19. SOLUTIONS OF TRIANGLE

1. INTRODUCTION

In any triangle ABC, the side BC, opposite to the angle A is denoted by a; the side CA and AB, opposite to the angles B and C respectively are denoted by b and c respectively. The semi-perimeter of the triangle is denoted by s and its area by Δ or S. In this chapter, we shall discuss various relations between the sides a, b, c and the angles A, B, C of Δ ABC.

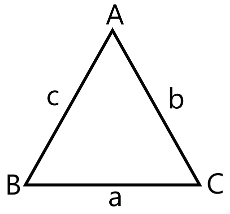


Figure 19.1

2. SINE RULE

The sides of a triangle (any type of triangle) are proportional to the sines of the angle opposite to them in triangle ABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

Note: (i) The above rule can also be written as $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

(ii) The sine rule is a very useful tool to express the sides of a triangle in terms of sines of the angle and vice-versa in the following manner: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k(\text{Let});\Rightarrow a=k\sin A,\ b=k\sin B,\ c=k\sin C$

Similarly, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\lambda(\text{Let});\ \Rightarrow \sin A=\lambda a,\ \sin B=\lambda b,\ \sin C=\lambda c$

3. COSINE RULE

In any ΔABC, $\cos A=\frac{b^2+c^2-a^2}{2bc}$; $\cos B=\frac{c^2+a^2-b^2}{2ac}$; $\cos C=\frac{a^2+b^2-c^2}{2ab}$

Note: In particular

$\angle A=60^\circ \Rightarrow b^2+c^2-a^2=bc$

$\angle B=60^\circ \Rightarrow c^2+a^2-b^2=ca$

$\angle C=60^\circ \Rightarrow a^2+b^2-c^2=ab$

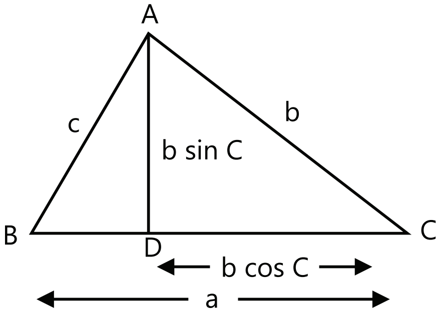


Figure 19.2

4. PROJECTION FORMULAE

If any ΔABC :(i) $a=b\cos C+c\cos B$ (ii) $b=c\cos A+a\cos C$ (iii) $c=a\cos B+b\cos A$

i.e. any side of a triangle is equal to the sum of the projection of the other two sides on it.

Case I: When ΔABC is an acute angled triangle,

$\cos B = \frac{BD}{AB} \Rightarrow BD = AB.\cos B \Rightarrow BD = c.\cos B$ and

$\cos C = \frac{CD}{AC} \Rightarrow CD = AC.\cos C \Rightarrow CD = b\cos C$

then, BD+DC=BC

$\therefore$ a = c cos B + b cos C

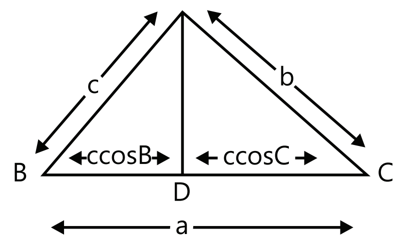


Figure 19.3

Case II: When ΔABC is an obtuse angled triangle,

$\cos C = \frac{CD}{AC} \Rightarrow CD = AC.\cos C$

$CD = b.\cos C$ and $\cos(180 - B) = \frac{BD}{AB} \Rightarrow BD = -c.\cos B$ then,

a = BC and CD – BD $\Rightarrow$ $a = b\cos C + c\cos B$

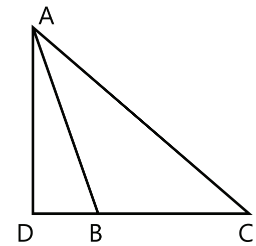


Figure 19.4

Illustration 1: If $A = 75^o$, $B = 45^o$, then what is the value of $b + c\sqrt{2}$? **(JEE MAIN)**

Sol: Here, $c = 180^o - 120^o = 60^o$. Therefore by using sine rule, we can solve the above problem.

Use sine rule $\frac{a}{\sin 75^o} = \frac{b}{\sin 45^o} = \frac{c}{\sin 60^o} = K \Rightarrow \begin{matrix} a = k\ \sin 75^o \\ b = k\ \sin 45^o \\ c = k\ \sin 60^o \end{matrix}$

consider, $\left(b + c\sqrt{2}\right) = k\left(\sin 45^o + \sqrt{2}\sin 60^0\right) = k\frac{\sqrt{3}+1}{\sqrt{2}} = 2k\frac{\sqrt{3}+1}{2\sqrt{2}} = 2k\sin 75^o = 2k\sin A = 2a$

Illustration 2: In a Δ ABC, if $B = 30^o$ and $c = \sqrt{3}b$, then find the value of A. **(JEE MAIN)**

Sol: Here, by using cosine rule $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ we can easily solve the above problem.

We have $\cos B = \frac{c^2 + a^2 - b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3b^2 + a^2 - b^2}{2 \times \sqrt{3}b \times a}$; $\Rightarrow a^2 - 3ab + 2b^2 = 0 \Rightarrow (a-2b)(a-b) = 0$

$\Rightarrow a - b = 0$ OR a – 2b = 0

$\Rightarrow$ Either $a = b \Rightarrow A = 30^o$ or $a = 2b \Rightarrow a^2 \Rightarrow 4b^2 \Rightarrow b^2 + c^2 \Rightarrow A = 90^o$.

Illustration 3: Prove that $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$ **(JEE MAIN)**

Sol: By sine rule i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ we can simply prove the above equation.

In equation $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$, putting a = k sin A, b = k sin B, c = k sin C

= $k\left(\sin A\sin(B-C) + \sin B\sin(C-A) + \sin C\sin(A-B)\right)$ =0 (expanding all terms gets cancelled)

(Using $\sin(\alpha - \beta) = \sin\alpha.\cos\beta - \sin\beta\cos\alpha$)

Illustration 4: Prove that $\sin(B-C) = \frac{b^2 - c^2}{a^2}\sin A$ **(JEE MAIN)**

Sol: Given, $\sin(B-C)=\frac{b^2-c^2}{a^2}\sin A \Rightarrow a^2\sin(B-C)=\left(b^2-c^2\right)\sin A$

Takin L.H.S., $a^2\sin(B-C)$ = a2(sinBcosC – cosBsinC)

Now using sine rule, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$ (say) and cosine rule, $\cos B=\frac{a^2+c^2-b^2}{2\ ac}$, and $\cos C=\frac{a^2+b^2-c^2}{2\ ab}$

$$=a^2\left(kb\frac{a^2+b^2-c^2}{2\ ab}-\frac{a^2+c^2-b^2}{2\ ac}\times kc\right)=ka\left(\frac{a^2+b^2-c^2-a^2+c^2-b^2}{2}\right)=\sin A\times\left(b^2-c^2\right)=\text{RHS}.$$

Illustration 5: The angles of a triangle are in 4:1:1 ratio. Find the ratio between its greater side and perimeter?

(JEE ADVANCED)

Sol: Here, the angles are 120^o, 30^o, 30^o. Therefore, by using sine rule, we will get the required ratio.

Angles are 120^o, 30^o, 30^o.

If the sides opposite to these angles are a, b and c respectively, a will be the greatest side.

Now from sine formula, $\frac{a}{\sin 120^o}=\frac{b}{\sin 30^o}=\frac{c}{\sin 30^o}$; $\Rightarrow \frac{a}{\sqrt{3}/2}=\frac{b}{1/2}=\frac{c}{1/2}$; $\Rightarrow \frac{a}{\sqrt{3}}=\frac{b}{1}=\frac{c}{1}=k$ (say)

then $a=\sqrt{3}k$, perimeter= $\left(2+\sqrt{3}\right)k$; $\therefore$ Required ratio = $\frac{\sqrt{3}k}{\left(2+\sqrt{3}\right)k}=\frac{\sqrt{3}}{2+\sqrt{3}}$.

Illustration 6: Solve $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}$ in term of k where k is perimeter of the ΔABC. **(JEE ADVANCED)**

Sol: We can solve the given problem simply by $\cos^2\theta=\frac{1+\cos 2\theta}{2}$

Here, $b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{b}{2}\left(1+\cos C\right)+\frac{c}{2}(1+\cos B)$ [using projection formula]

$=\frac{b+c}{2}+\frac{1}{2}a=\frac{a+b+c}{2}$; $\therefore b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{k}{2}$ [where k=a+b+c, given]

Illustration 7: In any triangle ABC, show that $\frac{a-b}{a+b}=\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$ **(JEE ADVANCED)**

Sol: We can derive the values of a, b and c using sine rule and putting it to L.H.S. we can prove the above problem.

We know that, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

$\Rightarrow a=k\sin A$, $b=k\sin B$, $c=k\sin C$ ...(i)

On putting the values of a and b from (1) on L.H.S., we get

$$\text{L.H.S.}=\frac{a-b}{a+b}=\frac{k\sin A-k\sin B}{k\sin A+k\sin B}=\frac{\sin A-\sin B}{\sin A+\sin B}=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$$

$$=\cot\left(\frac{A+B}{2}\right)\tan\left(\frac{A-B}{2}\right)=\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}=\text{R.H.S.}$$

5. NAPIER'S ANALOGY (LAW OF TANGENTS)

In any ΔABC, (i) $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$ (ii) $\tan\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right)$ (iii) $\tan\left(\frac{C-A}{2}\right)=\left(\frac{c-a}{c+a}\right)\cot\left(\frac{B}{2}\right)$

Proof:(i) R.H.S. $=\frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)=\frac{k\sin B-k\sin c}{k\sin B+k\sin c}\cot\left(\frac{A}{2}\right)$ [By the sine rule] $=\left(\frac{\sin B-\sin c}{\sin B+\sin c}\right)\cot\left(\frac{A}{2}\right)$

$=\left(\frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right).\cot\left(\frac{A}{2}\right)$ [By C & D formulae]

$=\tan\left(\frac{B-C}{2}\right).\cot\left(\frac{B+C}{2}\right)\Rightarrow\cot\left(\frac{A}{2}\right)=\tan\left(\frac{B-C}{2}\right).\tan\left(\frac{A}{2}\right).\cot\left(\frac{A}{2}\right)$ [By condition identities]

$=\tan\left(\frac{B-C}{2}\right)=\text{LHS}$

Similarly, (ii) and (iii) can be proved.

Illustration 8: In any triangle ABC, if $A=30^o$, b=3 and $c=3\sqrt{3}$, then find $\angle B$ and $\angle C$. **(JEE MAIN)**

Sol: By using formula, $\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}$, we can easily obtain the values of $\angle B$ and $\angle C$.

Here $\angle A=30^o$ $\therefore$ $\frac{B+C}{2}=90^o-\frac{A}{2}=90-15^o=75^o$ ... (i)

Since $c>b\Rightarrow\angle C>\angle B$ and $B+C=150^o$ ... (ii)

$\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}=\frac{c-b}{c+b}\tan\left(\frac{B+C}{2}\right);\Rightarrow\tan\frac{C-B}{2}=\frac{3\sqrt{3}-3}{3\left(\sqrt{3}+1\right)}\tan 75^o$

$\Rightarrow\tan\left(\frac{C-B}{2}\right)=\frac{c-b}{c+b}\cot\left(\frac{\pi}{2}-A\right)=\frac{c-b}{c+b}\tan\left(\frac{A}{2}\right)$

[Using (1)] $\Rightarrow\tan\frac{C-B}{2}=\frac{3\left(\sqrt{3}-1\right)}{3\left(\sqrt{3}+1\right)}\tan\left(45^o+30^o\right)=\frac{\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)}\left(\frac{\tan 45^o+\tan 30^o}{1-\tan 45^o\tan 30^o}\right)$

$=\frac{\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)}\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right)=\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)=1$; $\Rightarrow\frac{C-B}{2}=45^o$; $\left[\because\tan 45^o=1\right]$

$\Rightarrow C-B=90^o$ ... (iii)

Solving (ii) and (iii), we get $\angle B=30^o$ and $\angle C=120^o$

6. TRIGONOMETRIC RATIOS OF HALF ANGLES

Sine, cosine and tangent of half the angles of any triangle are related to their sides as given below. Note that the perimeter of ΔABC will be denoted by 2s i.e. 2s=a+b+c and the area denoted by Δ.

Formulae for $\sin\left(\frac{A}{2}\right), \sin\left(\frac{B}{2}\right), \sin\left(\frac{C}{2}\right)$ for any ΔABC

(i) $\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{ac}}$ (iii) $\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Formulae for $\cos\left(\frac{A}{2}\right), \cos\left(\frac{B}{2}\right), \cos\left(\frac{C}{2}\right)$ for any ΔABC

(i) $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\left(\frac{B}{2}\right) = \sqrt{\frac{s(s-b)}{ac}}$ (iii) $\cos\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-c)}{ab}}$

Formulae for $\tan\left(\frac{A}{2}\right), \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$ for any ΔABC

(i) $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ (iii) $\tan\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Illustration 9: In a triangle ABC, if $\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13}$, then find the value of $\tan^2(A/2)$. **(JEE MAIN)**

Sol: As we know, $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$. Therefore, by using this formula, we can solve the above problem.

$\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13} = \frac{3s-(a+b+c)}{11+12+13} = \frac{s}{36}$; Now $\tan^2\left(\frac{A}{2}\right) = \frac{(s-b)(s-c)}{s(s-a)} = \frac{12\times13}{36\times11} = \frac{13}{33}$

Illustration 10: In a triangle ABC, prove that $(a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$. **(JEE MAIN)**

Sol: Here by using $\tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ we can prove the above problem.

$$\text{L.H.S.} = (a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2s\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\right] = 2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right]$$

$$= 2s\sqrt{\frac{(s-c)}{s}}\left[\frac{s-b+s-a}{\sqrt{s-a}\sqrt{s-b}}\right] = \frac{2\sqrt{s}\sqrt{s-c}}{\sqrt{s-a}\sqrt{s-b}}(a+b+c-b-a) = 2c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{2c}{\tan\frac{C}{2}} = 2c\cot\frac{C}{2} = \text{R.H.S.}$$

Alternate: L.H.S. = $2s\left[\frac{\Delta}{s(s-a)} + \frac{\Delta}{s(s-b)}\right]$

$$= 2\Delta\left[\frac{1}{s-a} + \frac{1}{s-b}\right] = \frac{2\Delta(2s-a-b)}{(s-a)(s-b)} = \frac{2c\Delta^2}{(s-a)(s-b)\Delta} = \frac{2cs(s-c)}{\Delta} = 2c.\cot\frac{c}{2} = \text{R.H.S.}$$

Illustration 11: In a ΔABC, if $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in AP, then prove that the sides of ΔABC are in A.P. **(JEE MAIN)**

Sol: Here by using trigonometric ratios of half angles formula, we can prove the above illustration.

Given $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in A.P.

$\Rightarrow \frac{s(s-a)}{\Delta}, \frac{s(s-b)}{\Delta}, \frac{s(s-c)}{\Delta}$ are in A.P.

$\Rightarrow (s-a), (s-b), (s-c)$ are in A.P.

$\Rightarrow$ a, b, c are in A.P. **Proved**

Illustration 12: In a ΔABC, the sides a, b and c are in A.P. Then what is the value of $\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right):\cot\frac{B}{2}$? **(JEE ADVANCED)**

Sol: Simply by using formula of $\tan\frac{A}{2}$, $\tan\frac{C}{2}$ and $\cot\frac{B}{2}$ we can easily get the required result.

$$\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right):\cot\frac{B}{2} \Rightarrow \left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\right]:\sqrt{\frac{s(s-b)}{(s-c)(s-a)}} \Rightarrow \frac{(s-c)+(s-a)}{\sqrt{s}}:\sqrt{s}$$

$$= 2s-(a+c):s;\ \Rightarrow b:\frac{a+b+c}{2};\ \Rightarrow 2b:a+b+c \Rightarrow \text{ a.b.c are in A.P.}$$

$\therefore$ 2b:a+b+c = 2 : 3

Illustration 13: In any triangle ABC, show that $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{a+b-c}\cot\frac{C}{2}$. **(JEE ADVANCED)**

Sol: Similar to the above problem, by putting the values of $\cot\frac{A}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ we can prove the above problem.

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$\text{L.H.S.}\ \frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left(\sqrt{s^2(s-a)^2}+\sqrt{s^2(s-b)^2}+\sqrt{s^2(s-c)^2}\right)=\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left[s(s-a+s-b+s-c)\right]$$

$$=\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left[s\{3s-(a+b+c)\}\right]=\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}\left[s(3s-2s)\right]$$

$$\Rightarrow \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{... (i)}$$

Now, R.H.S. = $\frac{a+b+c}{a+b-c}\cot\frac{C}{2}=\frac{2s}{a+b+c-2c}\cot\frac{C}{2}=\frac{2s}{2s-2c}\cot\frac{C}{2}=\frac{s}{s-c}\cot\frac{C}{2}$

L.H.S = R.H.S

$$=\frac{s}{s-c}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}};\Rightarrow \frac{a+b+c}{a+b-c}\cot\frac{C}{2}=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{... (ii)}$$

From (i) and (ii), we have $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{a+b-c}\cot\frac{C}{2}$. **Proved**

7. AREA OF TRIANGLE

If Δ be the area of a triangle ABC, then $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

Proof: Let ABC be a triangle. Then the following cases arise.

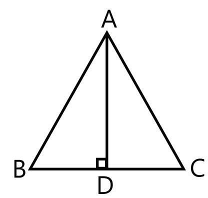


Figure 19.5

Case I: When ΔABC is an acute angled triangle, $\sin B = \frac{AD}{AB}$

AD=AB sin B ; AD=c sin B; $\Delta = \text{Area of } \Delta ABC$; $\Delta = \frac{1}{2}(BC)(AD)$; $\Delta = \frac{1}{2}ac\sin B$

Case-II: When Δ ABC is an obtuse angled triangle, $\sin(180-B) = \frac{AD}{AB}$;

AD=AB sin B $\Rightarrow$ AD=c sin B

Δ =Area of ΔABC; $\Delta = \frac{1}{2}(BC)(AD)$; $\Delta = \frac{1}{2}ac\sin B$; So in each case, $\Delta = \frac{1}{2}ac\sin B$

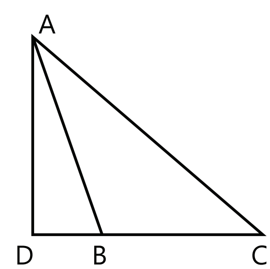


Figure 19.6

(ii) Heron's formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Proof: $\Delta = \frac{1}{2}bc\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right) = bc\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)$

$= bc\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$ [By half angle formula] = $\sqrt{s(s-a)(s-b)(s-c)}$

(iii) $\Delta = \frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)} = \frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)} = \frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$

From the above results, we obtain the following values of sinA, sinB and sinC

(iv) $\sin A = \frac{2\Delta}{bc} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$ (v) $\sin B = \frac{2\Delta}{ca} = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

(vi) $\sin C = \frac{2\Delta}{ab} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

Further with the help of (iv), (v), (vi) we obtain $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc}$

Illustration 14: In any triangle ABC, prove that $4\Delta\cot A = b^2 + c^2 - a^2$. **(JEE MAIN)**

Sol: We can prove the above problem by using formula of area of triangle i.e. $\Delta = \frac{1}{2}bc\sin A$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}$.

L.H.S.= $4\Delta\cot A = 4.\frac{1}{2}bc\sin A.\frac{\cos A}{\sin A} = 2bc\cos A = 2bc.\frac{b^2+c^2-a^2}{2bc} = b^2+c^2-a^2$ = R.H.S.

Illustration 15: In any triangle ABC, prove that $\frac{a^2-b^2}{2}.\frac{\sin A\sin B}{\sin(A-B)} = \Delta$ **(JEE MAIN)**

Sol: By putting $a = k\sin A$ and $b = k\sin B$ we can prove the above illustration.

L.H.S. $= \frac{a^2-b^2}{2}.\frac{\sin A\sin B}{\sin(A-B)} = \frac{(k^2\sin^2 A - k^2\sin^2 B)\sin A\sin B}{2\sin(A-B)} = \frac{k^2(\sin^2 A - \sin^2 B)\sin A\sin B}{2\sin(A-B)}$

[using sine formula a=k sinA etc.]= $\frac{k^2\sin(A+B)\sin(A-B)\sin A\sin B}{2\sin(A-B)}=\frac{k^2}{2}.\sin(A+B)\sin A\sin B$

$=\frac{1}{2}(k\sin A)(k\sin B)\sin(\pi-c);\ [\because A+B=\pi-C];\ \frac{1}{2}ab\sin C=\Delta=\text{R.H.S.}$

Illustration 16: A tree stands vertically on a hill side which make an angle of 15° with the horizontal. From a point on the ground 35m down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. Find the height of the tree. **(JEE MAIN)**

Sol: We can simply obtain the height of the tree from the given figure.

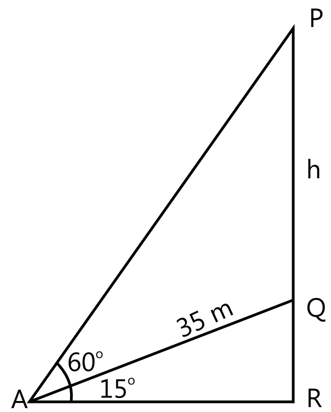


Figure 19.7

In Δ AQR, $QR=AQ\sin15^o=35\sin15^o$; $AR=AQ\cos15^o=35\cos15^o$

In ΔAPR, $\tan60^o=\frac{PR}{AR};\Rightarrow PR=AR.\sqrt{3};\ \Rightarrow PQ+QR=\sqrt{3}AR$

$\Rightarrow h+35\sin15^o=\sqrt{3}.35\cos15^o$

$\Rightarrow h=35\left(\sqrt{3}\cos15^o-\sin15^o\right)=35\left(\sqrt{3}.\frac{\sqrt{3}+1}{2\sqrt{2}}-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

$=35\left(\frac{3+\sqrt{3}-\sqrt{3}+1}{2\sqrt{2}}\right)=\frac{35}{2\sqrt{2}}(4)=35\sqrt{2}m$

Hence, the height of the tree= $35\sqrt{2}m$

Illustration 17: In any triangle ABC, prove that $a\cos A+b\cos B+c\cos C=2a\sin B\sin C=\frac{8\Delta^2}{abc}$. **(JEE ADVANCED)**

Sol: we can solve this illustration by substituting $a=k\sin A$, $b=k\sin B$, and $c=k\sin C$.

As $a=k\sin A,\ b=k\sin B,\ c=k\sin C$ ...(i)

L.H.S.= $a\cos A+b\cos B+c\cos C=k\sin A\cos A+k\sin B\cos B+k\sin C\cos C$ [using sine formula]

$=\frac{k}{2}\left[\{\sin2A+\sin2B\}+\sin2C\right]=\frac{k}{2}\left[2\sin(A+B)\cos(A-B)+2\sin C\cos C\right]$

$=\frac{k}{2}\left[2\sin C\cos(A-B)+2\sin C\cos C\right]=\frac{k}{2}\left[2\sin C\left\{\cos(A-B)+\cos\left(\pi-\overline{A+B}\right)\right\}\right]$

$=k\sin C\left[\cos(A-B)-\cos(A+B)\right]=k\sin C\left[2\sin A\sin B\right]$

$=2(k\sin A).\sin B\sin C=2a\sin B\sin C=2a\left(\frac{2\Delta}{ac}\right)\left(\frac{2\Delta}{ab}\right);\quad\left[\Delta=\frac{1}{2}ac\sin B\Rightarrow\sin B=\frac{2\Delta}{ac}\text{ and }\sin C=\frac{2\Delta}{ab}\right]$

$=\frac{8\Delta^2}{abc}$ =R.H.S.

Illustration 18: The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the point A measured along the line AB which makes an angle 30^o with AQ. Prove that $h=d\left(\sqrt{3}-1\right)$ **(JEE ADVANCED)**

Sol: By using sine rule in ΔABP, we can prove that $h=d\left(\sqrt{3}-1\right)$

In the figure, PQ represents a tower of height h. The angle of elevation of the point P from the point A on the ground is

$\Rightarrow \angle PAQ = 45^o$; $\Rightarrow \angle PAB + \angle BAQ = 45^o$; $\Rightarrow \angle PAB + 30^o = 45^o$

$\angle PAB = 45^o - 30^o$ [Given $\angle BAQ = 30^o$]

$\angle PAB = 15^o$ ... (i)

(Given) $\angle APH = 45^o$; $\Rightarrow \angle APB + \angle BPH = 45^o$ (given) $\Rightarrow \angle APB + 30 = 45^o \Rightarrow \angle APB = 15^o$ ... (ii)

From (i) and (ii), we have $\angle PAB = \angle APB$ So BP=AB=d; $\Rightarrow BP = d$

[Given AB=d]

Again $\angle PAQ = 45^o$, $\angle Q = 90^o \Rightarrow \angle APQ = 45^o$

In ΔAPQ, $\angle PAQ = \angle APQ \Rightarrow AQ = PQ = h$

$AP^2 = PQ^2 + AQ^2 = h^2 + h^2 \Rightarrow AP^2 = 2h^2 \Rightarrow AP = \sqrt{2}h$

Applying sine formula in ΔABP, we get $\frac{AB}{\sin 15^o} = \frac{AP}{\sin 150^o}$

$\Rightarrow \frac{d}{\sin 15^o} = \frac{\sqrt{2}h}{\sin 150^o} \Rightarrow d = \frac{\sqrt{2}h}{\frac{1}{2}}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \Rightarrow d = \left(\sqrt{3}-1\right)h$

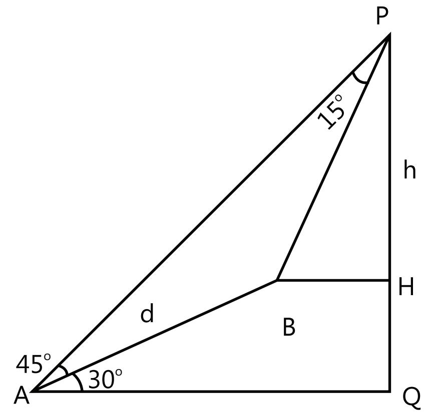


Figure 19.8

Illustration 19: A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC=7m, CA=8m and AB=9m. This lamp post subtends an angle $\tan^{-1}(3)$ at the point B. Determine the height of the lamp post. **(JEE ADVANCED)**

Sol: Here in $\Delta BMP \Rightarrow \tan\angle PBM = \frac{PM}{BM}$, therefore by obtaining the value of BM we can find out the height of lamp post.

Here, ABC is a triangular plot. A lamp post PM is situated at the mid-point M of the side AC. Here PM subtends an angle $\tan^{-1}(3)$ at the point B. a=7m, b=8m and c=9m

In ΔABC, $\cos C = \frac{a^2+b^2-c^2}{2ab}$ or $\cos C = \frac{BC^2+CA^2-AB^2}{2BC.CA}$

$\cos C = \frac{7^2+8^2-9^2}{2\times7\times8} = \frac{49+64-81}{112} = \frac{32}{112} = \frac{2}{7}$ ... (i)

In ΔBCM, $\cos C = \frac{BC^2+CM^2-BM^2}{2BC.CM}$; $\cos C = \frac{7^2+4^2-BM^2}{2\times7\times4} = \frac{65-BM^2}{56}$

$\Rightarrow \frac{2}{7} = \frac{65-BM^2}{56} \Rightarrow 65 - BM^2 = \frac{2}{7}\times56 = 16$ [Using(i)]

$\Rightarrow BM^2 = 65 - 16 = 49 \Rightarrow BM = 7m$

In $\Delta BMP \Rightarrow \tan\angle PBM = \frac{PM}{BM} \Rightarrow \tan\left(\tan^{-1}3\right) = \frac{PM}{BM} \Rightarrow 3 = \frac{PM}{7} \Rightarrow PM = 21m$

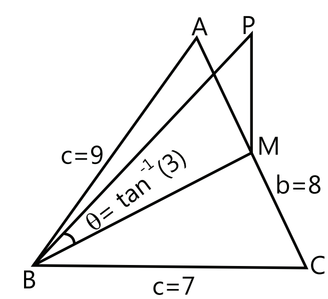


Figure 19.9

Hence, the height of the lamp post =21m.

8. PROPERTIES OF TRIANGLE

8.1 Circumcircle

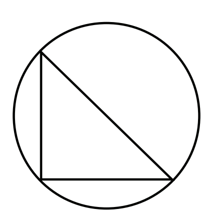


Figure 19.10

A circle passing through the vertices of a triangle is called a circumcircle of the triangle. The centre of the circumcircle is called the circumcentre of the triangle and it is the point of intersection of the perpendicular bisectors of the sides of the triangle. The radius of the circumcircle is called the circumradius of the triangle and is usually denoted by R and is given by the following formulae: $R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta S}$ Where Δ is area of triangle and $s=\frac{a+b+c}{2}$.

8.2 Incircle

The circle which can be inscribed within the triangle so as to touch all the three sides of the triangle is called the incircle of the triangle. The centre of the incircle is called the incentre of the triangle and it is the point of intersection of the internal bisectors of the angles of the triangle. The radius of the circle is called the inradius of the triangle and is usually denoted by rin-Radius: The radius r of the inscribed circle of a triangle ABC is given by

(a) $r=\frac{\Delta}{s}$ (ii) $r=(s-a)\tan\left(\frac{A}{2}\right)$, $r=(s-b)\tan\left(\frac{B}{2}\right)$ and $r=(s-c)\tan\left(\frac{C}{2}\right)$

(b) $r=\frac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$, $r=\frac{b\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$ and $r=\frac{c\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

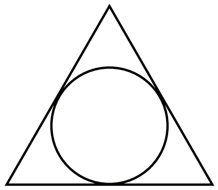


Figure 19.11

(c) $r=4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$

8.3 Centroid

In ΔABC, the mid-points of the sides BC, CA and AB are D,E and F respectively. The lines AD, BE and CF are called medians of the triangle ABC. The points of concurrency of three medians is called the centroid. Generally it is represented by G.

Also, $AG=\frac{2}{3}AD, BG=\frac{2}{3}BE$ and $CG=\frac{2}{3}CF.$

Length of medians from Figure 9.12

$\Rightarrow AD^2=b^2+\frac{a^2}{4}-ab\left(\frac{b^2+a^2-c^2}{2ab}\right)$

$\Rightarrow AD^2=\frac{2b^2+2c^2-a^2}{4}\Rightarrow AD=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Similarly, $BE=\frac{1}{2}\sqrt{2a^2+2c^2-b^2}$ and $CF=\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$

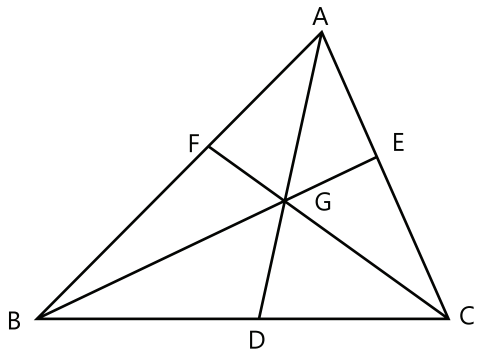


Figure 19.12

8.4 Apollonius Theorem

$AB^2+AC^2=2\left(AD^2+BD^2\right)$

Proof: $2\left(AD^2+BD^2\right)=2\left[\frac{1}{2}\left(2b^2+2c^2-a^2\right)+\frac{a^2}{4}\right]=b^2+c^2=AB^2+AC^2$

8.5 Orthocentre

The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called its orthocentre.Let the perpendicular AD,BE and CF from the vertices A,B and C on the opposite sides BC,CA and AB of ABC, respectively meet at O.

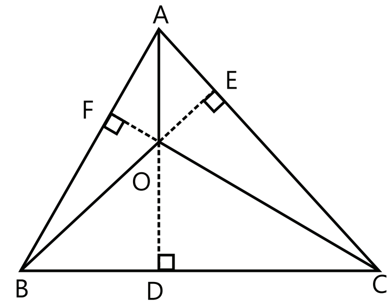


Figure 19.13

Then O is the orthocentre of the ΔABC. The triangle DEF is called the pedal Triangle of the ΔABC.

Centroid (G) of a triangle is situated on the line joining its circumcentre (O) and orthocenter (H) show that the line divides joining its circumcentre (O) and orthocenter (H) in the ratio 1:2.

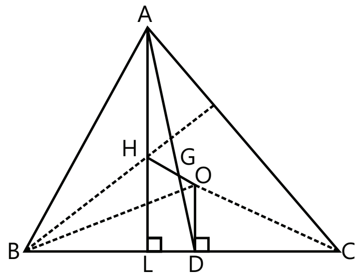


Figure 19.14

Proof: Let AL be a perpendicular from A on BC, then H lies on AL. If OD is perpendicular from O on BC, then D is mid-point of BC.

$\therefore$ AD is a median of ΔABC. Let the line HO meet the median AD at G. Now, we shall prove that G is the centroid of the ΔABC. Obviously, ΔOGD and ΔHGA are similar triangles.

$$\therefore \frac{OG}{HG} = \frac{GD}{GA} = \frac{OD}{HA} = \frac{R\cos A}{2R\cos A} = \frac{1}{2}$$

$\therefore GD = \frac{1}{2}GA \Rightarrow G \Rightarrow$ is centroid of ΔABC and $OG : HG = 1 : 2$

The distances of the orthocenter from the vertices and the sides: If O is the orthocenter and DEF the pedal triangle of the ΔABC, where AD, BE, CF are the perpendiculars drawn from A,B,C on the opposite sides BC,CA,AB respectively, then

(i) OA = 2R cosA, OB = 2R cosB and OC = 2R cosC

(ii) OD = 2R cosBcosC, OE=2R cosCcosA and OF =2R cosAcosB, where R is circumradius.

(iii) The circumradius of the pedal triangle = $\frac{R}{2}$

(iv) The area of pedal triangle $= 2\Delta\cos A\cos B\cos C$.

(v) The sides of the pedal triangle are $a\cos A$, $b\cos B$ and $c\cos C$ and its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

(vi) Circumradii of the triangles OBC, OCA, OAB and ABC are equal.

PLANCESS CONCEPTS

- The circumcentre, centroid and orthocentre are collinear.
- In any right angled triangle, the orthocentre coincides with the vertex containing the right angle.
- The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of a triangle.
- The mid-point of the hypotenuse of a right angled triangle is the circumcentreof the triangle.
- The centroid of the triangle lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1:2

Vaibhav Krishnan (JEE 2009,AIR 22)

9. PEDAL TRIANGLE

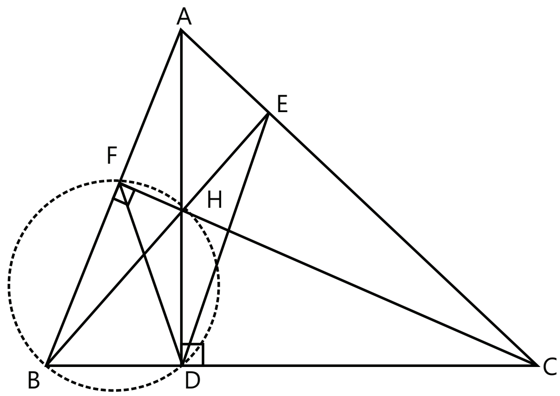


Figure 19.15

The triangle formed by the feet of the altitudes on the side of a triangle is called a pedal triangle.

In an acute angled triangle, orthocentre of ΔABC is the in-centre of the pedal triangle DEF.

Proof: Points F,H,D and B are concyclic

$\Rightarrow \angle FDH = \angle FBH = \angle ABE = \frac{\pi}{2} - A$

Similarly, points D, H, E and C are concyclic

$\Rightarrow \angle HDE = \angle HCE = \angle ACF = \frac{\pi}{2} - A$

Thus, $\angle FDH = \angle HDE \Rightarrow AD$ is the angle bisector of $\angle FDE$. Hence, altitudes of ΔABC are internal angle bisectors of the pedal triangle. Thus, the orthocentre of Δ ABC is the incentre of the pedal triangle DEF.

Sides of pedal triangle in acute angled triangle

In $\Delta AEF, AF = b\cos A, AE = c\cos A$

By cosine rule, $EF^2 = AE^2 + AF^2 - 2AE \times AF\cos(\angle EAF)$

$\Rightarrow EF^2 = b^2\cos^2 A + c^2\cos^2 A - 2bc\cos^3 A$

$\Rightarrow EF^2 = \cos^2 A\left(b^2 + c^2 - 2bc\cos A\right) = \cos^2 A\left(a^2\right) \quad \Rightarrow EF = a\cos A$

Circumradius of pedal triangle

Let circumradius be $R' \Rightarrow 2R' = \frac{EF}{\sin(\angle EDF)} = \frac{a\cos A}{\sin(\pi - 2A)} = \frac{a\cos A}{2\sin A\cos A} = \frac{a}{2\sin A} = R \Rightarrow R' = R/2$

PLANCESS CONCEPTS

- The circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocenter of the given triangle. This circle is known as "Nine point circle".
- The circumcentre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to the orthocentre.
- It also passes through midpoint of the line segment from each vertex to the orthocenter.
- Orthocenter of triangle is in centre of pedal triangle.

Shrikant Nagori (JEE 2009, AIR 30)

10. ESCRIBED CIRCLES OF THE TRIANGLE

The circle which touches the sides BC and two sides AB and AC produced of a triangle ABC is called the escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C respectively. The centres of the escribed circles are called the ex-centres. The centre of the escribed circle opposite to the angle A is the point of intersection of the external bisector of angles B and C. The internal bisector also passes through the same point. This centre is generally denoted by I_1.

Formulae for r_1, r_2, r_3

In any ΔABC, we have (i) $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2},\ r_2 = s\tan\frac{B}{2},\ r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = a\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}},\ r_2 = b\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}},\ r_3 = c\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},\ r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2},\ r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

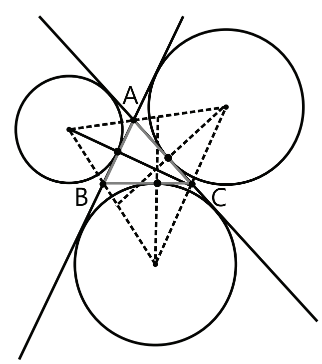


Figure 19.16

PLANCESS CONCEPTS

If I_1 is the centre of the escribed circle opposite to the angle B, then

$OI_1 = R\sqrt{1+8\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$; $OI_2 = R\sqrt{1+8\cos\frac{A}{2}.\sin\frac{B}{2}.\cos\frac{C}{2}}$; $OI_3 = R\sqrt{1+8\cos\frac{A}{2}.\cos\frac{B}{2}.\sin\frac{C}{2}}$

Where R is circum radius

- The Sum of the opposite angles of a cyclic quadrilateral is 180^{o}.
- In a cyclic quadrilateral, the sum of the products of the opposites is equal to the product of diagonals. This is known as Ptolemy's theorem.
- If the sum of the opposite sides of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.
- If I_1, I_2 and I_3 are the centres of escribed circles which are opposite to A,B and C respectively and I is the centre of the incircle, then triangle ABC is the pedal triangle of the triangle $I_1I_2I_3$ and I is the orthocenter of triangle $I_1I_2I_3$.
- The circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocenter of the given triangle. This circle is also known as nine point circle.
- The circumradius of a cyclic quadrilateral, $R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$

Nitish Jhawar (JEE 2009, AIR 7)

11. LENGTH OF ANGLE BISECTOR AND MEDIANS

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$ and $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$. Note that $m_a^2+m_b^2+m_c^2 = \frac{3}{4}\left(a^2+b^2+c^2\right)$

Illustration 20: The ratio of the circumradius and in radius of an equilateral triangle is_______ **(JEE MAIN)**

Sol: Here, as we know, all angles of an equilateral triangle are 60^0, therefore by using formula of Circumradius and In radius we can obtain the required ratio.

$\frac{r}{R} = \frac{a\cos A + b\cos B + c\cos C}{a+b+c}$. In an equilateral triangle, $60^0 = A = B = C = \frac{(a+b+c)\cos 60^0}{(a+b+c)} = \frac{1}{2}$

Illustration 21: In a ΔABC, a=18 and b=24cm and c=30cm then find the value of r_1, r_2 and r_3. **(JEE MAIN)**

Sol: As we know, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$ and $r_3 = \frac{\Delta}{s-c}$. Hence, we can solve the above problem by using this formula.

a=18cm, b=24cm,c=30cm; $\therefore 2s = a+b+c = 72cm$; s=36cm But, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = 216$ sq. units Then, $r_1 = \frac{\Delta}{s-a} = \frac{216}{18} = 12cm$; $r_2 = \frac{\Delta}{s-b} = \frac{216}{12} = 18cm$; $r_3 = \frac{\Delta}{s-c} = \frac{216}{6} = 36cm$

So, r_1, r_2, r_3 are 12cm, 18cm, and 36cm respectively.

Illustration 22: If the exradii of a triangle are in HP, the corresponding sides are in ____ **(JEE MAIN)**

Sol: Here, in this problem, r_1, r_2 and r_3 are in H.P.

$$\Rightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3} \text{ are in A.P.} \Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta} \text{ are in A.P.} \Rightarrow s-a, s-b, s-c \text{ are in A.P.}$$

$$\Rightarrow -a, -b, -c \text{ are in A.P.} \Rightarrow a, b, c \text{ are in A.P.}$$

Illustration 23: Find the value of $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$. **(JEE ADVANCED)**

Sol: By using $r_1=\frac{\Delta}{s-a}$, $r_2=\frac{\Delta}{s-b}$ and $r_3=\frac{\Delta}{s-c}$, we can solve the above problem.

$$\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3}=(b-c)\left(\frac{s-a}{\Delta}\right)+(c-a)\left(\frac{s-b}{\Delta}\right)+(a-b)\left(\frac{s-c}{\Delta}\right)$$

$$=\frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$$

$$=\frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}=\frac{0}{\Delta}=0$$

Illustration 24: Find the value of the $r\cot\frac{B}{2}\cot\frac{C}{2}$. **(JEE ADVANCED)**

Sol: Here, in this problem, r=4RsinA/2.sinB/2.sinC/2. By putting this value, we can solve the above problem.

$$r\cot B/2.\cot C/2 = 4R\sin A/2\sin B/2.\sin C/2.\frac{\cos B/2}{\sin B/2}.\frac{\cos C/2}{\sin C/2} \quad [\text{as } r=4R\sin A/2.\sin B/2.\sin C/2]$$

$$= 4R.\sin A/2.\cos B/2.\cos C/2 = r_1 \{\text{as}, r_1 = 4R\sin A/2.\cos B/2.\cos C/2\}$$

$$\therefore r\cot B/2.\cot C/2 = r_1$$

12. EXCENTRAL TRIANGLE

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle. Note that:

(i) The incentre I of ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

(ii) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(iii) The sides of the excentral triangle are

$4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$ and

Its angles are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

(iv) Distance between the incentre and excentre

$$II_1 = 4R\sin\frac{A}{2};\ II_2 = 4R\sin\frac{B}{2};\ II_3 = 4R\sin\frac{C}{2}.$$

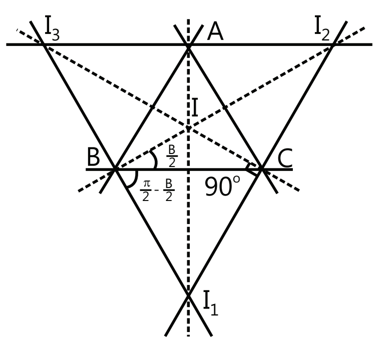


Figure 19.17

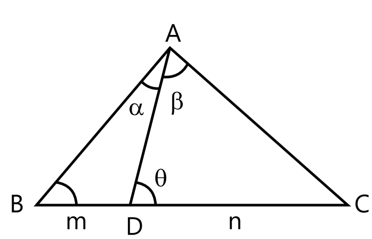


Figure 19.18

13. M-N THEOREM (RATIO FORMULA)

If D be a point on the side BC of a ΔABC such that BD:DC = m:n and $\angle ADC = \theta$, $\angle BAC = \alpha$ and $\angle DAC = \beta$.

(a) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$ (b) $(m+n)\cot\theta = n\cot B - m\cot C$

Proof: (a) Given that, $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$

$\because \angle ADB = (180^o - \theta)$; $\angle BAD = \alpha$ and $\angle DAC = \beta$

$\therefore \angle ABD = 180^o - (\alpha + 180^o - \theta) = \theta - \alpha$ and $\angle ACD = 180^o - (\theta + \beta)$

From $\Delta ABD, \frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta-\alpha)}$ ... (i)

From $\Delta ADC, \frac{DC}{\sin\beta} = \frac{AD}{\sin\left[180^o - (\theta+\beta)\right]}$ or $\frac{DC}{\sin\beta} = \frac{AD}{\sin(\theta+\beta)}$ ... (ii)

dividing (i) by (ii), then $\frac{BD\sin\beta}{DC\sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$ or $\frac{m}{n}\frac{\sin\beta}{\sin\alpha} = \frac{\sin\theta.\cos\beta + \cos\theta.\sin\beta}{\sin\theta.\cos\alpha - \cos\theta\sin\alpha}$

or $m\sin\theta\sin\beta\cos\alpha - m\cos\theta.\sin\alpha.\sin\beta = n\sin\alpha\sin\theta\cos\beta + n\sin\alpha\cos\theta\sin\beta$

$m\cot\alpha - m\cot\theta = n\cot\beta + n\cot\theta$ [dividing both sides by $\sin\alpha\sin\beta\sin\theta$] or $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(b) Given $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$; $\therefore \angle ADB = 180^o - \theta$; $\angle ABD = B$ and $\angle ACD = C$

and $\angle BAD = 180^o - (180^o - \theta + B) = \theta - B$; $\therefore \angle DAC = 180^o - (\theta + C)$

and now from $\Delta ABD. \frac{BD}{\sin(\theta - B)} = \frac{AD}{\sin B}$ ... (i)

and from $\Delta ADC, \frac{DC}{\sin\left[180^o - (\theta + C)\right]} = \frac{AD}{\sin C}$ or $\frac{DC}{\sin(\theta+C)} = \frac{AD}{\sin C}$ ... (ii)

dividing (i) by (ii), then $\frac{BD}{DC}.\frac{\sin(\theta+C)}{\sin(\theta-B)} = \frac{\sin C}{\sin B}$ or, $\frac{m}{n}\frac{\sin\theta\cos C + \cos\theta\sin C}{\sin\theta\cos B - \cos\theta\sin B} = \frac{\sin C}{\sin B}$

or, $m\sin\theta\cos C\sin B + m\cos\theta\sin C\sin B = n\sin\theta\sin C\cos B - n\cos\theta\sin B\sin C$

or, $m\cot C + m\cot\theta = n\cot B - n\cot\theta$ [dividing both sides by $\sin B\sin C\sin\theta$]

or, $(m+n)\cot\theta = n\cot B - m\cot C$

Illustration 25: In a triangle ABC, if $\cot\frac{A}{2}\cot\frac{B}{2} = c$, $\cot\frac{B}{2}\cot\frac{C}{2} = a$ and $\cot\frac{C}{2}\cot\frac{A}{2} = b$,

then find the value of $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}$. **(JEE MAIN)**

Sol: Here, by using trigonometric ratios of half angle, we can solve above problem.

$$\cot\frac{A}{2}\cot\frac{B}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)} \times \frac{s(s-b)}{(s-c)(s-a)}} = c;\ \frac{s}{s-c} = c \Rightarrow \frac{1}{s-c} = \frac{c}{s}$$

Similarly $\frac{1}{s-a} = \frac{a}{s}$ and $\frac{1}{s-b} = \frac{b}{s}$

So that $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{a+b+c}{s} = \frac{2s}{s} = 2$

14. SOLUTION OF DIFFERENT TYPES OF TRIANGLE

In a triangle, there are six elements- three sides and three angles. In plane geometry, we have done that if three of the elements are given, at least one of which must be side, then the other three elements can be uniquely determined. The procedure of determining unknown elements from the known elements is called solving a triangle.

Solution of a right angled triangle

Case I: When two sides are given: Let the triangle be right angled at C. Then we can determine the remaining elements as given in the table.

	Given	Required
(i)	a, b	$\tan A = \frac{a}{b}$, $B = 90^o - A$, $c = \frac{a}{\sin A}$
(ii)	a, c	$\sin A = \frac{a}{c}$, $b = c\cos A$, $B = 90^o - A$

Case II: When a side and an acute angle are given: In this case, we can determine the remaining elements as given in the table.

	Given	Required
(i)	a, A	$B = 90^o - A$, $b = a\cot A$, $c = \frac{a}{\sin A}$
(ii)	c, A	$B = 90^o - A$, $a = c\sin A$, $b = c\cos A$

Solution of a triangle in general

Case I: When three sides a, b, c are given: In this case, the remaining elements are determined by using the following formulae. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where 2s = a + b + c

$$\sin A = \frac{2\Delta}{bc}, \sin B = \frac{2\Delta}{ac}, \sin C = \frac{2\Delta}{ab}. \text{ OR } \tan\left(\frac{A}{2}\right) = \frac{\Delta}{s(s-a)}, \tan\left(\frac{B}{2}\right) = \frac{\Delta}{s(s-b)}, \tan\left(\frac{c}{2}\right) = \frac{\Delta}{s(s-c)}$$

Case II: When two sides a, b and the included angle C are given: In this case, we use the following formulae:

$$\Delta = \frac{1}{2}ab\sin C,\ \tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{2C}{2}\right);\ \frac{A+B}{2} = 90^o - \frac{C}{2} \text{ and } c = \frac{a\sin C}{\sin A}$$

Case III: When one side a and two angle A and B are given: In this case, we use the following formulae to determine the remaining elements.

$$A + B + C = 180^o;\ C = 180^o - (A+B) \text{ and } c = \frac{a\sin C}{\sin A};\ \Delta = \frac{1}{2}ca\sin B$$

Case IV: When two sides a, b and the A opposite to one side is given: In this case, we use the following formulae.

$$\sin B = \frac{b}{a}\sin A \qquad \text{... (i)}$$

$$C = 180^o - (A+B),\ c = \frac{a\sin C}{\sin A}$$

From (i), the following possibilities will arise:

When A is an acute angle and $a < b\sin A$.

In this case, the relation $\sin B = \frac{b}{a}\sin A$ gives that $\sin B > 1$, which is impossible. Hence no triangle is possible.

When A is an acute angle and $a = b\sin A$.

In this case, only one triangle is possible which is right angled at B.

When A is an acute angle and $a > b\sin A$

In this case, there are two values of B given by $\sin B = \frac{b\sin A}{a}$ say B_1 and B_2 such that $B_1 + B_2 = 180^o$ and side c can be obtained by using $c = \frac{a\sin C}{\sin A}$

Some useful results:

Solution of oblique triangles:

The triangle which are not right angled are known as oblique triangles. The problems on solving an oblique triangle lie in the following categories:

(a) When three sides are given

(b) When two side and included angle are given

(c) When one side and two angles are given

(d) When all the three angles are given

(e) Ambiguous case in solution of triangle

When the three sides are given: When three sides a, b, c of a triangle are given, then to solve it, we have to find its three angles A,B,C. For this cosine rule can be used.

When two sides and included angle are given: Problem based on finding the angles when any two sides and the angles between them or any two sides and the difference of the opposite angles to them are given, Napier's analogy can be used.

When one side and two angles are given: Problems based on finding the sides and angles when any two and side opposite to one of them are given, then sine rule can be used.

When all the three angles are given: In this case unique solution of triangle is not possible. In this case only the ratio of the sides can be determined.

For this the formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ can be used

Ambiguous case in solution of triangles: When any two sides and one of the corresponding angles are given, under certain additional conditions, two triangles are possible. The case when two triangles are possible is called the ambiguous case.

In fact, when any two sides and the angle opposite to one of them are given either no triangle is possible or only one triangle is possible or two triangles are possible.

Now, we will discuss the case when two triangles are possible.

Illustration 26: Solve the triangle, if $b = 72.95$, $c = 82.31$, $B = 42^o47'$ **(JEE MAIN)**

Sol: By using sine rule i.e. $\frac{\sin C}{c} = \frac{\sin B}{b}$, we can solve the given triangle.

(i) To find C $\frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow \sin C = \frac{c\sin B}{b} = \frac{82.31 \times \sin 42^o 47'}{72.95} = 0.7663$

$C = \sin^{-1}(0.7663)$ $C_1 = 50^o 1'12''$ and $C_2 = 129^o 58'48''$

I solution	II solution
$C = 50^o1'12"$ $A = 180^o - (B + C)$ $A = 180^o - (42^o47' + 50^o1'12") = 87^o11'48"$ To find a $\frac{a}{\sin A} = \frac{b}{\sin B}$ $a = \frac{b\sin A}{\sin B} = \frac{72.95 \times \sin 87^o11'48"}{\sin 42^o27'}$ $a = 107.95$	$C = 129^o58'48"$ $A = 180^o - (B + C)$ $= 180^o - (42^o47' + 129^o58'48") = 7^o14'12"$ To find a $\frac{a}{\sin A} = \frac{b}{\sin B}$ $a = \frac{b\sin A}{\sin B} = \frac{72.95 \times \sin 7^o14'12"}{\sin 42^o27'}$ $a = 13.62$

$\therefore$ Two solutions are

$C_1 = 50^o1'12"$ $A_1 = 87^o11'48"$ $a_1 = 107.95$ $C_2 = 129^o58'48"$ $A_2 = 7^o14'12"$ $a_2 = 13.62$

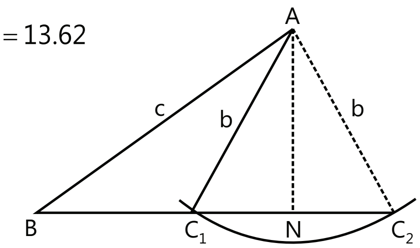


Figure 19.19

Geometrically, we draw the triangle with given data c,b and angle B.

(a) If $AN(= c\sin B) = b$ (exactly). The triangle is a right angled triangle.

(b) If $AN(= c\sin B) > b$, the triangle cannot be drawn.

(c) If $AN(= c\sin B) < b < c$, two triangles are possible.

(d) b>c, only one triangle is possible.

Illustration 27: In a triangle ABC, b=16cm, c=25cm, and $B = 33^o15'$. Find the angle C. **(JEE MAIN)**

Sol: Simply by using sine rule, we can find out the angle C.

We know that, $\frac{\sin C}{c} = \frac{\sin B}{b}$ [Here, b=16cm, c=25cm, B= $33^o15'$]

$\sin C = \frac{c}{b}\sin B = \frac{25\sin 33^o15'}{16} = 0.8567$; $C = \sin^{-1}(0.8567) = 58^o57'$; $C_1 = 58^o57'$; $C_2 = 180^o - 58^o57' = 121^o3'$

PROBLEM SOLVING TACTICS

In the application of sine rule, the following points are to be noted. We are given one side a and some other side x is to be found. Both these are in different triangles. We choose a common side y of these triangles. Then apply sine rule for a and y in one triangle and for x and y for the other triangle and eliminate y. Thus, we will get the unknown side x in terms of a.

In the adjoining figure, a is the known side of ΔABC and x is the unknown side of triangle ACD. The common side of these triangles is AC=y(say). Now, apply sine rule.

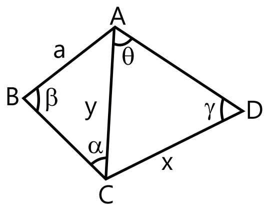


Figure 19.20

$\therefore \frac{a}{\sin\alpha} = \frac{y}{\sin\beta}$ ...(i) and $\frac{x}{\sin\theta} = \frac{y}{\sin\gamma}$ ... (ii)

Dividing (ii) by (i) we get, $\frac{x\sin\alpha}{a\sin\theta} = \frac{\sin\beta}{\sin\gamma}$; $\therefore x = \frac{a\sin\beta\sin\theta}{\sin\alpha\sin\gamma}$

In case of generalized triangle problems, option verification is very useful using equilateral, isosceles or right angle triangle properties. So, it is advised to remember properties of these triangles.

FORMULAE SHEET

(a) In ΔABC, $\angle A + \angle B + \angle C = \pi$

(a) $\sin(B+C) = \sin(\pi - A) = \sin A$

(b) $\cos(C+A) = \cos(\pi - B) = -\cos B$

(c) $\sin\frac{A+B}{2} = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$

(d) $\cos\frac{B+C}{2} = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2}$

(b) **Sine rule:** In, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ Where R = Circumradius and a, b, c are sides of triangle.

(c) **Cosine rule:** $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

(d) **Trigonometric ratios of half – angles:**

(a) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ where 2s = a + b + c; **(b)** $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; **(c)** $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(e) **Area of a triangle :** $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

(f) **Heron's formula :** $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.

(g) **Circumcircle Radius :** $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$

(h) **Incircle Radius :** **(a)** $r = \frac{\Delta}{s}$; **(b)** $r = (s-a)\tan\left(\frac{A}{2}\right)$, $r = (s-b)\tan\left(\frac{B}{2}\right)$ and $r = (s-c)\tan\left(\frac{C}{2}\right)$

(i) **Radius of the Escribed Circle :**

(a) $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

(b) $r_1 = s\tan\frac{A}{2}$, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$

(c) $r_1 = a\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$, $r_2 = b\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$, $r_3 = c\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(d) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, $r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$, $r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(j) **Length of Angle bisector and Median:**

$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ and $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$ $\Rightarrow m_a$ - length of median, β_a - length of bisector.

Solved Examples

JEE Main/Boards

Example 1: In any triangle PQR, prove that, $(b+c)\cos P+(c+a)\cos Q+(a+b)\cos R = a+b+c$.

Sol: Simply, by using projection rule, we can solve the above problem.

given L.H.S. $(b+c)\cos P+(c+a)\cos Q+(a+b)\cos R$

$= b\cos P + c\cos P + c\cos Q + a\cos Q + a\cos R + b\cos R$

$= (b\cos P + a\cos Q) + (c\cos P + a\cos R) + (c\cos Q + b\cos R)$

$= c + b + a = $ R.H.S. [By using projection Rule]

Example 2: In any ΔABC,

If $a = 2, b = \sqrt{3}+1$ and $C = 60^o$, solve the triangle.

Sol: Here, by using $\tan\frac{B-A}{2} = \frac{b-a}{b+a}\cot\frac{C}{2}$,

we can obtain the value of B – A.

Two sides and the included angle is given.

$\therefore \tan\frac{B-A}{2} = \frac{b-a}{b+a}\cot\frac{C}{2} = \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2}\cot 30^o = \frac{\sqrt{3}-1}{\sqrt{3}+3}\sqrt{3}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\tan 60^o - \tan 45^o}{1+\tan 60^o \tan 45^o}$

$\tan(60^o - 45^o) = \tan 15^o$

$\therefore \frac{B-A}{2} = 15^o$ or $B - A = 30^o$ ...(i)

We know, $A + B + C = 180^o$

$\Rightarrow A + B = 120^o$ ...(ii)

Solving (i) and (ii), we get $B = 75^o$ & $A = 45^o$

To find side c, we use sine rule

$\frac{a}{\sin A} = \frac{c}{\sin 60^o}$; or $c = 2\sqrt{2}\frac{\sqrt{3}}{2} = \sqrt{6}$

Thus $A = 45^o$, $B = 75^o$ and $c = \sqrt{6}$.

Example 3: If $A = 30^o, a = 100, c = 100\sqrt{2}$, solve the triangle

Sol: Here, simply by using sine rule, we can obtain the required values.

$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \sin C = \frac{1}{\sqrt{2}} \quad \therefore C = 135^o \text{ or } 45^o$

$C = 45^o \Rightarrow B = 105^o$; $b = \frac{c\sin B}{\sin C} = \frac{100}{\sqrt{2}}(\sqrt{3}+1)$

$C = 135^o \Rightarrow B = 15^o$; $b = \frac{c\sin B}{\sin C} = \frac{100}{\sqrt{2}}(\sqrt{3}-1)$

Example 4: In a triangle ABC, if a=3, b=4 and $\sin A = \frac{3}{4}$, then find the value of $\angle B$.

Sol: By using sine rule, we can obtain $\angle B$.

We have, $\frac{\sin A}{a} = \frac{\sin B}{b}$ or $\sin B = \frac{b}{a}\sin A$

Since, a=3, b=4, sinA=3/4,

We get, $\sin B = \frac{4}{3}\times\frac{3}{4} = 1$

$\therefore \angle B = 90^o$

Example 5: Find the smallest angle of the triangle whose sides are $6+\sqrt{12}, \sqrt{48}, \sqrt{24}$.

Sol: The smallest angles of a triangle are those angles whose opposite sides are small.

Let $a = 6+\sqrt{12}, b = \sqrt{48}, c = \sqrt{24}$

Here, c is the smallest side.

$\angle C$ is the smallest angle of the triangle.

Now $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$= \frac{(48+24\sqrt{3})+48-24}{4(3+\sqrt{3}).4\sqrt{3}} = \frac{\sqrt{3}}{2}$

So, $\angle C = \pi/6$

Example 6: In a ΔABC, $\tan A \tan B \tan C = 9$. For such triangles, if $\tan^2 A + \tan^2 B + \tan^2 C = \lambda$ then find the value of λ.

Sol: Here, by solving

$(\tan A - \tan B)^2 + (\tan B - \tan C)^2 + (\tan C - \tan A)^2 > 0$,

we can obtain the value of λ.

$\Rightarrow 2\begin{pmatrix}\tan^2 A + \tan^2 B + \tan^2 C - \tan A \tan B \\ -\tan B \tan C - \tan C \tan A\end{pmatrix} > 0$

$\Rightarrow 3\left(\tan^2 A+\tan^2 B+\tan^2 C\right)-\left(\tan A+\tan B+\tan C\right)^2>0$

$\Rightarrow 3\lambda-\left(\tan A\cdot\tan B\cdot\tan C\right)^2>0;$

$\Rightarrow 3\lambda-81>0\ \therefore \lambda>27$

Example 7: In a triangle, a, b and A are given and c_1, c_2 are two values of the third side c. Find the sum of the areas of two triangles with side a, b, c_1 and a, b, c_2.

Sol: Here, as we know $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$, therefore, by solving this equation we can obtain c_1 and c_2.

$\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ or $c^2-(2b\cos A)c+\left(b^2-a^2\right)=0$

Which is a quadratic in c, whose roots are c_1 and c_2;

$\therefore c_1+c_2=2b\cos A$ and $c_1c_2=b^2-a^2$

$\therefore$ Sum of areas of two Δ's with sides a, b, c_1 & a, b, c_2;

$=\frac{1}{2}bc_1\sin A+\frac{1}{2}bc_2\sin A$

$=\frac{1}{2}b(c_1+c_2)\sin A=\frac{1}{2}b.2b\cos A.\sin A=\frac{1}{2}b^2\sin 2A$

Example 8: In any triangle ABC, if $\tan\theta=\dfrac{2\sqrt{ab}}{a-b}\sin\dfrac{C}{2}$

Prove that $c=(a-b)\sec\theta$

Sol: As given, $\tan\theta=\dfrac{2\sqrt{ab}}{a-b}\sin\dfrac{C}{2}$. Hence, by solving this and using formula $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$, we can solve the above problem.

$\because \tan\ \theta=\dfrac{2\sqrt{ab}}{a-b}\sin\dfrac{C}{2}$

$\therefore (a-b)^2\tan^2\theta=4ab\sin^2\dfrac{C}{2}$

or $(a-b)^2\left(\sec^2\theta-1\right)=4ab\sin^2\dfrac{C}{2}$

or $(a-b)^2\sec^2\theta=(a-b)^2+4ab\sin^2\dfrac{C}{2}$

or $(a-b)^2\sec^2\theta=a^2+b^2-2ab.\left(1-2\sin^2\dfrac{C}{2}\right)$

or $(a-b)^2\sec^2\theta=a^2+b^2-2ab\cos C$

$\left[\because \cos C=\dfrac{a^2+b^2-c^2}{2ab}\right]$

or $(a-b)^2\sec^2\theta=c^2;\ \therefore c=(a-b)\sec\theta$

Example 9: In any Δ, prove that,

$\dfrac{1}{a}\cos^2\dfrac{A}{2}+\dfrac{1}{b}\cos^2\dfrac{B}{2}+\dfrac{1}{c}\cos^2\dfrac{C}{2}=\dfrac{s^2}{abc}$

Sol: Here, simply by using trigonometric ratios of half angle formula we can prove the above example.

Given L.H.S.= $\dfrac{1}{a}\cos^2\dfrac{A}{2}+\dfrac{1}{b}\cos^2\dfrac{B}{2}+\dfrac{1}{c}\cos^2\dfrac{C}{2}$

$=\dfrac{1}{a}.\left(\dfrac{s(s-a)}{bc}\right)+\dfrac{1}{b}.\left(\dfrac{s(s-b)}{ac}\right)+\dfrac{1}{c}.\left(\dfrac{s(s-c)}{ab}\right)$

$=\dfrac{s}{abc}(s-a+s-b+s-c)$

$=\dfrac{s}{abc}\{3s-(a+b+c)\}=\dfrac{s}{abc}(3s-2s)$

$==\dfrac{s^2}{abc}$ =R.H.S

Hence proved.

Example 10: In any ΔABC, prove that

$a\cos A+b\cos B+c\cos C=\dfrac{8\Delta^2}{abc}$.

Sol: By using sine rule, we can obtain values of a, b and c and then by substituting these values in L.H.S. we can prove this.

$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$ (let)[by sine rule]

Then, $a=k\sin A,\ b=k\sin B$ and $c=k\sin C$

Now, $a\cos A+b\cos B+c\cos C$

$=k\sin A\cos A+k\sin B\cos B+k\sin C\cos C$

$=\dfrac{k}{2}\left[\sin 2A+\sin 2B+\sin 2C\right]$

$=\dfrac{k}{2}\left[4\sin A\sin B\sin C\right]=2k\sin A\sin B\sin C$

$=2a\sin B\sin C=2a.\dfrac{2\Delta}{ac}.\dfrac{2\Delta}{ab}$

$\left[\therefore \Delta=\dfrac{1}{2}ab\sin C=\dfrac{1}{2}ac\sin B\quad \therefore \sin B=\dfrac{2\Delta}{bc},\ \sin C=\dfrac{2\Delta}{ab}\right]$

$=\dfrac{8\Delta^2}{abc}$ = R.H.S

Example 11: In a ΔABC, $\angle c=90^o$, $a=3$, $b=4$ and D is a point on AB so that $\angle BCD=30^o$. Find the length of CD.

Sol: Here, by using Pythagoras theorem and sine rule, we can obtain the length of CD.

In ΔABC, $\angle C = 90^o$

$$\therefore AB^2 = 3^2 + 4^2 = 25$$

$$AB = \sqrt{3^2 + 4^2} = 5$$

and

$$\therefore \sin A = \frac{3}{5}, \sin B = \frac{4}{5}$$

in ΔBCD, by sine rule

$$\frac{BD}{\sin 30^o} = \frac{CD}{\sin B}$$

$$\therefore BD = \frac{\sin 30^o}{\sin B}.CD = \frac{1}{2} \times \frac{5}{4} CD = \frac{5}{8} CD \quad \text{... (i)}$$

And in ΔACD, by sine rule

$$\frac{AD}{\sin 60^o} = \frac{CD}{\sin A}$$

$$\Rightarrow AD = \frac{\sin 60^o}{\sin A} CD = \frac{\sqrt{3}}{2} \times \frac{5}{3} CD = \frac{5\sqrt{3}}{6} CD \quad \text{... (ii)}$$

But BD+AD=AB $\therefore \frac{5}{8} CD + \frac{5\sqrt{3}}{6} CD = 5$

Or $\frac{15 + 20\sqrt{3}}{24} CD = 5$

Or $CD = \frac{24}{3 + 4\sqrt{3}} = \frac{24\left(4\sqrt{3} - 3\right)}{48 - 9}$

Hence $CD = \frac{24}{39}\left(4\sqrt{3} - 3\right) = \frac{8}{13}\left(4\sqrt{3} - 3\right)$

JEE Advanced/Boards

Example 1: Prove that, $a\cos\frac{B - C}{2} = \left(b + C\right)\sin\frac{A}{2}$.

Sol: By using sine rule i.e.

$\frac{b + c}{a} = \frac{\sin B + \sin C}{\sin A}$, we can prove the above example.

$\frac{b + c}{a} = \frac{\sin B + \sin C}{\sin A}$, [using sine Rule]

$$= \frac{2\sin\frac{B + C}{2}\cos\frac{B - C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{2\cos\frac{A}{2}\cos\frac{B - C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\cos\frac{B - C}{2}}{\sin\frac{A}{2}}$$

$$\therefore \left(b + c\right)\sin\frac{A}{2} = a\cos\frac{B - C}{2}.$$

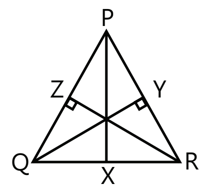
Example 2: If a_1, a_2, a_3 are the altitudes of the triangle PQR,

prove that, $\frac{\cos P}{a_1} + \frac{\cos Q}{a_2} + \frac{\cos R}{a_3} = \frac{1}{R}$.

Sol: Here, simply by using sine rule i.e. $\left(\cos P\right)a = 2R\sin P\cos P$ and so on, we can prove the above problem.

In ΔPQR, consider, $PX = a_1$, $QY = a_2$, $RZ = a_3$.

Area of triangle PQR



$$= \Delta = \frac{1}{2} a a_1 = \frac{1}{2} b a_2 = \frac{1}{2} c a_3$$

$$a_1 = \frac{2\Delta}{a},\ a_2 = \frac{2\Delta}{b},\ a_3 = \frac{2\Delta}{c}$$

$$\frac{\cos P}{a_1} + \frac{\cos Q}{a_2} + \frac{\cos R}{a_3}$$

$$= \frac{\left(\cos P\right)a}{2\Delta} + \frac{\left(\cos Q\right)b}{2\Delta} + \frac{\left(\cos R\right)c}{2\Delta}$$

$$= \frac{1}{2\Delta}\left[2R\sin P\cos P + 2R\sin Q\cos Q + 2R\sin R\cos R\right]$$

[using sine rule]

$$= \frac{R}{2\Delta}\left[\sin 2P + \sin 2Q + \sin 2R\right]$$

$$= \frac{R}{2\Delta}\left[2\sin\left(P + Q\right)\cos\left(P - Q\right) + 2\sin R\cos R\right]$$

$$= \frac{2R\sin R}{2\Delta}\left[\cos\left(P - Q\right) - \cos\left(P + Q\right)\right]$$

[using P+Q+R= π]

$$= \frac{R\sin R}{\Delta} 2\sin P\sin Q = \frac{2R}{\Delta}\sin P\sin Q\sin R$$

$$= \frac{2R}{\Delta}\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R} \quad \text{[using sine rule]}$$

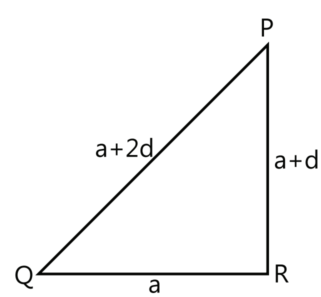
$$= \frac{abc}{4\Delta} \times \frac{1}{R^2} = \frac{R}{R^2} = \frac{1}{R} \quad \left[\text{by using } R = \frac{abc}{4\Delta}\right]$$

Example 3: If the sides of a triangle PQR are in A.P and if its greatest angle exceeds the least angle by α,

show that the sides are in the ratio $1 - x : 1 : 1 + x$

where $x = \sqrt{\frac{1 - \cos\alpha}{7 - \cos\alpha}}$

Sol: As the sides of a given triangle are in A.P., by considering the sides to be a, a + d, a+2d and using sine rule, we can obtain the required result.



Consider the sides to be a, a + d, a + 2 d : d > 0

Let P be the least angle and R be the greatest angle.

Let $P=\phi$; then $R=\phi+\alpha$ and $Q=180^{o}-(2\phi+\alpha)$

$$\frac{a}{\sin\phi}=\frac{a+2d}{\sin(\phi+\alpha)}=\frac{a+d}{\sin(\pi-(2\phi+\alpha))}$$

$$\Rightarrow\frac{a}{\sin\phi}=\frac{a+2d}{\sin(\phi+\alpha)}=\frac{a+d}{\sin(2\phi+\alpha)}=\frac{2\,(a+d)}{\sin\phi+\sin\,(\phi+\alpha)} \quad ...(i)$$

From the first and the second term,

$$\frac{a}{\sin\phi}=\frac{a+2d}{\sin(\phi+\alpha)}=\frac{a}{a+2d}=\frac{\sin\phi}{\sin(\phi+\alpha)}$$

By componendo and dividendo, we get,

$$\frac{a+a+2d}{a+2d-a}=\frac{\sin\phi+\sin(\phi+\alpha)}{-\sin\phi+\sin(\phi+\alpha)}$$

$$\Rightarrow\frac{2(a+d)}{2d}=\frac{2\sin\left(\phi+\frac{\alpha}{2}\right).\cos\frac{\alpha}{2}}{2\cos\left(\phi+\frac{\alpha}{2}\right).\sin\frac{\alpha}{2}}\Rightarrow\frac{a+d}{d}=\frac{\tan\left(\phi+\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}}$$

$$\Rightarrow\frac{d}{a+d}=\frac{\tan\frac{\alpha}{2}}{\tan\left(\phi+\frac{\alpha}{2}\right)} \quad ...\text{(ii)}$$

From the third and the fourth term of equation (i) we get

$$\frac{a+d}{\sin(2\phi+\alpha)}=\frac{2(a+d)}{\sin\phi+\sin\,(\phi+\alpha)}\Rightarrow 2=\frac{\sin\phi+\sin\,(\phi+\alpha)}{\sin(2\phi+\alpha)}$$

$$\Rightarrow 2=\frac{\cos\frac{\alpha}{2}}{\cos\left(\phi+\frac{\alpha}{2}\right)}\Rightarrow\cos\left(\phi+\frac{\alpha}{2}\right)=\frac{\cos\frac{\alpha}{2}}{2}$$

$$\therefore\tan\left(\phi+\frac{\alpha}{2}\right)=\frac{\sqrt{4-\cos^{2}\frac{\alpha}{2}}}{\cos\frac{\alpha}{2}} \quad ...\text{(iii)}$$

From (ii) and (iii) we get,

$$\frac{d}{a+d}=\frac{\sin\frac{\alpha}{2}}{\sqrt{4-\cos^{2}\frac{\alpha}{2}}}=\sqrt{\frac{\sin^{2}\frac{\alpha}{2}}{4-\cos^{2}\frac{\alpha}{2}}}$$

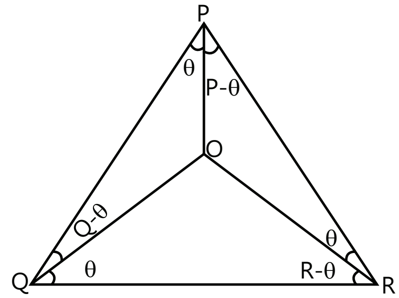
$$\Rightarrow\frac{d}{a+d}=\sqrt{\frac{\frac{1-\cos\alpha}{2}}{4-\frac{1+\cos\alpha}{2}}}=\sqrt{\frac{1-\cos\,\alpha}{7-\cos\,\alpha}}=x$$

Required ratio is a : a + d : a + 2d

$$=1-\frac{d}{a+d}:1:1+\frac{d}{a+d}=1-x:1:1+x$$

Example 4: Let O be a point inside a triangle PQR such that $\angle OPQ=\angle OQR=\angle ORP=\theta$. Show that $\cot\theta=\cot P+\cot Q+\cot R$.

Sol: Simply, by applying sine rule in ΔPOQ and ΔQOR, we can prove the above problem.



$\angle POQ=\pi-Q$ and $\angle QOR=\pi-R$

Applying the sine rule in ∠POQ,

we have $\frac{c}{\sin(\pi-Q)}=\frac{OQ}{\sin\theta}\Rightarrow OQ=\frac{c\sin\theta}{\sin Q}$ ... (i)

Applying the sine rule in ΔQOR,

we have $\frac{OQ}{\sin(R-\theta)}=\frac{a}{\sin(\pi-R)}\Rightarrow OQ=\frac{a\sin(R-\theta)}{\sin R}$ ... (ii)

From (i) and (ii), we have $\frac{c\sin\theta}{\sin R}=\frac{a\sin(R-\theta)}{\sin R}$

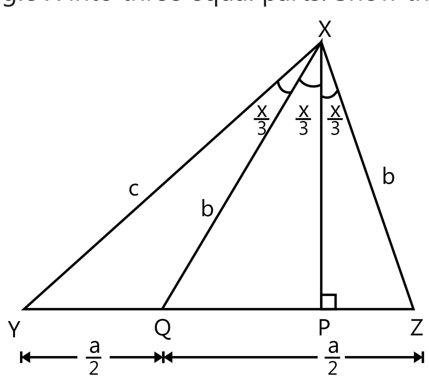
Using Sine Rule we have

$$\frac{2R\sin R\sin\theta}{\sin Q}=\frac{2R\sin P\sin(R-\theta)}{\sin R}$$

$$\frac{\sin R}{\sin P\sin Q}=\frac{\sin(R-\theta)}{\sin R\sin\theta};\frac{\sin(P+Q)}{\sin P\sin Q}=\cot\theta-\cot R$$

$\cot Q+\cot P=\cot\theta-\cot R$ or $\cot\theta=\cot P+\cot Q+\cot R$.

Example 5: In a triangle XYZ, the median XQ and the perpendicular XP from the vertex X to the side QR divide angle X into three equal parts. Show that



$$\cos\frac{X}{3}\sin^{2}\frac{X}{3}=\frac{3a^{2}}{32bc}.$$

Sol: By using the cosine rule in ΔXYQ and ΔPQR and then subtracting them, we'll get the result.

As given, $\angle YXQ = \angle QXP = \angle PXZ = \frac{X}{3}$; $YQ = ZQ = \frac{a}{2}$

$QP = PZ = \frac{a}{4}$ [Since ΔXQP and ΔXPZ are congruent]

XQ = XZ =b $\therefore$ In ΔXYQ

$$\cos\frac{X}{3} = \frac{c^2 + b^2 - \frac{a^2}{4}}{2bc} = \frac{4c^2 + 4b^2 - a^2}{8bc} \quad ...(i)$$

InΔPQR, we have $\cos P = \frac{c^2 + b^2 - a^2}{2bc}$

$$4\cos^3\frac{X}{3} - 3\cos\frac{X}{3} = \frac{c^2 + b^2 - a^2}{2bc} \quad ...(ii)$$

Subtracting (ii) from (i) we get

$$\cos\frac{X}{3} - 4\cos^3\frac{X}{3} + 3\cos\frac{X}{3}$$

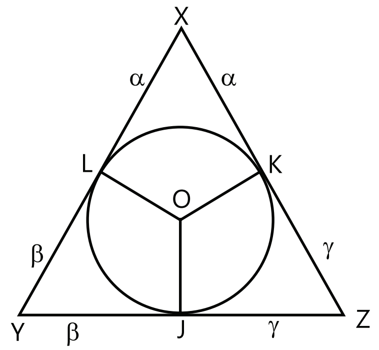
$$= \frac{4c^2 + 4b^2 - a^2}{8bc} - \frac{c^2 + b^2 - a^2}{2bc}$$

$$4\cos\frac{X}{3}\left(1 - \cos^2\frac{X}{3}\right) = \frac{3a^2}{8bc};$$

$$\cos\frac{X}{3}\sin^2\frac{X}{3} = \frac{3a^2}{32bc}.$$

Example 6: If α, β and γ are the distances of the vertices of a triangle XYZ from nearest points of contact of the incircle with sides of ΔXYZ,

prove that $r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}$.



Sol: Here, as we know, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ and $r = \frac{\Delta}{s}$.

Given XL = XK = α ,YL = YJ = β ,ZJ = ZK = γ

2s =XY + YZ + ZX =c + a + b = $2\alpha + 2\beta + 2\gamma$

Area of triangle XYZ = $\sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{(\alpha + \beta + \gamma)\alpha\beta\gamma}$$

$$\therefore r = \frac{\Delta}{s} = \frac{\sqrt{(\alpha + \beta + \gamma)\alpha\beta\gamma}}{\alpha + \beta + \gamma}$$

$$\Rightarrow r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}.$$

Example 7: In any triangle ABC, if

$$\cos\theta = \frac{a}{b+c}, \cos\phi = \frac{b}{a+c}, \cos\varphi = \frac{c}{a+b}$$

where θ, ϕ, φ lie between 0 and π,

prove that $\tan\frac{\theta}{2}\tan\frac{\phi}{2}\tan\frac{\varphi}{2} = \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

Sol: Here, by using formula

$\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$ and $\frac{a}{b+c} = \cos\theta$,we can solve the above problem.

$$\frac{a}{b+c} = \cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$$

By componendo and dividendo,

$$\frac{2}{2\tan^2\frac{\theta}{2}} = \frac{a+b+c}{b+c-a}; \Rightarrow \tan^2\frac{\theta}{2} = \frac{2s - 2a}{2s} = \frac{s-a}{s}$$

Similarly, $\tan^2\frac{\phi}{2} = \frac{s-b}{s}$ and $\tan^2\frac{\varphi}{2} = \frac{s-c}{s}$

$$\therefore \tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}\tan^2\frac{\varphi}{2} = \frac{(s-a)(s-b)(s-c)}{s^3} = \frac{\Delta^2}{s^4}$$

$$\therefore \tan\frac{\theta}{2}\tan\frac{\phi}{2}\tan\frac{\varphi}{2} = \frac{\Delta}{s^2} \quad ...(i)$$

Now $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

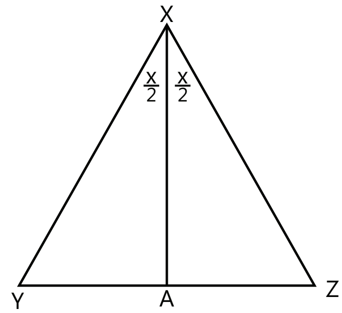
$$= \sqrt{\frac{(s-b)(s-c)(s-c)(s-a)}{s(s-a)s(s-b)}\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}} = \frac{\Delta}{s^2} \quad ...(ii)$$

From (i) and (ii)

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}\tan\frac{\varphi}{2} = \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

Example 8: The bisector of angle X of triangle XYZ meets YZ at A. If XA= ℓ then, prove that



(i) $\ell = \frac{2bc}{b+c}\cos\frac{X}{2}$ (ii) $\frac{a}{b+c} = \sqrt{1-\frac{1^2}{bc}}$

Sol: (i) Simply by using area of triangle formula, we can prove the above equation.

(ii) Here, by using cosine rule, we can prove it.

(i) Area of ΔXYZ =Area of ΔXYA+ Area of ΔXAZ

$$\frac{1}{2}bc\sin X = \frac{1}{2}c\ell\sin\frac{X}{2} + \frac{1}{2}b\ell\sin\frac{X}{2}$$

$$2bc\cos\frac{X}{2} = \ell(b+c);\ \Rightarrow \ell = \frac{2bc}{b+c}\cos\frac{X}{2}$$

(ii) $\frac{YA}{AZ} = \frac{XY}{XZ} = \frac{c}{b}$ or $\frac{YA}{c} = \frac{ZA}{b} = \frac{YA+ZA}{c+b} = \frac{a}{c+b}$

$$\Rightarrow \frac{YA}{c} = \frac{a}{c+b} \qquad \text{... (i)}$$

In triangle XYA, $\cos\frac{X}{2} = \frac{c^2+\ell^2-YA^2}{2c\ell}$

$$\Rightarrow -2c\ell\cos\frac{X}{2} + c^2 + \ell^2 = YA^2$$

Substituting value of $\cos\frac{X}{2}$ from (i) we get

$$QA^2 = -2c\ell\frac{\ell(b+c)}{2bc} + c^2 + \ell^2 = \frac{c^2b-\ell^2c}{b}$$

Equation (A) gives $\frac{(QA)^2}{c^2} = \left(\frac{a}{c+b}\right)^2$

or $\frac{c^2b-\ell^2c}{bc^2} = \left(\frac{a}{c+b}\right)^2$ or $\sqrt{1-\frac{\ell^2}{bc}} = \frac{a}{c+b}$.

Example 9: A cyclic quadrilateral ABCD of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides AB=1 and $\angle A$ is acute and the diagonal $BD = \sqrt{3}$,find the lengths of the other sides.

Sol: Here, area of cyclic quadrilateral = area of triangle ABC + area of triangle BCD. Therefore by using cosine rule in triangle ABD and BCD, we will be solving the above example.

Given AB=1, BD= $\sqrt{3}$

OA=OB=OD=OC=1=R (O being center of the circle),

in triangle ABD,

$\frac{BD}{\sin A} = 2R \Rightarrow \frac{\sqrt{3}}{2} = \sin A$ ($\because$ Given circle is circumcircle of ΔABD) $\Rightarrow A = \frac{\pi}{3}$;Hence $C = \frac{2\pi}{3}$

($\because$ ABCD is a cyclic quadrilateral)

Using cosine rule in triangle ABD,

$$\cos A = \frac{(AB)^2 + (AD)^2 - (BD)^2}{2AB.AD}$$

$$\frac{1}{2} = \frac{1+(AD)^2-3}{2AD} \text{ or } AD^2 - AD - 2 = 0$$

or $(AD-2)(AD+1) = 0$; $\therefore AD = 2$

Using cosine rule in triangle BCD, we have

$$\cos C = \frac{(BC)^2+(CD)^2-(BD)^2}{2(BC).(CD)} \Rightarrow -\frac{1}{2} = \frac{(BC)^2+(CD)^2-3}{2(BC).(CD)}$$

or $(BC)^2 + (CD)^2 + (BC)(CD) - 3 = 0$ ... (i)

Area of cyclic quadrilateral = Area of triangle ABC + Area of triangle BCD

$$\frac{3\sqrt{3}}{4} = \frac{1}{2}.1.2\sin\frac{\pi}{3} + \frac{1}{2}.BC.CD\sin\frac{2\pi}{3}$$

3=2+BC.CD or BC.CD=1 ... (ii)

Solving (i) and (ii), we get BC=CD=1

Hence length of sides of cyclic quadrilateral are

AD=2, BC=CD=1.

Example 10: The sides of a triangle are in A.P. and its area is $\frac{3}{5}$th of an equilateral triangle of the same perimeter. Prove that the sides are in the ratio 3:5:7.

Sol: Here, sides of triangle are in A.P. Hence, by considering the sides to be a – d, a and a + d and then by using area of triangle formula and the given conditions, we can prove the given ratios.

Let the sides be a – d, a and a + d.

2s=sum of the sides = 3a

$\therefore s=\frac{3a}{2}$

Now, Δ_1 = Area of the triangle whose sides are in A.P.

$$=\sqrt{\frac{3a}{2}\left[\frac{3a}{2}-a+d\right]\left[\frac{3a}{2}-a\right]\left[\frac{3a}{2}-a-d\right]}$$

$$=\frac{\sqrt{3}a}{4}\sqrt{(a+2d)(a-2d)} \text{ or } \Delta_1=\frac{\sqrt{3}a}{4}\sqrt{a^2-4d^2} \quad \text{... (i)}$$

Perimeter of equilateral triangle = Perimeter of the given triangle

$\therefore 3\times$ One side of equilateral triangle = 3a

$\Rightarrow$ Side of the equilateral triangle = a

Now, Δ_2 = Area of the equilateral triangle

$$=\frac{\sqrt{3}}{4}\times(\text{side})^2=\frac{\sqrt{3}}{4}a^2 \quad \text{... (ii)}$$

From question, $=\frac{\Delta_1}{\Delta_2}=\frac{3}{5}$ or $=\frac{\sqrt{a^2-4d^2}}{a}=\frac{3}{5}$

or $25a^2-100d^2=9a^2$ or $16a^2=100d^2$

$\Rightarrow \frac{a}{d}=\frac{5}{2}$

Ratio of the sides= a – d : a + d

$$=\frac{a}{d}-1:\frac{a}{d}:\frac{a}{d}+1=\frac{5}{2}-1:\frac{5}{2}:\frac{5}{2}+1$$

$$=\frac{3}{2}:\frac{5}{2}:\frac{7}{2}=3:5:7$$

JEE Main/Boards

Exercise 1

Q.1. If a=13, b=14, c=15, find r and R.

Q.2. In an equilateral triangle, find the relation between the in radius and the circum radius.

Q.3. If in a triangle $\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)}$, prove that it is either a right angled or an isosceles triangle.

Q.4. If ΔABC is scalene and $\cos A+\cos B=4\sin^2 C/2$ then prove that A, B, C are in A.P.

Q.5. Solve the triangle, if $a=2,\ b=\sqrt{6},\ c=\sqrt{3}-1$.

Q.6. If a=5, b=7 and $\sin A=\frac{3}{4}$, solve the triangle, if possible.

Q.7. Two sides of the triangle are of length $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30^o. How many such triangles are possible? Find the length of their third side and area.

Q.8. If in a triangle ABC, $\angle A=\frac{\pi}{3}$ and AD is a median then prove that $4AD^2=b^2+bc+c^2$.

Q.9. If A=30^o, b=8 and a=6, find c.

Q.10. The angles of a triangle are in the ratio 2:3:7. Find the ratio of its sides.

Q.11. In a triangle ABC, if 3a = b + c, prove that: $\cot\frac{B}{2}\cot\frac{C}{2}=2$.

Q.12. D is the mid point of BC in a triangle ABC. If AD is perpendicular to AC, prove that $\cos A\cos C=\frac{2(c^2-a^2)}{3ac}$.

Q.13. In ΔABC, prove that:

$$\frac{\cos A}{b\cos C+c\cos B}+\frac{\cos B}{c\cos A+a\cos C}+\frac{\cos C}{a\cos B+b\cos A}=\frac{a^2+b^2+c^2}{2abc}.$$

Q. 14. Prove that $(b+c-a)\{\cot(B/2)+\cot(C/2)\}=2a\cot(A/2)$.

Q.15. If $p_1,\ p_2,\ p_3$ be the altitudes of a triangle ABC from the vertices A, B, C respectively and Δ be the area of the triangle ABC, prove that $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{2ab\cos^2\frac{C}{2}}{\Delta(a+b+c)}$

Q.16. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Q.17. Prove that

$$a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$$

Q.18. With usual notations, if in a triangle,

ABC $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ then prove that

$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$.

Q.19. In a triangle ABC, prove that

$$\frac{a}{\cos B\cos C}+\frac{b}{\cos C\cos A}+\frac{c}{\cos A\cos B}$$

=2a tan B.tan C.sec A

Q.20. Prove that the radius of the circle passing through the centre of the inscribed circle of the triangle ABC and through the end points of the base BC is $\frac{a}{2}\sec\frac{A}{2}$.

Q.21. In a triangle ABC, Prove that

$$(b+c-a)\tan\frac{A}{2}=(c+a-b)\tan\frac{B}{2}=(a+b-c)\tan\frac{C}{2}$$

Q.22. 3 circles of radius 3, 4, 5 touches externally. Find the distance from point of contact to intersection point of tangents.

Q.23. Perpendiculars are drawn from the vertices A, B, C of an acute angled triangle on the opposite sides, and produced to meet the circumcircle of the triangle. If these produced parts be α, β, γ respectively.

Show that $\frac{a}{\alpha}+\frac{a}{\beta}+\frac{a}{\gamma}=2(\tan A+\tan B+\tan C)$.

Q.24. If in a triangle $8R^2=a^2+b^2+c^2$, prove that the triangle is right angled.

Q.25. If Δ is the area of a triangle with side length a, b, c then show that $\Delta\le\frac{1}{4}\sqrt{(a+b+c)abc}$.

Q.26. In any triangle ABC, prove that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

Q.27. In a triangle of base a, the ratio of the other two sides is r(<1). Show that the altitude of the triangle is less than or equal to $\frac{a}{1-r^2}$.

Q.28. Let ABC be a triangle having O and I as its circumcentre and incentre respectively if R and r be the circumradius and the inradius respectively, then prove that $(IO)^2=R^2-2Rr$. Further show that the triangle BIO is a right angled if and only if b is the A.M. of a and c.

Q.29. If α, β and γ are the altitudes of the ΔABC from the vertices A, B and C respectively then

show that $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{1}{\Delta}(\cot A+\cot B+\cot C)$

Exercise 2

Single Correct Choice Type

Q.1. If A is the area and 2s the sum of the 3 sides of a triangle, then

(A) $A\le\frac{s^2}{3\sqrt{3}}$ (B) $A=\frac{s^2}{2}$

(C) $A>\frac{s^2}{\sqrt{3}}$ (D) None

Q.2. In a triangle ABC, CH and CM are the lengths of the altitude and median to the base AB. If a=10, b=26, c=32 then length (HM) is

(A) 5 (B) 7 (C) 9 (D) None

Q.3. In a triangle ABC, CD is the bisector of the angle C. If $\cos\frac{C}{2}$ has the value $\frac{1}{3}$ and $l(CD)=6$, then $\left(\frac{1}{a}+\frac{1}{b}\right)$ has the value equal to

(A) $\frac{1}{9}$ (B) $\frac{1}{12}$ (C) $\frac{1}{6}$ (D) None

Q.4. With usual notations in a triangle ABC, $(II_1).(II_2).(II_3)$ has the value equal to

(A) R^2r (B) $2R^2r$ (C) $4R^2r$ (D) $16R^2r$

Q.5. With usual notation in a ΔABC

$$\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)=\frac{KR^3}{a^2b^2c^2}$$

Where K has the value equal to

(A) 1 (B) 16 (C) 64 (D) 128

Q.6. If the incircle of the ΔABC touches its sides respectively at L, M and N and if x,y,z be the circumradii of the triangles MIN, NIL and LIM where I is the incentre then the product xyz is equal to

(A) Rr^2 (B) rR^2 (C) $\frac{1}{2}Rr^2$ (D) $\frac{1}{2}rR^2$

Q.7. The product of the distances of the incentre from the angular points of a ΔABC is

(A) $4R^2r$ (B) $4Rr^2$ (C) $\frac{(abc)R}{s}$ (D) $\frac{(abc)s}{R}$

Q.8. If x,y and z are the distances of incentre from the vertices of the triangle ABC respectively then $\frac{abc}{xyz}$ is equal to

(A) $\prod \tan\frac{A}{2}$ (B) $\sum \cot\frac{A}{2}$

(C) $\sum \tan\frac{A}{2}$ (D) $\sum \sin\frac{A}{2}$

Q.9. For each natural number k, let C_k denotes the circle with radius k centimeters and centre at the origin. On the circle C_k, a particle moves k centimeters in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1,0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n then n equal to

(A)6 (B)7 (C)8 (D)9

Q.10. If in a triangle ABC $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$ then the value of the angle is

(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Previous Years' Questions

Q.1. In a triangle ABC, $\angle B=\frac{\pi}{3}$ and $\angle C=\frac{\pi}{4}$. Let D divides BC internally in the ratio 1:3, then $\frac{\sin\angle BAD}{\sin\angle CAD}$ is equal to ***(1995)***

(A) $\frac{1}{\sqrt{6}}$ (B) $\frac{1}{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\sqrt{\frac{2}{3}}$

Q.2. If in a triangle PQR, $\sin P, \sin Q, \sin R$ are in AP, then ***(1998)***

(A) The altitudes are in AP.

(B) The altitudes are in HP.

(C) The medians are in GP.

(D) The medians are in AP.

Q.3. In a triangle PQR, $\angle R=\frac{\pi}{2}$, if $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2+bx+c=0\,(a\neq 0)$, then ***(1999)***

(A) a + b = c (B) b + c = a

(C) a + c = b (D) b = c

Q.4. In a triangle ABC, $2ac\sin\frac{1}{2}(A-B+C)=$ ***(2000)***

(A) $a^2+b^2-c^2$ (B) $c^2+a^2-b^2$

(C) $b^2-c^2-a^2$ (D) $c^2-a^2-b^2$

Q.5. In a triangle ABC, let $\angle C=\pi/2$, if r is the inradius and R is the circumradius of the triangle, then 2(r + R) is equal to ***(2000)***

(A) a + b (B) b + c

(C) c + a (D) a + b + c

Q.6. The number of integral points (integral point means both the coordinates should be integers) exactly in the interior of the triangle with vertices (0, 0), (0, 21) and (21, 0) is ***(2003)***

(A) 133 (B) 190 (C) 233 (D) 105

Q.7. The sides of a triangle are in the ratio $1:\sqrt{3}:2$, then the angles of the triangle are in the ratio ***(2004)***

(A) 1:3:5 (B) 2:3:2 (C) 3:2:1 (D) 1:2:3

Q.8. There exists a triangle ABC satisfying the conditions ***(1986)***

(A) $b\sin A=a,\ A<\frac{\pi}{2}$ (B) $b\sin A>a,\ A>\frac{\pi}{2}$

(C) $b\sin A>a,\ A<\frac{\pi}{2}$ (D) $b\sin A<a,\ A<\frac{\pi}{2},\ b>a$

Q.9. A polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is ..... ***(1987)***

Q.10. The sides of a triangle inscribed in a given circle subtend angles α, β and γ at the centre. The minimum value of the arithmetic mean of $\cos\left(\alpha+\frac{\pi}{2}\right)$, $\cos\left(\beta+\frac{\pi}{2}\right)$ and $\cos\left(\gamma+\frac{\pi}{2}\right)$ is equal to ...... ***(1987)***

Q.11. If the angles of a triangle are 30^o and 45^o and the included side is $\left(\sqrt{3}+1\right)$ cm, then the area of the triangle is....... ***(1988)***

Q.12. If in a triangle ABC $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$ then the value of the angle A is .... degree. ***(1993)***

Q.13. If p_1, p_2, p_3 are the perpendiculars from the vertices of a triangle to the opposite sides,

prove that $p_1p_2p_3=\frac{a^2b^2c^2}{8R^3}$ ***(1978)***

Q.14. If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C and Δ the area of the triangle, then

prove that $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{2ab}{\left(a+b+c\right)\Delta}\cos^2\frac{C}{2}$ ***(1978)***

Q.15. If in a triangle ABC, $a=1+\sqrt{3}$ cm, b=2cm and $\angle C=60^o$, then find the other two angles and the third side**.** ***(1978)***

Q.16. The ex-radii r_1, r_2, r_3 of ΔABC are in HP, show that its sides a,b,c are in AP. ***(1983)***

Q.17. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle. ***(1991)***

Q.18. Prove that a triangle ABC is equilateral if and only if $\tan A+\tan B+\tan C=3\sqrt{3}$. ***(1998)***

Q.19. I_n is the area of n sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle,

prove that $I_n=\frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$ ***(2003)***

Q.20. Circle with radii 3,4 and 5 touch each other externally, if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the point of contact. ***(2005)***

Q.21 Consider a ΔABC and let a, b and c denote the lengths of the sides opposite to vertices A,B,C respectively. Suppose a = 6, b = 10 and the area of the triangles is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to......... ***(2010)***

Q.22 In a Δ PQR, if 3 sin P + 4 cos Q = 6 and 4 sin Q + 3 cos P = 1, then the angle R is equal to ***(2012)***

(A) $\frac{6\pi}{6}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{3\pi}{4}$

Q.23 ABCD is a trapezium such that AB and CD are parallel and BC $\perp$ CD. If $\angle ABD=\theta$, BC = p and CD = q then AB is equal to : ***(2013)***

(A) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (B) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

(C) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (D) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

Q.24 If the angles of elevation of the top of lower from three collinear points A, B and C, on a line leading to the foot of the lower, are 30°, 45° and 60° respectively, then the ratio, AB : BC, is **(2015)**

(A) $\sqrt{3}:1$ (B) $\sqrt{3}:\sqrt{2}$

(C) $1:\sqrt{3}$ (D) 2 : 3

JEE Advanced/Boards

Exercise 1

Q.1 Given a triangle ABC with AB=2 and AC=1. Internal Bisector of $\angle BAC$ intersects BC at D. If AD=BD and Δ is the area of triangle ABC, then find the value of $12\Delta^2$.

Q.2 In a triangle ABC, let angles A, B, C are in G.P. with common ratio 2. If circumradius of triangle ABC is 2, then find the value of $\left(b^{-1}+c^{-1}-a^{-1}\right)$

Q.3 In a triangle ABC, BD is a median.

If $l\left(BD\right)=\frac{\sqrt{3}}{4}.l\left(AB\right)$ and $\angle DBC=\frac{\pi}{2}$.

Determine the $\angle ABC$.

Q.4 In an isosceles Δ ABC, if the altitudes intersect on the inscribed circle then find the secant of the vertical angle 'A'.

Q.5 ABCD is a rhombus. The circumradii of ΔABD and ΔACD are 12.5 and 25 respectively. Find the area of rhombus.

Q.6 In a triangle ABC if $a^2+b^2=101c^2$ then find the value of $\frac{\cot C}{\cot A+\cot B}$.

Q.7 The two adjacent sides of a cyclic quadrilateral are 2 & 5 and the angle between them is 60^o. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

Q.8 If in a ΔABC, a = 6, b = 3 and cos(A-B) = 4/5 then find its area.

Q.9 The triangle ABC (with side lengths a,b,c as usual) satisfies $\log a^2=\log b^2+\log c^2-\log\left(2bc\cos A\right)$. What can you say about this triangle?

Q.10 The sides of a triangle are consecutive integers n, n+1 and n+2 and the largest angle is twice the smallest angle. Find n.

Q.11 The triangle ABC is a right angled triangle, right angle at A. The ratio of the radius of the circle circumscribed to the radius of the circle escribed to the radius of the circle escribed to the hypotenuse is, $\sqrt{2}:\left(\sqrt{3}+\sqrt{2}\right)$. Find the acute angles B & C. Also find the ratio of the two sides of the triangle other than the hypotenuse.

Q.12 If a, b, c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a-\log b-\log 2$.

Also $a\left(1-x^2\right)+2bx+c\left(1+x^2\right)=0$ has two equal roots.

Find the value of $\sin A+\sin B+\sin C$.

Q.13 Given a triangle ABC with sides a= 7, b = 8 and c = 5. If the value of the expression $\left(\sum\sin A\right)/\left(\sum\cot\frac{A}{2}\right)$ can be expressed in the form $\frac{P}{q}$ where p, q $\in$ N and $\frac{p}{q}$ is in its lowest form find the value of (p + q).

Q.14 If $r_1=r+r_2+r_3$, then prove that the triangle is a right angled triangle.

Q.15 If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

Q.16 In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC, then prove that, $\frac{2}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$

Q.17 In a right angled triangle ABC, $\angle C=90^o$ and sides AC, AB are roots of the equation $2+y^2=3y$. If the internal angle bisector of angle A intersects BC at D such that $BD:CD=x^2+1:2x$, then find the sum of all possible values of $\tan\frac{A}{2}$.

Q.18 Given a right triangle with $\angle A=90^o$. Let M be the mid-point of BC. If the inradii of the triangle ABM and ACM are r_1 and r_2 then find the range of $\frac{r_1}{r_2}$.

Q.19 If the length of the perpendiculars from the vertices of a triangle A,B,C on the opposite sides are

p_1, p_2, p_3, then prove that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$.

Q.20 Tangents parallel to the three sides of ΔABC are drawn to its incircle. If x, y, z be the lengths of the parts of the tangents within the triangle (with respect to the sides a, b, c) then find the value of $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$.

Exercise 2

Single Correct Choice Type

Q.1 If the median of a triangle ABC through A is perpendicular to AB then $\frac{\tan A}{\tan B}$ has the value equal to

(A) $\frac{1}{2}$ (B) 2 (C) –2 (D) $-\frac{1}{2}$

Q.2 In a ΔABC, the value of $\frac{a\cos A+b\cos B+c\cos C}{a+b+c}$ is equal to

(A) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$

Q.3 With usual notation in a ΔABC, if R=K $\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{r_1r_2+r_2r_3+r_3r_1}$ where k has the value equal to

(A) 1 (B) 2 (C)1/4 (D)4

Q.4 In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

(A) $\frac{\pi}{3}$ & $\frac{\pi}{6}$ (B) $\frac{\pi}{8}$ & $\frac{3\pi}{8}$ (C) $\frac{\pi}{4}$ & $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ & $\frac{3\pi}{10}$

Q.5 Let f, g, h be the lengths of the perpendiculars from the circumcentre of the ΔABC on the sides a, b and c respectively.

If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h}=\lambda\frac{abc}{fgh}$ then the value of λ is

(A)1/4 (B)1/2 (C)1 (D)2

Q.6 If 'O' is the circumcentre of the ΔABC and R_1, R_2 and R_3 are the radii of the circumcircles of triangle OBC, OCA and OAB respectively then $\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}$ has the value equal to

(A) $\frac{abc}{2R^2}$ (B) $\frac{R^3}{abc}$ (C) $\frac{4\Delta}{R^2}$ (D) $\frac{\Delta}{4R^2}$

Q.7 In a Δ ABC, a semicircle is inscribed, whose diameter lies on the side C. Then the radius of the semicircle is

(A) $\frac{2\Delta}{a+b}$ (B) $\frac{2\Delta}{a+b-c}$ (C) $\frac{2\Delta}{s}$ (D) $\frac{c}{2}$

Where Δ is the area of the triangle ABC.

Q.8 If in a Δ ABC, $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$ then the triangle is

(A) Right angled (B) Isosceles

(C) Equilateral (D) Obtuse

Q.9 If cosA + cosB + 2cosC = 2 then the sides of the Δ ABC are in

(A) A.P. (B)G.P. (C) H.P. (D)None

Q.10 The product of the arithmetic mean of the lengths of the sides of a triangle and harmonic mean of the lengths of the altitudes of the triangle is equal to:

(A) Δ (B) 2 Δ (C) 3 Δ (D) 4 Δ

[where Δ is the area of the triangle ABC]

Q.11 If in a triangle sinA : sinC = sin(A–B) : sin(B–C) then $a^2 : b^2 : c^2$

(A) Are in A.P. (B) Are in G.P.

(C) Are in H.P. (D) None of these

Q.12 The sines of two angles of a triangle are equal to $\frac{5}{13}$ & $\frac{99}{101}$. The cosine of the third angle is

(A) 245/1313 (B) 255/1313

(C) 735/1313 (D) 765/1313

Multiple correct choice type

Q.13 If the side of a right angled triangle are $\{\cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)\}$ and $\{\sin 2\alpha+\sin 2\beta+2\sin(\alpha+\beta)\}$, then the length of the hypotenuse is

(A) $2[1+\cos(\alpha-\beta)]$ (B) $2[1+\cos(\alpha+\beta)]$

(C) $4\cos^2\frac{\alpha-\beta}{2}$ (D) $4\sin^2\frac{\alpha+\beta}{2}$

Match the Columns

Q.14 Let P be an interior point of ΔABC. Match the correct entries for the ratios of the Area of ΔPBC : Area of Δ PCA: Area of Δ PAB depending on the position of the point P w.r.t. Δ ABC.

	Column I		Column II
(A)	If P is centroid (G)	(p)	tanA:tanB:tanC
(B)	If P is incentre (I)	(q)	sin2A:sin2B:sin2C
(C)	If P is orthocenter (H)	(r)	sinA:sinB:sinC
(D)	If P is circumcentre (S)	(s)	1:1:1
		(t)	cosA:cosB:cosC

Q.15 In a Δ ABC, BC=2, CA= $1+\sqrt{3}$ and $\angle C = 60^o$. Feet of the perpendicular from A, B and C on the opposite sides BC, CA and AB are D, E and F respectively and are concurrent at P. Now match the entries of Column I with respective entries of Column II.

	Column I		Column II
(A)	Radius of the circle circumscribing the ΔDEF, is	(p)	$\frac{\sqrt{6}-\sqrt{2}}{4}$
(B)	Area of the Δ DEF, is	(q)	$\frac{1}{\sqrt{2}}$
(C)	Radius of the circle inscribed in the Δ DEF, is	(r)	$\frac{\sqrt{3}}{4}$
		(s)	$\frac{\sqrt{6}+\sqrt{2}}{4}$

Previous Years' Questions

Q.1 Which of the following pieces of data does not uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)? ***(2002)***

(A) a, sinA, sinB (B) a, b, c

(C) a, sinB, R (D) a, sinA, R

Q.2 If the angles of a triangle are in the ratio 4:1:1, then the ratio of the longest side to the perimeter is ***(2003)***

(a) $\sqrt{3}:\left(2+\sqrt{3}\right)$ (b)1:3:

(c) $1:2+\sqrt{3}$ (d)2:3

Q.3 Orthocentre of triangle with vertices (0,0), (3,4) and (4,0) is ***(2003)***

(A) $\left(3,\frac{5}{4}\right)$ (B) $\left(3,12\right)$

(C) $\left(3,\frac{3}{4}\right)$ (D) $\left(3,9\right)$

Q.4 In a Δ ABC, among the following which one is true? ***(2005)***

(A) $\left(b+c\right)\cos\frac{A}{2}=a\sin\left(\frac{B+C}{2}\right)$

(B) $\left(b+c\right)\cos\left(\frac{B+C}{2}\right)=a\sin\frac{A}{2}$

(C) $\left(b-c\right)\cos\left(\frac{B-C}{2}\right)=a\cos\left(\frac{A}{2}\right)$

(D) $\left(b-c\right)\cos\left(\frac{A}{2}\right)=a\sin\left(\frac{B-C}{2}\right)$

Q.5 In radius of a circle which is inscribed in a isosceles triangle one of whose angle is $2\pi/3$, is $\sqrt{3}$, then area of triangle is ***(2006)***

(A) $4\sqrt{3}$ (B) $12-7\sqrt{3}$

(C) $12+7\sqrt{3}$ (D)None of these

Q.6 If the angles A,B and C of a triangle are in an arithmetic progression and if a,b and c denote the lengths of the sides opposite to A,B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is **(2010)**

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C)1 (D) $\sqrt{3}$

Paragraph (Q.7 to Q.9):

Read the following Paragraph and answer the questions.

Consider the circle $x^2+y^2=9$ and the parabola $y^2=8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. ***(2007)***

Q.7 The ratio of the areas of the triangle PQS and PQR is

(A) $1:\sqrt{2}$ (B)1:2 (C)1:4 (D)1:8

Q.8 The radius of the circumcircle of the triangle PRS is

(A)5 (B) $3\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{3}$

Q.9 The radius of the incircle of the triangle PQR is

(A) $\frac{4}{8}$ (B)3 (C) $\frac{8}{3}$ (D)2

Q.10 Internal bisector of $\angle A$ of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and side AB at F. If a,b,c represent sides of Δ ABC, then ***(2006)***

(A) AE is HM of b and c (B) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

(C) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (D)The Δ AEF is isosceles

Q.11 A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then ***(2008)***

(A) $\frac{1}{PS}+\frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ (B) $\frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(C) $\frac{1}{PS}+\frac{1}{ST} < \frac{4}{QR}$ (D) $\frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}$

Q.12 Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a,b and c denote the lengths of the sides opposite to A,B and C respectively. The value(s) of x for which $a = x^2+x+1$, $b = x^2-1$ and $c = 2x+1$ is (are) ***(2010)***

(A) $-\left(2+\sqrt{3}\right)$ (B) $1+\sqrt{3}$

(C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

Q.13 In a triangle ABC, AD is the altitude from A. Given b>c, $\angle C = 23^o$ and $AD = \frac{abc}{b^2-c^2}$, then $\angle B = \ldots\ldots$ ***(1994)***

Q.14 In a triangle ABC, a:b:c = 4:5:6. The ration of the radius of the circumcircle to that of the incircle is .... ***(1996)***

Q.15 ABC is a triangle, D is the middle point of BC. If AD is perpendicular to AC, then prove that

$\cos A \cos C = \frac{2\left(c^2-a^2\right)}{3ac}$ ***(1980)***

Q.16 For a triangle ABC it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that the triangle is equilateral. ***(1984)***

Q.17 With usual notation, if in a triangle ABC $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$. **(1984)**

Q.18 In a triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle A into angles 30^o and 45^o. Find the length of the side BC. ***(1985)***

Q.19 If in a triangle ABC $\cos A \cos B + \sin A \sin B \sin C = l$, show that a: b : c $= 1:1:\sqrt{2}$. ***(1986)***

Q.20 In a triangle of base a, the ratio of the other two sides is $r\left(<1\right)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$. ***(1991)***

Q.21 Let A_1, A_2, ... A_n be the vertices of an n-sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3}+\frac{1}{A_1A_4}$. Find the value of n. ***(1994).***

Q.22 Consider the following statements concerning a triangle ABC

(i) The sides a, b, c and area of Δ are rational

(ii) a, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are rational

(iii) a, sinA, sinB, sinC are rational.

Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) ***(1994)***

Q.23 Let A, B, C be three angles such that $A = \frac{\pi}{4}$ and $\tan B \tan C = p$. Find all positive values of P such that A, B, C are the angles of triangle. ***(1997)***

Q.24 Let ABC be a triangle having O and I as its circumcentre and incentre, respectively. If R and r are the circumradius and the inradius, respectively, then prove that $\left(IO\right)^2 = R^2 - 2Rr$. Further show that the triangle BIO is a right angled triangle if and only if b is the arithmetic mean of a and c. ***(1999)***

Q.25 Let ABC be a triangle with incentre I and radius r. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals AFIE, BDIF and CEID respectively,

prove that: $\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$. ***(2000)***

Q.26 If Δ is the area of a triangle with side lengths a, b, c then show that $\Delta \le \frac{1}{4}\sqrt{(a+b+c)abc}$.

Also show that the equality occurs in the above inequality if and only if a=b=c. ***(2001)***

Q.27 Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and AB = 2CD. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, then its radius is ***(2007)***

(A) 3 (B) 2 (C) 3/2 (D) 1

Paragraph (Q.28 to 30)

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F. respectively. The line PQ is given by the equation $\sqrt{3}x+y-6=0$ and the point D is $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ. ***(2008)***

Q.28 The equation of circle C is

(A) $(x-2\sqrt{3})^2+(y-1)^2=1$

(B) $(x-2\sqrt{3})^2+\left(y+\frac{1}{2}\right)^2=1$

(C) $(x-\sqrt{3})^2+(y+1)^2=1$

(D) $(x-\sqrt{3})^2+(y-1)^2=1$

Q.29 Points E and F are given by

(A) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$, $(\sqrt{3},0)$ (B) $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$, $(\sqrt{3},0)$

(C) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$, $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ (D) $\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right)$, $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

Q.30 Equation of the sides QR, RP are

(A) $y=\frac{2}{\sqrt{3}}x+1,\ y=-\frac{2}{\sqrt{3}}x-1$

(B) $y=\frac{1}{\sqrt{3}}x,\ y=0$

(C) $y=\frac{\sqrt{3}}{2}x+1,\ y=-\frac{\sqrt{3}}{2}x-1$

(D) $y=\sqrt{3}x,\ y=0$

Q.31 Let $z=\cos\theta+i\sin\theta$. Then the value of $\sum_{m=1}^{15}\text{Im}\,(z^{2m-1})$ at $\theta=2^\circ$ is ***(2009)***

(A) $\frac{1}{\sin 2^\circ}$ (B) $\frac{1}{3\sin 2^\circ}$

(C) $\frac{1}{2\sin 2^\circ}$ (D) $\frac{1}{4\sin 2^\circ}$

Q.32 In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B+\cos C=4\sin^2\frac{A}{2}$. If a, b and c denote the length of the sides of the triangle opposite to the angles A, B and C, respectively, then ***(2009)***

(A) b + c = 4a

(B) b + c = 2a

(C) Locus of point A is an ellipse

(D) Locus of point A is a pair of straight lines

Q.33 For $0<\theta<\frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6}\text{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\text{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2}$$

is (are) ***(2009)***

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

Q.34 The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is ***(2009)***

Q.35 Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is **(2009)**

Q.36 Let ABC be a rectangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + 1, b = x^2 - 1$ and c = 2x + 1 is (are) **(2010)**

(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$

(C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

Q.37 Let $\theta, \varphi \in [0, 2\pi]$ be such that

$$2\cos\theta(1-\sin\varphi) = \sin^2\theta\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)\cos\varphi - 1,$$

$\tan(2\pi-\theta) > 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$.

Then φ cannot satisfy **(2012)**

(A) $0 < \varphi < \frac{\pi}{2}$ (B) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$

(C) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$ (D) $\frac{3\pi}{2} < \varphi < 2\pi$

Q.38 Let PQR be a triangle of area Δ with $a = 2,\ b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are lengths of the sides of the triangle opposite to the angle at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **(2012)**

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

Q.39 In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the length of PN, QL and RM are consecutive even integer. Then possible length(s) of the side(s) of the triangle is (are) **(2013)**

(A) 16 (B) 18 (C) 24 (D) 22

Q.40 In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circumradius of the triangle is **(2014)**

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

Q.41 Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangles ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P? **(2015)**

(A) $(4,\ 2\sqrt{2})$ (B) $(9,\ 3\sqrt{2})$

(C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $(1,\ \sqrt{2})$

Q.42 The circle $C_1 : x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then **(2016)**

(A) $Q_2Q_3 = 12$

(B) $R_2R_3 = 4\sqrt{6}$

(C) Area of the triangle OR_2R_3 is $6\sqrt{2}$

(D) Area of the triangle PQ_2Q_3 is $4\sqrt{2}$

Q.43 In a triangle XYZ, let x, y, z be the length of sides opposite to the angle X, Y, Z, respectively, and 2s = x + y + z. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of triangle XYZ is $\frac{8\pi}{3}$, then **(2016)**

(A) Area of the triangle XYZ is $6\sqrt{6}$

(B) The radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Q.44 The orthocenter of the triangle F_1MN is **(2016)**

(A) $\left(-\frac{9}{10}, 0\right)$ (B) $\left(\frac{2}{3}, 0\right)$

(C) $\left(\frac{9}{10}, 0\right)$ (D) $\left(\frac{2}{3}, \sqrt{6}\right)$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.10 Q.13 Q.15 Q.22
Q.24 Q.29

Exercise 2

Q.3 Q.4 Q.6 Q.9

Previous Years' Questions

Q.5 Q.12 Q.14 Q.19
Q.22

JEE Advanced/Boards

Exercise 1

Q.8 Q.13 Q.17 Q.20
Q.27

Exercise 2

Q.3 Q.6 Q.13 Q.14

Previous Years' Questions

Q.4 Q.6 Q.11 Q.12
Q.13 Q.22 Q.25

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $r = 4, R = \frac{65}{8}$

Q.2 2r=R

Q.5 $A = \frac{\pi}{4}, B = \frac{\pi}{3}, C = \frac{5\pi}{12}$

Q.6 No triangle can be formed

Q.7 sides: $2\sqrt{3} \pm \sqrt{2}$; Area: $2\sqrt{3} - \sqrt{2}$, $2\sqrt{3} \pm \sqrt{2}$

Q.9 $c = 4\sqrt{3} \pm 2\sqrt{5}$

Q.10 $\sqrt{2} : 2 : \sqrt{3} + 1$

Q.16 4,5 and 6

Q.22. $\sqrt{5}$

Exercise 2

Single Correct Choice Type

Q.1 A **Q.2** C **Q.3** A **Q.4** D **Q.5** C **Q.6** C
Q.7 B **Q.8** B **Q.9** B **Q.10** D

Previous Years' Questions

Q.1 A **Q.2** B **Q.3** A **Q.4** B **Q.5** A **Q.6** B

Q.7 D

Q.8. (A, D)

Q.9. $\operatorname{cosec} 20^{\circ}$

Q.10. $-\frac{\sqrt{3}}{2}$

Q.11. $\frac{1+\sqrt{3}}{2}$ sq. unit

Q.12. 90°

Q.15 $c = \sqrt{6}, \angle B = 45^{\circ}$ and $\angle A = 75^{\circ}$

Q.17 4, 5, 6 unit

Q.20 $\sqrt{5}$ **Q.21** 3 **Q.22** B **Q.23** A **Q.24** A

JEE Advanced/Boards

Exercise 1

Q.1 9 **Q.2** $\frac{1}{4\sin A}$ **Q.3** 120^o

Q.4 $1/(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})$ **Q.5** 400 **Q.6** 50

Q.6 3 cms & 2 cms **Q.7** two triangles: $\left(2\sqrt{3}-\sqrt{2}\right)$ & $\left(2\sqrt{3}+\sqrt{2}\right)$; $\left(2\sqrt{3}-\sqrt{2}\right)$ & $\left(2\sqrt{3}+\sqrt{2}\right)$ sq. units

Q.9 Isosceles **Q.10** 4, 5, 6 **Q.11** $B=\frac{5\pi}{12};\ C=\frac{\pi}{12};\ \frac{b}{c}=2+\sqrt{3}$

Q.12 $\frac{12}{5}$ **Q.13** 10 **Q.14** 90^o

Q.15 right angled triangle **Q.16** $\frac{2}{r}$ **Q.17** 8

Q.18 $bc\left[\frac{\sin C\sin B}{(1+\sin C)(1+\sin B)}\right]$ **Q.19** $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$ **Q.20** 1

Exercise 2

Single Correct Choice Type

Q.1 C **Q.2** A **Q.3** C **Q.4** B **Q.5** A **Q.6** C

Q.7 A **Q.8** C **Q.9** A **Q.10** B **Q.11** A **Q.12** B

Multiple correct choice type

Q.13 A,C

Match the Columns

Q.14 $A\to s;\ B\to r;\ C\to p;\ D\to q$ **Q.15** $A\to q;\ B\to r;\ C\to p$

Previous Years' Questions

Q.1 D **Q.2** A **Q.3** C **Q.4** D **Q.5** C **Q.6** D

Q.7 C **Q.8** B **Q.9** D **Q.10** A,B,C,D **Q.11** B,D **Q.12** B

Q.13. 113^o **Q.14**. $\frac{16}{7}$ **Q.15** $\frac{2(c^2-a^2)}{3ac}$ **Q.16** Equilateral **Q.17** 7 : 19 : 25 **Q.18.** 2

Q.19 $1:1:\sqrt{2}$ **Q.20** $p\le\frac{ar}{1-r^2}$ **Q.21.** 7 **Q.22** (iii) ⇒ (i). **Q.23.** $p\in\left(-\infty,0\right)\cup\left(3+2\sqrt{2},\infty\right)$

Q.24 2b = a + c **Q.25** $\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$ **Q.26** Equilateral **Q.27** B **Q.28** D

Q.29 A **Q.30** D **Q.31** D **Q.32** B, C **Q.33** C, D **Q.34** 8

Q.35 4 **Q.36** A, B **Q.37** A, C, D **Q.38** C **Q.39** B, D **Q.40** B

Q.41 A, D **Q.42** A, B, C **Q.43** A, C, D **Q.44** A

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $r = \frac{\Delta}{S}$; $R = \frac{abc}{4\Delta}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(8)(7)(6)} = 84$

$R = \frac{abc}{4\Delta} = \frac{(13)(14)(15)}{4\times 84} = \frac{65}{8}$

$r = \frac{84}{21} = 4$

Sol 2: $r = \frac{\Delta}{S} = \frac{\frac{\sqrt{3}}{4}a^2}{3a} \times a^2 = \frac{a}{2\sqrt{3}}$

$R = \frac{abc}{4\Delta} = \frac{a^3}{4\frac{\sqrt{3}}{4}a^2} = \frac{a}{\sqrt{3}}$

$2r = R$

Sol 3: $\frac{a^2-b^2}{a^2+b^2} = \frac{\sin A\cos B - \cos A\sin B}{\sin A\cos B + \cos A\sin B}$

$\frac{a^2}{b^2} = \frac{\sin A\cos B}{\cos A\sin B} = \frac{\sin A}{\sin B}.\frac{\cos B}{\cos A}$

$\Rightarrow \frac{a}{b} = \frac{\cos B}{\cos A} \Rightarrow a\cos A = b\cos B$

$\Rightarrow$ either $\cos A = 0$ or $\cos B$ or $a = b$

Sol 4: ΔABC

$\cos A + \cos B = 4\sin^2\frac{C}{2}$

$2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 4\sin^2\frac{C}{2}$

$2\sin\frac{C}{2}\cos\frac{A-B}{2} = 4\sin^2\frac{C}{2}$

$\cos\frac{A-B}{2} = 2\sin\frac{C}{2} \Rightarrow \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}} = 2$

$\Rightarrow \frac{2\cos\frac{C}{2}\cos\frac{A-B}{2}}{2\cos\frac{C}{2}\sin\frac{C}{2}} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\sin C} = 2$

$\Rightarrow \frac{\sin A+\sin B}{\sin C} = 2 \Rightarrow \frac{a+b}{c} = 2$

Sol 5: $\cos A = \frac{b^2+c^2-a^2}{2bc}$

Using cosine formula, we get

$A = \frac{\pi}{4}, B = \frac{\pi}{3}, C = \frac{5\pi}{12}$

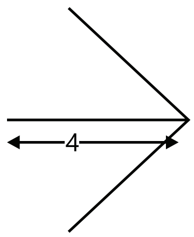
Sol 6: $a = 5, b = 7, \sin A = \frac{4}{3}$

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$\sin B = \frac{28}{15} > 1$

no triangle can be formed.

Sol 7: $\frac{a}{\sin A} = \frac{b}{\sin B}$



$\frac{\sqrt{6}}{\sin 30^o} = \frac{4}{\sin B}$

$\sin B = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$

$\sin B = \sqrt{\frac{2}{3}}$ $\cos B = \pm\frac{1}{\sqrt{3}}$

$\sin A = \frac{1}{2}\cos A = \frac{\sqrt{3}}{2}$

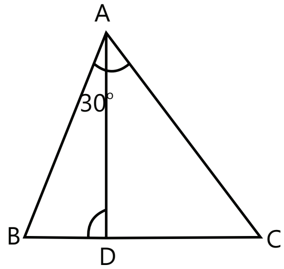
$\cos C = \frac{a^2+b^2-c^2}{2ab} = \cos(180 - (A + B))$

$= -\cos A\cos B + \sin A\sin B$

$\Rightarrow \frac{22-c^2}{8\sqrt{6}} = \frac{\sqrt{2}\pm\sqrt{3}}{2\sqrt{3}}$; we get $c = 2\sqrt{3} \pm \sqrt{2}$

Area $= \frac{1}{2}ab\sin C = 2\sqrt{3} - \sqrt{2}$ or $2\sqrt{3} + 2$

Sol 8: $\angle A = \frac{\pi}{3}$



Length of median

$AD^2 = AC^2+CD^2 - 2AC\times CD\times C$

$AD^2 = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

$4AD^2 = 2b^2 + 2c^2 - a^2$

$4AD^2 = (b^2 + c^2 + bc)+(b^2 + c^2 - a^2 - ac)$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{1}{2}$

$b^2 + c^2 - a^2 - bc = 0$

Sol 9: $\dfrac{6}{\sin 30^o} = \dfrac{8}{\sin B} = \dfrac{c}{\sin C}$

$\sin B = \dfrac{2}{3} \Rightarrow \cos B = \dfrac{\pm\sqrt{5}}{3}$

$\sin 30^o = \dfrac{1}{2}$

$\Rightarrow \sin C = \sin(180 - 30 - B) = \sin(150 - B)$

$= \sin 150 \cos B - \cos 150 \sin B$

$= \pm\dfrac{\sqrt{5}}{6} + \dfrac{\sqrt{3}}{2} \times \dfrac{2}{3} = \dfrac{\pm\sqrt{5}+2\sqrt{3}}{6}$

$C = \dfrac{6}{\sin 30}.\sin C = 4\sqrt{3} \pm 2\sqrt{5}$

Sol 10: $\angle A = 2x; \angle B = 3x; \angle C = 7x$

$\angle A + \angle B + \angle C = 180 \Rightarrow 12x = 180 \Rightarrow x = 15^o$

$\Rightarrow \angle A = 30^o; \angle B = 45^o; \angle C = 105^o$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

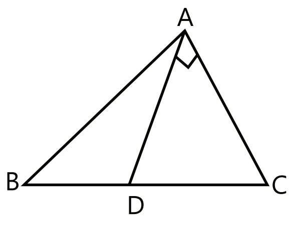
$a : b : c = \sin A : \sin B : \sin C$

$= \dfrac{1}{2} : \dfrac{1}{\sqrt{2}} : \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sqrt{2} : 2 : \sqrt{3} + 1$

Sol 11: $a = \dfrac{b+c}{3}$, $S = 2a$

$\cot\dfrac{B}{2}\cot\dfrac{C}{2} = \dfrac{S}{S-a} = \dfrac{2a}{2a-a} = 2$

Sol 12: $AD = \dfrac{1}{2}\sqrt{2b^2+2c^2-a^2}$; $AC = b$



$CD = \dfrac{a}{2} = \sqrt{b^2 + \dfrac{2b^2+2c^2-a^2}{4}}$

$\dfrac{a^4}{4} = \dfrac{6b^2+2c^2-a^2}{4}$

$a^2 - c^2 = 3b^2$

$\cos A \cos C = \dfrac{(b^2+c^2-a^2)(a^2+b^2-c^2)}{4b^2ac}$

$= \dfrac{(-2b^2)(4b^2)}{4b^2ac} = \dfrac{-2b^2}{ac} = \dfrac{2}{3}\dfrac{(-3b^2)}{ac} = \dfrac{2(c^2-a^2)}{3ac}$

Sol 13: From projection formula

$a \cos B + b \cos A = C$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

So L.H.S. $= \dfrac{b^2+c^2-a^2}{abc} + \dfrac{c^2+a^2-b^2}{abc} + \dfrac{a^2+b^2-c^2}{abc}$

$= \dfrac{(a^2+b^2+c^2)}{abc}$

Sol 14: $\tan\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}$

$\cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \sqrt{\dfrac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}$

$\dfrac{(s-b)\sqrt{s}+(s-c)\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}} = \dfrac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$

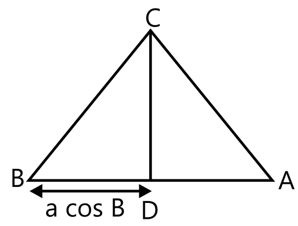
L.H.S. $= (b+c-a)\dfrac{a\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}$

$= \dfrac{2a\sqrt{s}(s-a)}{\sqrt{(s-a)(s-b)(s-c)}} = 2a\sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}$

$= 2a \cot\dfrac{A}{2}$ = R.H.S.

Sol 15: Length of CD = (a sin B)

$\sin C = \sin(180 - (A + B)) = \sin A \cos B$



L.H.S. $= \dfrac{1}{a\sin B} + \dfrac{1}{c\sin B} - \dfrac{1}{c\sin A}$

$= \frac{2R}{bc} + \frac{2R}{ca} - \frac{2R}{ab} = 2R\frac{(a+b-c)}{abc}$

$= 2R\frac{2(s-c)}{abc} = \frac{4R(s-c)}{abc} = \frac{s-c}{\Delta}$

$= \frac{ab\cos^2\frac{C}{2}}{4\Delta S} = \frac{2ab\cos^2\frac{C}{2}}{(a+b+c)\Delta}$

Sol 16: $\frac{a}{\sin A} = \frac{a+1}{\sin(180-A-2A)} = \frac{a+2}{\sin 2A}$

A < 180 – A – 2A < 2A

A < 180 – 3A < 2A

4A < 180 < 5A

A < 45º; A > $\frac{180}{5}$ ⇒ A > 36º

a = $\frac{a+1}{3-4\sin^2 A} = \frac{a+2}{2\cos A}$

a = $\frac{a+1}{4\cos^2 A-1} = \frac{a+2}{2\cos A}$

a = $\frac{a+1}{\left(\frac{a+2}{a}\right)^2-1} = \frac{(a+1)a^2}{4+4a}$

⇒ 4 + 4a = (a + 1)a

⇒ a = 4

Sol 17: L.H.S. = a sin(B – C) + b sin(C – A) + c sin(A – B)

= a sinB cosC – a cosB sinC + b sinC cosA – b cosC sinA + C sinA cosB – C cosA sinB

= cosC (a sin B – b sin A) + cos B(C sin A – a sinC) + cos C(a sin B – b sinA)

= 0 = R.H.S..(as a sin B = b cosA)

Hence Proved.

Sol 18: $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$

$\left.\begin{matrix} b+c=11x \\ c+a=12x \\ a+b=13x \end{matrix}\right\} \begin{matrix} a=7x \\ b=6x \\ c=5x \end{matrix}$

cosA = $\frac{b^2+c^2-a^2}{2bc} = \frac{36+25-49}{2(6)(5)} = \frac{1}{5}$

cosB = $\frac{a^2+c^2-b^2}{2ac} = \frac{49+25-36}{2(7)(5)} = \frac{38}{70} = \frac{19}{35}$

cosC = $\frac{a^2+b^2-c^2}{2ab} = \frac{49+36-25}{2(6)(7)} = \frac{5}{7}$

5 cosA = $\frac{35\cos B}{19} = \frac{75\cos B}{75}$

$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

Sol 19: $\frac{a\cos A + b\cos B + c\cos C}{\cos A\cos B\cos C}$

= 2a tanB tanC secA

⇒ a cosA + b cosB + c cosC = 2a sinB sinC

⇒ a cosA + b cosB + cosC = 2(2R sinA) sinB sinC

⇒ a cosA + b cosB + c cosC = 4R sinA sinB sinC

L.H.S. = a cosA + b cosB + c cosC

= (2R sinA) cosA + 2RsinBcosB + 2RsinCcosC

= R(sin 2A + sin 2B + sin 2C)

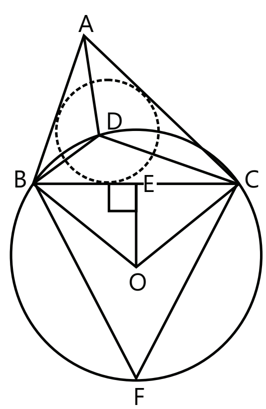
from property of a triangle

sin2A + sin2B + sin2C = 4 sinA sinB sinC

(we can prove this property using basic trigonometric formulas)

= 4R sinA sinB sinC

Sol 20:



BE = EC = a/2

∠BOE = ∠EOC

∠DBE = ∠B/2

∠DCE = ∠C/2

∠BDC = 180° – $\left(\frac{B+C}{2}\right)$ = 180° – $\left(\frac{180-A}{2}\right)$ = 90° + $\frac{A}{2}$

∠BFC = 180° – $\left(90+\frac{A}{2}\right)$ = 90° – $\frac{A}{2}$

$\angle BOC = 2\left(90 - \frac{A}{2}\right) = 180° - A$

$\angle BOE = 90° - \frac{A}{2}$

$\angle EBO = 90° - \left(90 - \frac{A}{2}\right) = \frac{A}{2}$

$\Rightarrow BE = \frac{a}{2} \Rightarrow BO = \frac{a}{2}\sec\frac{A}{2}$

Sol 21: $(b + c - a)\tan\frac{A}{2} = (c + a - b)\tan\frac{B}{2}$

$= (a + b - c)\tan\frac{C}{2}$

$b + c - a = 2(s - a)$

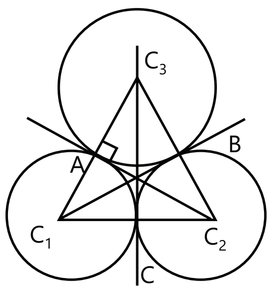
$c + a - b = 2(s - b)$

$a + b - c = 2(s - c)$

$r = (s - a)\tan\left(\frac{A}{2}\right)$

Hence proved.

Sol 22: $C_1C_3 = 3 + 5 = 8$



$C_3C_2 = 5 + 4 = 9$

$C_1C_2 = 3 + 4 = 7$

OA = OB = OC = k(let's say)

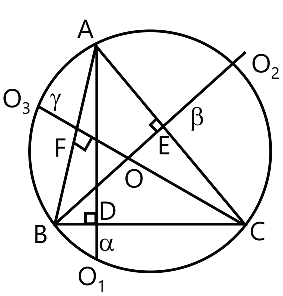
since C_3C is angle bisector of angle $\angle C_1C_3C_2$.

So OA is radius of incircle of triangle $\Delta C_1C_2C_3$.

$OA = r = \frac{\Delta}{S} = \frac{\sqrt{12(12-7)(12-8)(12-9)}}{12}$

$= \sqrt{5}$

Sol 23: $DO_1 = a$, $EO_2 = b$, $FO_3 = g$



$\angle BOA = 2C$

$\angle A$

Sol 24: $8R^2 = a^2 + b^2 + c^2$

$a^2 = 4R^2 \sin^2 A$

So, $2 = \sin^2 A + \sin^2 B + \sin^2 C$

$2 = \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$

$4 = 3 - \cos 2A - \cos 2B - \cos 2C$

$\cos 2A + \cos 2B + \cos 2C = 1$

$2\cos(A + B)\cos(A - B) + 2\cos^2 C - 1 = 1$

$2\cos C[\cos C - \cos(A - B)] = 0$

$\cos C\,[\cos(A + B) + \cos(A - B)] = 0$

$\cos A \cos B \cos C = 0$

So one angle should be 90°.

Sol 25: $\Delta \le \frac{1}{4}\sqrt{(a+b+c)abc}$

R.H.S. $= \frac{1}{4}\sqrt{(a+b+c)abc} = \sqrt{S\frac{(abc)}{8}}$

$= \sqrt{\frac{S}{2}\left(\frac{abc}{4\Delta}\right)\Delta} = \sqrt{\left(\frac{S}{\Delta}\right)\left(\frac{abc}{4\Delta}\right)\frac{\Delta^2}{2}} = \Delta\sqrt{\frac{R}{2r}}$

$= \Delta\sqrt{\frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}$

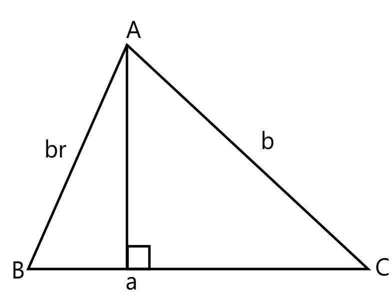
Sol 26: $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2}$

$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

L.H.S. $= \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = \frac{3s-(a+b+c)}{s} = 1$

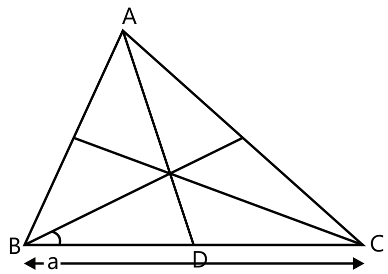
Sol 27: $b\cos B \le \frac{a}{1-r^2}$



$$b\left(\frac{a^2+b^2r^2-b^2}{2abr}\right) \le \frac{a}{1-r^2}$$

$$\frac{a^2+b^2(r^2-1)}{2a^2r} \le \frac{1}{1-r^2}$$

Sol 28:

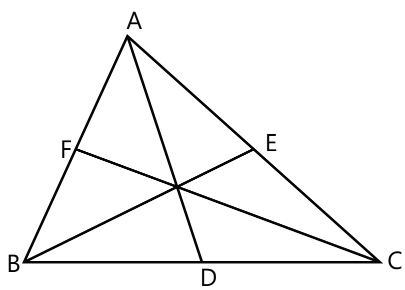


let's say B lie on center and BC lies on x-axis B(0, 0), C(a, 0)

A(c cosB, csinB)

$$BD = \left(\frac{c}{b+c}\right)a$$

Sol 29: AD = α = C sinB = b sinC = (2R) sinB sinC



$$\text{R.H.S.} = \sum\frac{1}{\Delta}(\cot A) = \sum\frac{1}{\Delta^2}\Delta\cot A$$

$$= \frac{1}{\Delta^2}\sum\frac{1}{2}bc\ \sin A\frac{\cos A}{\sin A} = \frac{1}{\Delta^2}\sum\frac{1}{2}bc\ \cos A$$

$$= \frac{1}{2\Delta^2}\sum\frac{b^2+c^2-a^2}{2} = \frac{1}{4\Delta^2}\sum a^2$$

$$= \sum\left(\frac{a}{2\Delta}\right)^2 = \sum\frac{1}{\alpha^2}$$

Exercise 2

Sol 1: (A) Area = $\sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}ab\sin C$

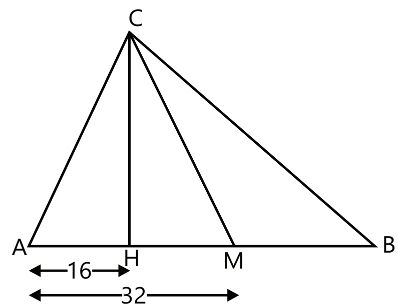
$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge [(s-a)(s-b)(s-c)]^{1/3}$$

$$\left(\frac{s}{3}\right)^{3/2} \ge [(s-a)(s-b)(s-c)]^{1/2}$$

$$\frac{\sqrt{5}(s)^{3/2}}{3\sqrt{3}} \ge [s(s-a)(s-b)(s-c)]^{1/2}$$

$$A \le \frac{s^2}{3\sqrt{3}}$$

Sol 2: (C) CH =b sin C = 2R sin B sin C



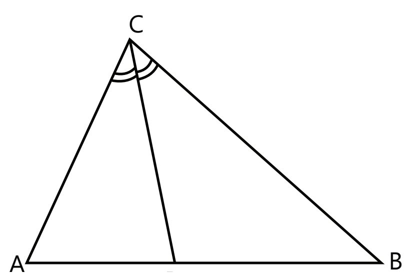
AM = 16, AC = b = 26, CB = 10 = a

$$AH = b\cos A = b\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{(26)^2+(32)^2-(10)^2}{2(32)} = 25$$

HM = 25 – 16 = 9

Sol 3: (A) AB = C; AD = $\left(\frac{b}{b+a}\right)C$



length of angle bisector = 6

$$= \frac{2ab\cos\frac{c}{2}}{a+b} \Rightarrow \frac{1}{a}+\frac{1}{b}=\frac{1}{9}$$

Sol 4: (D) $II_1 = 4R\sin\frac{A}{2}$

$$(II_1)(II_2)(II_3) = 64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 16R^2\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = 16R^2r$$

Sol 5: (C) $r_1 = \dfrac{\Delta}{s-a}$

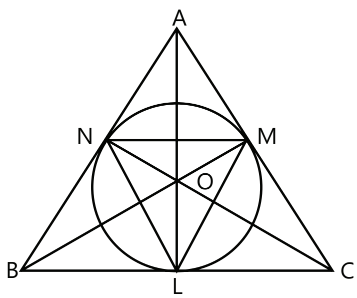
$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{s-a+s-b}{\Delta} = \frac{c}{\Delta}$$

$$\text{L.H.S.} = \frac{abc}{\Delta^3} = \frac{a^3b^3c^3}{64\Delta^3}\,\frac{64}{a^2b^2c^2} = \frac{64R^2}{a^2b^2c^2}$$

K = 64

Sol 6: (C) ON = r = OL = OM

$\angle NOL = 180 - B$



$$\angle ONL = \frac{180-(180-B)}{2} = \frac{B}{2}$$

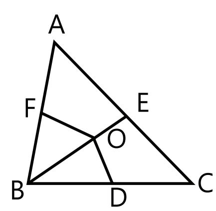
$$\text{R of } \Delta NOL = \frac{r}{2\sin\angle ONL}$$

$$xyz = \frac{r^3}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$xyz = \frac{r^2R}{2}$$

Sol 7: (B) $\angle ABO = \angle B/2$



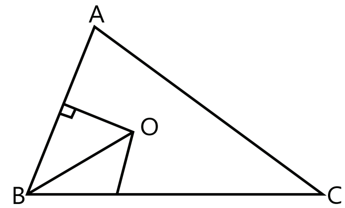
OF = r; $OB = \dfrac{r}{\sin\frac{B}{2}}$

$$(OB)(OC)(OA) = \frac{4r^3R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{4r^3R}{r} = 4r^2R$$

Sol 8: (B) $OB = x = \dfrac{r}{\sin\frac{B}{2}}$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$



$$R = \frac{a}{2\sin A}$$

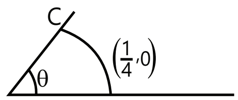
$$xyz = abc\left(\frac{\sin^2\frac{A}{2}}{\sin A}\,\frac{\sin\frac{B}{2}}{\sin B}\,\frac{\sin\frac{C}{2}}{\sin C}\right)$$

$$xyz = abc\left(\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\right)$$

$$\frac{abc}{xyz} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \Sigma\cot\frac{A}{2}$$

Sol 9: (B) arc length = k

radius of circle = k



$k(\theta) = k \Rightarrow \theta = 1$ radius

Particle moves to next circle after completing arc length = k.

To cross positive x-axis

$n(\theta) > 2\pi$

$n(\theta) > 6.28$

$\theta = 1$ radian

n > 6.28

n = 7

Sol 10: (D) $\text{L.H.S.} = \dfrac{2\cos A}{a} + \dfrac{2\cos B}{b} + \dfrac{2\cos C}{c}$

by cosine rule

$$= \frac{2b^2+2c^2-2a^2+2a^2+2c^2-2b^2+2a^2+2b^2-2c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{abc} = \frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab} = \frac{a}{bc}+\frac{b}{ca}$$

then we can conclude that there should nor $\dfrac{c}{ab}$ term in L.H.S.. It is possible only if cosA = 0

$$A = \frac{\pi}{2}$$

Previous Years' Questions

Sol 1: (A) In Δ ABD, applying sine rule, we get

$$\frac{AD}{\sin\pi/3} = \frac{x}{\sin\alpha}$$

$$\Rightarrow AD = \frac{\sqrt{3}}{2} x\sin\alpha \qquad \text{... (i)}$$

And in Δ ACD, applying sine rule, we get

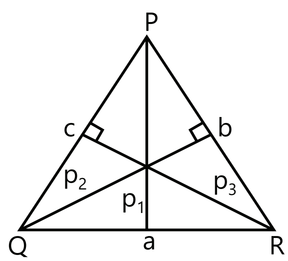
$$\frac{AD}{\sin\pi/4} = \frac{x}{\sin\beta}$$

$$\Rightarrow AD = \frac{3}{\sqrt{2}} x\sin\beta \qquad \text{... (ii)}$$

From Eqs. (i) and (ii)

$$\frac{\sqrt{3}x}{2\sin\alpha} = \frac{3x}{\sqrt{2}\sin\beta} \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{1}{\sqrt{6}}$$

Sol 2: (B) By the law of sine rule



$$\frac{a}{\sin P} = \frac{b}{\sin Q} = \frac{c}{\sin R} = k \text{ (say)}$$

Also, $\frac{1}{2}ap_1 = \Delta \Rightarrow \frac{2\Delta}{a} = p_1 \Rightarrow p_1 = \frac{2\Delta}{k\sin P}$

Similarly, $p_2 = \frac{2\Delta}{k\sin Q}$ and $p_3 = \frac{2\Delta}{k\sin R}$

Since, sin P, sin Q, sin R are in AP, we get that p_1, p_2, p_3 are in HP.

Sol 3: (A) It is given that tan (P/2) and tan (Q/2) are the roots of the quadratic equation

$ax^2 + bx + c = 0$ and $\angle R = \frac{\pi}{2}$

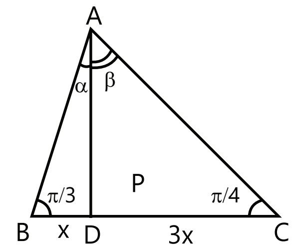
$\therefore$ tan (P/2) + tan (Q/2) = –b/a

and tan (P/2) tan (Q/2) = c/a

Since, P + Q + R = 180°

$\Rightarrow P + Q = 90° \Rightarrow \frac{P+Q}{2} = 45°$

$$\Rightarrow \tan\left(\frac{P+Q}{2}\right) = \tan 45°$$



$$\Rightarrow \frac{\tan(P/2)+\tan(Q/2)}{1-\tan(P/2)\tan(Q/2)} = 1$$

$$\Rightarrow \frac{-b/a}{1-c/a} = 1 \Rightarrow \frac{-b/a}{\frac{a-c}{a}} = 1 \Rightarrow \frac{-b}{a-c} = 1$$

$\Rightarrow -b = a - c \Rightarrow a + b = c$

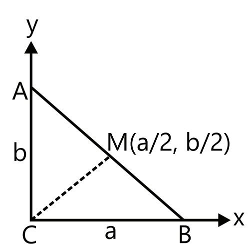
Sol 4: (B) We know than, A + B + C = 180°

$\Rightarrow$ A + C – B = 180 – 2B.

Now, 2ac sin$\left[\frac{1}{2}(A-B+C)\right]$ = 2ac sin (90° – B)

$= 2ac \cos B = \frac{2ac\cdot(a^2+c^2-b^2)}{2ac} = a^2+c^2-b^2$

Sol 5: (A)



Here, $R^2 = MC^2 = \frac{1}{4}(a^2+b^2)$ (by distance from origin)

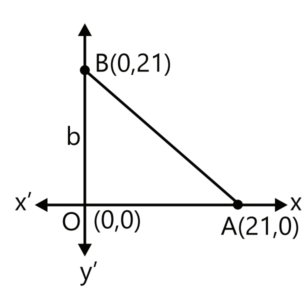
$\frac{1}{4}c^2$ (Pythagorus theorem)

$\Rightarrow R = \frac{c}{2}$

Next, r = (s – c) tan (C/2) = (s – c) tan π/4 = s – c

$\therefore$ 2(r+R) = 2r + 2R = 2s – 2c + c = a + b + c – c = a + b

Sol 6: (B)



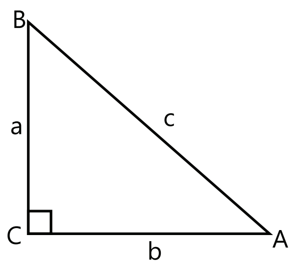
The given vertices of triangle are (0, 0) (0, 21) and (21, 0). To find number of integral points inside the ΔAOB, then

$x > 0,\ y > 0$ and $x + y < 21$

∴Number of points exactly in the interior of the triangle

$$=\frac{20\times 20-20}{2}=190$$

Sol 7: (D)



Let $a : b : c = 1 : \sqrt{3} : 2 \Rightarrow c^2 = a^2 + b^2$

∴ Triangle is right angled at C

or $\angle C = 90^o$ and $\frac{a}{b}=\frac{1}{\sqrt{3}}$

In ΔBAC, $\tan A = \frac{a}{b} = \frac{1}{\sqrt{3}}$

$\Rightarrow A = 30^o$ and $B = 60^o$ $\quad (A + B = 90^o)$

∴ Ratio of angles, $A : B : C = 30^o : 60^o : 90^o$

$\Rightarrow A : B : C = 1 : 2 : 3$

Sol 8: (A, D) The sine formula is

$$\frac{a}{\sin A}=\frac{b}{\sin B}\Rightarrow a\sin B = b\sin A$$

(a) $b \sin A = a \Rightarrow a \sin B = a \Rightarrow B = \frac{\pi}{2}$

Since, $\angle A < \frac{\pi}{2}$ therefore, the triangle is possible

(b) and **(c)** : $b \sin A > a \Rightarrow a \sin B > a \Rightarrow \sin B > 1$

∴ ΔABC is not possible

(d): $b \sin A < a \Rightarrow a \sin B < a$

$\Rightarrow \sin B < 1 \Rightarrow \angle B$ exists

Now, $b > a \Rightarrow B > A$ since $A < \frac{\pi}{2}$

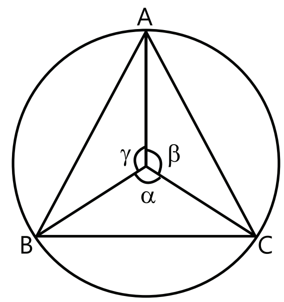
∴ The triangle is possible.

Sol 9: Here, central angle $= \frac{360^o}{9} = 40^o$

In ΔACM, $\frac{1}{r} = \sin 20^o \Rightarrow r = \text{cosec } 20^o$

∴ Radius of circle = cosec 20^o

Sol 10:



Since, sides of a triangle subtends α, β, γ at the center.

∴ $\alpha+\beta+\gamma = 2\pi$ ... (i)

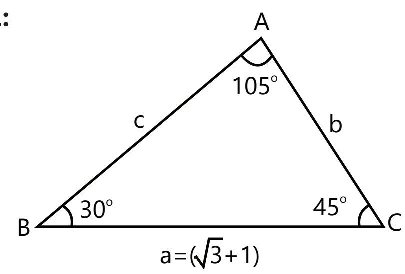
∴ Arithmetic mean

$$=\frac{\cos\left(\frac{\pi}{2}+\alpha\right)+\cos\left(\frac{\pi}{2}+\beta\right)+\cos\left(\frac{\pi}{2}+\gamma\right)}{3}$$

The value is minimum when the three angles are equal.

The minimum value of the AM is $-\frac{\sqrt{3}}{2}$.

Sol 11:



By sine rule, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$$\Rightarrow \frac{\sqrt{3}+1}{\sin(105^o)}=\frac{b}{\sin 30^o} \Rightarrow b=\frac{(\sqrt{3}+1)\sin 30^o}{\sin 105^o}$$

∴ Area of triangle

$$=\frac{1}{2}ab\sin 45^o=\frac{1}{2}(\sqrt{3}+1)\frac{(\sqrt{3}+1)\sin 30^o\sin 45^o}{\sin 105^o}$$

$$=\frac{1}{2}\cdot\frac{(\sqrt{3}+1)^2}{(\sin 45^o\cos 60^o+\cos 45^o\sin 60^o)}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

$$=\frac{1}{4\sqrt{2}}\frac{(3+1+2\sqrt{3})}{\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{2}+\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\right)}=\frac{(4+2\sqrt{3})}{4\sqrt{2}(1+\sqrt{3})}\cdot 2\sqrt{2}$$

$$=\frac{(1+\sqrt{3})^2}{2(1+\sqrt{3})}=\frac{1+\sqrt{3}}{2}\text{sq unit}$$

Sol 12: Given, $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$ ... (i)

We know that, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

$\cos B = \dfrac{c^2+a^2-b^2}{2ac}$

and $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

On putting these values in Eq. (i), we get

$$\frac{2(b^2+c^2-a^2)}{2abc}+\frac{c^2+a^2-b^2}{2abc}+\frac{2(a^2+b^2-c^2)}{2abc}$$

$$=\frac{a}{bc}+\frac{b}{ca}$$

$$\Rightarrow \frac{2(b^2+c^2-a^2)+c^2+a^2-b^2+2(a^2+b^2-c^2)}{2abc}$$

$=\dfrac{a^2+b^2}{abc}$ $\Rightarrow 3b^2 + c^2 + a^2 = 2a^2 + 2b^2$

$\Rightarrow b^2 + c^2 = a^2$

Hence, the angle A is 90º

Sol 13: $p_1p_2p_3 = \dfrac{8\Delta^3}{abc}$

Since, $\Delta = \dfrac{abc}{4R}$

$\therefore\ p_1p_2p_3 = \dfrac{8}{abc}\cdot\dfrac{(abc)^3}{64R^3} = \dfrac{(abc)^2}{8R^3}$

Sol 14: Since, $\Delta = \dfrac{1}{2}ap_1 \Rightarrow \dfrac{1}{p_1} = \dfrac{a}{2\Delta}$

Similarly, $\dfrac{1}{p_2} = \dfrac{b}{2\Delta}, \ \dfrac{1}{p_3} = \dfrac{c}{2\Delta}$

$$\therefore\ \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3} = \frac{1}{2\Delta}(a+b-c)$$

$$= \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta} = \frac{s(s-c)}{ab}\cdot\frac{ab}{s\Delta}$$

$$= \frac{ab}{\left(\frac{a+b+c}{2}\right)\Delta}\cdot\cos^2\frac{C}{2} = \frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}$$

Sol 15: Given that,

$a = 1 + \sqrt{3}$, $b = 2$ and $\angle C = 60^\circ$

We have, $c^2 = a^2 + b^2 - 2ab\cos C$

$\Rightarrow c^2 = (1+\sqrt{3})^2 + 4 - 2(1+\sqrt{3})\cdot 2\cos 60^\circ$

$\Rightarrow c^2 = 1+2\sqrt{3} + 3 + 4 - 2 - 2\sqrt{3}$

$\Rightarrow c^2 = 6 \Rightarrow c = \sqrt{6}$

Using sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \frac{1+\sqrt{3}}{\sin A} = \frac{2}{\sin B} = \frac{\sqrt{6}}{\sin 60^\circ}$$

$$\therefore \sin B = \frac{2\sin 60^\circ}{\sqrt{6}} = \frac{2\times\frac{\sqrt{3}}{2}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$$

$\therefore\ \angle B = 45^\circ$

$\Rightarrow \angle A = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$

Sol 16: Since, r_1, r_2, r_3 are ex-radii of ΔABC are in HP.

$\therefore \dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3}$ are in AP

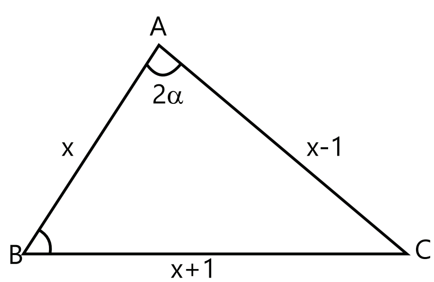
$\Rightarrow \dfrac{s-a}{\Delta}, \dfrac{s-b}{\Delta}, \dfrac{s-c}{\Delta}$ are in AP

$\Rightarrow s - a, s - b, s - c$ are in AP

$\Rightarrow -a, -b, -c$ are in AP

$\Rightarrow a, b, c$ are in AP.

Sol 17: Let ABC be the triangle such that the lengths of its sides CA, AB and BC are $x - 1$, x and $x + 1$ respectively where $x \in N$ and $x > 1$. Let $\angle B = \alpha$ be the smallest angle and $\angle A = 2\alpha$ be the largest angle.



Then, by sine rule, we have

$$\frac{\sin\alpha}{x-1} = \frac{\sin 2\alpha}{x+1}$$

$$\Rightarrow \frac{\sin 2\alpha}{\sin\alpha} = \frac{x+1}{x-1} \Rightarrow 2\cos\alpha = \frac{x+1}{x-1}$$

$\therefore \cos\alpha = \dfrac{x+1}{2(x-1)}$ ... (i)

Also, $\cos\alpha = \dfrac{x^2+(x+1)^2-(x-1)^2}{2x(x+1)}$, using cosine law

$$\Rightarrow \cos\alpha = \frac{x+4}{2(x-1)} \quad \text{... (ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x+1}{2(x-1)} = \frac{x+4}{2(x+1)}$$

$\Rightarrow (x+1)^2 = (x+4)(x-1)$

$\Rightarrow x^2 + 2x + 1 = x^2 + 3x - 4 \Rightarrow x = 5$

Hence, the lengths of the sides of the triangle are 4, 5 and 6 unit.

Sol 18: If the triangle is equilateral, then

$A = B = C = 60^o$

$\Rightarrow \tan A + \tan B + \tan C = 3\tan 60^o = 3\sqrt{3}$

Conversely assume that

$\tan A + \tan B + \tan C = 3\sqrt{3}$

But in triangle ABC, $A + B = 180^o - C$

Taking tan on both sides, we get

$\tan(A+B) = \tan(180^o - C)$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \tan A + \tan B + \tan C + \tan A \tan B \tan C = 3\sqrt{3}$

$\Rightarrow$ none of the tanA, tanB, tanC can be negative

$\Rightarrow \Delta ABC$ cannot be obtuse angle triangle

Also, AM $\geq$ GM

$$\frac{1}{3}[\tan A + \tan B + \tan C] \geq [\tan A \tan B \tan C]^{1/3}$$

$$\Rightarrow \frac{1}{3}(3\sqrt{3}) \geq (3\sqrt{3})^{1/3} \Rightarrow \sqrt{3} \geq \sqrt{3}$$

So, the equality can hold if and only if

tanA = tanB = tanC or A = B = C or when the triangle is equilateral.

Sol 19: We know, $I_n = \frac{n}{2} r^2 \sin\frac{2\pi}{n}$ (I_n is area of regular polygon)

$$\Rightarrow \frac{2I_n}{n} = \sin\frac{2\pi}{n} \quad (r=1) \quad \text{...(i)}$$

and $O_n = nr^2 \tan\frac{\pi}{2}$ {O_n is area of circumscribing polygon}

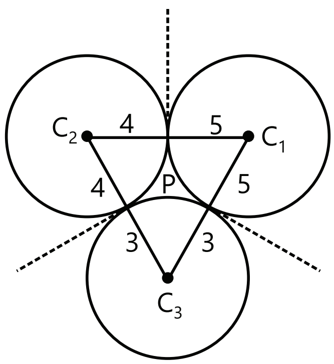
$$\frac{O_n}{n} = \tan\frac{\pi}{n}, \quad \text{... (ii)}$$

$$\therefore \frac{2I_n}{O_n} = \frac{\sin\frac{2\pi}{n}}{\tan\frac{\pi}{n}} \therefore \frac{I_n}{O_n} = \cos^2\frac{\pi}{n} = \frac{1+\cos\frac{2\pi}{n}}{2}$$

$$\frac{I_n}{O_n} = \frac{1+\sqrt{1-(2I_n/n)^2}}{2} \quad \text{[from Eq. (i)]}$$

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-(2I_n/n)^2}\right)$$

Sol 20: Since, the circles with radii 3, 4 and 5 touch each other externally and P is the point of intersection of tangents



$\Rightarrow$ P is incentre of $\Delta C_1C_2C_3$

Thus, distance of point P from the points of contact

= In radius (r) of $\Delta C_1C_2C_3$

ie, $r = \frac{\Delta}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$

where 2s = 7 + 8 + 9

$\therefore s = 12$

Hence, $r = \sqrt{\frac{(12-7)(12-8)(12-9)}{12}} = \sqrt{\frac{5\cdot4\cdot3}{12}} = \sqrt{5}$

Sol 21: $\Delta = \frac{1}{2}ab\sin C \sin C = \frac{2\Delta}{ab} = \frac{2\times15\sqrt{3}}{6\times10} = \frac{\sqrt{3}}{2} C = 120^o$

$\Rightarrow c = \sqrt{a^2+b^2-2ab\cos C}$

$= \sqrt{6^2+10^2-2\times6\times10\times\cos120^o} = 14$

$$r = \frac{\Delta}{s} \Rightarrow r^2 = \frac{225\times3}{\left(\frac{6+10+14}{2}\right)} = 3$$

Sol 22 : (B)

$3 \sin P + 4 \cos Q = 6$ ... (i)

$4 \sin Q + 3 \cos P = 1$ ... (ii)

From (1) and (2) $\angle P$ is obtuse,

$(3\sin P + 4\cos Q)^2 + (4\sin Q + 3\cos P)^2 = 37$

$\Rightarrow 9 + 16 + 24(\sin P \cos Q + \cos P \sin Q) = 37$

$\Rightarrow 24 \sin(P+Q) = 12$

$\Rightarrow \sin(P+Q) = \frac{1}{2} \Rightarrow P + Q = \frac{5\pi}{6} \Rightarrow R = \frac{\pi}{6}$

Sol 23: (A) Let AB = x

$\tan(\pi - \theta - \alpha) = \frac{p}{x-q} \Rightarrow \tan(\theta+\alpha) = \frac{p}{q-x}$

$\Rightarrow q - x = p\cot(\theta+\alpha)$

$\Rightarrow x = q - p\cot(\theta+\alpha) = q - p\left(\frac{\cot\theta\cot\theta - 1}{\cot\alpha + \cot\theta}\right)$

$= q - p\left(\frac{\frac{q}{p}\cot\theta - 1}{\frac{q}{p} + \cot\theta}\right) = q - p\left(\frac{q\cot\theta - p}{q + p\cot\theta}\right)$

$= q - p\left(\frac{q\cos\theta - p\sin\theta}{q\sin\theta + p\cos\theta}\right)$

Sol 24: (A) $\tan 30^\circ = \frac{h}{AD} \Rightarrow AD = h\sqrt{3}$

$BD = h\ ;\ CD = \frac{h}{\sqrt{3}}$

$\frac{AB}{BC} = \frac{AD - BD}{BD - CD} = \frac{\sqrt{3}-1}{1 - \frac{1}{\sqrt{3}}} = \frac{3-\sqrt{3}}{\sqrt{3}-1} = \sqrt{3}$

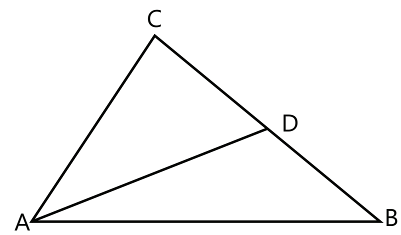
JEE Advanced/Boards

Exercise 1

Sol 1: AB = 2, AC = 1, BC = a

$CD = \frac{1}{1+2}(a) = \frac{a}{3}$, $DB = \frac{2a}{3}$, $AD = \frac{2a}{3}$

Length of angle bisector $= \frac{2bc\cos\frac{A}{2}}{b+c} = \frac{2a}{3}$



$\Rightarrow \frac{2(2)\cos\frac{A}{2}}{3} = \frac{2a}{3} \Rightarrow \cos\frac{A}{2} = \frac{a}{2}$

$\Rightarrow \cos A = \frac{a^2}{2} - 1 = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5 - a^2}{4}$

$\Rightarrow a^2 = 3$

$\therefore 12\Delta^2 = 12 \times \frac{1}{2} bc\sin A = 9$

Sol 2: A = x, B = 2x, C = 4x

$\Rightarrow 7x = 180$

$x = \frac{180}{7}$; C is obtuse angle

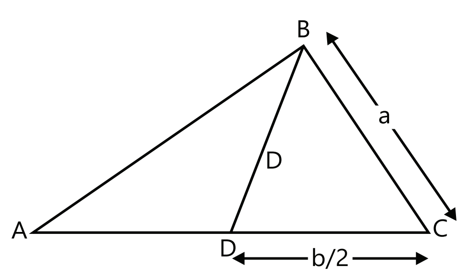
$2R = 4 = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$;

$\sin C = \sin 3A$

$-\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{4}\left[\frac{1}{\sin B} + \frac{1}{\sin C} - \frac{1}{\sin A}\right]$

$= \frac{1}{4}\left[\frac{1}{3A - 4\sin^3 A} + \frac{1}{\sin^2 A} - \frac{1}{\sin A}\right] = \frac{1}{4\sin A}$

Sol 3: Length of median BD $= \frac{\sqrt{3}}{4}\ell(AB)$



$BD = \frac{\sqrt{3}}{4}c = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$

$\frac{\sqrt{3}}{2}c = \sqrt{2a^2 + 2c^2 - b^2}$

$\frac{3}{4}c^2 = 2a^2 + 2c^2 - b^2 \Rightarrow b^2 = 2a^2 + \frac{5}{4}c^2$

$\Rightarrow 4b^2 = 8a^2 + 5c^2$ ... (i)

$BD = \frac{\sqrt{3}}{4}c = \sqrt{\frac{b^2}{4} - a^2}$

$\Rightarrow \frac{3}{16}c^2 = \frac{b^2}{4} - a^2 \Rightarrow 3c^2 = 4b^2 - 16a^2$

$4b^2 = 3c^2 + 16a^2$ ... (ii)

From (1) & (2)

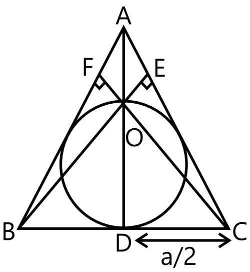
$2c^2 = 8a^2;\ c = 2a$

$\Rightarrow 4b^2 = 12a^2 + 16a^2 \Rightarrow b = \sqrt{7}\,a$

$\cos\angle ABC = \frac{a^2 + c^2 - b^2}{2ac} = \frac{a^2 + 4a^2 - 7a^2}{2a(2a)} = \frac{-3}{4}$

$\Rightarrow \angle ABC = 120^o$

Sol 4: $OD = 2r$



secA = ? = we need to find

$OA = 2R\cos A$

$OA + OD = AD = \frac{a}{2}\cot\frac{A}{2}$

$2r + 2R\cos A = \frac{a}{2}\cot\frac{A}{2}$

$2\left(R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) + 2R\cos A = \frac{2R\sin A}{2}\cot\frac{A}{2}$

$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 2\cos A = 2\sin\frac{A}{2}\cos\frac{A}{2}\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}$

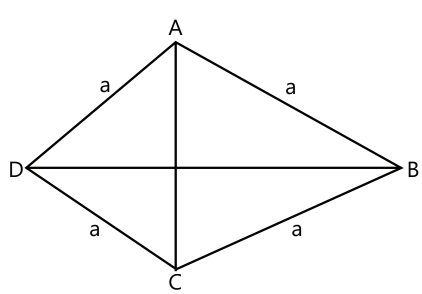
$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 2\cos A = 2\cos^2\frac{A}{2} - 1 + 1$

$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 2\cos A = \cos A + 1$

$\cos A = 1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore \sec A = 1/(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2})$

Sol 5: $\angle DAB = A$



$\angle DAC = \frac{A}{2}$

$DB = 2a\sin\frac{A}{2}$

$\angle BAC = 180 - A$

$AC = 2a\cos\frac{A}{2}$

$R_1(\text{of } \Delta ABD) = \frac{2a\sin\frac{A}{2}}{2\sin A} = 2a\frac{a}{2\cos\frac{A}{2}}$

$\cos\frac{A}{2} = \frac{a}{25}$

R_2 (of ΔACB) = 25

$= \frac{2a\cos\frac{A}{2}}{2\sin A} = \frac{a}{2\sin\frac{A}{2}}$

$\sin\frac{A}{2} = \frac{A}{50}$

$a^2\left[\frac{1}{(25)^2} + \frac{1}{(50)^2}\right] = 1$

$a^2 = (25)^2\frac{4}{5} = 500$

Area of rhombus = 2(Area ΔABD)

$= 2\frac{1}{2}a^2\sin A = a^2\sin A = 500\left(\frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}\right) = 400$

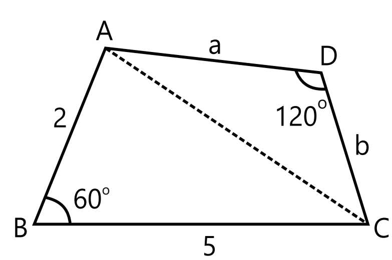
Sol 6: $\frac{\cot C}{\cot A + \cot B} = \frac{\frac{\cos C}{\sin C}}{\frac{\cos A\sin B + \sin A\cos B}{\sin A\sin B}}$

$= \frac{\cos C}{\sin(A+B)}\cdot\frac{\sin A\sin B}{\sin C} = \frac{\cos C}{\sin C}\cdot\frac{\sin A\sin B}{\sin C}$

Applying sine & cosine rule

$= \left(\frac{a^2 + b^2 - c^2}{2ab}\right)\frac{1}{c}\frac{a.b}{c} = \frac{100c^2}{2ab}\left(\frac{ab}{c^2}\right) = 50$

Sol 7: We know that in cyclic quadrilateral, sum of opposite angle is 180^o



$\angle ABC = 60^o$, AD = a, CD = b

$\angle ADC = 120^o$

Area $(\angle ABC) = \frac{1}{2}(2)(5)\sin 60^o = \frac{5\sqrt{3}}{2}$

Total area = $4\sqrt{3}$

Area $(\Delta ADC) = 4\sqrt{3} - \frac{5\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2} = \frac{1}{2}ab\sin 120^o$

$\Rightarrow ab = 6$ ... (i)

$\cos\angle ABC = \frac{(5)^2+(2)^2-(AC)^2}{2(5)(2)}$

$(AC)^2 = 19$

$\cos\angle ADC = \frac{a^2+b^2-19}{2(a)(b)}$

$-\frac{1}{2} = \frac{a^2+b^2-19}{2(6)}$

$a^2 + b^2 = 13$ ... (ii)

From (1) & (2)

a = 3cm, b = 2cm

Sol 8: $\cos 30^o = \frac{(4)^2+c^2-6}{2(4)(c)} = \frac{\sqrt{3}}{2}$

$\frac{10+c^2}{4c} = \sqrt{3}$

$c^2 - 4\sqrt{3}c + 10 = 0 \Rightarrow c = 2\sqrt{3} \pm \frac{\sqrt{48-40}}{2}$

$\Rightarrow c = 2\sqrt{3} \pm \sqrt{2}$

Two triangles are possible.

Sol 9: $a^2 = \frac{b^2c^2}{2b\cos A}$

$\cos A = \frac{bc}{2a^2} = \frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow b^2c^2 = a^2b^2 + a^2c^2 - a^4$

$\Rightarrow b^2(c^2 - a^2) - a^2(c^2 - a^2) = 0$

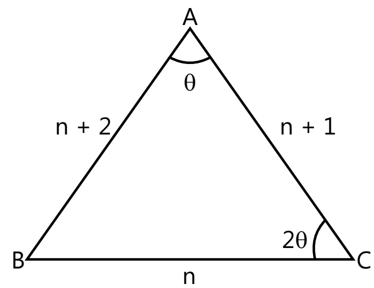
$\Rightarrow (c^2 - a^2)(b^2 - a^2) = 0$

$\Rightarrow$ either c = a or a = b

Hence triangle is isosceles.

Sol 10: Let the sides of Δbe n, n + 1, n + 2 where n ∈ N.

Let a = n, b = n + 1, c = n + 2



Let the smallest, angle ∠A = θ then the greatest ∠C = 2θ. In ΔABC by applying Sine Law we get,

Sin θ/n = sin 2 θ/n + 2

$\Rightarrow$ sin θ/n = 2 sin θ cos θ/ n + 2

$\Rightarrow$ 1/n = 2 cos θ/n + 2 (as sin θ ≠ 0)

$\Rightarrow$ cos θ = n + 2/2n ... (i)

In Δ ABC by Cosine Law, we get

Cos θ = $(n + 1)^2 + (n + 2)^2 - n^2/2(n + 1)(n + 2)$ ... (ii)

Comparing the values of cos θ from (i) and (ii), we get

$(n + 1)^2 + (n + 2)^2 - n^2/2(n + 1)(n + 2) = n + 2/2n$

$\Rightarrow (n + 2)^2 (n + 1) = n(n + 2)^2 + n(n + 1)^2 - n^2$

$\Rightarrow n (n + 2)^2 (n + 2)^2 = n(n + 2)^2 + n(n + 1)^2 - n^3$

$\Rightarrow n^2 + 4n + 4 = n^3 + 2n^2 + n - n^3$

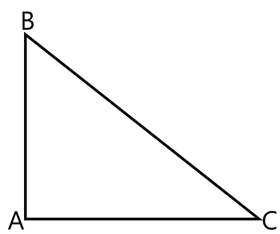
$\Rightarrow n^2 - 3n - 4 = 0 \Rightarrow (n + 1)(n - 4) = 0$

$\Rightarrow$ n = 4 (as n ≠ – 1)

∴ Sides of Δ are 4, 4 + 1, 4 + 2, i.e. 4, 5, 6.

Sol 11: $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$\frac{R}{r_1} = \frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}$



$4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$

$\Rightarrow 4\frac{1}{\sqrt{2}}\sin\frac{B}{2}\sin\left(45-\frac{B}{2}\right) = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$

$\Rightarrow \frac{1}{\sqrt{2}}\sin\frac{B}{2}\left[\cos\frac{B}{2}-\sin\frac{B}{2}\right] = \frac{\sqrt{3}+\sqrt{2}}{4}$

$\Rightarrow \sin\frac{B}{2}\cos\frac{B}{2} - \sin^2\frac{B}{2} = \frac{\sqrt{6}+2}{4}$

$\Rightarrow \sin B - 1 + \cos B = \frac{\sqrt{6}+2}{2}$

$\Rightarrow \sin B + \cos B = \frac{\sqrt{6}+4}{2}$

$\Rightarrow \sin(B + 45^o) = \frac{\sqrt{6}+4}{2\sqrt{2}}$

$\Rightarrow B = \frac{5\pi}{12}, C = \frac{\pi}{12}$

$\Rightarrow \frac{b}{c} = \frac{\sin B}{\sin C} = 2 + \sqrt{3}$

Sol 12: $\left(\frac{a+c}{a}\right)\left(\frac{a}{b}\right) = 2$

$a + c = 2b$ ... (i)

$\Rightarrow a(1 - x^2) + 2bx + c(1 + x^2) = 0$

$x^2(c - a) + 2bx + c + a = 0$

for equal roots $b^2 - c^2 + a^2 = 0$

$a^2 + b^2 = c^2$ ... (ii)

Hence it is right angle at $\angle C$

Putting value in equation (2) from equation (1)

$\Rightarrow a^2 + b^2 = (2b - a)^2 \Rightarrow a^2 + b^2 = 4b^2 + a^2 - 4ab$

$\Rightarrow 3b^2 = 4ab \Rightarrow b = \frac{4}{3}a;\ c = \frac{5}{4}b \Rightarrow c = \frac{5}{3}a$

Hence $a = 3k, b = 4k, c = 5k$

$\sin A + \sin B + \sin C = \frac{3}{5} + \frac{4}{5} + 1 = \frac{12}{5}$

Sol 13: $a = 7, b = 8, c = 5$

$= \frac{\sin A + \sin B + \sin C}{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}} = \frac{\frac{2\Delta}{bc} + \frac{2\Delta}{ac} + \frac{2\Delta}{ab}}{\frac{s}{s-c}\cot\frac{c}{2} + \cot\frac{c}{2}}$

$= \frac{2\Delta}{abc}\frac{(2)s(s-c)}{(a+b)\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}}$

$= \frac{4\Delta}{abc}\sqrt{\frac{s(s-a)(s-b)(s-c)}{(a+b)^2}}$

$= \frac{4\Delta^2}{(a+b)abc}$

Calculating value $p + q = 10$

Sol 14: $r_1 - r = r_2 + r_3$

$\frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$

$\Rightarrow \frac{\Delta a}{s(s-a)} = \frac{\Delta a}{(s-b)(s-c)}$

$\Rightarrow s(s - a) = (s - b)(s - c) \Rightarrow \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = 1$

$\Rightarrow A = 90^o$

Sol 15: $2(2R)^2 = a^2 + b^2 + c^2$

$2 = \left(\frac{a}{2R}\right)^2 + \left(\frac{b}{2R}\right)^2 + \left(\frac{c}{2R}\right)^2$

$\Rightarrow \sin^2A + \sin^2B + \sin^2C = 2$

$\Rightarrow 3 - \cos 2A - \cos 2B - \cos 2C = 4$

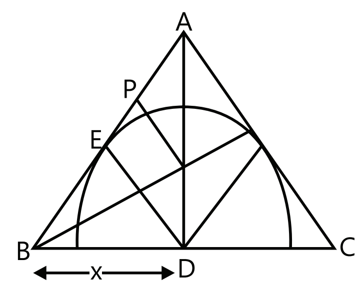
$\Rightarrow \cos 2A + \cos 2B + \cos 2C = -1$

$\Rightarrow 2\cos(A + B)[\cos(A - B) + 1] = -1$

$\Rightarrow \cos C \cos A \cos B = 0$

Hence it is right angled triangle.

Sol 16: $\frac{3}{r} - \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$



$OD = \frac{r}{\sin\frac{A}{2}}$; $AD = \frac{2bc\cos\frac{A}{2}}{b+c}$

$r_a = DE = \left(\frac{AD}{OD}\right) OP = \frac{2bc\cos\frac{A}{2}\sin\frac{A}{2}r}{(b+c)r} = \frac{\Delta}{b+c}$

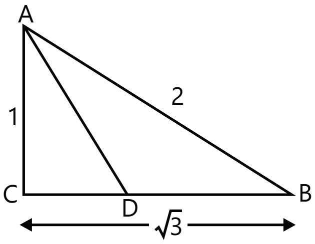
$\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$

$\frac{1}{r} = \left(\frac{1}{r} - \frac{1}{r_a}\right) + \left(\frac{1}{r} - \frac{1}{r_b}\right) + \left(\frac{1}{r} - \frac{1}{r_c}\right) = \frac{s}{\Delta}$

$\frac{1}{r_a} = \frac{b+c}{\Delta}$

R.H.S. $= \frac{b+c+c+a+a+b}{\Delta} = \frac{2s}{\Delta} = \frac{2}{r}$

Sol 17 : $y^2 + 2 = 3y$



$y = 1, 2$

AC = 1, AB = 2

$$\frac{BD}{CD} = \frac{x^2+1}{2x}$$

$$\frac{BD+CD}{CD} = \frac{x^2+1+2x}{2x}$$

$$\frac{\sqrt{3}}{CD} = \frac{(x+1)^2}{2x}$$

$$CD = \frac{2\sqrt{3}x}{(x+1)^2} = \frac{b}{c+b}(\sqrt{3})$$

Similarly $BD = \dfrac{\sqrt{3}(x^2+1)}{(x+1)^2}$

$$\frac{2\sqrt{3}x}{(x+1)^2} = \frac{1}{\sqrt{3}}$$

$6x = (x+1)^2 \Rightarrow x^2 + 1 - 4x = 0$

$\Rightarrow x_1 + x_2 = 4$

$$\tan\frac{A_1}{2} + \tan\frac{A_2}{2} = \frac{2x_1}{1} + \frac{2x_2}{1} = 8$$

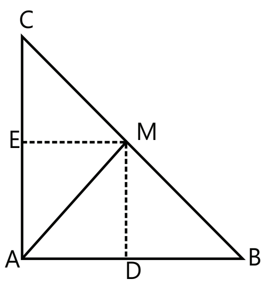
Sol 18: Area (ΔAMC) = Area(ΔAMB)

$$\frac{(BC)}{2} = AM = MB = \frac{a}{2}$$

$AC = 2(AE) = b,\ AD = DB = \dfrac{a}{2}$

In radii of $\Delta AMB = \dfrac{\Delta}{s}$

$$= \frac{1}{2}\frac{\text{Area }(\Delta ABC)}{\left(\frac{a+c}{2}\right)} = \frac{\text{Area }(\Delta ABC)}{a+c} = \frac{bc}{a+c}$$



In radius of $\Delta AMC = \dfrac{1}{2}\dfrac{\text{Area }(\Delta ABC)}{\left(\frac{a+b}{2}\right)}$

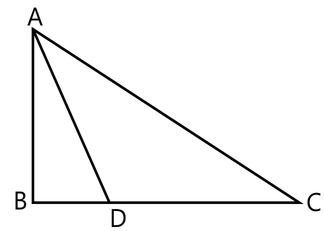
$$= \frac{\text{Area }(\Delta ABC)}{a+b} = \frac{bc}{a+b}$$

$$r_1 r_2 = \frac{bc}{a+c}\,\frac{bc}{a+b}$$

$$\frac{r_1}{r_2} = bc\left[\frac{1}{\frac{a}{c}+1}\cdot\frac{1}{\frac{a}{b}+1}\right] = bc\left[\frac{\sin C\sin B}{(1+\sin C)(1+\sin B)}\right]$$

Sol 19: AD = (AB) sinB = C sinB

$P_1 = AD = 2R\sin C\sin B$



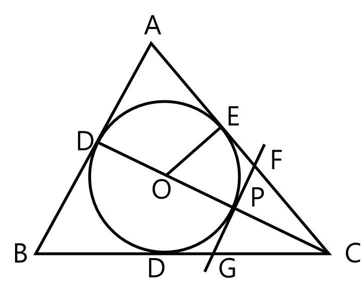
$$\sum\frac{1}{A} = \frac{(\sin A+\sin B+\sin C)}{2R\sin A\sin B\sin C}$$

$$= \frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2R\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}$$

$$= \frac{1}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{1}{r} = \text{R.H.S.}$$

$$= \frac{1}{r} = \frac{\Delta}{s} = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

Sol 20: Area of ΔABC = Area of ABGF + Area of ΔGFC



$$\frac{1}{2}cP_3 = \frac{z+c}{2}(2r) + \frac{1}{2}z(P_3 - 2r)$$

$$\Rightarrow zP_3 + 2cr = cP_3 \Rightarrow \frac{z}{c} = 1 - \frac{2r}{P_3}$$

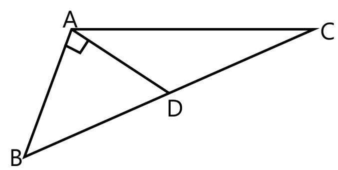
$$\therefore \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 - 2r\left(\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3}\right)$$

$= 3 - 2 = 1$

Exercise 2

Single Correct Choice Type

Sol 1: (C) BD = $\frac{1}{2}$BC



Pythagoras theorem in ΔBAD

$c^2 + (AD)^2 = (BD)^2$

$c^2 + \frac{1}{4}(2b^2 + 2c^2 - a^2) = \frac{a^2}{4}$

$\Rightarrow a^2 = b^2 + 3c^2$

$$\frac{\tan A}{\tan B} = \frac{\sin A}{\sin B}\frac{\cos B}{\cos A}$$

$$= \frac{a}{b}\left(\frac{a^2+c^2-b^2}{2ac}\right)\left(\frac{2bc}{b^2+c^2-a^2}\right) = \frac{a^2+c^2-b^2}{b^2+c^2-a^2} = -2$$

Sol 2: (A) Put a = 2R sin A

$$= \frac{2(\sin A\cos A + \cos B\sin B + \sin C\cos C)}{2(\sin A + \sin B + \sin C)}$$

$$= \frac{\sin 2A + \sin 2B + \sin 2C}{2(\sin A + \sin B + \sin C)}$$

$$= \frac{2\sin(A+B)\cos(A-B) + 2\sin C\cos C}{2(\sin A + \sin B + \sin C)}$$

$$= \frac{\sin C[\cos(A-B) - \cos(A+B)]}{(\sin A + \sin B + \sin C)}$$

$$= \frac{2\sin A\sin B\sin C}{\sin A + \sin B + \sin C}$$

$$= \frac{2\sin A\sin B\sin C}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \frac{2\sin A\sin B\sin C}{2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]}$$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{R}$$

Sol 3: (C) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

R.H.S. =

$$\frac{64R^3\cos\frac{C}{2}\left[\sin\frac{A+B}{2}\right]\left[\cos\frac{B}{2}\sin\frac{A+C}{2}\right]\left[\cos\frac{A}{2}\sin\frac{B+C}{2}\right]}{\sum 16R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right]}$$

$$= \frac{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sum_{i=1}^{3}\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \frac{4R\frac{s\sqrt{s(s-a)(s-b)(s-c)}}{abc}}{\sum\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}(s-c)}$$

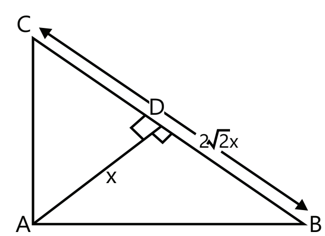
$$= \frac{\left(\frac{4RS}{abc}\Delta\right)}{\left[(s-c)+(s-b)+(s-a)\right]\frac{\Delta}{abc}} = 4R \Rightarrow k = \frac{1}{4}$$

Sol 4: (B) AC = $\frac{x}{\sin\angle ACD} = \frac{x}{\sin C}$

AB = $\frac{x}{\sin B}$

$x^2\left(\frac{1}{\sin^2 C} + \frac{1}{\sin^2 B}\right) = 8x^2$

$\sin^2 B + \sin^2 C = 8\sin^2 C\sin^2 B$



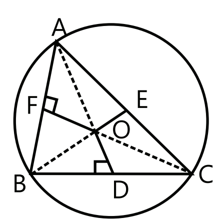
$1 = 8\sin^2 C\cos^2 C$

$(\sin 2C) = \frac{1}{\sqrt{2}} \Rightarrow C = \frac{\pi}{8} \Rightarrow B = \frac{3\pi}{8}$

Sol 5: (A) OB = R

$\angle$AOC = 2A, $\angle$BOD = A

f = OD = R cos A



L.H.S. $= \frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \frac{a}{R\cos A}+\frac{b}{R\cos B}+\frac{c}{R\cos C}$

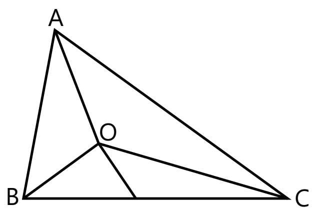
Put R $= \frac{a}{2\sin A}$ = 2(tanA+tanB+tanC) = 2tanA tanB tanC

R.H.S. $= \lambda \frac{abc}{R^3 \cos A \cos B \cos B}$

Put R $= \frac{a}{2\sin A}$ = 8λ tanA tanB tanC

$\Rightarrow \lambda = \frac{1}{4}$

Sol 6: (C) ∠AOB = 2∠C = 2C



OA = OB = R

$\frac{AB}{\sin\angle AOB} = 2R_1$

$R_1 = \frac{C}{2\sin 2C}$

L.H.S. = 2[sin2A + sin2B + sin2C]

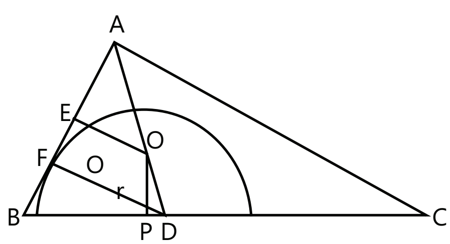
= 2[2 sin(A + B) cos(A – B) + 2 sin C cos C]

= 4 sin C[cos(A – B) ± cos(A + B)] = 8 sinA sinB sinC

$= 8\left(\frac{1}{2}ab\sin C\right)\left(\frac{1}{2}bc\sin A\right)\left(\frac{1}{2}ca\sin B\right)$

$= \frac{8a}{a^2b^2c^2} = \frac{64\Delta^3}{a^2b^2c^2} = \frac{4\Delta}{R^2}$

Sol 7: (A) DF = ?; OE = r; OD $= \frac{r}{\sin\frac{A}{2}}$



$AD = \frac{2bc\cos\frac{A}{2}}{b+c}$

$FD = \left(\frac{AD}{OD}\right)$

$FD = \frac{2bc\cos\frac{A}{2}}{r(b+c)} r\sin\frac{A}{2} = \frac{bc\sin A}{b+c} = \frac{2\Delta}{b+c}$

If C is the base $= \frac{2\Delta}{a+b}$

Sol 8: (C) From sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Given situation

$\frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos B} = k$

2R sinA = k cosA

tanA = tanB = tanC

Hence it is equilateral triangle.

Sol 9: (A) cosA + cosB = 2(1 – cosC)

$\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2\sin^2\frac{C}{2}$

$\Rightarrow \cos\left(\frac{A-B}{2}\right) = 2\sin\frac{C}{2} = 2\cos\left(\frac{A+B}{2}\right)$

$\Rightarrow \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} = 2$

$\Rightarrow \frac{\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)} = \frac{2+1}{2-1}$

$\Rightarrow \cot\frac{A}{2}\cot\frac{B}{2} = 3 = \frac{5}{5-c}$

a + b + c = 3a + 3b – 3c⇒2c = a + b

Sol 10: (B) Arithmetic mean $= \frac{a+b+c}{3}$

$= \frac{2R}{3}$(sinA + sinB + sinC)

Length of altitude = b sinC = 2R sinBsinC

Harmonic mean $= \frac{3}{\sum \frac{1}{2R\sin B\ \sin C}}$

$= \frac{6R\sin A\ \sin B\ \sin C}{\sin A+\sin B+\sin C}$

Product $= (6R)\left(\frac{2R}{3}\right)(\sin A \sin B \sin C)$

= $4R^2$ sinA sin B sinC

$= \left(\frac{a}{\sin A}\right)\left(\frac{b}{\sin B}\right)\sin A \sin B \sin C = 2\Delta$

Sol 11: (A) $\dfrac{\sin A\,\sin(B-C)}{\sin C\,\sin(A-B)}=1$

L.H.S.= $\dfrac{\sin(B+C)\sin(B-C)}{\sin(A+B)\sin(A-B)}=\dfrac{\cos 2B-\cos 2C}{\cos 2A-\cos 2B}=1$

$\Rightarrow 2\sin^2 B=\sin^2 A+\sin^2 C \Rightarrow 2b^2=a^2+c^2$

Sol 12: (B) $\sin A=\dfrac{5}{13}$ A = acute angle

$\sin B=\dfrac{99}{101}\sim 1$

$B \sim 90$

C will be acute angle.

$\cos C > 0$

$\cos C=\sqrt{1-\sin^2 C}=\sqrt{1-\sin^2(A+B)}$

$=\sqrt{1-[\sin A\cos B+\cos A\sin B]^2}$

$\Rightarrow \cos B=\sqrt{1-\left(\dfrac{99}{101}\right)^2}=\dfrac{20}{101}$

$\Rightarrow \cos A=\dfrac{12}{13}$

$\Rightarrow \cos C=\sqrt{1-\left[\dfrac{5}{12}\left(\dfrac{20}{101}\right)+\dfrac{12}{13}\left(\dfrac{99}{101}\right)\right]^2}=\dfrac{255}{1313}$

Multiple correct choice type

Sol 13: (A, C) Hypotenuse

$=\sqrt{\begin{array}{l}[(\cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)]^2\\ \qquad +[\sin 2\alpha+\sin 2\beta+2\sin(\alpha+\beta)]^2\end{array}}$

$\cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta)$

$=\cos^2\alpha-\sin^2\alpha+\cos^2\beta-\sin^2 b+2\cos\alpha\cos\beta-2\sin\alpha\sin b$

$=(\cos\alpha+\cos\beta)^2-(\sin\alpha+\sin\beta)^2$

$\sin 2\alpha+\sin 2\beta+2\sin(\alpha+\beta)$

$=2\sin\alpha\cos\alpha+2\sin\beta\cos\beta+2\sin\alpha\cos b+2\cos\alpha\sin b$

$=2\sin\alpha[\cos\alpha+\cos\beta]+2\sin\beta[\cos\beta+\cos\alpha]$

$=2(\cos\alpha+\cos\beta)(\sin\alpha+\sin\beta)$

Hypotenuse

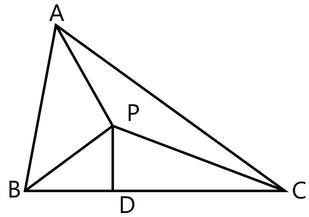
$=(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2$

$=2+2\cos^2\alpha\cos\beta+2\sin\beta\cos b$

$=2+2\cos(\alpha-\beta)=4\cos^2\dfrac{\alpha-\beta}{2}$

Match the columns type

Sol 14: $A\to s;\ B\to r;\ C\to p;\ D\to q$



(A) Centroid divides the triangle into three equal part so 1 : 1 : 1

(B) P is incentre

PD = r

BC = a

Area (ΔPBC) = ar

hence ratio = a : b : c = sinA : sinB : sinC

(C) P is orthocentre

Area of (ΔPBC) = $\dfrac{1}{2}\times PD\times BC=\dfrac{a}{2}\times(AD+AP)$

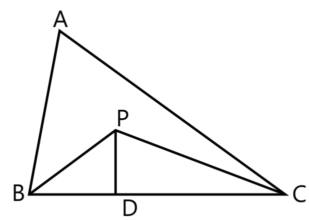
$=\dfrac{a}{2}(2R\sin B\sin C-2R\cos A)$

$=aR(\sin B\sin C-\cos A)=aR[\sin B\sin C+\cos(B+C)$

$=2R\cos B\cos C=2R\sin A\cos B\cos C$

Hence ratio tanA : tanB: tanC

(D) If P is circumcentre



PB = R

$BD=\dfrac{a}{2}$

$PD=\sqrt{R^2-\dfrac{a^2}{4}}$

Area $\Delta PBC=\dfrac{1}{2}\times PD\times BC$

$=\dfrac{a}{2}\sqrt{R^2-\dfrac{a^2}{4}}=\dfrac{a}{2}\sqrt{\dfrac{a^2}{4\sin 2A}-\dfrac{a^2}{4}}$

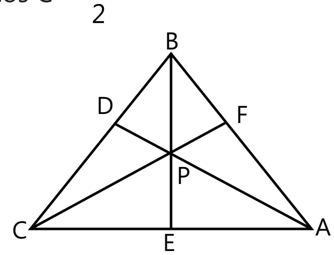
$=\dfrac{a^4}{4}\dfrac{\cos A}{\sin B}=(2R)^2\sin 2A$

Ratio = sin2A : sin2B : sin2C

Sol 15: $A\to q;\ B\to r;\ C\to p;\ D\to q$

$CA=b=1+\sqrt{3}$

$BC=a=2,\ \cos C=\dfrac{1}{2}$



$(AB) = \sqrt{6} = C$

$\Rightarrow \cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{(2)^2 + (6) - (\sqrt{3}+1)^2}{2(2)(\sqrt{6})}$

$\Rightarrow \cos B = \dfrac{6 - 2\sqrt{3}}{4\sqrt{6}} = \dfrac{3 - \sqrt{3}}{2\sqrt{3}} = \dfrac{\sqrt{3} - 1}{2}$

$B = 75^o$

$A = 45^o$

$\Rightarrow (BE) = a \sin C = 2\left(\dfrac{\sqrt{3}}{2}\right) = \sqrt{3}$

$\Rightarrow DA = C \sin B = \sqrt{6}\,(\sin 75^o)$

$\Rightarrow CF = b \sin A = (\sqrt{3}+1)\dfrac{1}{\sqrt{2}}$

ΔDEF is pedal triangle of ΔABC

$\angle DEB = \dfrac{\pi}{2} - A \Rightarrow \angle DEF = \pi - 2A$

(A) Circum scribing circle $= \dfrac{EF}{\sin(EDF)}$

$= \dfrac{a\cos A}{2\sin(\pi - 2a)} = \dfrac{R}{2} = \dfrac{a}{\Delta \sin A} = \dfrac{1}{\sqrt{2}}$

(B) Area $= \dfrac{1}{2} \times (DE)(DF) \times \sin(EDF)$

$= \dfrac{1}{2}(c \cos C \times b \cos B) \times \sin(\pi - 2A)$

$= bc \sin A \cos A \cos B \cos C$

$= \left(\sqrt{3}+1\right)\sqrt{6}\dfrac{1}{\sqrt{2}}\left(\dfrac{\sqrt{3}-1}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{2\sqrt{6}}{8} = \dfrac{\sqrt{3}}{4}$

(C) $r = \dfrac{\Delta}{s} = \dfrac{\sqrt{3}/4}{\Sigma a\cos A} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

Previous Years' Questions

Sol 1: (D) Here, for

(a) If a, sinA, sinB are given, then we can determine b $= \dfrac{a}{\sin A}\sin B$, $c = \dfrac{a}{\sin A}\sin C$. So, all the three sides are unique.

So, option (A) is incorrect.

(b)The three sides can uniquely make an acute angled triangle. So, option (B) is incorrect.

(c)If a, sin B, R are given, then we can determine b = 2R sinB, sinA $= \dfrac{a \sin B}{b}$. So, sin C can be determined.

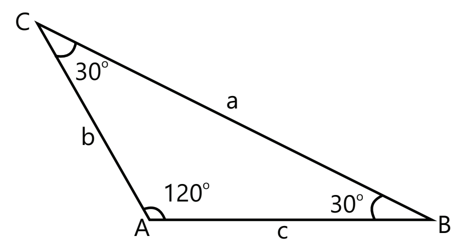
Hence, side c can also be uniquely determined.

(d) If a, sin A, R are given, then

$\dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

But this could not determine the exact values of b and c.

Sol 2: (A)



Given, ratio of angles are 4 : 1 : 1

$\Rightarrow 4x + x + x = 180^o$

$\Rightarrow x = 30^o$

$\therefore \angle A = 120^o,\ \angle B = \angle C = 30^o$

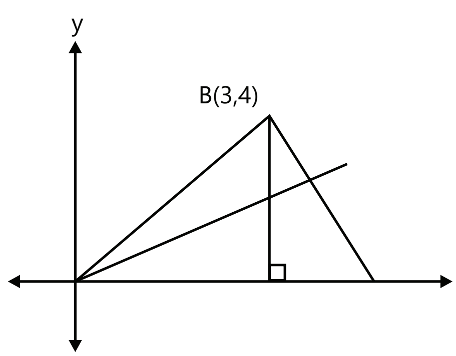
Thus, the ratio of longest side to perimeter $= \dfrac{a}{a+b+c}$

Let b = c = x

$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$

$\Rightarrow a^2 = 2x^2 - 2x^2 \cos A = 2x^2(1 - \cos)$

$\Rightarrow a^2 = 4x^2 \sin^2 A/2 \Rightarrow a = 2x \sin A/2$



$\Rightarrow a = 2x \sin 60^o = \sqrt{3}x$

Thus, required ratio is

$\dfrac{a}{a+b+c} = \dfrac{\sqrt{3}}{x + x + \sqrt{3}x} = \dfrac{\sqrt{3}}{2+\sqrt{3}}$

Sol 3: (C) To find orthocentre of the triangle formed by

(0, 0), (3, 4) and (4, 0).

Let H be the orthocentre of ΔOAB

∴ (Slope of OP ie, OH) · (slope of BA) = –1

$$\Rightarrow \left(\frac{y-0}{3-0}\right)\cdot\left(\frac{4-0}{3-4}\right) = -1 \;\Rightarrow -\frac{4}{3}y = -1 \;\Rightarrow y = \frac{3}{4}$$

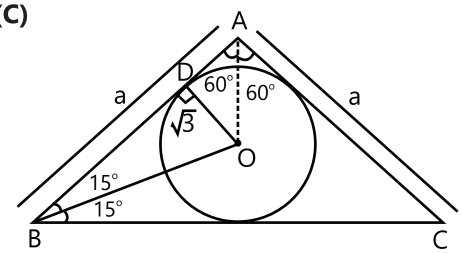
∴Required orthocentre = (3, y) = $\left(3, \frac{3}{4}\right)$

Sol 4: (D) Let a, b, c are the sides of triangle ABC.

$$\text{Now, } \frac{b+c}{a} = \frac{k(\sin B + \sin C)}{k\sin A} = \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$\Rightarrow \frac{b+c}{a} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}} \quad \text{Also, } \frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}}$$

Sol 5: (C)



Let AB = AC = a and ∠A =120°

∴ Area of triangle = $\frac{1}{2}$ $a^2 \sin 120°$

where, a = AD = BD

= $\sqrt{3}\tan 30° + \sqrt{3}\cot 15°$

$$= 1 + \frac{\sqrt{3}}{\tan(45°-15°)} = 1 + \sqrt{3}\left(\frac{1+\tan 45°\tan 30°}{\tan 45° - \tan 30°}\right)$$

$$= 1 + \sqrt{3}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) \qquad \therefore a = 4 + 2\sqrt{3}$$

$$\Rightarrow \text{Area of triangle } = \frac{1}{2}(4+2\sqrt{3})^2\left(\frac{\sqrt{3}}{2}\right) = 12 + 7\sqrt{3}$$

Sol 6: (D) Since, A, B, C are in AP

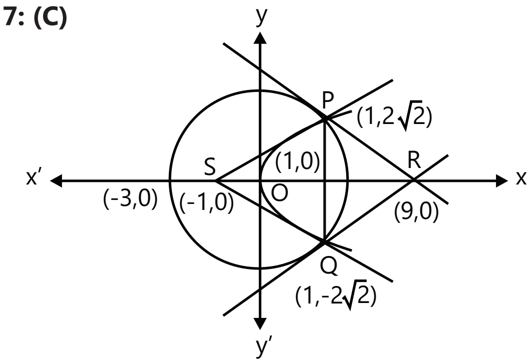
⇒ 2B = A + C ie, ∠B = 60°

$$\therefore \frac{a}{c}(2\sin C\cos C) + \frac{a}{c}(2\sin A\cos A) = 2k\,(a\cos C + c\cos A)$$

$$\text{Using, } \left[\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{k}\right]$$

= 2k (b) = 2sinB [using, b = a cosC + c cosA] = $\sqrt{3}$

Sol 7: (C)



Coordinates of P and Q are $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$

Now, PQ = $\sqrt{(4\sqrt{2})^2 + 0^2} = 4\sqrt{2}$

Area of Δ PQR = $\frac{1}{2}\cdot 4\sqrt{2}\cdot 8 = 16\sqrt{2}$

Area of Δ PQS = $\frac{1}{2}\cdot 4\sqrt{2}\cdot 2 = 4\sqrt{2}$

Ratio of area of triangle PQS and PQR is 1 : 4

Sol 8: (B) Equation of circumcircle of Δ PRS is

$(x + 1)(x + 9) + y^2 + \lambda y = 0$

It will pass through $(1, 2\sqrt{2})$, then

$-16 + 8 + \lambda\cdot 2\sqrt{2} = 0$

$$\Rightarrow \lambda = \frac{8}{2\sqrt{2}} = 2\sqrt{2}$$

∴Equation of circumcircle is

$x^2 + y^2 - 8x + 2\sqrt{2}\,y - 9 = 0$

Hence, its radius is $3\sqrt{3}$.

Alternate Solution

$$\text{Let } \angle PSR = \theta \Rightarrow \sin\theta = \frac{2\sqrt{2}}{2\sqrt{3}} \qquad \therefore \sin\theta = \frac{PR}{2R}$$

$\Rightarrow PR = 6\sqrt{2} = 2R\cdot\sin\theta \Rightarrow R = 3\sqrt{3}$

Sol 9: (D) Radius of incircle is, $r = \frac{\Delta}{s}$

Since, $\Delta = 16\sqrt{2}$

$$\text{Now, } s = \frac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$$

$$\therefore r = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$$

Sol 10: (A, B, C, D) Since, $\Delta ABC = \Delta ABD + \Delta ACD$

$$\Rightarrow \frac{1}{2}bc\sin A = \frac{1}{2}c\,AD\sin\frac{A}{2} + \frac{1}{2}b\,AD\sin\frac{A}{2}$$

$$\Rightarrow AD = \frac{2bc}{b+c}\cos\frac{A}{2}$$

Again, $AE = AD\sec\frac{A}{2} = \frac{2bc}{b+c} \Rightarrow$ AE is HM of b and c.

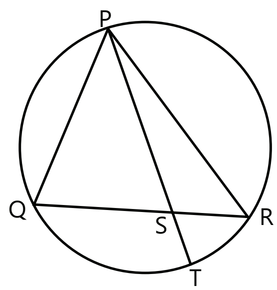
$$EF = ED + DF = 2DE = 2AD\tan\frac{A}{2}$$

$$= 2\frac{2bc}{b+c}\cos\frac{A}{2}\tan\frac{A}{2} = \frac{4bc}{b+c}\sin\frac{A}{2}$$

Since, $AD \perp EF$ and DE = DF and AD is bisector.

$\Rightarrow \Delta AEF$ is isosceles.

Sol 11: (B, D)



Let a straight line through the vertex P of a given Δ PQR intersects the side QR at the point S and the circumcircle of Δ PQR at the point T.

Points P, Q, R, T are concyclic, then $PS \cdot ST = QS \cdot SR$

Now, $\frac{PS+ST}{2} > \sqrt{PS\cdot ST}$ (AM > GM)

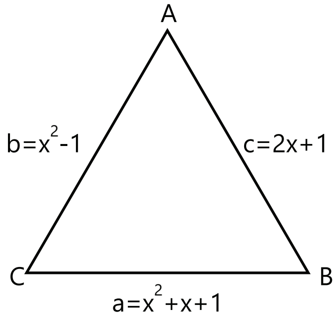
and $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{PS\cdot ST}} \frac{2}{\sqrt{QS\cdot SR}}$

Also, $\frac{SQ+QR}{2} > \sqrt{SQ\cdot SR}$

$$\Rightarrow \frac{QR}{2} > \sqrt{SQ\cdot SR} \Rightarrow \frac{1}{\sqrt{SQ\cdot SR}} > \frac{2}{QR} \Rightarrow \frac{2}{\sqrt{SQ\cdot SR}} > \frac{4}{QR}$$

Hence, $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS\cdot SR}} > \frac{4}{QR}$

Sol 12: (B)



Using, $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(x^2+x+1)^2+(x^2-1)^2-(2x+1)^2}{2(x^2+x+1)(x^2-1)}$$

$\Rightarrow (x+2)(x+1)(x-1)x + (x^2-1)^2 = \sqrt{3}\,(x^2+x+1)(x^2-1)$

$\Rightarrow x^2 + 2x + (x^2-1) = \sqrt{3}\,(x^2+x+1)$

$\Rightarrow (2-\sqrt{3})x^2 + (2-\sqrt{3})x - (\sqrt{3}+1) = 0$

$\Rightarrow x = -(2+\sqrt{3})$ and $x = 1+\sqrt{3}$

But, $x = -(2+\sqrt{3}) \Rightarrow c$ is negative

$\therefore x = 1+\sqrt{3}$ is the only solution.

Sol 13: In ΔADC, $\frac{AD}{b} = \sin 23^\circ$

$\Rightarrow AD = b\sin 23^\circ$

But $AD = \frac{abc}{b^2-c^2}$ (given)

$$\Rightarrow \frac{abc}{b^2-c^2} = b\sin 23^\circ \Rightarrow \frac{a}{b^2-c^2} = \frac{\sin 23^\circ}{c} \quad \text{...(i)}$$

Again, in ΔABC,

$$\frac{\sin A}{a} = \frac{\sin 23^\circ}{c}$$

$$\Rightarrow \frac{\sin A}{a} = \frac{a}{b^2-c^2} \text{ [from Eq. (i)]}$$

$$\Rightarrow \sin A = \frac{a^2}{b^2-c^2} \Rightarrow \sin A = \frac{k^2\sin^2 A}{k^2\sin^2 B - k^2\sin^2 C}$$

$$\Rightarrow \sin A = \frac{\sin^2 A}{\sin^2 B - \sin^2 C}$$

$$\Rightarrow \sin A = \frac{\sin^2 A}{\sin(B+C)\sin(B-C)}$$

$$\Rightarrow \sin A = \frac{\sin^2 A}{\sin A\cdot\sin(B-C)}$$

$\Rightarrow \sin(B-C) = 1$ $(\sin A \neq 0)$

$\Rightarrow \sin(B-23^\circ) = \sin 90^\circ \Rightarrow B - 23^\circ = 90^\circ$

$\Rightarrow B = 113^\circ$

Sol 14: We have, $R = \frac{abc}{4\Delta}$ and $r = \frac{\Delta}{s}$

$$\frac{R}{r} = \frac{abc}{4\Delta}\cdot\frac{s}{\Delta} = \frac{abc\cdot s}{4\Delta^2} = \frac{abc}{4(s-a)(s-b)(s-c)}$$

But a : b : c = 4 : 5 : 6(given)

$\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k(\text{let})$

$\Rightarrow a = 4k,\ b = 5k,\ c = 6k$

Now, $s = \frac{1}{2}(a + b + c) = \frac{1}{2}(4k + 5k + 6x) = \frac{15k}{2}$

$$\therefore \frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{15k}{2}-4k\right)\left(\frac{15k}{2}-5k\right)\left(\frac{15k}{2}-6k\right)}$$

$$= \frac{30k^3}{k^3\left(\frac{15-8}{2}\right)\left(\frac{15-10}{2}\right)\left(\frac{15-12}{2}\right)} = \frac{30\cdot 8}{7\cdot 5\cdot 3} = \frac{16}{7}$$

Sol 15: In Δ ADC, we have $\cos C = \frac{AC}{CD}$

$\cos C = \frac{2b}{a}$ ... (i)

Applying cosine formula in Δ ABC, we have

$\cos A = \frac{b^2+c^2-a^2}{2bc}$

and $\cos C = \frac{a^2+b^2-c^2}{2ab}$ ... (ii)

From Eqs. (i) and (ii), we get

$\frac{a^2+b^2-c^2}{2ab} = \frac{2b}{a}$

$\Rightarrow a^2 + b^2 - c^2 = 4b^2$

$\Rightarrow a^2 - c^2 = 3b^2$ ... (iii)

Now, $\cos A \cos C = \frac{b^2+c^2-a^2}{2bc}\cdot\frac{2b}{a} = \frac{b^2+c^2-a^2}{ac}$

$= \frac{3b^2+3(c^2-a^2)}{3ac}$

$= \frac{(a^2-c^2)+3(c^2-a^2)}{3ac} = \frac{2(c^2-a^2)}{3ac}$

Sol 16: Let a, b, c are the sides of a ΔABC.

Given, $\cos A + \cos B + \cos C = \frac{3}{2}$

$\Rightarrow \frac{b^2+c^2-a^2}{2bc} + \frac{a^2+c^2-b^2}{2ac} + \frac{a^2+b^2-c^2}{2ab} = \frac{3}{2}$

$\Rightarrow ab^2 + ac^2 - a^3 + ba^2 + bc^2 - b^3 + ca^2 + cb^2 - c^3 = 3abc$

$\Rightarrow a(b-c)^2 + b(c-a)^2 + c(a-b)^2$

$= \frac{(a+b+c)}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$

$\Rightarrow (a + b - c)(a - b)^2 + (b + c - a)(b - c)^2 + (c + a - b)(c - a)^2 = 0$

(as we know, a+b–c > 0, b+c–a > 0, c+a–b > 0)

$\therefore$ Each term on the left of equation has positive coefficient multiplied by perfect square, each term must be separately zero.

$\Rightarrow a = b = c$

$\therefore$Triangle is an equilateral.

Sol 17: Let $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = \lambda$

$\Rightarrow (b + c) = 11\lambda,\ c + a = 12\lambda,\ a + b = 13\lambda$ ... (i)

$\Rightarrow 2(a + b + c) = 36\lambda$

$\Rightarrow a + b + c = 18\lambda$ ... (ii)

On solving the Eqs. (i) and (ii), we get

$a=7\lambda$, $b=6\lambda$ and $c=5\lambda$

$\therefore \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36\lambda^2+25\lambda^2-49\lambda^2}{2(30)\lambda^2} = \frac{1}{5}$

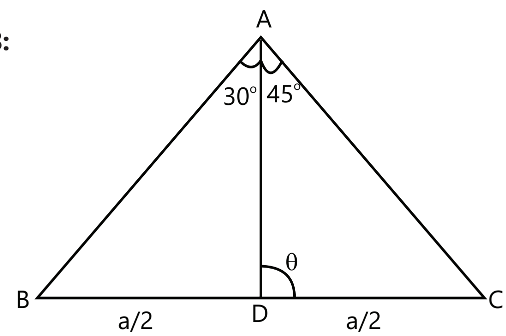
$\cos B = \frac{a^2+c^2-b^2}{2ac}$

$= \frac{49\lambda^2+25\lambda^2-36\lambda^2}{70\lambda^2} = \frac{19}{35}$

$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49\lambda^2+36\lambda^2-25\lambda^2}{84\lambda^2} = \frac{5}{7}$

$\therefore \cos A : \cos B : \cos C = \frac{1}{5} : \frac{19}{35} : \frac{5}{7} = 7 : 19 : 25$

Sol 18:



Let AD be the median to the base BC = a of ΔABC, and let $\angle ADC = \theta$, then

$\left(\frac{a}{2}+\frac{a}{2}\right)\cot\theta = \frac{a}{2}\cot 30^o - \frac{a}{2}\cot 45^o \Rightarrow \cot\theta = \frac{\sqrt{3}-1}{2}$

Applying sine rule in ΔADC, we get

$\frac{AD}{\sin(\pi-\theta-45^o)} = \frac{DC}{\sin 45^o}$

$$\Rightarrow \frac{AD}{\sin(\theta+45^\circ)} = \frac{\frac{a}{2}}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow AD = \frac{a}{\sqrt{2}}\left(\sin 45^\circ \cos\theta + \cos 45^\circ \sin\theta\right)$$

$$\Rightarrow AD = \frac{a}{\sqrt{2}}\left(\frac{\cos\theta+\sin\theta}{\sqrt{2})}\right) = \frac{a}{2}(\cos\theta+\sin\theta)$$

$$\Rightarrow \frac{1}{\sqrt{11-6\sqrt{3}}} = \frac{a}{2}\left(\frac{\sqrt{3}-1}{\sqrt{8-2\sqrt{3}}} + \frac{2}{\sqrt{8-2\sqrt{3}}}\right)$$

$$\Rightarrow a = \frac{2\sqrt{8-2\sqrt{3}}}{\sqrt{44-24\sqrt{3}+22\sqrt{3}-36}} = 2\frac{\sqrt{8-2\sqrt{3}}}{\sqrt{8-2\sqrt{3}}} = 2$$

Sol 19: Given,

cosA cosB + sinA sinB sinC = 1

$$\Rightarrow \sin C = \frac{1-\cos A\cos B}{\sin A \sin B} \quad \text{... (i)}$$

$$\Rightarrow \frac{1-\cos A\cos B}{\sin A\sin B} \le 1 \quad (\sin C \le 1)$$

$\Rightarrow$ 1–cosA cosB ≤ sinA sinB

$\Rightarrow$ 1 ≤ cos(A – B)

$\Rightarrow$ cos(A – B) ≥ 1

$\Rightarrow$ cos(A – B) = 1 [as cos(θ)≤1)

$\Rightarrow$ A – B = 0

On putting A = B in Eq. (i), we get

$$\sin C = \frac{1-\cos^2 A}{\sin^2 A}$$

$$\Rightarrow \sin C = 1 \Rightarrow C = \frac{\pi}{2}$$

Now, A + B + C = π

$$\Rightarrow A + B = \frac{\pi}{2}$$

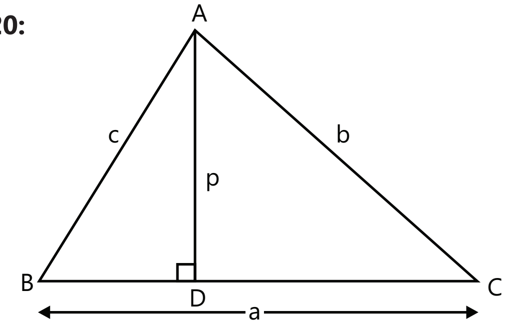
$$\Rightarrow A = \frac{\pi}{4}\left(\because\ A = B \text{ and } C = \frac{\pi}{2}\right)$$

∴ sinA : sinB : sinC

$$= \sin\frac{\pi}{4} : \sin\frac{\pi}{4} : \sin\frac{\pi}{2}$$

$$\Rightarrow a : b : c = \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 1 = 1 : 1 : \sqrt{2}$$

Sol 20:



Let ABC be a triangle with base BC=a and altitude AD =P

then, Area of ΔABC = $\frac{1}{2}$bc sinA

Also, area of ΔABC = $\frac{1}{2}$ap

$$\therefore \frac{1}{2}ap = \frac{1}{2}bc\sin A$$

$$\Rightarrow p = \frac{bc\sin A}{a} \Rightarrow p = \frac{abc\sin A}{a^2}$$

$$\Rightarrow p = \frac{abc\sin A\cdot(\sin^2 B - \sin^2 C)}{a^2(\sin^2 B - \sin^2 C)}$$

$$\Rightarrow p = \frac{abc\sin A\cdot\sin(B+C)\sin(B-C)}{(b^2\sin^2 A - c^2\sin^2 A)}$$

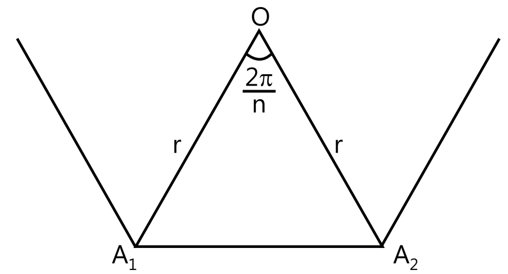
$$\left(\because \text{sin rule}: \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$$

$$= \frac{abc\sin^2 A\cdot\sin(B-C)}{(b^2-c^2)\cdot\sin^2 A} = \frac{abc\sin(B-C)}{b^2-c^2}$$

$$= \frac{ab^2 r\sin(B-C)}{b^2-b^2r^2} = \frac{ar\sin(B-C)}{1-r^2}$$

$$\Rightarrow p \le \frac{ar}{1-r^2} \quad [\sin(B-C) \le 1]$$

Sol 21:



Let O be the centre and r be the radius of the circle passing

through the vertices $A_1, A_2, ..., A_n$

Then, $\angle A_1 OA_2 = \frac{2\pi}{n}$, also $OA_1 = OA_2 = r$

Again by cos formula we know that,

$$\cos\left(\frac{2\pi}{n}\right)=\frac{OA_1^2+OA_2^2-A_1A_2^2}{2(OA_1)(OA_2)}$$

$$\Rightarrow \cos\left(\frac{2\pi}{n}\right) = \frac{r^2+\ r^2-A_1A_2^2}{2\,(r)\,(r)}$$

$$\Rightarrow 2r^2\cos\left(\frac{2\pi}{n}\right) = 2r^2-\ A_1A_2^2$$

$$\Rightarrow A_1A_2^2 = 2r^2-2r^2\cos\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow A_1A_2^2 = 2r^2\left(1-\cos\left(\frac{2\pi}{n}\right)\right)$$

$$\Rightarrow A_1A_2^2 = 2r^2{\cdot}2\sin^2\left(\frac{\pi}{n}\right)$$

$$\Rightarrow A_1A_2^2 = 4r^2\sin^2\left(\frac{\pi}{n}\right) \quad \Rightarrow A_1A_2 = 2r\sin\left(\frac{\pi}{n}\right)$$

Similarly, $A_1A_3 = 2r\sin\left(\frac{2\pi}{n}\right)$

and $A_1A_4 = 2r\sin\left(\frac{3\pi}{n}\right)$

Since, $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$ (given)

$$\Rightarrow \frac{1}{2r\sin(\pi/n)}=\frac{1}{2r\sin(2\pi/n)}+\frac{1}{2r\sin(3\pi/n)}$$

$$\Rightarrow \frac{1}{\sin(\pi/n)}=\frac{1}{\sin(2\pi/n)}+\frac{1}{\sin(3\pi/n)}$$

$$\Rightarrow \frac{1}{\sin(\pi/n)}=\frac{\sin\left(\frac{3\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)}{\sin(2\pi/n)\,\sin(3\pi/n)}$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right)\cdot\sin\left(\frac{3\pi}{n}\right)=\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)+\sin\left(\frac{\pi}{n}\right)\cdot\sin\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right)\left[\sin\left(\frac{3\pi}{n}\right)-\sin\left(\frac{\pi}{n}\right)\right] = \sin\left(\frac{\pi}{n}\right)\cdot\sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right)\left[\left\{2\cos\left(\frac{3\pi+\pi}{2n}\right)\sin\left(\frac{3\pi-\pi}{2n}\right)\right\}\right]$$

$$= \sin\left(\frac{\pi}{n}\right)\cdot\sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow 2\sin\left(\frac{2\pi}{n}\right)\cdot\cos\left(\frac{2\pi}{n}\right)\cdot\sin\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \frac{4\pi}{n}=\pi-\frac{3\pi}{n} \Rightarrow \frac{7\pi}{n}=\pi \Rightarrow n=7$$

Sol 22 : It is given that a, b, c and area of Δ are rational.

We have, $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$$=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-b)}=\frac{\Delta}{s(s-b)}$$

Again, a, b, c are rational given, $s=\frac{a+b+c}{2}$ are rational,

Also, (s – b) is rational since, Δ is rational, therefore, we get

$\tan\left(\frac{B}{2}\right)=\frac{\Delta}{s(s-b)}$ is rational.

Similarly, $\tan\left(\frac{C}{2}\right)=\frac{\Delta}{s(s-c)}$ is rational,

Therefore a, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are rational.

Which shows that (i) $\Rightarrow$ (ii)

Again, it is given that

a, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are rational, then

$$\tan\frac{A}{2} = \tan\left(\frac{\pi}{2}-\frac{B+C}{2}\right)$$

$$\Rightarrow \cot\left(\frac{B+C}{2}\right)=\frac{1}{\tan\left(\frac{B}{2}+\frac{C}{2}\right)} = \frac{1-\tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{C}{2}\right)}{\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)}$$

Since, tan (B/2) and tan(C/2) are rational numbers, we get tan (A/2) is a rational number.

Now, $\sin A = \frac{2\tan A/2}{1+\tan^2 A/2}$ as tan (A/2) is a rational number, sinA is a rational number. Similarly, sinB and sinC are rational numbers. Thus, a, sinA, sinB, sinC are rational numbers therefore (ii) $\Rightarrow$ (iii)

Now again, a, sinA, sinB, sinC are rational.

By the sine rule $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$$\Rightarrow b=\frac{a\sin B}{\sin A} \text{ and } c=\frac{a\sin C}{\sin A}$$

since a, sin A, sin B and sin C are rational numbers,

$\Rightarrow$ b and c are also rational.

Also, $\Delta = \frac{1}{2}bc\sin A$

As b, c and sinA are rational numbers, Δ is a rational numbers.

Therefore, a, b, c and Δ are rational numbers.

Therefore (iii) $\Rightarrow$ (i).

Sol 23: Since, A + B + C = π

$\Rightarrow$ B + C = π − π/4 = 3π/4 ...(i) (A= π/4, given)

$\therefore 0 < B, C < 3\pi/4$.

Also, given tanB · tanC = p

$\Rightarrow \frac{\sin B\cdot\sin C}{\cos B\cdot\cos C} = \frac{p}{1}$ ... (i)

$\Rightarrow \frac{\sin B\cdot\sin C+\cos B\cos C}{\sin B\cdot\sin C-\cos B\cdot\cos C} = \frac{p+1}{p-1}$

$\Rightarrow \frac{\cos(B-C)}{\cos(B+C)} = \frac{1+p}{1-p}$

$\Rightarrow \cos(B-C) = -\frac{(1+p)}{\sqrt{2}(1-p)}$ ... (ii) (B+C = 3π/4)

Since, B or C can vary from 0 to 3π/4

$0 \le B - C < 3\pi/4$

$-\frac{1}{\sqrt{2}} < \cos(B-C) \le 1$ ... (iii)

From Eqs. (ii) and (iii), we get

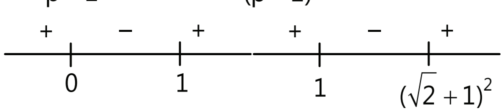
$-\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)} \le 1$

$\Rightarrow -\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}(p-1)}$ and $\frac{1+p}{\sqrt{2}(p-1)} \le 1$

$\Rightarrow \frac{1+p}{p-1}+1 \ge 0$ and $\frac{1+p-\sqrt{2}p+\sqrt{2}}{\sqrt{2}(p-1)} \le 0$

$\Rightarrow \frac{2p}{p-1} \ge 0$ and $\frac{(1-\sqrt{2})\left(p-\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)}{\sqrt{2}(p-1)} \le 0$

$\Rightarrow \frac{2p}{p-1} > 0$ and $\frac{(p-(\sqrt{2}+1)^2)}{(p-1)} \ge 0$



$\Rightarrow$ (p< 0 or p > 1

and(p < 1 or p>($\sqrt{2}$ + 1)^2)

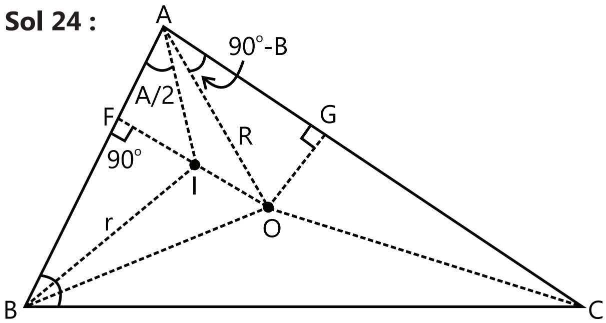
Combing above expressions;

p< 0 or p ≥ ($\sqrt{2}$ +1)^2

ie, $P \in (-\infty, 0) \cup [(\sqrt{2}+1)^2, \infty)$

or $P \in (-\infty, 0) \cup [3+2\sqrt{2}, \infty)$

Sol 24 :



It is clear from figure that OA = R.

$AI = \frac{IF}{\sin(A/2)}$

$\therefore$ Δ AIF is right angled triangle = $\frac{r}{\sin(A/2)}$

Butr = 4R sin (A/2) sin (B/2) sin(C/2)

$\therefore$ AI = 4R sin (B/2) sin(C/2)

Again, ∠GOA = B $\Rightarrow$ OAG = 90° − B

Therefore, ∠IAO = ∠IAC − ∠OAC

= A/2 − (90° − B) = $\frac{1}{2}$(A+2B−180°)

= $\frac{1}{2}$(A + 2B − A − B − C) = $\frac{1}{2}$(B − C)

In ΔOAI, $OI^2 = OA^2 + AI^2 - 2(OA)(AI)\cos(\angle IAO)$

= R^2 + [4R sin (B/2)sin (C/2)]^2 − 2R· [4Rsin(B/2) sin(C/2)] $\cos\left(\frac{B-C}{2}\right)$

= [R^2+16R^2 sin^2(B/2) sin^2(C/2) − 8R^2 sin(B/2) sin(C/2)] $\cos\left(\frac{B-C}{2}\right)$]

= R^2 [1+16 sin^2(B/2) sin^2(C/2) − 8sin(B/2) sin(C/2)] $\cos\left(\frac{B-C}{2}\right)$]

= R^2 [1+8 sin(B/2)

sin(C/2) $\left\{2\sin(B/2)\ \sin(C/2)\ \cos\left(\frac{B-C}{2}\right)\right\}$]

= R^2 [1+8 sin(B/2)

$\sin(C/2)\left\{\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{B+C}{2}\right)-\cos\left(\frac{B-C}{2}\right)\right\}\Bigg]$

$= R^2\left[1-8\sin(B/2)\sin(C/2)\cos\left(\frac{B+C}{2}\right)\right]$

$= R^2\left[1-8\sin(B/2)\sin(C/2)\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\right]$

$\left[\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\right]$

$= R^2\,[1-8\sin(A/2)\sin(B/2)\sin(C/2)]$

$= R^2\left[1-8\left(\frac{r}{4R}\right)\right] = R^2 - 2Rr$

Now, in right ΔBIO

$\Rightarrow OB^2 = BI^2 + IO^2$

$\Rightarrow R^2 = BI^2 + R^2 - 2Rr$

$\Rightarrow 2Rr = BI^2$

$\Rightarrow 2Rr = r^2/\sin^2(B/2)$

$\Rightarrow 2R = r/\sin^2(B/2)$

$\Rightarrow 2R\sin^2 B/2 = r$

$\Rightarrow R(1-\cos B) = r$

$\Rightarrow \frac{abc}{4\Delta}(1-\cos B) = \frac{\Delta}{s}$

$\Rightarrow abc(1-\cos B) = \frac{4\Delta^2}{s}$

$\Rightarrow abc\left[1-\frac{a^2+c^2-b^2}{2ac}\right]=\frac{4\Delta^2}{s}$

$\Rightarrow abc\left[\frac{2ac-a^2-c^2+b^2}{2ac}\right]=\frac{4\Delta^2}{s}$

$\Rightarrow b[b^2-(a-c)^2] = \frac{4\Delta^2}{s}$

$\Rightarrow b[b^2-(a-c)^2] = 8(s-a)(s-b)(s-c)$

$\Rightarrow b[\{b-(a-c)\}\{b+(a-c)\}] = 8(s-a)(s-b)(s-c)$

$\Rightarrow b[(b+c-a)(b+a-c)] = 8(s-a)(s-b)(s-c)$

$\Rightarrow b[(2s-2a)(2s-2c)] = 8(s-a)(s-b)(s-c)$

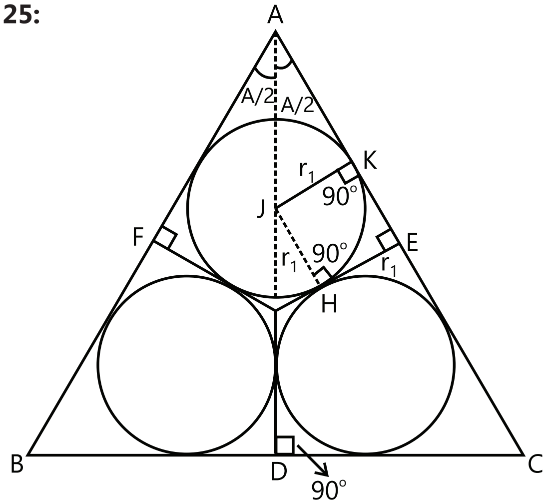
$\Rightarrow b[(2\cdot 2(s-a)(s-c)] = 8(s-a)(s-b)(s-c)$

$\Rightarrow b = 2s-2b$

$\Rightarrow 2b = a+c$

Which shows that b is arithmetic mean between a and c.

Sol 25:



The quadrilateral HEKJ is a square because all four angles are right angle and JK = JH.

Therefore, HE = JK = r_1 and IE = r(given)

$\Rightarrow IH = r - r_1$

Now, in right angled triangle IHJ,

$\angle JIH = \pi/2 - A/2$

[$\angle IEA = 90^o$, $\angle IAE = A/2$ and $\angle JIH = \angle AIE$] in triangle JIH

$\tan\left(\frac{\pi}{2}-\frac{A}{2}\right) = \frac{r_1}{r-r_1} \Rightarrow \cot\frac{A}{2}=\frac{r_1}{r-r_1}$

Similarly, $\cot\frac{B}{2} = \frac{r_2}{r-r_2}$ and $\cot\frac{C}{2}=\frac{r_3}{r-r_3}$

On adding above results, we get

cot A/2 + cot B/2 + cot C/2 = cot A/2 cot B/2 cot C/2

$\Rightarrow \frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3} = \frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$

Sol 26: Given $\Delta \le \frac{1}{4}\sqrt{(a+b+c)abc}$

$\Rightarrow \frac{1}{4\Delta}\sqrt{(a+b+c)abc} \ge 1$

$\Rightarrow \frac{(a+b+c)abc}{16\Delta^2} \ge 1 \Rightarrow \frac{2s\,abc}{16\Delta^2} \ge 1$

$\Rightarrow \frac{s\,abc}{8\cdot s(s-a)(s-b)(s-c)} \ge 1$

$\Rightarrow \frac{abc}{8(s-a)(s-b)(s-c)} \ge 1$

$\Rightarrow \frac{abc}{8} \ge (s-a)(s-b)(s-c)$

Now, puts – a = x ≥ 0, s – b = y ≥ 0, s – c = z ≥ 0

s – a + s – b = x + y

2s – a – b = x + y

c = x + y

Similarly,a = y + z, b = x + z

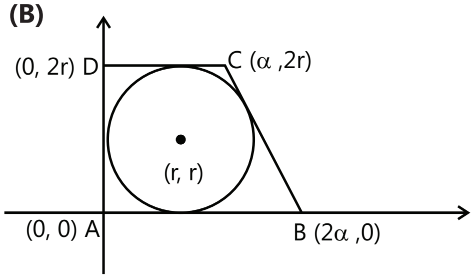
$$\Rightarrow \frac{(x+y)}{2}\cdot\frac{(y+z)}{2}\cdot\frac{(x+z)}{2} \geq xyz$$

which it true

Now quality will hold if

x = y = z ⇒ a = b = c ⇒ triangle is equilateral.

Sol 27: (B)



$$18 = \frac{1}{2}(3\alpha)(2r) \Rightarrow \alpha r = 6$$

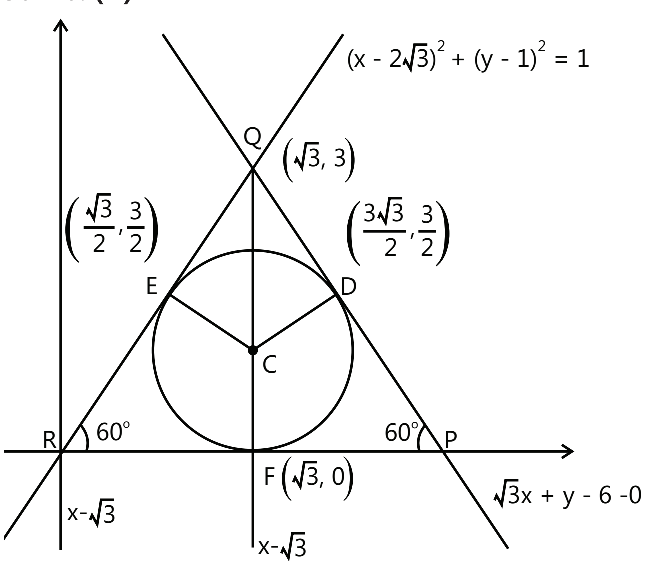
Line $y = -\frac{2r}{\alpha}(x - 2\alpha)$ is tangent to

$(x - r)^2 + (y - r)^2 = r^2$

$2\alpha = 3r$ and $\alpha r = 6$

r = 2.

Sol 28: (D)



Equation of CD is $\dfrac{x - \frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{y - \frac{3}{2}}{\frac{1}{2}} = -1 \quad \Rightarrow \; C \equiv (\sqrt{3}, 1)$

Equation of the circle is $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Sol 29: (A) Since the radius of the circle is 1 and C $(\sqrt{3}, 1)$, coordinates of $F \equiv (\sqrt{3}, 0)$

Equation of CE is $\dfrac{x - \sqrt{3}}{-\frac{\sqrt{3}}{2}} = \dfrac{y - 1}{\frac{1}{2}} = 1 \quad \Rightarrow \quad E \equiv \left(\dfrac{\sqrt{3}}{2}, \dfrac{3}{2}\right)$

Sol 30: (D) Equation of QR is $y - 3 = \sqrt{3}(x - \sqrt{3})$

$\Rightarrow \quad y = \sqrt{3x}$

Equation of RP is y = 0.

Sol 31: (D) $X = \sin\theta + \sin 3\theta + ... + \sin 29\theta$

$2(\sin\theta)X = 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta + ... +$

$\cos 28\theta - \cos 30\theta$

$$X = \frac{1 - \cos 30\,\theta}{2\sin\theta} = \frac{1}{4\sin 2^{o}}$$

Sol 32: (B, C)

$$2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$$

$$\cos\left(\frac{B-C}{2}\right) = 2\sin(A/2)$$

$$\Rightarrow \frac{\cos\left(\frac{B-C}{2}\right)}{\sin A/2} = 2 \;\Rightarrow\; \frac{\sin B + \sin C}{\sin A} = 2$$

$\Rightarrow b + c = 2a$ (constant)

Sol 33: (C, D) Given solutions

$$\frac{1}{\sin(\pi/4)}\left[\frac{\sin(\theta + \pi/4 - \theta)}{\sin\theta\cdot\sin(\theta+\pi/4)} + \frac{\sin(\theta + \pi/2 - (\theta+\pi/4))}{\sin(\theta+\pi/4)\cdot\sin(\theta+\pi/2)} + ... + \frac{\sin((\theta+3\pi/2) - (\theta+5\pi/4))}{\sin(\theta+3o/2)\cdot\sin(\theta+5\pi/4)}\right]$$

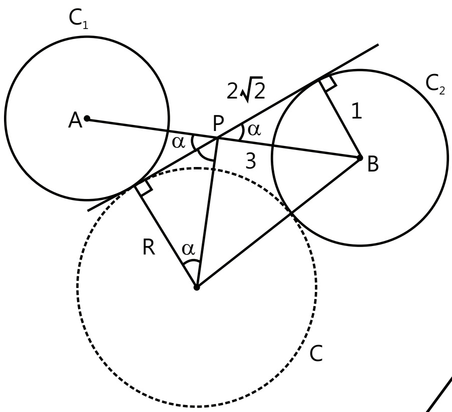
$= 4\sqrt{2}$

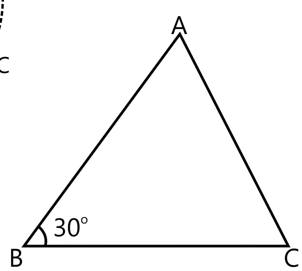
$\Rightarrow \quad \sqrt{2}[\cos\theta - \cot(\theta+\pi/4) + \cot(\theta+\pi/4)$

$-\cot(\theta+\pi/2) + ... + \cot(\theta+5\pi/4) - \cot(\theta+3\pi/2)] = 4\sqrt{2}$

Sol 34: $\cos\alpha = \frac{2\sqrt{2}}{3}$, $\sin\alpha = \frac{1}{3}$, $\tan\alpha = \frac{2\sqrt{2}}{R}$

$\Rightarrow R = \frac{2\sqrt{2}}{\tan\theta} = 8$ units





Sol 35:

$\cos\beta = \frac{a^2+16-8}{2\times a\times 4} \quad \Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{a^2+8}{8a}$

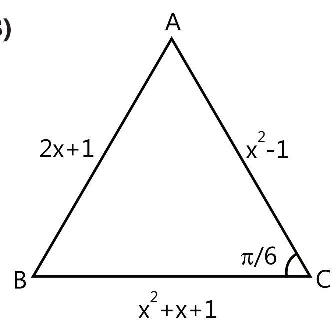
$\Rightarrow \quad a^2 + 4\sqrt{3}\,a + 8 = 0$

$\Rightarrow \quad a_1 + a_2 = 4\sqrt{3},\ a_1a_2 = 8$

$\Rightarrow \quad |a_1 - a_2| = 4$

$\Rightarrow \quad |\Delta_1 - \Delta_2| = \frac{1}{2}\times 4\sin 30^o \times 4 = 4$

Sol 36: (A, B)



$\cos\frac{\pi}{6} = \frac{(x^2-1)^2 + (x^2+x+1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$

$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x^2+3x+2)(x^2-x)}{2(x^2+x+1)(x^2-1)}$

$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x+1)(x+2)x(x-1)}{2(x^2+x+1)(x^2-1)}$

$\Rightarrow \quad \sqrt{3} = \frac{x^2-1+x(x+2)}{x^2+x+1}$

$\Rightarrow \quad \sqrt{3}(x^2+x+1) = 2x^2+2x-1$

$\Rightarrow \quad (\sqrt{3}-2)x^2 + (\sqrt{3}-2)x + (\sqrt{3}+1) = 0$

On solving

$\Rightarrow \quad x^2 + x - (3\sqrt{3}+5) = 0$

$x = \sqrt{3}+1,\ -(2+\sqrt{3})$

Sol 37: (A, C, D)

$2\cos\theta(1-\sin\varphi) = \frac{2\sin^2\theta}{\sin\theta}\cos\varphi - 1 = 2\sin\theta\varphi - 1$

$2\cos\theta - 2\cos\theta\sin\varphi = 2\sin\theta\cos\varphi - 1$

$2\cos\theta + 1 = 2\sin(\theta+\varphi)$

$\tan(2\pi-\theta) > 0 \Rightarrow \tan\theta < 0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$

$\frac{1}{2} < \sin(\theta+\varphi) < 1$

$\Rightarrow \quad 2\pi + \frac{\pi}{6} < \theta + \varphi < \frac{5\pi}{6} + 2\pi$

$2\pi + \frac{\pi}{6} - \theta_{max} < \varphi < 2\pi + \frac{5\pi}{6} - \theta_{min}$

$\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$

Sol 38: (C)

$a = 2 = QR,\ b = \frac{7}{2} = PR,\ c = \frac{5}{2} = PQ$

$s = \frac{a+b+c}{2} = \frac{8}{4} = 4$

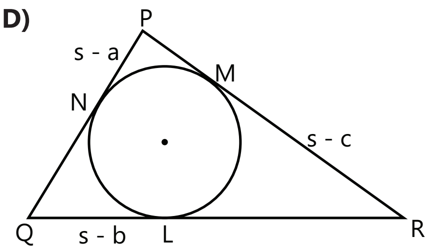
$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)}$

$= \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$

$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2}$

$= \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$

Sol 39: (B, D)



Let s – a = 2k – 2, s – b = 2k, s – c = 2k + 2, k ∈ I, k > 1

Adding we get,

s = 6k

So, a = 4k + 2, b = 4k, c = 4k – 2

Now, $\cos P = \frac{1}{3}$

$$\Rightarrow \frac{b^2+c^2-a^2}{2bc} = \frac{1}{3}$$

$$\Rightarrow 3[(4k)^2+(4k-2)^2-(4k+2)^2] = 2\times 4k(4k-2)$$

$$\Rightarrow 3[16k^2-4(4k)\times 2] = 8k\,(4k-2)$$

$$\Rightarrow 48k^2-96k = 32k^2-16k \Rightarrow 16k^2 = 80k \Rightarrow k = 5$$

So, sides are 22, 20, 18

Sol 40: (B) x = a + b, y = ab

$x^2 - c^2 = y$

$$\Rightarrow \frac{a^2+b^2-c^2}{2ab} = -\frac{1}{2} = \cos(120^\circ)$$

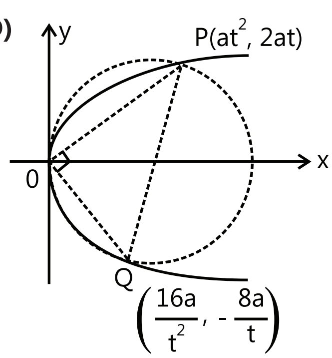
$$\Rightarrow \angle C = \frac{2\pi}{3}$$

$$\Rightarrow R = \frac{abc}{4\Delta}, r = \frac{\Delta}{s}$$

$$\Rightarrow \frac{r}{R} = \frac{4\Delta^2}{s(abc)} = \frac{4\left[\frac{1}{2}ab\sin\left(\frac{2\pi}{3}\right)\right]^2}{\frac{x+c}{2}\cdot y\cdot c}$$

$$\frac{r}{R} = \frac{3y}{2c(x+c)}$$

Sol 41: (A, D)



P(at², 2at)

$$Q\left(\frac{16a}{t^2}, -\frac{8a}{t}\right)$$

$$\Delta OPQ = \frac{1}{2}OP\cdot OQ$$

$$\Rightarrow \frac{1}{2}\left|at\sqrt{t^2+4}\cdot\frac{a(-4)}{t}\sqrt{\frac{16}{t^2}+4}\right| = 3\sqrt{2}$$

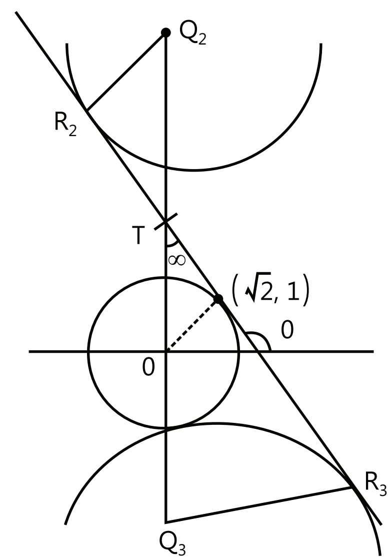
$$\Rightarrow t^2-3\sqrt{2}t+4=0 \Rightarrow t=\sqrt{2}, 2\sqrt{2}$$

$$P(at^2, 2at) = P\left(\frac{t^2}{2}, t\right)$$

$$t=\sqrt{2} \Rightarrow P(1, \sqrt{2})$$

$$t=2\sqrt{2} \Rightarrow P(4, 2\sqrt{2})$$

Sol 42: (A, B, C)



$$x^2+y^2=3$$

$$x^2=2y$$

Intersection point is $P \equiv (\sqrt{2}, 1)$

Equation of tangent is $\sqrt{2}x+y=3$

$$\tan(\theta) = -\sqrt{2}$$

$$\tan(\alpha) = \tan(\theta-90) = -\cot\theta = \frac{1}{\sqrt{2}}$$

$$\sin(\alpha) = \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{Q_3T}$$

$$\Rightarrow Q_3T = 6$$

$\therefore \quad Q_2Q_3 = 2Q_3T = 12$

$\tan(\alpha) = \frac{1}{\sqrt{2}} = \frac{2\sqrt{3}}{R_3T} \Rightarrow R_3T = 2\sqrt{6}$

$\therefore \quad R_2R_3 = 2R_3T = 4\sqrt{6}$

$\perp$ distance of o from R_2 R_3 is $\left|\frac{3}{\sqrt{(\sqrt{2})^2+1^2}} = \sqrt{3}\right|$

$\therefore$ Area $(OR_2R_3) = \frac{1}{2}\times\sqrt{3}\times 4\sqrt{6} = 6\sqrt{2}$ square units

Similarly, Area $(PQ_2Q_3) = \frac{1}{2}\times\sqrt{2}\times 12 = 6\sqrt{2}$ square units

Sol 43: (A, C, D)

$\frac{s-X}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{3s-(x+y+z)}{9} = \frac{s}{9}$

$\therefore \quad x = \frac{5s}{9},\ y = \frac{2s}{3},\ z = \frac{7s}{9}$

$A = \pi r^2 = \frac{8\pi}{3}$

$\Rightarrow \quad \frac{\Delta}{s} = \sqrt{\frac{8}{3}} \quad \Rightarrow \quad \Delta^2 = \frac{8s^2}{3}$

$\Rightarrow \quad s\cdot\frac{4s}{9}\cdot\frac{s}{3}\cdot\frac{2s}{9} = \frac{8}{3}s^2$

$\Rightarrow \quad s = 9$

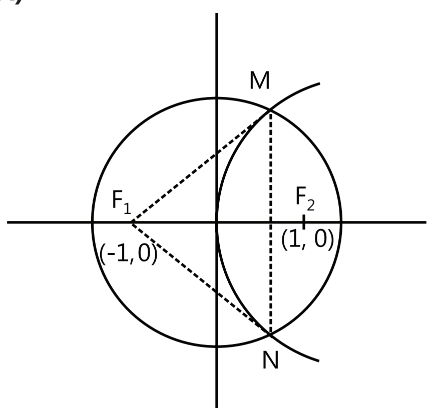
$\therefore \quad \Delta = \sqrt{\frac{8}{3}}\times 9 = 6\sqrt{6}$ square units

$R = \frac{xyz}{4\Delta} = \frac{\frac{5s}{9}\cdot\frac{2s}{3}\cdot\frac{7s}{9}}{4\times 6\sqrt{6}} = \frac{35}{24}\sqrt{6}$

$\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\sin\left(\frac{z}{2}\right) = \frac{r}{4R} = \frac{\sqrt{\frac{8}{3}}}{4\times\frac{35\sqrt{6}}{24}} = \frac{4}{35}$

$\sin^2\left(\frac{x+y}{z}\right) = \cos^2\left(\frac{z}{2}\right) = \frac{1+\cos(z)}{2} = \frac{3}{5}$

Sol 44: (A)



a = 3

$e = \frac{1}{3}$

$\therefore \quad F_1 \equiv (-1, 0)$

$F_2 \equiv (1, 0)$

So, equation of parabola is $y^2 = 4x$

Solving simultaneously, we get $\left(\frac{3}{2}, \pm\sqrt{6}\right)$

$\therefore$ Orthocentre is $\left(\frac{-9}{10}, 0\right)$