5√5 < 9√2

∴ $|z_1 - z_2| > |z_1 - z_3|$

∴ 11 + 6i is closest to 1 + i

Question 4.
If $|z| = 3$, show that $7 \leq |z + 6 - 8i| \leq 13$.
Solution:

$|z| = 3$, To find the lower bound and upper bound we have

$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

$||z| - |6 - 8i|| \leq |z + 6 - 8i| \leq |z| + |6 - 8i|$

$|3 - \sqrt{36 + 64}| \leq |z + 6 - 8i| \leq 3 + \sqrt{36 + 64}$

$|3 - 10| \leq |z + 6 - 8i| \leq 3 + 10$

$7 \leq |z + 6 - 8i| \leq 13$

Question 5.

If $|z| = 1$, show that $2 \leq |z^2 - 3| \leq 4$.

Solution:

given $|z| = 1$

Now $|z^2 - 3| \leq |z^2| + |-3|$

$= |z|^2 + 3$

$= 1 + 3 = 4$

$\therefore |z^2 - 3| \leq 4$ .... (1)

$|z^2 - 3| \geq ||z^2|-|-3||$

$= |1 - 3| = |-2| = 2$

$|z^2 - 3| \geq 2$ .... (2)

From 1 and 2

$2 \leq |z^2 - 3| \leq 4$

Hence Proved.

Question 6.

If $\left|z - \frac{2}{z}\right|$ = 2, show that the greatest and least value of |z| are √3 + 1 and √3 − 1 respectively.

Solution:

$\left|z - \frac{2}{z}\right| = 2$

We know that

We know that

$$\left||z| - \left|\frac{2}{z}\right|\right| \le \left|z - \frac{2}{z}\right| = 2$$

$$|z| - \left|\frac{2}{z}\right| \le 2$$

Case (*i*) Let $t = |z|$ $\Rightarrow \left|t - \frac{2}{t}\right| \le 2$

$$-2 < t - \frac{2}{t} < 2 \quad \Rightarrow t - \frac{2}{t} > -2 \text{ (or) } t - \frac{2}{t} < 2$$

$$t^2 - 2 < 2t \quad \Rightarrow t^2 - 2t - 2 < 0$$

Case (*ii*)

$$t^2 - 2t - 2 < 0 \quad \Rightarrow t = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$$

The minimum value of |z| is |1 – √3| = √3 – 1

The greatest value of |z| is √3 + 1

Question 7.

If z_1, z_2 and z_3 are three complex numbers such that $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$, show that $|9z_1 z_2 + 4z_1 z_3 + z_2 z_3| = 6$.

Solution:

$$|z_1| = 1 \quad \Rightarrow |z_1|^2 = 1$$

$$z_1\bar{z}_1 = 1 \quad \Rightarrow z_1 = \frac{1}{\bar{z}_1}$$

$$|z_2| = 2 \quad \Rightarrow |z_2|^2 = 4$$

$$z_2\bar{z}_2 = 4 \quad \Rightarrow z_2 = \frac{4}{\bar{z}_2}$$

Similarly $z_3 = \frac{9}{\bar{z}_3}$

$$|z_1 + z_2 + z_3| = 1 \Rightarrow \left|\frac{1}{\bar{z}_1} + \frac{4}{\bar{z}_2} + \frac{9}{\bar{z}_3}\right| = 1$$

$$\left|\frac{\bar{z}_2\bar{z}_3 + 4\bar{z}_1\bar{z}_3 + 9\bar{z}_2\bar{z}_1}{\bar{z}_1\bar{z}_2\bar{z}_3}\right| = 1 \Rightarrow \frac{|\bar{z}_2\bar{z}_3 + 4\bar{z}_1\bar{z}_3 + 9\bar{z}_1\bar{z}_2|}{|\overline{z_1 z_2 z_3}|} = 1$$

$$\frac{|\overline{z_2 z_3 + 4z_1 z_3 + 9z_1 z_2}|}{|\overline{z_1 z_2 z_3}|} = 1 \Rightarrow |z_2 z_3 + 4z_1 z_3 + 9z_1 z_2| = |z_1||z_2||z_3|$$

$$|z_2 z_3 + 4z_1 z_3 + 9z_1 z_2| = 1 \times 2 \times 3 = 6$$

Question 8.
If the area of the triangle formed by the vertices z, iz, and z + iz is 50 square units, find the value of |z|.
Solution:
The vertices are z = x + iy.
iz = i(x + iy) = -y + ix
z + iz = (x – y) + i(x + y)
the points representing the complex numbers z, iz and z + iz are (x, y), (-y, x) and ((x – y), (x + y)) respectively.
given area of triangle = 50 square units

$$\left|\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right| = 50$$

$$|x(x - x - y) - y(x + y - y) + (x - y)(y - x)| = 100$$

$$|-xy - xy + 2xy - x^2 - y^2| = 100$$

$$x^2 + y^2 = 100$$

$$|z|^2 = 100$$

$$|z| = 10$$

Question 9.
Show that the equation z3 + 2z¯ = 0 has five solutions.
Solution:
Given that z3 + 2z¯ = 0
z3 = -2 z¯ ……. (1)
Taking modulus on both sides,

$$|z^3| = |-2\bar{z}| \qquad \Rightarrow |z|^3 = 2|\bar{z}|$$

$$|z|^3 = 2|z| \qquad \Rightarrow |z|^3 - 2|z| = 0$$

$$|z|(|z|^2 - 2) = 0$$

$$|z| = 0 \text{ or } |z|^2 - 2 = 0$$

$$|z| = 0 \Rightarrow z = 0 \text{ is a solution.}$$

$$|z|^2 - 2 = 0 \Rightarrow |z|^2 = 2$$

$$z\bar{z} = 2$$

From (1) $\Rightarrow$ $\dfrac{-z^3}{2} = \bar{z}$ $\therefore$ (2) $\Rightarrow z\left(\dfrac{-z^3}{2}\right) = 2$ $\Rightarrow -z^4 = 4$

$$z^4 = -4$$

z has four non-zero solution.
Hence including zero solution. There are five solutions.

Question 10.
Find the square roots of
(i) 4 + 3i
(ii) -6 + 8i
(iii) -5 – 12i
Solution:

(i) $z = 4 + 3i$

$|z| = |4 + 3i| = \sqrt{16 + 9} = 5$

We have $$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$$

$$\sqrt{4+3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\frac{3}{3}\sqrt{\frac{5-4}{2}}\right) = \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

(ii) $z = -6 + 8i \Rightarrow |z| = \sqrt{36+64} = 10$

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$$

$$\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10-6}{2}} + i\frac{8}{8}\sqrt{\frac{10+6}{2}}\right) = \pm\left(\sqrt{2} + i2\sqrt{2}\right) = \pm\sqrt{2}(1+2i)$$

(iii) $z = -5 - 12i$

$$|z| = \sqrt{25+144} = \sqrt{169} = 13$$

$$z = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$$

$$\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13-5}{2}} - i\frac{12}{12}\sqrt{\frac{13+5}{2}}\right) = \sqrt{\frac{8}{2}} - i\sqrt{\frac{18}{2}} = \pm\sqrt{4} - i\sqrt{9}$$

$$= \pm(2 - i3) = \pm(2 - 3i)$$

Ex 2.6

Question 1.

If z = x + iy is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$. show that the locus of z is real axis.

Solution:

$\left|\frac{z-4i}{z+4i}\right| = 1$

⇒ |z – 4i| = |z + 4i|

let z = x + iy

⇒ |x + iy – 4i| = |x + iy + 4i|

⇒ |x + i(y – 4)| = |x +(y + 4)|

⇒ $\sqrt{x^2 + (y-4)^2} = \sqrt{x^2 + (y+4)^2}$

Squaring on both sides, we get

$x^2 + y^2 - 8y + 16 = x^2 + y^2 + 16 + 8y$

⇒ -16y = 0

⇒ y = 0 in two equation of real axis.

Question 2.

If z = x + iy is a complex number such that Im $\left(\frac{2z+1}{iz+1}\right) = 0$ show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.

Solution:

Let z = x + iy

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0 \quad \Rightarrow \text{Im}\left(\frac{2(x+iy)+1}{i(x+iy)+1}\right) = 0$$

$$\Rightarrow \text{Im}\left[\frac{2x+i2y+1}{ix-y+1}\right] = 0 \quad \Rightarrow \text{Im}\left[\frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-(ix)}{(1-y)-(ix)}\right] = 0$$

Considering only the imaginary parts,

$$\frac{-x(2x+1)+2y(1-y)}{(1-y)^2+x^2} = 0 \quad \Rightarrow -2x^2 - x + 2y - 2y^2 = 0$$

$2x^2 + 2y^2 + x - 2y = 0.$

Hence proved.

Question 3.

Obtain the Cartesian form of the locus of z = x + iy in each of the following cases:

(i) $[\text{Re}(iz)]^2 = 3$

(ii) $\text{Im}[(1 - i)z + 1] = 0$
(iii) $|z + i| = |z - 1|$
(iv) $\bar{z} = z^{-1}$
Solution:
(i) Given $z = x + iy$
$iz = i(x + iy)$
$\therefore [\text{Re}(iz)]^2 = -y$
$(y)^2 = 3$
$y^2 = 3$

(ii) $\text{Im}[(1 - i)z + 1] = 0$
Solution:
Given $z = x + iy$
$= (1 - i)(x + iy) + 1$
$= x - ix + iy + y + 1 = 0$
$\Rightarrow (x + y + 1) + i(y - x)$
$\text{Im}[(1 - i)z + 1] = 0$
$\therefore y - x = 0$
$x - y = 0$

(iii) $|z + i| = |z - 1|$
Solution:
$|x + iy + i| = |x + iy - 1|$ [$\because z = x + iy$]

$|x + i(y + 1)| = |(x - 1) + iy|$

$\sqrt{x^2 + (y+1)^2} = \sqrt{(x-1)^2 + y^2}$

squaring on both sides

$x^2 + (y + 1)^2 = (x - 1)^2 + y^2$

$x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$

$2x + 2y = 0$

$x + y = 0$ [$\because z = x + iy$]

(iv) $\bar{z} = z^{-1}$

$\Rightarrow \bar{z} = \frac{1}{z}$

$\Rightarrow z\bar{z} = 1$

$\Rightarrow |z|^2 = 1$

$\Rightarrow |x + iy|^2 = 1$

$\Rightarrow x^2 + y^2 = 1$

Question 4.
Show that the following equations represent a circle, and, find its centre and radius.
(i) |z – 2 – i| = 3
(ii) |2z + 2 – 4i| = 2
(iii) |3z – 6 + 12i| = 8
Solution:

(i) |z – 2 – i| = 3

This can be written as

|z – (2 – i)| = 3

This is the from $|z - z_0| = r$ and it represents a circle.

∴ Centre is (2, 1) and radius = 3 units

Aliter:

Let z = x + iy

∴ |z – 2 – i| = 3

|x + iy – 2 – i| = 3

|(x – 2) + i(y – 1)| = 3

$\sqrt{(x+2)^2+(y-1)^2}$ = 3

squaring on both sides

$(x-2)^2 + (y-1)^2 = 9$

$x^2 - 4x + 4 + y^2 - 2y + 1 - 9 = 0$

$x^2 + y^2 - 4x - 2y - 4 = 0$

Comparing with general from of equation of circle

$ax^2 + by^2 + 2gx + 2fy + c = 0$

we get a = 1, b = 1, g = -2, f = -1, c = -4

Centre (-g, -f) = (2, 1)

radius = $\sqrt{g^2+f^2-c}$ = $\sqrt{4+1+4}$ = √9 = 3 units

(ii) |2z + 2 – 4i| = 2
Solution:

$|2z + 2 - 4i| = 2$
$2\,|z - (-1 + 2i)| = 2$
This can be written as
$|z - (-1 + 2i)| = 1$
which is in the from $|z - z_0| = r$
Centre of the circle = (-1, 2i)
radius = 1 unit

(iii) $|3z - 6 + 12i| = 8$.
Solution:
$3\,|z - (2 + 4i)| = 8$
$\Rightarrow |z - (2 - 4i)| = \frac{8}{3}$
This is in the form $|z - z_0| = r$
Centre of the circle (2 – 4i)
radius $\frac{8}{3}$ units.

Question 5.
Obtain the Cartesian equation for the locus of z = x + iy in each of the following cases.
(i) $|z - 4| = 16$
(ii) $|z - 4|^2 - |z - 1|^2 = 16$
Solution:
(i) $z = x + iy$
$|z - 4| = 16$
$\Rightarrow |x + iy - 4| = 16$
$\Rightarrow |(x - 4) + iy| = 16$
$\Rightarrow \sqrt{(x-4)^2 + y^2} = 16$
Squaring on both sides
$(x - 4)^2 + y^2 = 256$
$\Rightarrow x^2 - 8x + 16 + y^2 - 256 = 0$
$\Rightarrow x^2 + y^2 - 8x - 240 = 0$ represents the equation of circle
(ii) Given $z = x + iy$
$|x + iy - 4|^2 - |x + iy - 1|^2 = 16$
$|(x - 4) + iy|^2 - |(x - 1) + iy|^2 = 16$
$[(x - 4)^2 + y^2] - [(x - 1)^2 + y^2] = 16$
$x^2 - 8x + 16 + y^2 - x^2 + 2x - 1 - y^2 = 16$
$-6x - 1 = 0$
$6x + 1 = 0$

Ex 2.7

Question 1.

Write in polar form of the following complex numbers.

(i) 2 + i2√3

(ii) 3 – i√3

(iii) -2 – i2

(iv) i−1cosπ3+isinπ3

Solution:

(*i*) $$2 + i2\sqrt{3} = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{x^2 + y^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{2\sqrt{3}}{2}\right| = \tan^{-1}\left|\sqrt{3}\right| = \frac{\pi}{3}$$

$2 + 2i\sqrt{3}$ lies in I quadrant

$$\theta = \alpha = \frac{\pi}{3}$$

So $$(2 + i\,2\sqrt{3}) = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4\left[\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right], k \in z$$

(*ii*) $$3 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{x^2 + y^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{3}\right| = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = \frac{\pi}{6}$$

$(3 - i\sqrt{3})$ lies in IV quadrant

$$\theta = -\alpha = \frac{-\pi}{6}$$

$$(3 - i\sqrt{3}) = 2\sqrt{3}\left[\cos\frac{-\pi}{6} + i\sin\left(\frac{-\pi}{6}\right)\right]$$

$$= 2\sqrt{3}\left[\cos\left(2k\pi - \frac{\pi}{6}\right) + i\sin\left(2k\pi - \frac{\pi}{6}\right)\right], k \in z$$

(*iii*) $$-2 - i2 = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{x^2 \quad y^2} = \sqrt{4+4} = 2\sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-2}{-2}\right| = \tan^{-1}|1| = \frac{\pi}{4}$$

(–2 – i2) lies in III quadrant

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$$

$\therefore$
$$-2 - i2 = 2\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$$

$$= 2\sqrt{2}\left[\cos 2k\pi - \frac{3\pi}{4} + i\sin 2k\pi - \frac{3\pi}{4}\right] k \in z$$

(iv) Consider $-1 + i = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}|1| = \frac{\pi}{4}$$

$-1 + i$ lies in II quadrant

$$\theta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$-1 + i = \sqrt{2}\ \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

$$\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \frac{\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} \qquad \left[\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{9\pi - 4\pi}{12}\right) + i\sin\left(\frac{9\pi - 4\pi}{12}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right] = \sqrt{2}\left[\text{cis } 2k\pi + \frac{5\pi}{12}\right], k \in z$$

Question 2.
Find the rectangular form of the following complex numbers.

(*i*) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ **(*ii*)** $\dfrac{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$

Solution:

(i) $\arg(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

$$= \cos\left(\frac{\pi}{6}+\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{6}+\frac{\pi}{12}\right) = \cos\left(\frac{2\pi+\pi}{12}\right) + i\sin\left(\frac{2\pi+\pi}{12}\right)$$

$$= \cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i)$$

(ii) $$\frac{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)} = \frac{1}{2}\,\frac{\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)}{\cos\frac{\pi}{3} + i\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{1}{2}\cos\left(\frac{-\pi}{6}-\frac{\pi}{3}\right) + i\sin\left(\frac{-\pi}{6}-\frac{\pi}{3}\right) = \frac{1}{2}\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = \frac{1}{2}[-i] = 0 - \frac{i}{2} = -\frac{i}{2}$$

Question 3.

If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n) = a+ib$ **, show that**

(i) $(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\cdots(x_n^2+y_n^2) = a^2+b^2$

(ii) $\sum_{r=1}^{n} \tan^{-1}\left(\frac{y_r}{x_r}\right) = \tan^{-1}\left(\frac{b}{a}\right) + 2k\pi$, $k \in \mathbb{Z}$

Solution:

(i) $(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3) \ldots\ldots (x_n + iy_n) = a + ib$ …… (1)

Taking modulus on both sides,

$|(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3) \ldots\ldots (x_n + iy_n)| = |a + ib|$

$|x_1 + iy_1|\,|x_2 + iy_2|\,|x_3 + iy_3| \ldots\ldots |x_n + iy_n| = |a + ib|$

$$\sqrt{x_1^2+y_1^2}\ \sqrt{x_2^2+y_2^2}\ \sqrt{x_3^2+y_3^2}\ \ldots\ \sqrt{x_n^2+y_n^2} = \sqrt{a^2+b^2}$$

Squaring on both sides

$$(x_1^2+y_1^2)\ (x_2^2+y_2^2)\ (x_3^2+y_3^2)\ \ldots\ (x_n^2+y_n^2) = a^2+b^2$$

(ii) Taking Argument on both sides of (1)

$$\arg\left[(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\ldots(x_n+iy_n)\right] = \arg(a+ib)$$

$$\arg(x_1+iy_1) + \arg(x_2+iy_2) + \arg(x_3+iy_3) + \ldots \arg(x_n+iy_n) = \arg(a+ib)$$

$$\tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\tan^{-1}\left(\frac{y_3}{x_3}\right)+\ldots+\tan^{-1}\left(\frac{y_n}{x_n}\right) = 2k\pi+\tan^{-1}\left(\frac{b}{a}\right)$$

$$\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=2k\pi+\tan^{-1}\left(\frac{b}{a}\right)(k\in z)$$

Question 4.

If $\frac{1+z}{1-z}$ = cos 2θ + i sin 2θ, show that z = i tan θ.

Solution:

$$\frac{1+z}{1-z} = \frac{\cos 2\theta+i\sin 2\theta}{1} \qquad \left(\text{Use } \frac{\text{Nr}-\text{Dr}}{\text{Nr}+\text{Dr}} \begin{array}{c}\text{componendo and}\\ \text{dividendo rule}\end{array}\right)$$

$$\frac{1+z-1+z}{1+z+1-z} = \frac{\cos 2\theta+i\sin 2\theta-1}{\cos 2\theta+i\sin 2\theta+1}$$

$$\frac{2z}{2} = \frac{1-2\sin^2\theta+2i\sin\theta\cos\theta-1}{2\cos^2\theta-1+i2\sin\theta\cos\theta+1} \qquad [(x+iy)=i(y-xi)]$$

$$z=\frac{2\sin\theta[i\cos\theta-\sin\theta]}{2\cos\theta[\cos\theta+i\sin\theta]} = \frac{i\sin\theta[\cos\theta+i\sin\theta]}{\cos\theta[\cos\theta+i\sin\theta]}$$

$$z=i\tan\theta$$

Question 5.

If cos α + cos β + cos γ = sin α + sin β + sin γ = 0, then show that

(i) cos 3α + cos 3β + cos 3γ = 3 cos (α + β + γ)

(ii) sin 3α + sin 3β + sin 3γ = 3 sin (α + β + γ)

Solution:

cos α + cos β + cos γ = sin a + sin β + sin γ = 0

Let a = cos α + i sin α, b = cos β + i sin β, c = cos γ + i sin γ

Euler's form eiθ = cos θ + i sin θ,

Now a + b + c = (cos α + cos β + cos γ) + i(sin α + sin β + sin γ)

= 0 + i 0

= 0

If a + b + c = 0 then $a^3 + b^3 + c^3$ = 3 abc

= $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

= 3(cos α + i sin α) + (cos β + i sin β) + (cos γ + i sin γ)

By Euler's theorem

= $(ei\alpha)^3 + e(i\beta)^3 + e(i\gamma)^3$ = 3 eiα eiβ eiγ

ei3α + ei3β + ei3γ = 3 ei(α+β+γ)

= cos 3α + i sin 3α + cos 3β + i sin 3β + cos 3γ + i sin 3γ)

= 3[cos(α + β + γ) + i sin (α + β + γ)]

(cos 3α + cos 3β + cos 3γ) + i (sin 3α + sin 3β + sin 3γ)

$= 3 \cos(\alpha + \beta + \gamma) + i\, 3 \sin(\alpha + \beta + \gamma)$

Equating real and imaginary parts,

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$

$\cos 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$

Question 6.

If $z = x + iy$ and $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, then show that $x^2 + y^2 + 3x - 3y + 2 = 0$.

Solution:

$\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$

We have $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

$\arg(z - i) - \arg(z + 2) = \frac{\pi}{4}$

Let $z = x + iy$

$\arg(x + iy - i) - \arg(x + iy + 2) = \frac{\pi}{4}$

$\arg(x + i(y - 1)) - \arg(x + 2 + iy) = \frac{\pi}{4}$

$$\tan^{-1}\left(\frac{y-1}{x}\right) - \tan^{-1}\left(\frac{y}{x+2}\right) = \frac{\pi}{4} \quad \Rightarrow \tan^{-1}\left(\frac{\frac{y-1}{x} - \frac{y}{x+2}}{1+\frac{y-1}{x}\times\frac{y}{x+2}}\right) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\frac{(y-1)(x+2)-xy}{x(x+2)}}{\frac{x(x+2)+y(y-1)}{x(x+2)}}\right) = \frac{\pi}{4} \quad \Rightarrow \tan^{-1}\left(\frac{xy+2y-x-2-xy}{x^2+2x+y^2-y}\right) = \frac{\pi}{4}$$

$$\left(\frac{2y-x-2}{x^2+y^2+2x-y}\right) = \tan\frac{\pi}{4} = 1$$

$2y - x - 2 = x^2 + y^2 + 2x - y$

$x^2 + y^2 + 3x - 3y + 2 = 0$

Hence proved.

Ex 2.8

Question 1.

If to ω ≠ 1 is a cube root of unity, then show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=1$

Solution:

Since ω is a cube root of unity, we have $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$

$$\text{LHS} = \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = \frac{\omega^2\left(a+b\omega+c\omega^2\right)}{\omega^2\left(b+c\omega+a\omega^2\right)}+\frac{\omega^2\left(a+b\omega+c\omega^2\right)}{\omega^2\left(c+a\omega+b\omega^2\right)}$$

$$= \frac{a\omega^2+b+c\omega}{\omega^2\left(b+c\omega+a\omega^2\right)}+\frac{\omega^2\left(a+b\omega+c\omega^2\right)}{\left(c\omega^2+a+b\omega\right)} \qquad (\therefore\ \omega^3=1,\ \omega^4=\omega)$$

$$= \frac{1}{\omega^2}+\frac{\omega^2}{1} = \frac{1+\omega^4}{\omega^2} = \frac{1+\omega}{\omega^2} = \frac{-\omega^2}{\omega^2} = -1$$

Question 2.

Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$

Solution:

$$\text{LHS} = \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$$

Polar form of $\frac{\sqrt{3}}{2}+\frac{i}{2} = r(\cos\theta + i\sin\theta)$

$$r = \sqrt{x^2+y^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$$

$$\theta = \alpha = \frac{\pi}{6}$$

$$\frac{\sqrt{3}}{2}+\frac{i}{2} = 1\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

Similarly $\frac{\sqrt{3}}{2}-\frac{i}{2} = \left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$

$$(1) \Rightarrow \text{LHS} = \left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^5+\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^5$$

$$= \cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}+\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}$$

$$= 2\cos\frac{5\pi}{6} = 2\cos\left(\pi-\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = -2\times\frac{\sqrt{3}}{2} = -\sqrt{3} = \text{RHS}$$

Question 3.

Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$

Solution:

$$\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}=\left(\frac{1+z}{1+\frac{1}{z}}\right)^{10}$$

Let $z=\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}$

$\therefore \frac{1}{z}=\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}$

$$=\left(\frac{1+z}{(z+1)}\times z\right)^{10}=z^{10}$$

$\left[\because |z|=1\Rightarrow |z|^2=1\Rightarrow z\bar{z}=1\Rightarrow \bar{z}=\frac{1}{z}\right]$

$$=\left(\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}\right)^{10}=i^{10}\left[\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}\right]^{10}$$

$$=i^8\cdot i^2\left[\cos\pi-i\sin\pi\right]=-1\,[-1]=1$$

Question 4.

If 2 cos α = x + $\frac{1}{x}$ and 2 cos β = y + $\frac{1}{y}$, show that

(*i*) $\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$

(*ii*) $xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

(*iii*) $\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$

(*iv*) $x^m y^n+\frac{1}{x^m y^n}=2\cos(m\alpha+n\beta)$

Solution:

(i) $2\cos\alpha=x+\frac{1}{x}$

$\Rightarrow 2\cos\alpha=\frac{x^2+1}{x}$

$\Rightarrow 2x\cos\alpha=x^2+1$

$\Rightarrow x^2-2x\cos\alpha+1=0$

$$x = \frac{2\cos\alpha \pm \sqrt{4\cos^2 x - 4}}{2}$$

$$x = \frac{2\cos\alpha \pm 2\sqrt{-(1-\cos^2\alpha)}}{2}$$

$$x = \frac{2\cos\alpha \pm 2i\sin\alpha}{2}$$

$$x = \cos\alpha \pm i\sin\alpha$$

Let $x = \cos\alpha + i\sin\alpha$, Similarly $y = \cos\beta + i\sin\beta$

$$x = e^{i\alpha},\ y = e^{i\beta}$$

$$\frac{x}{y} = \frac{e^{i\alpha}}{e^{i\beta}} = e^{i\alpha}e^{-i\beta} = e^{i(\alpha-\beta)}$$

$$\frac{x}{y} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

Similarly $\frac{y}{x} = \cos(\alpha-\beta) - i\sin(\alpha-\beta)$

$$\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha-\beta)$$

(i) $xy = e^{i\alpha}.\ e^{i\beta} = e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

$$\frac{1}{xy} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$

$$xy - \frac{1}{xy} = [\cos(\alpha+\beta) + i\sin(\alpha+\beta)] - [\cos(\alpha+\beta) - i\sin(\alpha+\beta)]$$

$$= 2i\sin(\alpha+\beta)$$

(iii) $\frac{x^m}{y^n} = \frac{\left(e^{i\alpha}\right)^m}{\left(e^{i\beta}\right)^n} = \frac{e^{im\alpha}}{e^{in\beta}} = e^{i(m\alpha-n\beta)}$

$$\frac{x^m}{y^n} = \cos(m\alpha-n\beta) + i\sin(m\alpha-n\beta)$$

$$\frac{y^n}{x^m} = \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$$

$$\frac{x^m}{y^n} - \frac{y^n}{x^m} = [\cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)] - [\cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)]$$

$$= 2i\sin(m\alpha - n\beta)$$

(iv) $x^m y^n = \left(e^{i\alpha}\right)^m \left(e^{i\beta}\right)^n = e^{im\alpha} e^{in\beta} = e^{i(m\alpha + n\beta)}$

$$x^m y^n = \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$$

$$\frac{1}{x^m y^n} = \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$$

Question 5.

Solve the equation $z^3 + 27 = 0$

Solution:

z3 + 27 = 0

⇒ z3 = -27

⇒ z3 = 33(-1)

⇒ $z = 3(-1)^{1/3}$

$$z = 3\left[\cos(2k\pi + \pi) + i\ \sin 2k\pi + \pi\right]^{\frac{1}{3}}$$

$$= 3\left[\cos\left(\frac{2k\pi + \pi}{3}\right) + i\ \sin\left(\frac{2k\pi + \pi}{3}\right)\right]$$

$$k = 0,1,2$$

$$k = 0, z = 3\left[\cos\frac{\pi}{3} + i\ \sin\frac{\pi}{3}\right]$$

$$k = 1, z = 3\left[\cos\frac{3\pi}{3} + i\ \sin\frac{3\pi}{3}\right] = -3$$

$$k = 2, z = 3\left[\cos\frac{5\pi}{3} + i\ \sin\frac{5\pi}{3}\right]$$

Question 6.

If ω ≠ 1 is a cube root of unity, show that the roots of the equation $(z - 1)^3 + 8 = 0$ are -1, $1 - 2\omega$, $1 - 2\omega^2$

Solution:

$(z - 1)^3 + 8 = 0$

⇒ (z – 1)3 = -8

$$(z-1)^3 = (-2)^3 \times 1 \qquad \Rightarrow (z-1) = \left[(-2)^3\right]^{\frac{1}{3}}[1]^{\frac{1}{3}}$$

$$z-1 = -2\ [1]^{\frac{1}{3}} = -2\ [\cos 2k\pi + i\sin\ 2k\pi]^{\frac{1}{3}}$$

$$z-1 = -2\left[\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right];\ k = 0,1,2$$

$$k = 0,\ z-1 = -2\ (1) \qquad \Rightarrow z = -2+1 = -1$$

$$k = 1,\ z-1 = -2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right] = -2\omega$$

$$z = 1 - 2\ \omega$$

$$k = 2,\ z-1 = -2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right]$$

$$z-1 = -2\ \omega^2 \qquad \Rightarrow z = 1 - 2\ \omega^2$$

The roots are -1, $1 - 2\omega$, $1 - 2\omega^2$

Question 7.

Find the value of $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right)$

Solution:

$\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right)$

We know that 9th roots of unit are 1, ω, ω^2, ……., ω^8

Sum of the roots:

$1 + \omega + \omega^2 + ... + \omega^8 = 0 \Rightarrow \omega + \omega^2 + \omega^3 + ... + \omega^8 = -1$

If $k = 1$ $\Rightarrow \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9} = \omega$

If $k = 2$ $\Rightarrow \cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9} = \omega^2$

⋮ ⋮

If $k = 8$ $\Rightarrow \cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9} = \omega^8$

The sum of all the terms $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right) = -1$

Question 8.

If $\omega \neq 1$ is a cube root of unity, show that

(i) $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = 128$
(ii) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)\ldots\ldots(1 + \omega^{2n}) = 1$

Solution:

(i) $(1 - \omega + \omega^2)6 + (1 + \omega - \omega^2)6$
$= (-\omega + \omega)^6 + (-\omega - \omega^2)^6$
$= (-2\omega)^6 + (-2\omega^2)^6$
$= 26 [\omega^6 + \omega^{12}] [\because \omega^6 = 1]$
$= 64 [1 + 1]$
$= 64 \times 2$
$= 128$
(ii) $(1 - \omega) (1 + \omega^2) (1 + \omega) (1 + \omega2)$ ….. 2n factors
$(-\omega^2) (-\omega) (-\omega^2) (-\omega)$ ……. 2n factor
$(\omega^3) (\omega^3)$ …… 2n factor
$= 1.1$ …….. 2n factor
$= 1$

Question 9.

If z = 2 – 2i, find the rotation of z by θ radians in the counter clockwise direction about the origin when

(i) $\theta = \frac{\pi}{3}$

(ii) $\theta = \frac{2\pi}{3}$

(iii) $\theta = \frac{3\pi}{2}$

Solution:

(i) $z = 2 - 2i = 2(1 - i) = r(\cos\theta + i\sin\theta)$

$r = \sqrt{x^2 + y^2} = 2\sqrt{1+1} = 2\sqrt{2}$

$\alpha = \tan^{-1} = \left|\frac{y}{x}\right| = \tan^{-1}|1| = \frac{\pi}{4}$

(1 – i) lies in IV quadrant

$\theta = -\alpha = -\frac{\pi}{4}$

$\Rightarrow z = 2\sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$

z is rotated by $\theta = \frac{\pi}{3}$ in the counter clock wise direction.

$$z = 2\sqrt{2}\left[\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right] = 2\sqrt{2}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$$

(ii) z is rotated by $\theta = \frac{2\pi}{3}$ in the counter clockwise direction.

$$z = 2\sqrt{2}\left[\cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)\right]$$

$$= 2\sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$$

(iii) z is rotated by θ = 3π2 in the counter clockwise direction.

$$z = 2\sqrt{2}\left[\cos\left(\frac{3\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= 2\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$

Question 10.

Prove that the values of $\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1 \pm i)$

Solution:

Let $z = (-1)^{\frac{1}{4}}$

$$z = \left[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right]^{\frac{1}{4}}$$

$$z = \left[\cos\left(\frac{2k\pi + \pi}{4}\right) + i\sin\left(\frac{2k\pi + \pi}{4}\right)\right]$$

$k = 0,1,2,3$

$$k = 0,\ z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$k = 1,\ z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$k = 2,\ z = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$k = 3,\ z = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

The roots are $\pm\frac{1}{\sqrt{2}}(1 \pm i)$

Ex 2.9

Question 1.

$i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is ______

(a) 0
(b) 1
(c) -1
(d) i

Answer:
(a) 0

Question 2.

The value of $\sum_{i=1}^{13} \left(i^n + i^{n-1}\right)$ is ______

(a) 1 + i
(b) i
(c) 1
(d) 0

Answer:
(a) 1 + i

Hint. $\sum_{i=1}^{13} = (i + i^0) + (i^2 + i^1) + (i^3 + i^2) + (i^4 + i^3) + \ldots(i^{13} + i^{12})$

$= 1 + 2\,(i + i^2 + i^3 + \ldots + i^{12}) + i^{13} = 1 + 2(0) + i$

$= 1 + i$

Question 3.

The area of the triangle formed by the complex numbers z, iz, and z + iz in the Argand's diagrams is ______

(a) $\frac{1}{2}$ |z|
(b) $|z|^2$
(c) $\frac{3}{2}|z|^2$
(d) $2|z|^2$

Answer:

(a) $\frac{1}{2}$ |z|

Hint: Area of triangle = $\frac{1}{2}$ bh

= $\frac{1}{2}$ |z| |iz|

= $\frac{1}{2}$ $|z|^2$

Question 4.

The conjugate of a complex number is $\frac{1}{i-2}$. Then, the complex number is _____

(a) $\frac{1}{i+2}$

(b) $\frac{-1}{i+2}$

(c) $\frac{-1}{i-2}$

(d) $\frac{1}{i-2}$

Answer:

(b) $\frac{-1}{i+2}$

Hint:

$$\bar{z} = \frac{1}{i-2} \Rightarrow \overline{(\bar{z})} = \overline{\left(\frac{1}{i-2}\right)}$$

$$z = \frac{1}{-i-2} = \frac{-1}{i+2}$$

Question 5.

If $z = \frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$ then |z| is equal to ______

(a) 0
(b) 1
(c) 2
(d) 3

Answer:
(c) 2

Hint:

$$|z| = \left|\frac{(\sqrt{3}+i)^3 (3i+4)^2}{(8+6i)^2}\right| = \frac{|\sqrt{3}+i|^3 |4+3i|^2}{|8+6i|^2}$$

$$= \frac{(\sqrt{4})^3 (\sqrt{25})^2}{(\sqrt{100})^2} = \frac{2^3 \times 25}{100} = 2$$

Question 6.

If z is a non zero complex number, such that 2iz2 = z¯ then |z| is ______

(a) $\frac{1}{2}$

(b) 1

(c) 2

(d) 3

Answer:

(a) $\frac{1}{2}$

Hint:

$$2i\,z^2 = (\bar{z}) \qquad \Rightarrow |\,2i\,z^2\,| = |\,\bar{z}\,|$$

$$2\,|i|\,|z|^2 = |z| \qquad \Rightarrow 2\,|z|^2 = |z|$$

$$|z| = \frac{1}{2}$$

Question 7.

If |z – 2 + i| ≤ 2, then the greatest value of |z| is _______

(a) √3 – 2

(b) √3 + 2

(c) √5 – 2

(d) √5 + 2

Answer:

(d) √5 + 2

Hint:

$$|z - 2 + i| \le 2$$

$$\big||z_1| - |z_2|\big| \le |z_1 - z_2| \le 2 \qquad \Rightarrow \big||z_1| - |2-i|\big| \le |z - 2 + i| \le 2$$

$$|z_1| - |\sqrt{5}| \le 2 \qquad |z_1| - \sqrt{5} \le 2$$

$$|z_1| \le 2 + \sqrt{5}$$

Question 8.

If $\left|z - \frac{3}{z}\right|$, then the least value of |z| is _____

(a) 1
(b) 2
(c) 3
(d) 5

Answer:
(a) 1

Hint:

$$\left|z - \frac{3}{z}\right| = 2$$

$$\left||z| - \left|\frac{3}{z}\right|\right| \leq \left|z - \frac{3}{z}\right| = 2$$

$$|z| - \frac{3}{|z|} \leq 2 \qquad \text{Let } t = |z|$$

$$t - \frac{3}{t} \leq 2 \qquad \Rightarrow t^2 - 3 \leq 2t \qquad \Rightarrow t^2 - 2t - 3 \leq 0$$

$$t = \frac{2 \pm \sqrt{4+12}}{2} \qquad \Rightarrow t = \frac{2 \pm 4}{2}$$

$$t = \frac{2-4}{2}, t = \frac{2+4}{2}$$

$t = -1, 3$. The least value of $|z| = 1$

Question 9.

If |z| = 1, then the value of $\frac{1+z}{1+\bar{z}}$ is ______

(a) z

(b) $\bar{z}$

(c) $\frac{1}{z}$

(d) 1

Answer:
(a) z

Hint:

$|z| = 1 \qquad \Rightarrow \bar{z} = \frac{1}{z}$

$$\frac{1+z}{1+\bar{z}} = \frac{1+z}{1+\frac{1}{z}} = \frac{(1+z)}{(z+1)} \times z = z$$

Question 10.
The solution of the equation |z| – z = 1 + 2i is ______

(a) $\frac{3}{2} - 2i$
(b) $-\frac{3}{2} + 2i$
(c) $2 - \frac{3}{2}i$
(d) $2 + \frac{3}{2}i$

Answer:
(a) $\frac{3}{2} - 2i$

Hint.

$$|z| - z = 1 + 2i$$
$$\sqrt{x^2 + y^2} - (x + iy) = 1 + 2i$$
$$\sqrt{x^2 + y^2} - x - iy = 1 + 2i$$
$$y = -2 \qquad \Rightarrow \sqrt{x^2 + y^2} - x = 1$$
$$\sqrt{x^2 + 4} = (1 + x)$$

Squaring on both sides

$$x^2 + 4 = (1 + x)^2$$
$$x^2 + 4 = 1 + x^2 + 2x \qquad \Rightarrow 2x = 3 \qquad \Rightarrow x = \frac{3}{2}$$
$$z = \frac{3}{2} - 2i$$

Question 11.
If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1 z_2 + 4z_1 z_3 + z_2 z_3| = 12$, then the value of $|z_1 + z_2 + z_3|$ is ______

(a) 1
(b) 2
(c) 3
(d) 4

Answer:

(b) 2
Hint: $|z_1 + z_2 + z_3| = 2$

Question 12.
If z is a complex number such that z ∈ C/R and z + 1z ∈ R, then |z| is ______
(a) 0
(b) 1
(c) 2
(d) 3
Answer:
(b) 1
Hint: We have

$$z + \bar{z} = 2\,\text{Re}(z) \qquad \therefore \frac{1}{z} = \bar{z} \text{ only when } |z| = 1$$

$$z + \frac{1}{z} = z + \bar{z} = 2\,\text{Re}(z) \qquad \therefore |z| = 1$$

Question 13.
z_1, z_2 and z_3 are complex numbers such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ then $z_1^2 + z_2^2 + z_3^2$ is ______
(a) 3
(b) 2
(c) 1
(d) 0
Answer:
(d) 0
Hint:

$$|z_1| = 1 \qquad \Rightarrow |z_1|^2 = 1$$

$$\Rightarrow \quad z_1\,\bar{z}_1 = 1 \qquad \Rightarrow \bar{z}_1 = \frac{1}{z_1}$$

$$z_1 + z_2 + z_3 = 0 \qquad \Rightarrow \frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} = 0$$

$$\frac{\bar{z}_2\bar{z}_3 + \bar{z}_1\bar{z}_3 + \bar{z}_1\bar{z}_2}{\bar{z}_1\,\bar{z}_2\,\bar{z}_3} = 0 \qquad \Rightarrow \bar{z}_2\bar{z}_3 + \bar{z}_1\bar{z}_3 + \bar{z}_1\bar{z}_2 = 0$$

$$\Rightarrow \quad \overline{z_2 z_3} + \overline{z_1 z_3} + \overline{z_1 z_2} = 0$$

Question 14.

If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is _______

(a) $\frac{1}{2}$

(b) 1

(c) 2

(d) 3

Answer:

(b) 1

Hint. $\frac{z-1}{z+1}$ is purely imaginary.

$$\Rightarrow \text{Re}\left(\frac{z-1}{z+1}\right) = 0$$

$$\text{Re}\left(\frac{x+iy-1}{x+iy+1}\right) = 0$$

$$\Rightarrow \text{Re}\left(\frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{x+1-iy}\right) = 0$$

Considering only the real part

$$\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2} = 0$$

$$x^2 - 1 + y^2 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\Rightarrow |z|^2 = 1$$

$$\Rightarrow |z| = 1$$

Question 15.

If z = x + iy is a complex number such that |z + 2| = |z – 2|, then the locus of z is _____

(a) real axis

(b) imaginary axis

(c) ellipse

(d) circle

Answer:

(b) imaginary axis

Hint:

|z + 2| = | z – 2|

⇒ |x + iy + 2| = |x + iy – 2|

⇒ $|x + 2 + iy|^2 = |x - 2 + iy|^2$

⇒ $(x + 2)^2 + y^2 = (x - 2)^2 + y^2$

⇒ $x^2 + 4 + 4x = x2 + 4 - 4x$

⇒ 8x = 0

⇒ x = 0

Question 16.

The principal argument of $\frac{3}{-1+i}$ is ______

(a) $-\frac{5\pi}{6}$

(b) $-\frac{2\pi}{3}$

(c) $-\frac{3\pi}{4}$

(d) $\frac{-\pi}{2}$

Answer:

Hint:

$$\frac{3}{-1+i} = \frac{3(-1-i)}{(-1+i)(-1-i)} = \frac{3(-1-i)}{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}|1| = \frac{\pi}{4}$$

The complex number lies in III quadrant. $\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$

Question 17.

The principal argument of (sin 40° + i cos 40°)5 is ______

(a) -110°

(b) -70°

(c) 70°

(d) 110°

Answer:

(a) -110°

Hint:

(sin 40° + i cos 40°)5

= (cos 50° + i sin 50°)5

= (cos 250° + i sin 250°)

250° lies in III quadrant.

To find the principal argument the rotation must be in a clockwise direction which coincides with 250°

θ = -110°

Question 18.

If (1 + i) (1 + 2i) (1 + 3i)...(1 + ni) = x + iy, then 2.5.10......$(1 + n^2)$ is _____

(a) 1

(b) i

(c) $x^2 + y^2$

(d) $1 + n^2$

Answer:
(c) $x^2 + y^2$

Question 19.
If ω ≠ 1 is a cubic root of unity and (1 + ω)7 = A + B ω, then (A, B) equal to ______
(a) (1, 0)
(b) (-1, 1)
(c) (0, 1)
(d) (1, 1)
Answer:
(d) (1, 1)
Hint:

$$(1+\omega)^7 = A + B\omega$$
$$(-\omega^2)^7 = A + B\omega \qquad [\text{Since } 1+\omega+\omega^2 = 0;\ \omega^2 = -(1+\omega)]$$
$$-\omega^{14} = A + B\omega$$
$$-\omega^2 = A + B\omega$$
$$-[-(1+\omega)] = A + B\omega$$
$$1 + \omega = A + B\omega \qquad \Rightarrow A = 1, B = 1$$
$$(A, B) = (1, 1)$$

Question 20.
The principal argument of the complex number $\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is _____
(a) $\frac{2\pi}{3}$
(b) $\frac{\pi}{6}$
(c) $\frac{5\pi}{6}$
(d) $\frac{\pi}{2}$

Answer:
(d) $\frac{\pi}{2}$
Hint:

$$\frac{\left(1+i\sqrt{3}\right)^2}{4i\left(1-i\sqrt{3}\right)} = \frac{1-3+2i\sqrt{3}}{4(i+\sqrt{3})} = \frac{-2+2i\sqrt{3}}{4\left(\sqrt{3}+i\right)} = \frac{2\left(-1+i\sqrt{3}\right)}{4\left(\sqrt{3}+i\right)}$$
$$= \frac{i\left(\sqrt{3}+i\right)}{2\left(\sqrt{3}+i\right)} = \frac{i}{2} \qquad \Rightarrow \frac{\pi}{2}$$

Question 21.
If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is ______
(a) -2
(b) -1
(c) 1
(d) 2
Answer:
(b) -1
Hint:
$x^2 + x + 1 = 0$
α and β are the roots of the equation.
There are the two roots of cube roots of unity except 1.

$$\alpha = \omega, \beta = \omega^2$$
$$\alpha^{2020} + \beta^{2020} = \omega^{2020} + (\omega^2)^{2020} = \omega^{3(673)+1} + (\omega^{3(673)+1})^2$$
$$= \omega + \omega^2 = -1$$

Question 22.
The product of all four values of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$ is ____
(a) -2
(b) -1
(c) 1
(d) 2
Answer:
(c) 1

Hint. $$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = (\cos\pi + i\sin\pi)^{\frac{1}{4}} = \text{cis}\left(\frac{2k\pi + \pi}{4}\right)$$

The roots are $\text{cis}\left(\frac{\pi}{4}\right)$ $\text{cis}\left(\frac{3\pi}{4}\right)$ $\text{cis}\left(\frac{5\pi}{4}\right)$ $\text{cis}\left(\frac{7\pi}{4}\right)$

$$\text{Product of roots} = \text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) = \text{cis}\,(4\pi)$$
$$= \cos 0 + i\sin 0 = 1$$

Question 23.
If $\omega \neq 1$ is a cubic root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to _____
(a) 1
(b) -1
(c) $i\sqrt{3}$

(d) -i√3

Answer:

(d) -i√3

Hint:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k \qquad \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$(1+\omega+\omega^2=0) \qquad \omega = \frac{-1+i\sqrt{3}}{2}$$

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k\,(R_1 \to R_1+R_2+R_3)$$

$$3(-\omega-\omega^3-\omega^4) = 3k \qquad \Rightarrow 3(-2\omega-1) = 3k$$

$$-2\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}\right)-1 = k$$

$$1-i\sqrt{3}-1 = k \qquad \Rightarrow k = -i\sqrt{3}$$

Question 24.

The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is ______

(a) cis $\frac{2\pi}{3}$

(b) cis $\frac{4\pi}{3}$

(c) -cis $\frac{2\pi}{3}$

(d) -cis $\frac{4\pi}{3}$

Answer:

(a) cis $\frac{2\pi}{3}$

Hint:

$$1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{x^2+y^2} = \sqrt{4} = 2$$

$$\alpha = \theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\sqrt{3}\right| = \frac{\pi}{3}$$

$$\left(1+i\sqrt{3}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\left(1-i\sqrt{3}\right) = 2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$$

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left[\frac{2\text{cis}\left(\frac{\pi}{3}\right)}{2\text{cis}\left(\frac{-\pi}{3}\right)}\right]^{10} = \left[\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{3}\right)\right]^{10} = \left[\text{cis}\left(\frac{2\pi}{3}\right)\right]^{10} = \text{cis}\left(\frac{20\pi}{3}\right)$$

$$= \text{cis}\left(6\pi + \frac{2\pi}{3}\right) = \text{cis}\left(\frac{2\pi}{3}\right)$$

Question 25.

If ω = cis $\frac{2\pi}{3}$, then the number of distinct roots of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ are _____

(a) 1
(b) 2
(c) 3
(d) 4

Answer:
(a) 1
Hint:

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$ and $1 + \omega + \omega^2 = 0$

$$\begin{vmatrix} z & z & z \\ \omega & z+\omega & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$z\begin{vmatrix}1 & 1 & 1\\ \omega & z+\omega^2 & 1\\ \omega^2 & 1 & z+\omega\end{vmatrix}=0 \qquad \Rightarrow z\begin{vmatrix}1 & 0 & 0\\ \omega & z+\omega^2-\omega & 1-\omega\\ \omega^2 & 1-\omega^2 & z+\omega-\omega^2\end{vmatrix}=0$$

$$z\left[\left(z+\omega^2-\omega\right)\left(z+\omega-\omega^2\right)-\left(1-\omega^2\right)\left(1-\omega\right)\right]=0$$

$$z\left[\left(z^2-\left(\omega-\omega^2\right)^2\right)\right]-\left[1-\omega-\omega^2+\omega^3\right]$$

$$z\left\{z^2-\left(\omega^2+\omega-2\right)-3\right\} \qquad \Rightarrow z\ \left\{z^2\right\}=z^3=0$$

$z=0$ one distinct root