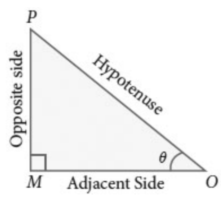
CHAPTER 6

TRIGONOMETRY

TRIGONOMETRIC RATIOS

Key Points

✓ 

✓ $$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{MP}{OP}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{OM}{OP}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \cot\theta = \frac{\cos\theta}{\sin\theta};$$

$$\text{cosec } \theta = \frac{1}{\sin\theta}; \sec\theta = \frac{1}{\cos\theta}$$

✓ $\sin(90^o - \theta) = \cos\theta$ $\cos(90^o - \theta) = \sin\theta$ $\tan(90^o - \theta) = \cot\theta$

✓ $\text{cosec}(90^o - \theta) = \sec\theta$ $\sec(90^o - \theta) = \text{cosec}\,\theta$ $\cot(90^o - \theta) = \tan\theta$

Trigonometric Ratio \ θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\text{cosec}\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

I. TRIGONOMETRIC IDENTITIES

Key Points

✓ $\sin^2\theta + \cos^2\theta = 1$	$\Rightarrow$	$\sin^2\theta = 1 - \cos^2\theta$ (or) $\cos^2\theta = 1 - \sin^2\theta$
✓ $1 + \tan^2\theta = \sec^2\theta$	$\Rightarrow$	$\tan^2\theta = \sec^2\theta - 1$ (or) $\sec^2\theta - \tan^2\theta = 1$
✓ $1 + \cot^2\theta = \text{cosec}^2\theta$	$\Rightarrow$	$\cot^2\theta = \text{cosec}^2\theta - 1$ (or) $\text{cosec}^2\theta - \cot^2\theta = 1$

Example 6.1

Prove that $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$

Solution :

$$\tan^2\theta - \sin^2\theta = \tan^2\theta - \frac{\sin^2\theta}{\cos^2\theta}.\cos^2\theta$$

$$= \tan^2\theta\,(1 - \cos^2\theta) = \tan^2\theta \sin^2\theta$$

Example 6.2

Prove that $\dfrac{\sin A}{1+\cos A} = \dfrac{1-\cos A}{\sin A}$

Solution :

$$\frac{\sin A}{1+\cos A} = \frac{\sin A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}$$

[multiply numerator and denominator by the conjugate of 1 + cosA]

$$= \frac{\sin A\,(1-\cos A)}{(1+\cos A)(1-\cos A)} = \frac{\sin A\,(1-\cos A)}{1-\cos^2 A}$$

$$= \frac{\sin A\,(1-\cos A)}{\sin^2 A} = \frac{1-\cos A}{\sin A}$$

Example 6.3

Prove that $1 + \dfrac{\cot^2\theta}{1+\text{cosec}\,\theta} = \text{cosec}\,\theta$

Solution :

$$= 1 + \frac{\cot^2\theta}{1+\text{cosec}\theta}$$

$$= 1 + \frac{\text{cosec}^2\theta - 1}{\text{cosec}\theta + 1} \quad \left[\text{since } \text{cosec}^2\theta - 1 = \cot^2\theta\right]$$

$$= 1 + \frac{(\text{cosec}\theta + 1)(\text{cosec}\theta - 1)}{\text{cosec}\theta + 1}$$

$$= 1 + (\text{cosec}\theta - 1) = \text{cosec}\theta$$

Example 6.4

Prove that $\sec\theta - \cos\theta = \tan\theta \sin\theta$

Solution :

$$\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta}{\cos\theta} \quad \left[\text{since } 1-\cos^2\theta = \sin^2\theta\right]$$

$$= \frac{\sin\theta}{\cos\theta} \times \sin\theta = \tan\theta \sin\theta$$

Example 6.5

Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \text{cosec}\theta + \cot\theta$

Solution :

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}}$$

[multiply numerator and denominator by the conjugate of 1 – cosθ]

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}}$$

$$[\text{since } \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{1+\cos\theta}{\sin\theta} = \text{cosec}\theta + \cot\theta$$

Example 6.6

Prove that $\dfrac{\sec\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta} = \cot\theta$

Solution :

$$= \frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \cot\theta$$

Example 6.7

Prove that $\sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B = 1$

Solution :

$= \sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A\cos^2 B + \sin^2 A\sin^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B$

$= \sin^2 A(\cos^2 B + \sin^2 B) + \cos^2 A(\sin^2 B + \cos^2 B)$

$= \sin^2 A(1) + \cos^2 A(1)$ (since $\sin^2 B + \cos^2 B = 1$)

$= \sin^2 A + \cos^2 A = 1$

Example 6.8

If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Solution :

Now, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

Squaring both sides,

$(\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$

$\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$

$2\cos^2\theta - \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$

$\cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$

$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$

$$\cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta} = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta}$$

$$= \sqrt{2}\sin\theta$$

[since $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$]

Therefore, $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Example 6.9

Prove that $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

Solution :

$(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta\sin^2\theta \times 1}{\sin^2\theta\cos^2\theta} = 1$$

Example 6.10

Prove that $\dfrac{\sin A}{1+\cos A} + \dfrac{\sin A}{1-\cos A} = 2\operatorname{cosec} A$

Solution :

$$= \frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A}$$

$$= \frac{\sin A(1-\cos A) + \sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$$

$$= \frac{\sin A - \sin A\cos A + \sin A + \sin A\cos A}{1-\cos^2 A}$$

$$= \frac{2\sin A}{1-\cos^2 A} = \frac{2\sin A}{\sin^2 A} = 2\operatorname{cosec} A$$

Example 6.11

If cosec θ + cot θ = P, then prove that

$$\cos\theta = \frac{P^2-1}{P^2+1}$$

Solution :

Given cosec θ + cot θ = P .... (1)

cosec² θ – cot² θ = 1 (identity)

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{\operatorname{cosec}\theta + \cot\theta}$$

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{P} \quad \text{........ (2)}$$

Adding (1) and (2) we get,

$$2\operatorname{cosec}\theta = P + \frac{1}{P}$$

$$2\operatorname{cosec}\theta = \frac{P^2+1}{P} \quad \text{........ (3)}$$

Subtracting (2) from (1), we get,

$$2\cot\theta = P - \frac{1}{P}$$

$$2\cot\theta = \frac{P^2-1}{P} \quad \text{........ (4)}$$

Dividing (4) by (3) we get,

$$\frac{2\cot\theta}{2\operatorname{cosec}\theta} = \frac{P^2-1}{P} \times \frac{P}{P^2+1}$$

$$\text{gives, } \cos\theta = \frac{P^2-1}{P^2+1}$$

Example 6.12

Prove that $\tan^2 A - \tan^2 B = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$

Solution :

$\tan^2 A - \tan^2 B$

$$= \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{\sin^2 A \cos^2 B - \sin^2 B \cos^2 A}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A\left(1-\sin^2 B\right) - \sin^2 B\left(1-\sin^2 A\right)}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$$

Example 6.13

Prove that

$$\left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right) = 2\sin A \cos A$$

Solution :

$$= \left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right)$$

$$= \left(\frac{(\cos A - \sin A)\,(\cos^2 A + \sin^2 A + \cos A \sin A)}{\cos A - \sin A}\right) -$$

$$= \left(\frac{(\cos A + \sin A)\,(\cos^2 A + \sin^2 A - \cos A \sin A)}{\cos A + \sin A}\right)$$

$$\left[\begin{array}{l}\text{since } a^3 - b^3 = (a-b)\,(a^2+b^2+ab) \\ \quad a^3 + b^3 = (a+b)\,(a^2+b^2-ab)\end{array}\right]$$

$$= (1 + \cos A \sin A) - (1 - \cos A \sin A)$$

$$= 2\cos A \sin A$$

Example 6.14

Prove that

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\operatorname{cosec} A + \cot A - 1} = 1$$

Solution :

$$=\frac{\sin A}{\sec A+\tan A-1}+\frac{\cos A}{\operatorname{cosec}A+\cot A-1}$$

$$=\frac{\sin A(\operatorname{cosec}A+\cot A-1)+\cos A(\sec A+\tan A-1)}{(\sec A+\tan A-1)(\operatorname{cosec}A+\cot A-1)}$$

$$=\frac{\sin A\operatorname{cosec}A+\sin A\cot A-\sin A+\cos A\sec A+\cos A\tan A-\cos A}{(\sec A+\tan A-1)(\operatorname{cosec}A+\cot A-1)}$$

$$=\frac{1+\cos A-\sin A+1+\sin A-\cos A}{\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}-1\right)\left(\frac{1}{\sin A}+\frac{\cos A}{\sin A}-1\right)}$$

$$=\frac{2}{\left(\frac{1+\sin A-\cos A}{\cos A}\right)\left(\frac{1+\cos A-\sin A}{\sin A}\right)}$$

$$=\frac{2\sin A\cos A}{(1+\sin A-\cos A)(1+\cos A-\sin A)}$$

$$=\frac{2\sin A\cos A}{[(1+\sin A-\cos A)][(1-\sin A-\cos A)]}$$

$$=\frac{2\sin A\cos A}{1-(1-2\sin A\cos A)}$$

$$=\frac{2\sin A\cos A}{1-1+2\sin A\cos A}=\frac{2\sin A\cos A}{2\sin A\cos A}=1$$

Example 6.15

Show that $\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right)=\left(\frac{1-\tan A}{1-\cot A}\right)^2$

Solution :

LHS

$$\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right)=\frac{1+\tan^2 A}{1+\frac{1}{\tan^2 A}}$$

$$=\frac{1+\tan^2 A}{\frac{\tan^2 A+1}{\tan^2 A}}=\tan^2 A \quad ......(1)$$

RHS

$$\left(\frac{1-\tan A}{1-\cot A}\right)^2=\left(\frac{1-\tan A}{1-\frac{1}{\tan A}}\right)$$

$$=\left(\frac{1-\tan A}{\frac{\tan A-1}{\tan A}}\right)^2=(-\tan A)^2=\tan^2 A$$

From (1) and (2),

$$\left(\frac{1+\tan^2 A}{1+\cot^2 A}\right)=\left(\frac{1-\tan A}{1-\cot A}\right)^2$$

Example 6.16

Prove that

$$\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}=\sin^2 A\cos^2 A$$

Solution :

$$\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$$

$$=\frac{\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)}{(\sec A-\operatorname{cosec}A)(\sec^2 A+\sec A\operatorname{cosec}A+\operatorname{cosec}^2 A)}$$

$$=\frac{\frac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A}}{(\sec A-\operatorname{cosec}A)\left(\frac{1}{\cos^2 A}+\frac{1}{\cos A\sin A}+\frac{1}{\sin^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)\left(\frac{\sin A}{\sin A\cos A}-\frac{\cos A}{\sin A\cos A}\right)}{(\sec A-\operatorname{cosec}A)\left(\frac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin^2 A\cos^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)(\sec A-\operatorname{cosec}A)}{(\sec A-\operatorname{cosec}A)(1+\sin A\cos A)}\times\sin^2 A\cos^2 A$$

$$=\sin^2 A\cos^2 A$$

Example 6.17

If $\frac{\cos^2\theta}{\sin\theta}=$ p and $\frac{\sin^2\theta}{\cos\theta}$ = q, then prove that

$p^2q^2(p^2+q^2+3)=1$

Solution :

We have $\frac{\cos^2\theta}{\sin\theta}=$ p .... (1) and $\frac{\sin^2\theta}{\cos\theta}$ = q.... (2)

$p^2q^2(p^2+q^2+3)=$

$$\left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\times\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2+\left(\frac{\sin^2\theta}{\cos\theta}\right)^2+3\right]$$

[from (1) and (2)]

$$=\left(\frac{\cos^4\theta}{\sin^2\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right)\times\left[\left(\frac{\cos^4\theta}{\sin^2\theta}\right)+\left(\frac{\sin^4\theta}{\cos^2\theta}\right)+3\right]$$

$$=(\cos^2\theta\times\sin^2\theta)\times$$

$$\left[\left(\frac{\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)\right]$$

$$=\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta$$

$$=(\cos^2\theta)^3\times(\sin^2\theta)^3+3\sin^2\theta\cos^2\theta$$

$$=\left[(\cos^2\theta\sin^2\theta)^3-3\cos^2\theta\sin^2\theta\,(\cos^2\theta\sin^2\theta)\right]$$

$$+3\sin^2\theta\cos^2\theta$$

$$=1-3\cos^2\theta\sin^2\theta\,(1)+3\cos^2\theta\sin^2\theta=1$$

EXERCISE 6.1

1. Prove the following identities.

i) cot θ + tan θ = sec θ cosec θ

ii) tan^4 θ + tan^2 θ = sec^4 θ – sec^2 θ

Solution:

i) LHS

$$=\cot\theta+\tan\theta$$

$$=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}$$

$$=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta.\cos\theta}$$

$$=\frac{1}{\sin\theta.\cos\theta}$$

$$=\frac{1}{\cos\theta}\times\frac{1}{\sin\theta}$$

$$=\sec\theta\,.\,\text{cosec}\,\theta$$

= RHS

ii) RHS

$$=\tan^4\theta+\tan^2\theta$$

$$=\tan^2\theta\,(\tan^2\theta+1)$$

$$=(\sec^2\theta-1)\,.\,(\sec^2\theta)$$

$$=\sec^4\theta-\sec^2\theta$$

= RHS

2. Prove the following identities.

i) $\dfrac{1-\tan^2\theta}{\cot^2\theta-1}=\tan^2\theta$

ii) $\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta-\tan\theta$

Solution :

i) LHS

$$=\frac{1-\tan^2\theta}{\cot^2\theta-1}$$

$$=\frac{1-\tan^2\theta}{\frac{1}{\tan^2\theta}-1}$$

$$=\frac{1-\tan^2\theta}{\frac{1-\tan^2\theta}{\tan^2\theta}}$$

$$=\tan^2\theta$$

= RHS

ii) LHS

$$=\frac{\cos\theta}{1+\sin\theta}$$

$$=\frac{\cos\theta}{1+\sin\theta}\times\frac{1-\sin\theta}{1-\sin\theta}$$

$$=\frac{\cos\theta\,(1-\sin\theta)}{\cos^2\theta}$$

$$=\frac{1-\sin\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\sec\theta-\tan\theta$$

= RHS

3. Prove the following identities.

i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$

Solution :

i) LHS

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$= \sec\theta + \tan\theta$

= RHS

ii) LHS

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= (\sec\theta + \tan\theta) + \frac{1}{\sec\theta + \tan\theta}$$

$= (\sec\theta + \tan\theta) + (\sec\theta - \tan\theta)$

$= 2\sec\theta$

= RHS

4. Prove the following identities.

i) $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1$

ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = 1 + (\sec\theta + \operatorname{cosec}\theta)^2$

Solution :

i) LHS

$= \sec^6\theta$

$= (\sec^2\theta)^3$

$= (1 + \tan^2\theta)^3$

$(\because (a+b)^3 = a^3 + b^3 + 3ab\,(a+b))$

$= 1 + \tan^6\theta + 3(1)\tan^2\theta)\,(1 + \tan^2\theta)$

$= 1 + \tan^6\theta + 3\tan^2\theta \,.\, \sec^2\theta$

= RHS

ii) LHS

$= (\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$= \sin^2\theta + \sec^2\theta + 2\sin\theta \,.\, \sec\theta + \cos^2\theta + \operatorname{cosec}^2\theta + 2\cos\theta \,.\, \operatorname{cosec}\theta$

$$= (\sin^2\theta + \cos^2\theta) + \sec^2\theta + \frac{2\sin\theta}{\cos\theta} + \operatorname{cosec}^2\theta + \frac{2\cos\theta}{\sin\theta}$$

$$= 1 + \sec^2\theta + \operatorname{cosec}^2\theta + 2\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= 1 + \sec^2\theta + \operatorname{cosec}^2\theta + 2\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)$$

$= 1 + \sec^2\theta + \operatorname{cosec}^2\theta + 2\sec\theta\operatorname{cosec}\theta$

$= 1 + (\sec\theta\operatorname{cosec}\theta)^2$

= RHS

5. Prove the following identities.

i) $\sec^4\theta\,(1 - \sin^4\theta) - 2\tan^2\theta = 1$

ii) $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\operatorname{cosec}\theta - 1}{\operatorname{cosec} + 1}$

Solution :

i) LHS

$= \sec^4\theta\,(1 - \sin^4\theta) - 2\tan^2\theta$

$$= \frac{1}{\cos^4\theta}(1+\sin^2\theta).(1-\sin^2\theta) - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{(1+\sin^2\theta)\cos^2\theta}{\cos^4\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1+\sin^2\theta}{\cos^2\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1-\sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta}{\cos^2\theta}$$

$= 1$

= RHS

ii) LHS

$$= \frac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta}$$

$$= \frac{\frac{\cos\theta}{\sin\theta} - \cos\theta}{\frac{\cos\theta}{\sin\theta} + 1}$$

$$= \frac{\cos\theta\left(\frac{1}{\sin\theta} - 1\right)}{\cos\theta\left(\frac{1}{\sin\theta} + 1\right)}$$

$$= \frac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1}$$

= RHS

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6. Prove the following identities.

$$i)\ \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$$

$$ii)\ \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$$

Solution :

i) LHS

$$= \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$= \frac{\left(\sin^2 A - \sin^2 B\right) + \left(\cos^2 A - \cos^2 B\right)}{\left(\cos A + \cos B\right).\left(\sin A + \sin B\right)}$$

$$= \frac{\left(\sin^2 A + \cos^2 A\right) - \left(\sin^2 B + \cos^2 B\right)}{\left(\cos A + \cos B\right).\left(\sin A + \sin B\right)}$$

$$= \frac{1-1}{\left(\cos A + \cos B\right).\left(\sin A + \sin B\right)}$$

$= 0$

= RHS

ii) LHS

$$= \frac{\sin^3 A - \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A + \cos^3 A}{\sin A - \cos A}$$

$$= \frac{\left(\sin A + \cos A\right).\left(\sin^2 A - \sin A\cos A + \cos^2 A\right)}{\sin A + \cos A}$$

$$+ \frac{\left(\sin A - \cos A\right).\left(\sin^2 A + \sin A\cos A + \cos^2 A\right)}{\sin A - \cos A}$$

$= (1 - \sin A\cos A) + (1 + \sin A\cos A)$

$= 2$

= RHS

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7. i) If sin θ + cos θ = $\sqrt{3}$, then prove that tan θ + cot θ = 1

ii) If $\sqrt{3}$ sin θ – cos θ = 0, then show that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Solution :

i) Given sin θ + cos θ = $\sqrt{3}$

$\Rightarrow (\sin\theta + \cos\theta)^2 = 3$

$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 3$

$\Rightarrow 1 + 2\sin\theta\cos\theta = 3$

$\Rightarrow \sin\theta\cos\theta = 1$ .... (1)

TP : tan θ + cot θ = 1

LHS : tan θ + cot θ

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{1} \qquad \text{(from (1))}$$

$$= 1$$

= RHS

Hence proved.

ii) Given Given $\sqrt{3}\ \sin\theta - \cos\theta = 0$

$$\Rightarrow \sqrt{3}\sin\theta = \cos\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^0$$

$$T.P : \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

LHS

$\tan 3\theta = \tan 3\,(30^0)$

$= \tan 90^0$

= undefined

RHS

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3\left(\frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{3}}\right)^3}{1 - 3\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{1 - 1}$$

$$= \frac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{0}$$

= undefined

= LHS = RHS

= Hence proved.

---

8. i) If $\frac{\cos\alpha}{\sin\beta} = m$ and $\frac{\cos\alpha}{\sin\beta} = n$, then prove that $(m^2 + n^2)\cos^2\beta = n^2$

ii) If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$, then prove that $(x^2y)^{\frac{2}{3}} - (xy^2)^{\frac{2}{3}} = 1$

Solution :

i) Given $\frac{\cos\alpha}{\sin\beta} = m, \ \frac{\cos\alpha}{\sin\beta} = n$

$$\therefore\ m^2 = \frac{\cos^2\alpha}{\cos^2\beta}, n^2 = \frac{\cos^2\alpha}{\sin^2\beta}$$

To Prove : $(m^2 + n^2)\cos^2\beta = n^2$

LHS :

$(m^2 + n^2)\cos^2\beta$

$$= \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta$$

$$= \cos^2\alpha\left(\frac{1}{\cos^2\beta} + \frac{1}{\sin^2\beta}\right).\cos^2\beta$$

$$= \cos^2\alpha\left(\frac{\sin^2\beta + \cos^2\beta}{\cos^2\beta\,.\sin^2\beta}\right).\cos^2\beta$$

$$= \frac{\cos^2\alpha}{\sin^2\beta}$$

$$= n^2$$

= RHS

ii) Given

$x = \cot\theta + \tan\theta$

$= \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$

$= \dfrac{1}{\sin\theta\,.\cos\theta}$

$y = \sec\theta - \cos\theta$

$= \dfrac{1}{\cos\theta} - \cos\theta$

$= \dfrac{1-\cos^2\theta}{\cos\theta}$

$= \dfrac{\sin^2\theta}{\cos\theta}$

To Prove :

$$\left(x^2y\right)^{2/3} - \left(xy^2\right)^{2/3} = 1$$

LHS

$$= \left(\frac{1}{\sin^2\theta\cos^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right)^{2/3}$$

$$-\left(\frac{1}{\sin\theta.\cos\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right)^{2/3}$$

$$= \left(\frac{1}{\cos^3\theta}\right)^{2/3} - \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{2/3}$$

$$= \left(\sec^3\theta\right)^{2/3} - \left(\tan^3\theta\right)^{2/3}$$

$= \sec^2\theta - \tan^2\theta$

$= 1$

$= 1$

$=$ RHS

9. i) If $\sin\theta + \cos\theta = p$ and $\sec\theta + \mathrm{cosec}\theta = q$, then prove that $q(p^2 - 1) = 2p$

ii) If $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$, then prove that $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

Solution :

i) Given $p = \sin\theta + \cos\theta$,

$q = \sec\theta + \mathrm{cosec}\,\theta$

To Prove : $q(p^2 - 1) = 2p$

LHS :

$q(p^2 - 1)$

$$(\sec\theta + \mathrm{cosec}\,\theta)\left[(\sin\theta + \cos\theta)^2 - 1\right]$$

$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)\left[\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1\right]$$

$$= \frac{\sin\theta + \cos\theta}{\cos\theta\,.\sin\theta}\left[1 + 2\sin\theta\cos\theta - 1\right]$$

$$= \frac{\sin\theta + \cos\theta}{\cos\theta\,.\sin\theta} \times 2\sin\theta\cos\theta$$

$= 2(\sin\theta\cos\theta)$

$= 2p$

$=$ RHS

ii) Given $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$

To Prove :

$\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

10. If $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1}{a}$, then prove that $\dfrac{a^2-1}{a^2+1} = \sin\theta$

Solution :

Given

$$\frac{\cos\theta}{1+\sin\theta} = \frac{1}{a}$$

To Prove : $\dfrac{a^2-1}{a^2+1} = \sin\theta$

$$\therefore a = \frac{1+\sin\theta}{\cos\theta}$$

$= (1+\sin\theta)\sec\theta$

$= \sec\theta + \tan\theta$

$\therefore$ LHS :

$$= \frac{a^2 - 1}{a^2 + 1}$$

$$= \frac{(\sec\theta + \tan\theta)^2 - 1}{(\sec\theta + \tan\theta)^2 + 1}$$

$$= \frac{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta - 1}{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 1}$$

$$= \frac{2\tan^2\theta + 2\sec\theta\tan\theta}{2\sec^2\theta + 2\sec\theta\tan\theta}$$

$$= \frac{\cancel{2}\tan\theta\ \cancel{(\tan\theta + \sec\theta)}}{\cancel{2}\sec\theta\ \cancel{(\sec\theta + \tan\theta)}}$$

$$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$

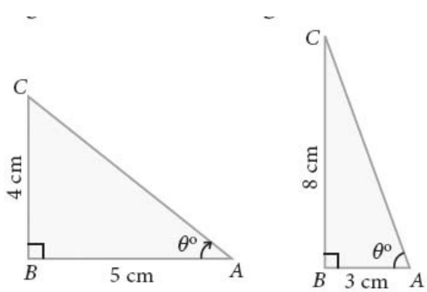
$= \sin\theta$

$=$ RHS

II. PROBLEMS INVOLVING ANGLE OF ELEVATION

Example 6.18

Calculate the size of $\angle$BAC in the given triangles.



Solution :

(i) In right triangle ABC [see Fig.]

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{4}{5}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right) = \tan^{-1}(0.8)$$

$$\theta = 38.7^0 \text{ (since } \tan 38.7^0 = 0.8011)$$

$$\angle BAC = 38.7^0$$

(ii) In right triangle ABC [see Fig.6.12(b)]

$$\tan\theta = \frac{8}{3}$$

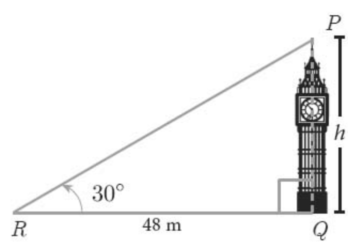
$$\theta = \tan^{-1}\left(\frac{8}{3}\right) = \tan^{-1}(2.66)$$

$$\theta = 69.4^0 \text{ (since } \tan 69.4^0 = 2.6604)$$

$$\angle BAC = 69.4^0$$

Example 6.19

A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is30°. Find the height of the tower.



Solution :

Let PQ be the height of the tower.

Take PQ = h and QR is the distance between the tower and the point R. In right triangle PQR, $\angle PRQ = 30^0$

$$\tan\theta = \frac{PQ}{QR}$$

$$\tan 30^0 = \frac{h}{48}$$

$$\text{gives, } \frac{1}{\sqrt{3}} = \frac{h}{48} \text{ so, } h = 16\sqrt{3}$$

Therefore the height of the tower is $16\sqrt{3}$ m

Example 6.20

A kite is flying at a height of 75 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

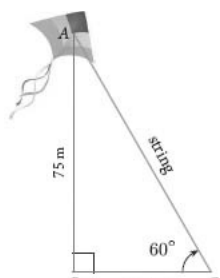


Fig. 6.14

Solution :

Let AB be the height of the kite above the ground. Then, AB = 75.

Let AC be the length of the string.

In right triangle ABC, $\angle ACB = 60^0$

$$\sin\theta = \frac{AB}{AC}$$

$$\sin 60^0 = \frac{75}{AC}$$

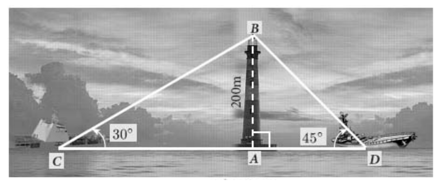
$$\text{gives, } \frac{\sqrt{3}}{2} = \frac{75}{AC} \text{ so, } AC = \frac{150}{\sqrt{3}} = 50\sqrt{3}$$

Hence, the length of the string is $50\sqrt{3}$ m

Example 6.21

Two ships are sailing in the sea on either sides of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are 30° and 45°respectively. If the lighthouse is 200 m high, find the distance between the two ships. ($\sqrt{3}$ = 1.732)

Solution :



Let AB be the lighthouse. Let C and D be the positions of the two ships.

Then, AB = 200 m.

$\angle ACB = 30^0$, $\angle ADB = 45^0$

In right triangle BAC,

$$\tan 30^0 = \frac{AB}{AC}$$

$$\frac{1}{\sqrt{3}} = \frac{200}{AC}$$

$$\text{gives, } AC = 200\sqrt{3} \quad ....(1)$$

In right triangle BAD,

$$\tan 45^0 = \frac{AB}{AD}$$

$$1 = \frac{200}{AD}$$

$$\text{gives, } AD = 200 \quad ....(2)$$

Now, CD = AC + AD

= $200\sqrt{3} + 200$ [by (1) and (2)]

CD = $200(\sqrt{3} + 1)$

$$= 200 \times 2.732 = 546.4$$

Distance between two ships is 546.4 m.

Example 6.22

From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 30 m high building are 45° and 60° respectively. Find the height of the tower. ($\sqrt{3}$ = 1.732)

Solution :

Fig. 6.16

Let AC be the height of the tower.

Let AB be the height of the building.

Then, AC = h metres, AB = 30 m

In right triangle CBP,

$\angle CPB = 60^0$

$$\tan\theta = \frac{BC}{BP}$$

$$\tan 60^0 = \frac{AB + AC}{BP}$$

$$so, \sqrt{3} = \frac{30 + h}{BP} \quad \text{.......(1)}$$

In right triangle ABP,

$\angle APB = 45^0$

$$\tan\theta = \frac{AB}{BP}$$

$$\tan 45^0 = \frac{30}{BP}$$

gives, $BP = 30$ ....(2)

Substituting (2) in (1), we get

$$\sqrt{3} = \frac{30 + h}{30}$$

$$h = 30\left(\sqrt{3} - 1\right)$$

$$= 30\,(1.732 - 1)$$

$$= 30(0.732) = 21.96$$

Hence, the height of the tower is 21.96 m.

Example 6.23

A TV tower stands vertically on a bank of a canal. The tower is watched from a point on the other bank directly opposite to it. The angle of elevation of the top of the tower is 58°. From another point 20 m away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal. (tan58° = 1.6003)

Solution :

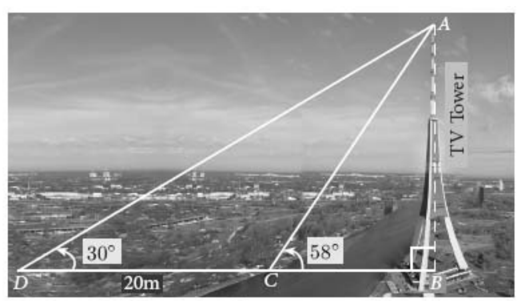


Fig. 6.17

Let AB be the height of the TV tower.

CD = 20 m.

Let BC be the width of the canal.

In right triangle ABC,

$$\tan 58^0 = \frac{AB}{BC}$$

$$1.6003 = \frac{AB}{BC} \qquad \text{..........(1)}$$

In right triangle ABD,

$$\tan 30^0 = \frac{AB}{BD} = \frac{AB}{BC+CD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC+20} \qquad \text{..........(2)}$$

Dividing (1) by (2) we get,

$$\frac{1.6003}{\frac{1}{\sqrt{3}}} = \frac{BC+20}{BC}$$

$$BC = \frac{20}{1.7791} = 11.24m \qquad \text{..........(3)}$$

$$1.6003 = \frac{AB}{11.24} \qquad \text{[from (1) and (3)]}$$

$$\text{AB} = 17.99$$

Hence, the height of the tower is 17.99 m and the width of the canal is 11.24 m.

Example 6.24

An aeroplane sets off from G on a bearing of 24° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 55° and a distance of 180 km away.

(i) How far is H to the North of G?

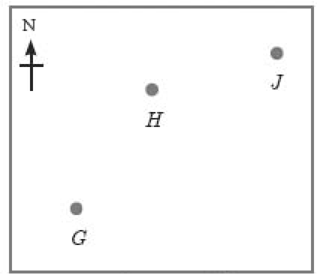
(ii) How far is H to the East of G?

(iii) How far is J to the North of H?

(iv) How far is J to the East of H?

$$\begin{pmatrix} \sin 24^0 = 0.4067 & \sin 11^0 = 0.1908 \\ \cos 24^0 = 0.9135 & \cos 11^0 = 0.9816 \end{pmatrix}$$

Solution :



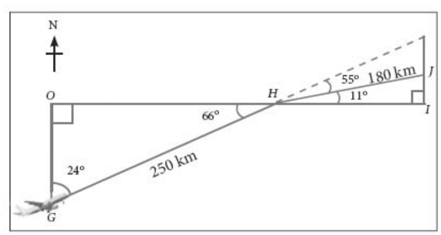
(i) In right triangle GOH,

$$\cos 24^0 = \frac{OG}{GH}$$

$$0.9135 = \frac{OG}{250}; \text{OG} = 228.38 \text{ km}$$

Distance of H to the North of G = 228.38 km

(ii) In right triangle GOH,



$$\sin 24^0 = \frac{OH}{GH}$$

$$0.4067 = \frac{OH}{250}; \text{OH} = 101.68$$

Distance of H to the East of G = 101.68 km

(iii) In right triangle HIJ,

$$\sin 11^0 = \frac{IJ}{HJ}$$

$$0.1908 = \frac{IJ}{180}; \text{ IJ} = 34.34 \text{ km}$$

Distance of J to the North of

H = 34.34 km

(iv) In right triangle HIJ,

$$\cos 11^0 = \frac{HI}{HJ}$$

$$0.9816 = \frac{HI}{180}; \text{ HI} = 176.69 \text{ km}$$

Distance of J to the East of

H = 176.69 km

Example 6.25

Two trees are standing on flat ground. The angle of elevation of the top of both the trees from a point X on the ground is 40°. If the horizontal distance between X and the smaller tree is 8 m and the distance of the top of the two trees is 20 m, calculate

(i) the distance between the point X and the top of the smaller tree.

(ii) the horizontal distance between the two trees. (cos 40° = 0.7660)

Solution :

Let AB be the height of the bigger tree and CD be the height of the smaller tree and X is the point on the ground.

(i) In right triangle XCD,

$$\cos 40^0 = \frac{CX}{XD}$$

$$XD = \frac{8}{0.7660} = 10.44 \text{ km}$$

Therefore the distance between X and top of the smaller tree = XD = 10.44 m

(ii) In right triangle XAB,

$$\cos 40^0 = \frac{AX}{BX}$$

$$= \frac{AC + CX}{BD + DX}$$

$$= 0.7660 = \frac{AC + 8}{20 + 10.44}$$

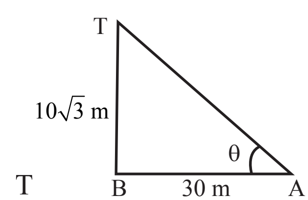
gives $AC = 23.32 - 8 = 15.32$ m

Therefore the horizontal distance between two trees = AC = 15.32 m

EXERCISE 6.2

1. Find the angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of a tower of height $10\sqrt{3}$ m.

Solution :



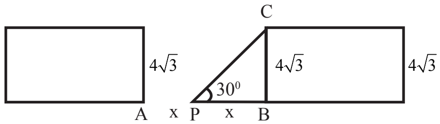
From the fig, $\tan\theta = \frac{10\sqrt{3}}{30}$

$$= \frac{1}{\sqrt{3}}$$

$$\therefore \ \theta = 30^0$$

2. A road is flanked on either side by continuous rows of houses of height $4\sqrt{3}$ m with no space in between them. A pedestrian is standing on the median of the road facing a row house. The angle of elevation from the pedestrian to the top of the house is 30°. Find the width of the road.

Solution :



Let AB = Width of theroad

P = Midpoint of AB

BC = height of the row houses

$= 4\sqrt{3}$ m

Let PB = PA = x m

In ΔPBC, $\tan 30^0 = \dfrac{BC}{PB}$

$\Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{4\sqrt{3}}{x}$

$\Rightarrow \quad x = 12$ m

$\therefore$ Width of the road = AB

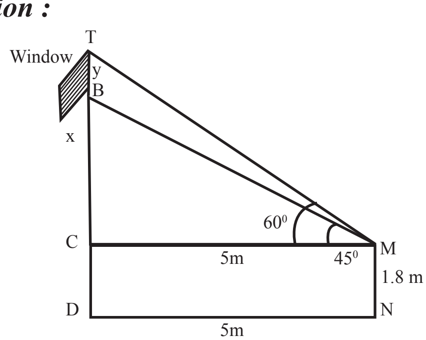
= 2x

= 2 (12)

= 24 m

3. To a man standing outside his house, the angles of elevation of the top and bottom of a window are 60° and 45° respectively. If the height of the man is 180 cm and if he is 5 m away from the wall, what is the height of the window? ($\sqrt{3}$ = 1.732)

Solution :



Let MN = 180 cm = 1.8 m
= height of the man
DN = 5 m = Dist. between Man & Wall
T, B $\rightarrow$ Top & Bottom of Window
BC = x, TB = y (Height of window)

In ΔCMB, $\tan 45^0 = \dfrac{x}{5}$

$\Rightarrow 1 = \dfrac{x}{5}$

$\Rightarrow x = 5$

In ΔCMT, $\tan 60^0 = \dfrac{x+y}{5}$

$\Rightarrow \quad \sqrt{3} = \dfrac{5+y}{5}$

$\Rightarrow \quad 5 + y = 5\sqrt{3}$

$\Rightarrow \quad y = 5\sqrt{3} - 5$

$= 5\left(\sqrt{3} - 1\right)$

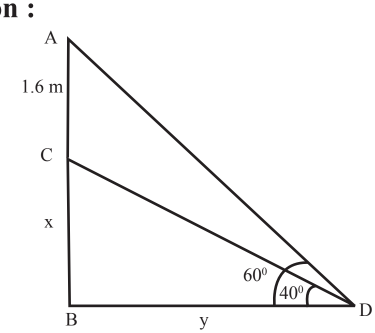
$= 5\,(0.732)$

$= 3.66$ m

$\therefore$ Height of window = 3,66 m

4. A statue 1.6 m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 40°. Find the height of the pedestal. (tan 40° = 0.8391, ($\sqrt{3}$ = 1.732)

Solution :



Let D $\rightarrow$ Point of observation on the ground

BC = x m = height of Pedestal

AC = 1.6m = height of statue

$\angle BDC = 40^0$, $\angle BDA = 60^0$, BD = y m

In ΔBCD, $\tan 40^0 = \dfrac{x}{y}$

$$\Rightarrow \quad y = \frac{x}{\tan 40^0}$$

$$= \frac{x}{0.8391} \qquad \text{..........} (1)$$

In ΔBAD, $\tan 60^0 = \dfrac{AB}{BD}$

$$\Rightarrow \quad \sqrt{3} = \frac{x+1.6}{y}$$

$$\Rightarrow \quad y = \frac{x+1.6}{\sqrt{3}} \qquad \text{..........} (2)$$

$\therefore$ From (1) & (2)

$$\frac{x}{0.8391} = \frac{x+1.6}{1.732}$$

$$\Rightarrow \quad 1.732\,x = 0.8391\,x + 1.6\,(0.8391)$$

$$\Rightarrow \quad 0.8929\,x = 1.343$$

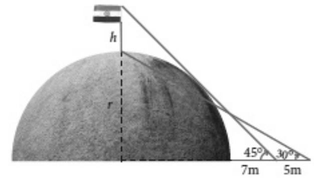
$$\Rightarrow \quad x = \frac{1.343}{0.8929}$$

$$= 1.5 \text{ m}$$

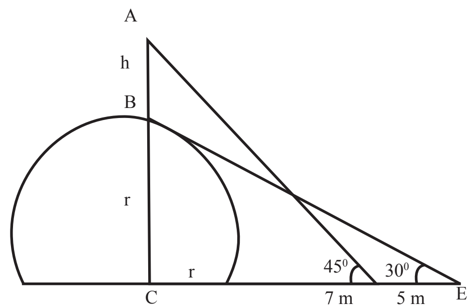
$\therefore$ Height of statue = 1.5 m

5. A flag pole 'h' metres is on the top of the hemispherical dome of radius 'r' metres. A man is standing 7 m away from the dome. Seeing the top of the pole at an angle 45° and moving 5 m away from the dome and seeing the bottom of the pole at an angle 30° . Find (i) the height of the pole (ii) radius of the dome.

($\sqrt{3}$ = 1.732)



Solution :



In ΔACD, $\tan 45^0 = \dfrac{AC}{CD}$

$$1 = \frac{h+r}{r+7}$$

$$\Rightarrow \quad r + 7 = h + r$$

$$\Rightarrow \quad h = 7$$

$\therefore$ Height of the pole = 7m

In ΔBCE, $\tan 30^0 = \dfrac{BC}{CE}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{r}{r+7+5}$$

$$\Rightarrow \quad \sqrt{3}r = r + 12$$

$$\Rightarrow \quad \sqrt{3}r - r = 12$$

$$\Rightarrow \quad r\left(\sqrt{3}-1\right) = 12$$

$$\Rightarrow \quad r = \frac{12}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{12\left(\sqrt{3}+1\right)}{2}$$

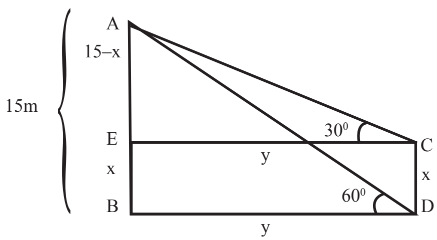
$$= 6\,(2.732)$$

$$= 16.392 \text{ m}$$

$\therefore$ Radius of dome = 16.39 m

6. The top of a 15 m high tower makes an angle of elevation of 60° with the bottom of an electronic pole and angle of elevation of 30° with the top of the pole. What is the height of the electric pole?

Solution :



Let AB = 15 m = Height of the tower

CD = x m = Height of the pole = BE

$\therefore$ AE = 15 – x

Let BD = EC = y

In ΔADB, $\tan 60^0 = \frac{AB}{BD}$

$$\sqrt{3} = \frac{15}{y}$$

$$\Rightarrow \quad y = \frac{15}{\sqrt{3}} = 5\sqrt{3} \quad \text{........ (1)}$$

In ΔACE, $\tan 30^0 = \frac{AE}{EC}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{15-x}{y}$$

$$\Rightarrow \quad 5\sqrt{3} = (15-x)\sqrt{3} \quad \text{(From (1))}$$

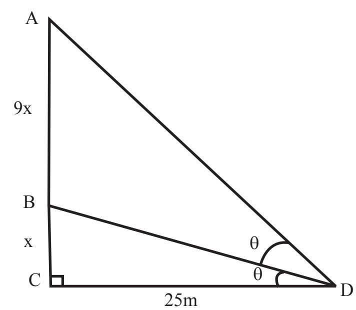
$$\Rightarrow \quad 5 = 15 - x$$

$$\Rightarrow \quad x = 10$$

$\therefore$ Height of the pole = 10m

7. A vertical pole fixed to the ground is divided in the ratio 1 : 9 by a mark on it with lower part shorter than the upper part. If the two parts subtend equal angles at a place on the ground, 25 m away from the base of the pole, what is the height of the pole?

Solution :



Let AC = height of the pole

B is a point on AC which divides it in the ratio 1 : 9 such that AB = 9x, BC = x

($\because$ lower part is shorter than upper part)

CD = 25 m = Dist. between pole and point of observation

In ΔBCD, $\tan\theta = \frac{BC}{CD} = \frac{x}{25}$

In ΔACD, $\tan 2\theta = \frac{AC}{CD} = \frac{10x}{25}$

$$\Rightarrow \quad \tan 2\theta = \frac{10x}{25}$$

$$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} = \frac{10x}{25}$$

$$\Rightarrow \quad \frac{2\left(x/25\right)}{1-x^2/25} = \frac{2x}{5}$$

$$\Rightarrow \quad \frac{\frac{2x}{25}}{\frac{625-x^2}{625}} = \frac{2x}{5}$$

$$\Rightarrow \frac{\cancel{2x}}{25} \times \frac{625}{625 - x^2} = \frac{\cancel{2x}}{5}$$

$$\Rightarrow \frac{25}{625 - x^2} = \frac{1}{5}$$

$$\Rightarrow 625 - x^2 = 125$$

$$\Rightarrow x^2 = 500$$

$$\Rightarrow x = 10\sqrt{5}$$

$\therefore$ Height of the pole $= 10x$

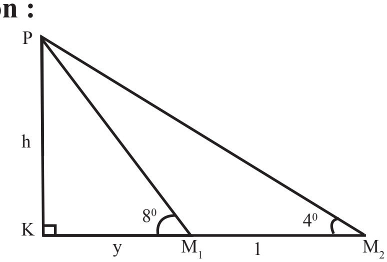
$$= 10\left(10\sqrt{5}\right)$$

$$= 100\sqrt{5} \text{ m}$$

---

8. A traveler approaches a mountain on highway. He measures the angle of elevation to the peak at each milestone. At two consecutive milestones the angles measured are 4° and 8°. What is the height of the peak if the distance between consecutive milestones is 1 mile. (tan 4° = 0.0699, tan 8° = 0.1405)

Solution :



Let PK = h = height of mountain

M_1, M_2 = Mile stones, M_1, M_2 = 1 mile

Let KM_1 = y

In ΔPKM_1,

$$\tan 8^0 = \frac{h}{y}$$

$$\Rightarrow y = \frac{h}{\tan 8^0} \qquad \text{...........(1)}$$

In ΔPKM_2,

$$\tan 4^0 = \frac{h}{y+1}$$

$$\Rightarrow y + 1 = \frac{h}{\tan 4^0}$$

$$\Rightarrow y = \frac{h}{\tan 4^0} - 1 \qquad \text{...........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{h}{\tan 8^0} = \frac{h}{\tan 4^0} - 1$$

$$\Rightarrow \frac{h}{\tan 4^0} - \frac{h}{\tan 8^0} = 1$$

$$\Rightarrow h\left[\frac{1}{\tan 4^0} - \frac{1}{\tan 8^0}\right] = 1$$

$$\Rightarrow h\left[\frac{\tan 8^0 - \tan 4^0}{\tan 8^0 \tan 4^0}\right] = 1$$

$$\Rightarrow h = \frac{\tan 8^0 \tan 4^0}{\tan 8^0 - \tan 4^0}$$

$$= \frac{0.14 \times 0.07}{0.14 - 0.07}$$

$$= \frac{0.0098}{0.07}$$

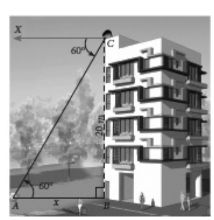
= 0.14 miles

III. PROBLEMS INVOLVING ANGLE OF DEPRESSION

Example 6.26

A player sitting on the top of a tower of height 20 m observes the angle of depression of a ball lying on the ground as 60° . Find the distance between the foot of the tower and the ball. ($\sqrt{3}$ = 1.732)

Solution :



Let BC be the height of the tower and A be the position of the ball lying on the ground. Then,

BC = 20 m and ∠XCA= 60° = ∠CAB

Let AB = x metres.

In right triangle ABC,

$$\tan 60^0 = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{20}{x}$$

$$x = \frac{20 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{20 \times 1.732}{3}$$

$$= 11.54\text{m}$$

Hence, the distance between the foot of the tower and the ball is 11.54 m.

Example 6.27

The horizontal distance between two buildings is 140 m. The angle of depression of the top of the first building when seen from the top of the second building is 30° . If the height of the first building is 60 m, find the height of the second building. ($\sqrt{3}$ = 1.732)

Solution :

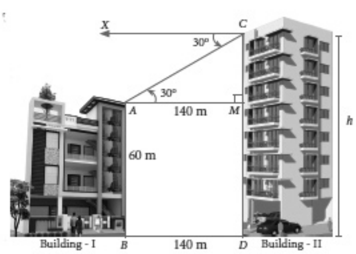


Fig. 6.22

The height of the first building
AB = 60 m. Now, AB = MD = 60 m
Let the height of the second building
CD = h. Distance BD = 140 m
Now, AM = BD = 140 m
From the diagram,

∠XCA = 30° = ∠CAM
In right triangle AMC,

$$\tan 30^0 = \frac{CM}{AM}$$

$$\frac{1}{\sqrt{3}} = \frac{CM}{140}$$

$$CM = \frac{140}{\sqrt{3}} = \frac{140\sqrt{3}}{3}$$

$$= \frac{140 \times 1.732}{3}$$

$$CM = 80.78$$

Now, h = CD
= CM + MD
= 80.78 + 60 = 140.78

Therefore the height of the second building is 140.78 m

Example 6.28

From the top of a tower 50 m high, the angles of depression of the top and bottom of a tree are observed to be 30° and 45° respectively. Find the height of the tree. ($\sqrt{3}$ = 1.732)

Solution :

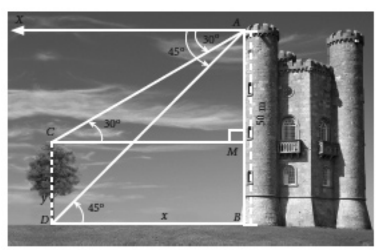


Fig. 6.23

The height of the tower AB = 50 m
Let the height of the tree
CD = y and BD = x
From the diagram,
∠XAC = 30° = ∠ACM and
∠XAD = 45° = ∠ADB
In right triangle ABD,

$$\tan 45^0 = \frac{AB}{BD}$$

$$1 = \frac{50}{x} \text{ gives } x = 50 \text{ m}$$

In right triangle AMC,

$$\tan 30^0 = \frac{AM}{CM}$$

$$\frac{1}{\sqrt{3}} = \frac{AM}{50} \quad [\text{since } DB = CM]$$

$$AM = \frac{50}{\sqrt{3}} = \frac{50\sqrt{3}}{3}$$

$$= \frac{50 \times 1.732}{3} = 28.85 \text{ m}$$

Therefore, height of the tree =

CD = MB = AB − AM

= 50 − 28.85 = 21.15 m

Example 6.29

As observed from the top of a 60 m high light house from the sea level, the angles of depression of two ships are 28° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. (tan 28° = 0.5317)

Solution :

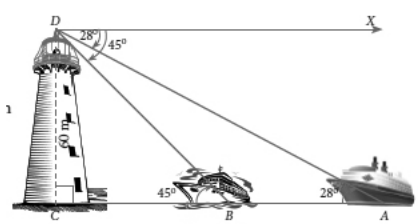


Fig. 6.24

Let the observer on the lighthouse CD be at D.

Height of the lighthouse CD = 60 m

From the diagram,

∠XDA = 28° = ∠DAC and

∠XDB = 45° = ∠DBC

In right triangle DCB,

$$\tan 45^0 = \frac{DC}{BC}$$

$$1 = \frac{60}{BC} \text{ gives } BC = 60 \text{ m}$$

In right triangle DCA,

$$\tan 28^0 = \frac{DC}{AC}$$

$$0.5317 = \frac{60}{AC}$$

$$\text{gives } AC = \frac{60}{0.5317}$$

$$= 112.85$$

Distance between the two ships

AB = AC − BC = 52.85 m

Example 6.30

A man is watching a boat speeding away from the top of a tower. The boat makes an angle of depression of 60° with the man's eye when at a distance of 200 m from the tower. After 10 seconds, the angle of depression becomes 45°. What is the approximate speed of the boat (in km / hr), assuming that it is sailing in still water? ($\sqrt{3}$ = 1.732)

Solution :

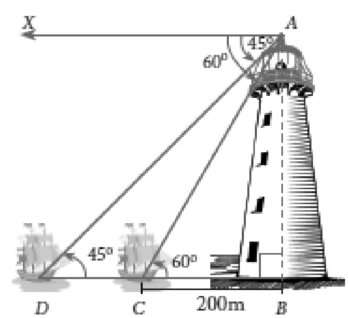


Fig. 6.25

Let AB be the tower.

Let C and D be the positions of the boat.

From the diagram,

$\angle XAC = 60° = \angle ACB$ and

$\angle XAD = 45° = \angle ADB$, BC = 200 m

In right triangle ABC, $\tan 60° = \dfrac{AB}{BC}$

gives $\sqrt{3} = \dfrac{AB}{200}$

we get $AB = 200\sqrt{3}$ .... (1)

In right triangle ABD, $\tan 45° = \dfrac{AB}{BD}$

gives $1 = \dfrac{200\sqrt{3}}{BD}$ [by (1)]

we get $BD = 200\sqrt{3}$

Now, $CD = BD - BC$

$CD = 200\sqrt{3} - 200$

$= 200(\sqrt{3} - 1) = 146.4$

It is given that the distance CD is covered in 10 seconds.

That is, the distance of 146.4 m is covered in 10 seconds.

Therefore, speed of the boat = $\dfrac{\text{distance}}{\text{time}}$

$= \dfrac{146.4}{10} = 14.64$ m/s

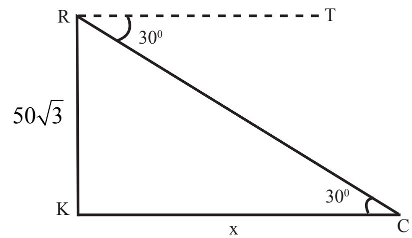
gives $14.64 \times \dfrac{3600}{1000}$ km/hr

= 52.704 km/hr

EXERCISE 6.3

1. From the top of a rock $50\sqrt{3}$ m high, the angle of depression of a car on the ground is observed to be 30°. Find the distance of the car from the rock.

Solution:



RK = $50\sqrt{3}$ m = height of the rock

C = Position of the car

KC = x m

$\angle TRC = \angle RCK = 30^0$

$$\tan 30^0 = \frac{RK}{KC}$$

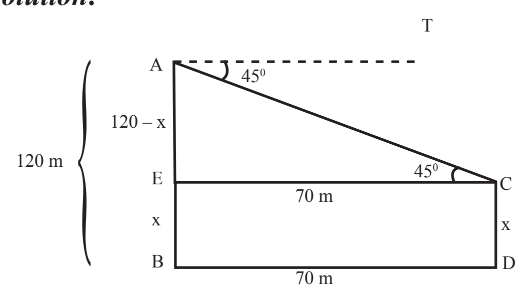
$$\frac{1}{\sqrt{3}} = \frac{50\sqrt{3}}{x}$$

$$\Rightarrow \quad x = 150\text{m}$$

$\therefore$ Dist. of the car from the rock = 150m

2. The horizontal distance between two buildings is 70 m. The angle of depression of the top of the first building when seen from the top of the second building is 45°. If the height of the second building is 120 m, find the height of the first building.

Solution:



Height of the 1st building = CD = x m

Height of the 2nd building = AB = 120 m

Distance between 2 buildings =

BD = EC = 70m

In Δ AEC,

$$\tan 45^0 = \frac{AE}{EC}$$

$$1 = \frac{120 - x}{70}$$

$$\Rightarrow \quad 70 = 120 - x$$

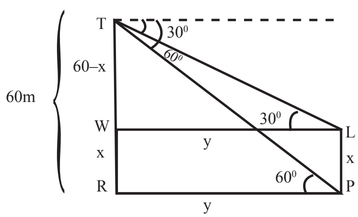
$$\Rightarrow \quad x = 120 - 70$$

$$\Rightarrow \quad x = 50$$

∴ Height of 1st building = 50 m

3. From the top of the tower 60 m high the angles of depression of the top and bottom of a vertical lamp post are observed to be 38° and 60° respectively. Find the height of the lamp post. (tan 38° = 0.7813, $\sqrt{3}$ = 1.732)

Solution:



TR = Height of the tower = 60 m

LP = Height of the lamp post = x m = WR

∴ TW = 60 – x

Let RP = WL = y

In ΔTRP,

$$\tan 60^0 = \frac{TR}{RP}$$

$$\Rightarrow \sqrt{3} = \frac{60}{y}$$

$$\Rightarrow \quad y = \frac{60}{\sqrt{3}}$$

$$\Rightarrow \quad y = 20\sqrt{3} \qquad \text{...........} (1)$$

In ΔTWL,

$$\tan 30^0 = \frac{TW}{WL}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{60 - x}{y}$$

$$\Rightarrow \quad y = \sqrt{3}\,(60 - x) \text{...........} (2)$$

∴ From (1) & (2)

$$\Rightarrow \quad 20\sqrt{3} = \sqrt{3}\,(60 - x)$$

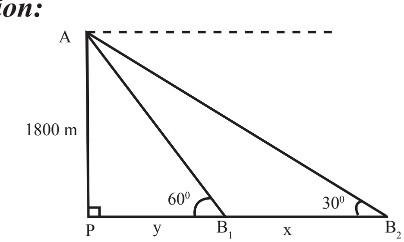
$$\Rightarrow \quad 20 = 60 - x$$

$$\Rightarrow \quad x = 60 - 20 \quad \Rightarrow \quad x = 40$$

∴ Height of the lamp post = 40m

4. An aeroplane at an altitude of 1800 m finds that two boats are sailing towards it in the same direction. The angles of depression of the boats as observed from the aeroplane are 60° and 30° respectively. Find the distance between the two boats. ($\sqrt{3}$ = 1.732)

Solution:



AP = 1800 m

= height of the plane from the ground

B_1, B_2 = Positions of 2 boats

Let B_1, B_2 = x m ; PB_1 = y

In ΔAPB_1,

$$\tan 60^0 = \frac{1800}{y}$$

$$\Rightarrow \quad \sqrt{3} = \frac{1800}{y}$$

$$\Rightarrow \quad y = \frac{1800}{\sqrt{3}} = 600\sqrt{3} \text{...........} (1)$$

In ΔAPB_2,

$$\tan 30^0 = \frac{1800}{y+x}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{1800}{y+x}$$

$$\Rightarrow \quad y + x = 1800\sqrt{3}$$

$$\Rightarrow \quad 600\sqrt{3} + x = 1800\sqrt{3}$$

(From (1))

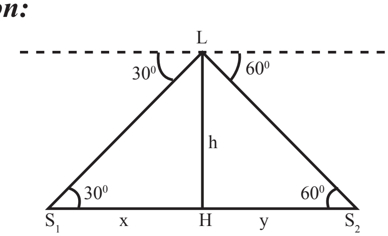
$$\Rightarrow \quad x = 1200\sqrt{3}$$

$$x = 1200\ (1.732)$$

$$= 2078.4 \text{ m}$$

5. From the top of a lighthouse, the angle of depression of two ships on the opposite sides of it are observed to be 30° and 60°. If the height of the lighthouse is h meters and the line joining the ships passes through the foot of the lighthouse, show that the distance between the ships is $\frac{4h}{\sqrt{3}}$ m.

Solution:



LH = h m = height of the light house

S_1, S_2 = Positions of 2 ships

S_1H = x m, S_2H = y m

To find : x + y

In ΔLS_1H,

$$\tan 30^0 = \frac{h}{x}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\Rightarrow \quad x = \sqrt{3}h \qquad \text{.........(1)}$$

In ΔLS_2H,

$$\tan 60^0 = \frac{h}{y}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{y}$$

$$\Rightarrow \quad y = \frac{h}{\sqrt{3}} \qquad \text{.........(2)}$$

$\therefore$ Adding (1) & (2)

$$x + y = \sqrt{3}h + \frac{h}{\sqrt{3}}$$

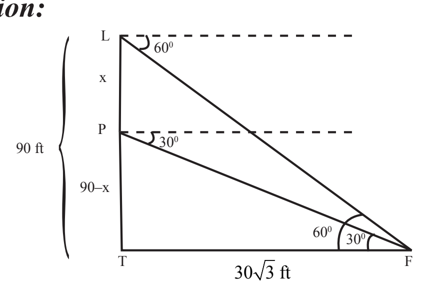
$$= \frac{3h+h}{\sqrt{3}} = \frac{4h}{\sqrt{3}}$$

$\therefore$ Distance between 2 ships

$$= \frac{4h}{\sqrt{3}} \text{ m}$$

6. A lift in a building of height 90 feet with transparent glass walls is descending from the top of the building. At the top of the building, the angle of depression to a fountain in the garden is 60°. Two minutes later, the angle of depression reduces to 30°. If the fountain is $30\sqrt{3}$ feet from the entrance of the lift, find the speed of the lift which is descending.

Solution:



LT= height of the lift = 90 ft

F = Position of the fountain

FT = Distance between fountain & lift

$= 30\sqrt{3}$ ft.

LP = x ft $\Rightarrow$ PT = $90^0 - x$

Time taken from L to P = 2 min.

In ΔPFT,

$$\tan 30^0 = \frac{90-x}{30\sqrt{3}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{90-x}{30\sqrt{3}}$$

$$\Rightarrow \quad 30 = 90 - x$$

$$\Rightarrow \quad x = 60 \text{ ft}$$

$$\therefore \text{ Speed of the lift} = \frac{\text{Dist.}}{\text{Time}}$$

$$= \frac{60}{2}$$

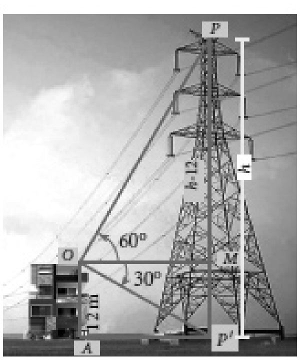
$$= 30 \text{ ft/min.}$$

PROBLEMS INVOLVING ANGLE OF ELEVATION AND DEPRESSION

Example 6.31

From the top of a 12 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 30°. Determine the height of the tower.

Solution :



As shown in Fig., OA is the building, O is the point of observation on the top of the building OA. Then, OA = 12 m.

PP' is the cable tower with P as the top and P' as the bottom.

Then the angle of elevation of P, $\angle$MOP = 60° .

And the angle of depression of P' , $\angle$MOP'= 30°

Suppose, height of the cable tower PP' = h metres.

Through O, draw OM $\perp$ PP'

MP = PP' − MP' = h −OA = h −12

In right triangle OMP, $\frac{MP}{OM} = \tan 60°$

$$\text{gives } \frac{h-12}{OM} = \sqrt{3}$$

$$\text{so, } OM = \frac{h-12}{\sqrt{3}} \qquad \text{............ (1)}$$

In right triangle OMP', $\frac{MP'}{OM} = \tan 30°$

$$\text{gives } \frac{12}{OM} = \frac{1}{\sqrt{3}}$$

$$\text{so, } OM = 12\sqrt{3} \qquad \text{............ (2)}$$

From (1) and (2) we have, $\frac{h-12}{\sqrt{3}} = 12\sqrt{3}$

$$\text{gives, } h - 12 = 12\sqrt{3} \times \sqrt{3} \text{ we get, } h = 48$$

Hence, the required height of the cable tower is 48 m.

Example 6.32

A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point ‘A’ on the ground is 60° and the angle of depression to the point ‘A’ from the top of the tower is 45°. Find the height of the tower. ($\sqrt{3}$ = 1.732)

Solution :

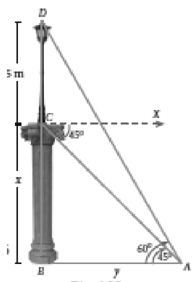


Fig. 6.28

Let BC be the height of the tower and CD be the height of the pole.

Let 'A' be the point of observation.

Let BC = x and AB = y.

From the diagram,

$\angle BAD = 60°$ and $\angle XCA = 45° = \angle BAC$

In right triangle ABC, $\tan 45° = \dfrac{BC}{AB}$

gives, $1 = \dfrac{x}{y}$ so, $x = y$ .... (1)

In right triangle ABD, tan60°

$$= \frac{BD}{AB} = \frac{BC + CD}{AB}$$

gives, $\sqrt{3} = \dfrac{x+5}{y}$ so, $\sqrt{3}y = x + 5$

we get, $\sqrt{3}x = x + 5$ [From (1)]

$$\text{so, } x = \frac{5}{\sqrt{3}-1} = \frac{5}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

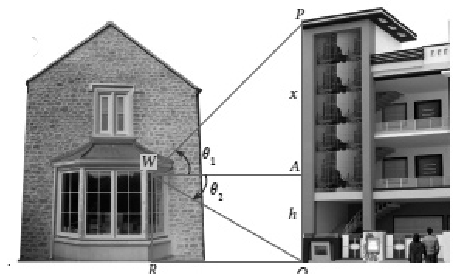
$$= \frac{5(1.732+1)}{2} = 6.83$$

Hence, height of the tower is 6.83 m.

Example6.33

From a window (h metres high above the ground) of a house in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are θ_1 and θ_2 respectively. Show that the height of the opposite house is $h\left(1 + \dfrac{\cot\theta_2}{\cot\theta_1}\right)$

Solution :



Let W be the point on the window where the angles of elevation and depression are measured. Let PQ be the house on the opposite side.

Then WA is the width of the street.

Height of the window = h metres

= AQ (WR = AQ)

Let PA = x metres.

In right triangle PAW, $\tan\theta_1 = \dfrac{AP}{AW}$

gives $\tan\theta_1 = \dfrac{x}{AW}$

so, $AW = \dfrac{x}{\tan\theta_1}$

we get, $AW = x\cot\theta$ ...... (1)

In right triangle QAW, $\tan\theta_2 = \dfrac{AQ}{AW}$

gives $\tan\theta_2 = \dfrac{h}{AW}$

we get, $AW = h\cot\theta_2$ ...... (2)

From (1) and (2) we get, $x\cot\theta_1 = h\cot\theta_2$

gives, $x = h\dfrac{\cot\theta_2}{\cot\theta_1}$

Therefore, height of the opposite house =

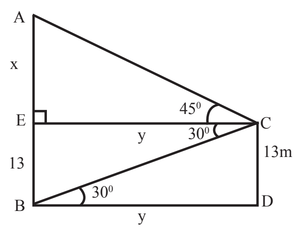
$$PA + AQ = x + h = h\frac{\cot\theta_2}{\cot\theta_1} + h = h\left(1 + \frac{\cot\theta_2}{\cot\theta_1}\right)$$

Hence Proved.

EXERCISE 6.4

1. From the top of a tree of height 13 m the angle of elevation and depression of the top and bottom of another tree are 45° and 30° respectively. Find the height of the second tree. ($\sqrt{3}$ = 1.732)

Solution:



CD = 13m = height of tree 1

AB = x + 13 = height of tree 2

BD = EC = y m = dist. between 2 trees

In ΔBCD.

$$\tan 30^0 = \frac{CD}{BD}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{13}{y}$$

$$\Rightarrow \quad y = 13\sqrt{3} \text{ m}$$

In ΔACE,

$$\tan 45^0 = \frac{AE}{EC}$$

$$\Rightarrow \quad 1 = \frac{x}{y}$$

$$\Rightarrow \quad y = x$$

$$\Rightarrow \quad x = 13\sqrt{3} \text{ m}$$

∴ Height of 2nd tree = x + 13

$$= 13\sqrt{3} + 13$$

$$= 13\left(\sqrt{3} + 1\right)$$

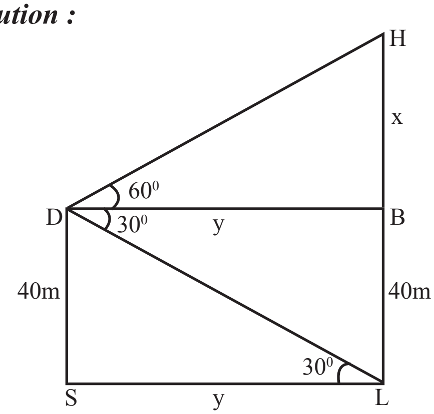
$$= 13 \times 2.732$$

$$= 35.516$$

$$\simeq 35.52 \text{ m}$$

2. A man is standing on the deck of a ship, which is 40 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill. ($\sqrt{3}$ = 1.732)

Solution :



D → Deck of a ship

DS = 40m = Deck of ship from water level

HL = Height of the hill = x + 40

SL = DB = Dist. between ship & hill

In ΔDLS,

$$\tan 30^0 = \frac{DS}{SL}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{40}{y}$$

$$\Rightarrow \quad y = 40\sqrt{3} \qquad \text{..........(1)}$$

In ΔDHB,

$$\tan 60^0 = \frac{HB}{DB}$$

$$\Rightarrow \sqrt{3} = \frac{x}{y}$$

$$\Rightarrow \quad y = \frac{x}{\sqrt{3}} \qquad \text{..........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{x}{\sqrt{3}} = 40\sqrt{3}$$

$$\Rightarrow \quad x = 40\sqrt{3} \times \sqrt{3}$$

$$= 120\text{m}$$

$\therefore$ Height of the hill = x + 40

= 120 + 40

= 160 m and

Distance between ship & hill

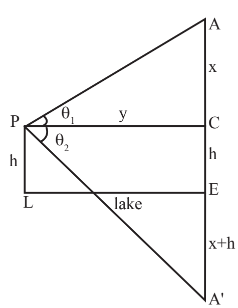
$$y = 40\sqrt{3}$$

$$= 40\,(1.732)$$

$$= 69.28 \text{ m}$$

3. If the angle of elevation of a cloud from a point 'h' metres above a lake is θ_1 and the angle of depression of its reflection in the lake is θ_2 . Prove that the height that the cloud is located from the ground is $\dfrac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 + \tan\theta_1}$

Solution :



LE $\rightarrow$ Surface of the lake

P $\rightarrow$ Point of observation 'h' mrs. from lake

A, A' $\rightarrow$ Positions of cloud & its reflection

AE = A'E, PL = CE = h m

To find :

$$EA = \frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$$

In ΔAPC,

$$\tan\theta_1 = \frac{x}{y}$$

$$\Rightarrow \quad y = \frac{x}{\tan\theta_1} \qquad \text{...........(1)}$$

In $\Delta PA'C$,

$$\tan\theta_2 = \frac{CA'}{PC}$$

$$\tan\theta_2 = \frac{x+2h}{y}$$

$$\Rightarrow \quad y = \frac{(x+2h)}{\tan\theta_2} \qquad \text{...........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{x}{\tan\theta_1} = \frac{x+2h}{\tan\theta_2}$$

$$\Rightarrow x\tan\theta_2 = x\tan\theta_1 + 2h\tan\theta_1$$

$$\Rightarrow x(\tan\theta_2 - \tan\theta_1) = 2h\tan\theta_1$$

$$\Rightarrow \qquad x = \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

$$\therefore AE = h + x$$

$$= h + \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

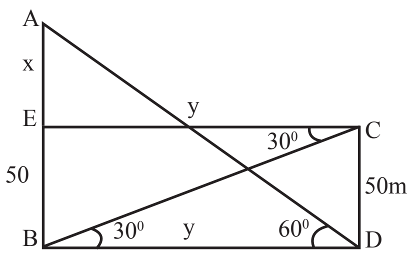
$$= h\left[1 + \frac{2\tan\theta_1}{\tan\theta_2 - \tan\theta_1}\right]$$

$$= h\left[\frac{\tan\theta_2 + \tan\theta_1}{\tan\theta_2 - \tan\theta_1}\right]$$

Hence proved.

4. The angle of elevation of the top of a cell phone tower from the foot of a high apartment is 60° and the angle of depression of the foot of the tower from the top of the apartment is 30° . If the height of the apartment is 50 m, find the height of the cell phone tower. According to radiations control norms, the minimum height of a cell phone tower should be 120 m. State if the height of the above mentioned cell phone tower meets the radiation norms.

Solution :



CD = 50m = height of the apartment = EB

AB = (x + 50) m = height of the cellphone tower

BD = EC = ym = dist. between tower & apartment

In ΔCDB,

$$\tan 30^0 = \frac{CD}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{50}{y}$$

$$\Rightarrow \quad y = 50\sqrt{3}\ m \qquad ....(1)$$

In ΔADB,

$$\tan 60^0 = \frac{AB}{BD}$$

$$\sqrt{3} = \frac{x+50}{y}$$

$$\Rightarrow \quad y = \frac{x+50}{\sqrt{3}}$$

$$\Rightarrow \quad 50\sqrt{3} = \frac{x+50}{\sqrt{3}} \qquad \text{(from (1))}$$

$$\Rightarrow \quad 150 = x+50$$

$$\therefore\ x = 100\text{m}$$

∴ Height of cell phone tower

= x + 50

= 150m > 120 m

∴ The tower does not meet the radiation norms.

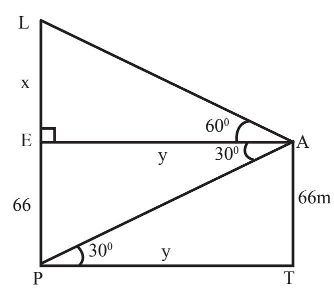
5. The angles of elevation and depression of the top and bottom of a lamp post from the top of a 66 m high apartment are 60° and 30° respectively. Find

(i) The height of the lamp post.

(ii) The difference between height of the lamp post and the apartment.

(iii) The distance between the lamp post and the apartment. ($\sqrt{3}$ = 1.732)

Solution :



AT = height of apartment = 66 m = EP

LP = height of the lamp post = x + 66

PT = EA = y m = dist. between post and apartment

In ΔAPT,

$$\tan 30^0 = \frac{AT}{y}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{66}{y}$$

$$\Rightarrow \quad y = 66\sqrt{3} \qquad \text{...........(1)}$$

In ΔALE,

$$\tan 60^0 = \frac{LE}{EA}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x}{y}$$

$$\Rightarrow \quad y = \frac{x}{\sqrt{3}} \qquad \text{...........(2)}$$

$\therefore$ From (1) & (2)

$$\frac{x}{\sqrt{3}} = 66\sqrt{3}$$

$$\Rightarrow \quad x = 66 \times 3$$

$$= 198$$

i) Height of the lamp post = x + 66

= 198 + 64

= 264 mrs

ii) Difference between height of lamp post & apartment = 264 – 66 = 198 m

iii) Dist. between lamp post & the apartment

$$\Rightarrow \quad y = 66\sqrt{3} \text{ m}$$

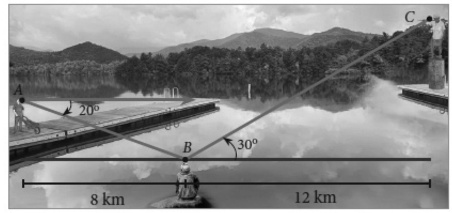
$$= 66\,(1.732)$$

$$= 114.312 \text{ m}$$

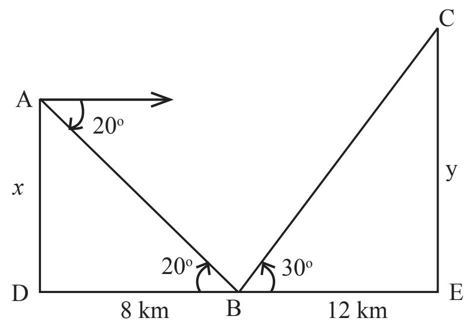
6. Three villagers A, B and C can see each other across a valley. The horizontal distance between A and B is 8 km and the horizontal distance between B and C is 12 km. The angle of depression of B from A is 20° and the angle of elevation of C from B is 30°. Calculate :

(i) the vertical height between A and B.

(ii) the vertical height between B and C.
($\tan 20° = 0.3640$, ($\sqrt{3} = 1.732$)



Solution :



A, B, C → Positions of 3 villagers

To find i) AD ii) CE

In ΔABD,

$$\tan 20^0 = \frac{x}{8}$$

$$\Rightarrow \quad x = 8 \,.\, \tan 20^0$$

$$= 8\,(0.3640)$$

$$= 2.912$$

$$\therefore \quad AD = 2.912 \text{ km}$$

In ΔCBE,

$$\tan 30^0 = \frac{y}{12}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{y}{12} \Rightarrow \quad y = \frac{12}{\sqrt{3}}$$

$$= 4\sqrt{3}$$

$$= 4\,(1.732)$$

$$= 6.928$$

$$\simeq 6.93$$

$$\therefore \quad CE = 6.93 \text{ km}$$

EXERCISE 6.5

Multiple choice questions

1. The value of $\sin^2\theta + \frac{1}{1+\tan^2\theta}$ is equal to

(1) $\tan^2\theta$ (2) 1 (3) $\cot^2\theta$
(4) 0

Hint : Ans : (2)

$$= \sin^2\theta + \frac{1}{1+\tan^2\theta}$$
$$= \sin^2\theta + \frac{1}{\sec^2\theta}$$
$$= \sin^2\theta + \cos^2\theta$$
$$= 1$$

2. $\tan\theta \operatorname{cosec}^2\theta - \tan\theta$ is equal to

(1) $\sec\theta$ (2) $\cot^2\theta$
(3) $\sin\theta$ (4) $\cot\theta$

Hint : Ans : (4)

$$= \tan\theta \,.\, \operatorname{cosec}^2\theta - \tan\theta$$
$$= \tan\theta\,(\operatorname{cosec}^2\theta - 1)$$
$$= \tan\theta \,.\, \cot^2\theta$$
$$= \frac{1}{\cot\theta} \times \cot^2\theta$$
$$= \cot\theta$$

3. If $(\sin\alpha + \operatorname{cosec}\alpha)^2 + (\cos\alpha + \sec\alpha)^2 = k + \tan^2\alpha + \cot^2\alpha$, then the value of k is equal to

(1) 9 (2) 7 (3) 5 (4) 3

Hint : Ans : (2)

$= (\sin\alpha + \operatorname{cosec}\alpha)^2 + (\cos\alpha + \sec\alpha)^2$
$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow \sin^2\alpha + \operatorname{cosec}^2\alpha + 2\sin\alpha \,.\, \operatorname{cosec}\alpha + \cos^2\alpha + \sec^2\alpha + 2\cos\alpha\sec\alpha$
$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow 1 + 2 + 2 + \operatorname{cosec}^2\alpha + \sec^2\alpha$
$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow 5 + 1 + \cot^2\alpha + 1 + \tan^2\alpha$
$= k + \tan^2\alpha + \cot^2\alpha$

$\Rightarrow 7 + \cot^2\alpha + \tan^2\alpha = k + \tan^2\alpha + \cot^2\alpha$

$\therefore k = 7$

4. If $\sin\theta + \cos\theta = a$ and $\sec\theta + \operatorname{cosec}\theta = b$, then the value of $b(a^2 - 1)$ is equal to

(1) 2a (2) 3a (3) 0
(4) 2ab

Hint : Ans : (1)

$b(a^2 - 1) = (\sec\theta + \operatorname{cosec}\theta)\,[(\sin\theta + \cos\theta)^2 - 1]$

$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)[2\sin\theta\cos\theta]$$
$$= 2\sin\theta + 2\cos\theta$$
$$= 2(\sin\theta + \cos\theta)$$
$$= 2a$$

5. If $5x = \sec\theta$ and $\frac{5}{x} = \tan\theta$, then $x^2 - \frac{1}{x^2}$ is equal to

(1) 25 (2) $\frac{1}{25}$ (3) 5 (4) 1

Hint : Ans : (2)

$$5x = \sec\theta,\ \frac{5}{x} = \tan\theta$$
$$\Rightarrow \sec^2\theta - \tan^2\theta = 1$$
$$\Rightarrow 25x^2 - \frac{25}{x^2} = 1$$
$$\Rightarrow 25\left(x^2 - \frac{1}{x^2}\right) = 1$$
$$\Rightarrow x^2 - \frac{1}{x^2} = \frac{1}{25}$$

6. If $\sin\theta = \cos\theta$, then $2\tan^2\theta + \sin^2\theta - 1$ is equal to

(1) $\frac{-3}{2}$ (2) $\frac{3}{2}$ (3) $\frac{2}{3}$ (4) $\frac{-2}{3}$

Hint : Ans : (2)

Given $\sin\theta = \cos\theta \Rightarrow \theta = 45^0$

$\therefore 2\tan^2\theta + \sin^2\theta - 1$

$= 2\tan^2 45^0 + \sin^2 45^0 - 1$

$= 2(1) + \left(\frac{1}{\sqrt{2}}\right)^2 - 1$

$= 2 + \frac{1}{2} - 1$

$= \frac{3}{2}$

7. If $x = a\tan\theta$ and $y = b\sec\theta$ then

(1) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Hint : Ans : (1)

$x = a\tan\theta,\ y = b\sec\theta$

$\therefore \tan\theta = \frac{x}{a},\ \sec\theta = \frac{y}{b}$

$\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

8. $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\theta)$ is equal to

(1) 0 (2) 1 (3) 2 (4) –1

Hint : Ans : (3)

$= (1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\theta)$

$= \left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right) \cdot \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)$

$= \left(\frac{(\cos\theta + \sin\theta) + 1}{\cos\theta}\right)\left(\frac{(\sin\theta + \cos\theta) - 1}{\sin\theta}\right)$

$= \left(\frac{(\cos\theta + \sin\theta)^2 - 1}{\cos\theta \cdot \sin\theta}\right) = \frac{2\sin\theta\cos\theta}{\cos\theta\sin\theta}$

$= 2$

9. $a\cot\theta + b\,\text{cosec}\,\theta = p$ and $b\cot\theta + a\,\text{cosec}\,\theta = q$ then $p^2 - q^2$ is equal to

(1) $a^2 - b^2$ (2) $b^2 - a^2$

(3) $a^2 + b^2$ (4) $b - a$

Hint : Ans : (2)

$p^2 - q^2 = (a\cot\theta + b\,\text{cosec}\,\theta)^2 - (b\cot\theta + a\,\text{cosec}\,\theta)^2$

$= a^2\cot^2\theta + b^2\text{cosec}^2\theta + 2ab\cot\theta\,\text{cosec}\,\theta) - b^2\cot^2\theta + a^2\text{cosec}^2\theta + 2ab\cot\theta\,\text{cosec}\,\theta)$

$= a^2(\cot^2\theta - \text{cosec}^2\theta) + b^2(\text{cosec}^2 - \cot^2\theta)$

$= a^2(-1) + b^2(1)$

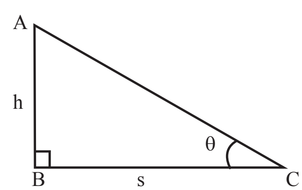
$= b^2 - a^2$

10. If the ratio of the height of a tower and the length of its shadow is $\sqrt{3} : 1$, then the angle of elevation of the sun has measure

(1) 45° (2) 30° (3) 90°

(4) 60°

Hint : Ans : (4)

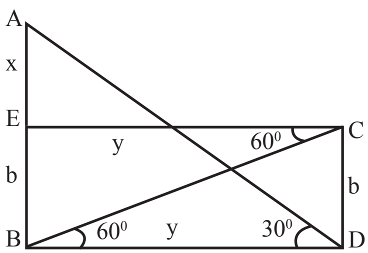


$\tan\theta = \frac{h}{s} = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \theta = 60^0$

11. The electric pole subtends an angle of 30° at a point on the same level as its foot. At a second point 'b' metres above the first, the depression of the foot of the tower is 60°. The height of the tower (in metres) is equal to

(1) $\sqrt{3}b$ (2) $\frac{b}{a}$ (3) $\frac{b}{2}$ (4) $\frac{b}{\sqrt{3}}$

Hint : Ans : (4)



$\tan 30^0 = \frac{x+b}{y}$	$\tan 60^0 = \frac{b}{y}$
$\frac{1}{\sqrt{3}} = \frac{x+b}{y}$	$\sqrt{3} = \frac{b}{y}$
$\Rightarrow \quad y = \sqrt{3}\,(x+b)$	$\Rightarrow \quad y = \frac{b}{\sqrt{3}}$

$$\therefore \sqrt{3}\,(x+b) = \frac{b}{\sqrt{3}}$$

$$\Rightarrow \quad 3(x+b) = b$$

$$\Rightarrow \quad b + x = \frac{b}{3}$$

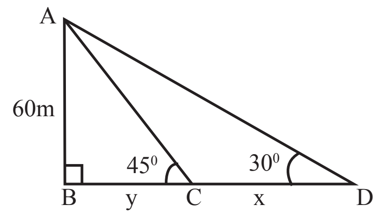
$$\Rightarrow \text{ height of tower} = \frac{b}{3} \text{ mts}$$

---

12. A tower is 60 m height. Its shadow is x metres shorter when the sun's altitude is 45° than when it has been 30° , then x is equal to

(1) 41.92 m (2) 43.92 m
(3) 43 m (4) 45.6 m

Hint : Ans : (3)



In ΔABC,

$$\tan 45^0 = \frac{AB}{BC} = \frac{60}{y}$$

$$\Rightarrow \quad 1 = \frac{60}{y}$$

$$\Rightarrow \quad y = 60^0$$

In ΔABD,

$$\tan 30^0 = \frac{AB}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{60}{x+y}$$

$$\Rightarrow \quad x + y = 60\sqrt{3}$$

$$\Rightarrow \quad x + 60 = 60\sqrt{3}$$

$$\Rightarrow \quad x = 60\sqrt{3} - 60$$

$$= 60\left(\sqrt{3} - 1\right)$$

$$= 60 \times 0.732$$

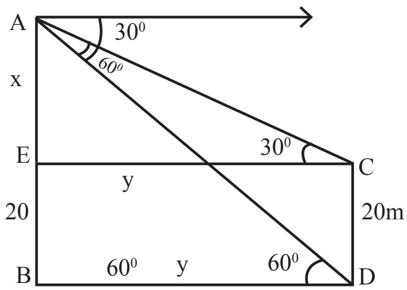
$$= 43.92 \text{ m}$$

---

13. The angle of depression of the top and bottom of 20 m tall building from the top of a multistoried building are 30° and 60° respectively. The height of the multistoried building and the distance between two buildings (in metres) is

(1) $20, 10\sqrt{3}$ (2) $30, 5\sqrt{3}$
(3) $20, 10$ (4) $30, 10\sqrt{3}$

Hint : Ans : (4)



In ΔACE,

$$\tan 30^0 = \frac{AE}{EC}$$

$$\frac{1}{\sqrt{3}} = \frac{x}{y}$$

$$\Rightarrow \quad y = \sqrt{3}x \quad ....(1)$$

In ΔADB,

$$\tan 60^0 = \frac{AB}{BD}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x+20}{y}$$

$$\Rightarrow \quad y = \frac{x+20}{\sqrt{3}} \quad ....(2)$$

$\therefore$ From (1) & (2)

$$\sqrt{3}\,x = \frac{x+20}{\sqrt{3}}$$

$$\Rightarrow \quad 3x = x + 20$$

$$\Rightarrow \quad 2x = 20$$

$$\therefore x = 10$$

$\therefore$ Height of multistoried building

= x + 20

= 10 + 20

= 30 m

$\therefore$ Distance between 2 buildings

$$y = \frac{x+20}{\sqrt{3}} = \frac{30}{\sqrt{3}} = 10\sqrt{3}$$

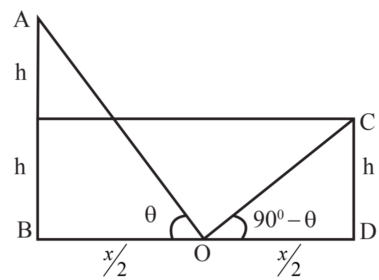
$= 10\,(1.732)$

$= 17.32$ m

14. Two persons are standing 'x' metres apart from each other and the height of the first person is double that of the other. If from the middle point of the line joining their feet an observer finds the angular elevations of their tops to be complementary, then the height of the shorter person (in metres) is

(1) $\sqrt{2}x$ (2) $\frac{x}{2\sqrt{2}}$ (3) $\frac{x}{\sqrt{2}}$ (4) $2x$

Hint : Ans : ( )



CD = h = height of shorter person

AB = 2h = height of taller person

BD = x mrs. $\Rightarrow$ BO = OD = $x/2$

In ΔAOB,

$$\tan\theta = \frac{2h}{x/2} = \frac{4h}{x} \quad ....(1)$$

In ΔCOD,

$$\tan(90^0 - \theta) = \frac{h}{x/2}$$

$$\Rightarrow \quad \cot\theta = \frac{h}{x/2} = \frac{2h}{x}$$

$$\Rightarrow \quad \tan\theta = \frac{x}{2h} \qquad .... (2)$$

$\therefore$ From (1) & (2)

$$\frac{4h}{x} = \frac{x}{2h}$$

$$\Rightarrow \quad 8h^2 = x^2$$

$$\Rightarrow \quad h^2 = \frac{x^2}{8}$$

$$\Rightarrow \quad h = \frac{x}{2\sqrt{2}}$$

$\therefore$ Height of the shorter person $= \dfrac{x}{2\sqrt{2}}$ mrs.

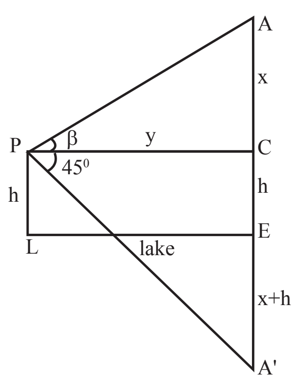
---

15. The angle of elevation of a cloud from a point h metres above a lake is b . The angle of depression of its reflection in the lake is 45° . The height of location of the cloud from the lake is

(1) $\dfrac{h(1+\tan\beta)}{1-\tan\beta}$ (2) $\dfrac{h(1-\tan\beta)}{1+\tan\beta}$

(3) h tan(45° − β) (4) none of these

Hint : Ans : (1)



LE → Surface of the lake

P → Point of observation

PL = h mrs = CE

A, A' → Positions of cloud & its reflection

$\therefore$ AE = A'E = x + h

In ΔAPC,

$$\tan\beta = \frac{x}{y}$$

$$\Rightarrow y = \frac{x}{\tan\beta} \qquad .... (1)$$

In ΔA'PC,

$$\tan 45 = \frac{x+2h}{y} \Rightarrow \frac{x+2h}{y} =$$

$$\Rightarrow \quad y = x + 2h \qquad .... (2)$$

$\therefore$ From (1) & (2),

$$x + 2h = \frac{x}{\tan\beta}$$

$$\Rightarrow 2h = \frac{x}{\tan\beta} - x$$

$$\Rightarrow 2h = x\left(\frac{1}{\tan\beta} - 1\right)$$

$$\Rightarrow 2h = x\left(\frac{1-\tan\beta}{\tan\beta}\right)$$

$$x = \frac{2h\tan\beta}{1-\tan\beta}$$

$\therefore$ Height of the cloud = h + x

$$= h + \frac{2h\tan\beta}{1-\tan\beta}$$

$$= h\left[1 + \frac{2\tan\beta}{1-\tan\beta}\right]$$

$$= h\left[\frac{1+\tan\beta}{1-\tan\beta}\right]$$

UNIT EXERCISE - 6

1. Prove that

(i) $\cot^2 A\left(\dfrac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\dfrac{\sin A-1}{1+\sec A}\right)=0$

(ii) $\dfrac{\tan^2\theta-1}{\tan^2\theta+1}=1-2\cos^2\theta$

Solution :

Solution :

i) LHS

$$\cot^2 A\left(\frac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\frac{\sin A-1}{1+\sec A}\right)$$

$$=\frac{1}{\tan^2 A}\left(\frac{\sec A-1}{1+\sin A}\right)-\frac{1}{\cos^2 A}\left(\frac{1-\sin A}{1+\sec A}\right)$$

$$=\frac{1}{(\sec A+1)\ (\sec A-1)}\cdot\left(\frac{\sec A-1}{1+\sin A}\right)-$$

$$\frac{1}{(1+\sin A)\ (1-\sin A)}\cdot\left(\frac{1-\sin A}{1+\sec A}\right)$$

$$=\frac{1}{(\sec A+1)\,(1+\sin A)}-\frac{1}{(1+\sin A)\,(1+\sec A)}$$

$= 0$

= RHS

ii) LHS

$$=\frac{\tan^2\theta-1}{\tan^2\theta+1}$$

$$=\frac{\tan^2\theta-1}{\sec^2\theta}$$

$$=(\tan^2\theta-1)\cos^2\theta$$

$$=\left(\frac{\sin^2\theta}{\cos^2\theta}-1\right)\cos^2\theta$$

$$=\sin^2\theta-\cos^2\theta$$

$$=1-\cos^2\theta-\cos^2\theta$$

$$=1-2\cos^2\theta$$

= RHS

2. Prove that

$$\left(\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2=\frac{1-\cos\theta}{1+\cos\theta}$$

Solution :

LHS

$$=\left(\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2$$

Take $1+\sin\theta=a,\ \cos\theta=b$

$$\therefore\ \frac{(a-b)^2}{(a+b)^2}$$

$$=\frac{a^2+b^2-2ab}{a^2+b^2+2ab}$$

$$=\frac{(1+\sin\theta)^2+\cos^2\theta-2(1+\sin\theta)\cos\theta}{(1+\sin\theta)^2+\cos^2\theta+2(1+\sin\theta)\cos\theta}$$

$$=\frac{1+\sin^2\theta+2\sin\theta+\cos^2\theta-2(1+\sin\theta)\cos\theta}{1+\sin^2\theta+2\sin\theta+\cos^2\theta+2(1+\sin\theta)\cos\theta}$$

$$=\frac{2+2\sin\theta-2(1+\sin\theta)\cos\theta}{2+2\sin\theta+2(1+\sin\theta)\cos\theta}$$

$$=\frac{2(1+\sin\theta)-2(1+\sin\theta)\cos\theta}{2(1+\sin\theta)+2(1+\sin\theta)\cos\theta}$$

$$=\frac{2(1+\sin\theta)\ [1-\cos\theta]}{2(1+\sin\theta)\ [1+\cos\theta]}$$

$$=\frac{1-\cos\theta}{1+\cos\theta}$$

= RHS

Hence proved.

3. If x $\sin^3\theta$ + y $\cos^3\theta$ = $\sin\theta\cos\theta$ and x $\sin\theta$ = y $\cos\theta$, then prove that $x^2+y^2=1$.

Solution :

Given $x\sin^3\theta+y\cos^3\theta=\sin\theta\cos\theta$,

$\Rightarrow x\sin\theta(\sin^2\theta)+y\cos\theta(\cos^2\theta)$

$=\sin\theta\cos\theta$

$\Rightarrow x\sin\theta(\sin^2\theta)+x\sin\theta(\cos^2\theta)$

$=\sin\theta\cos\theta$ (given $x\sin\theta=y\cos\theta$)

$\Rightarrow x \sin\theta (\sin^2\theta + \cos^2\theta) = \sin\theta \cos\theta$

$\Rightarrow \quad x = \cos\theta$

Also given $x \sin\theta = y \cos\theta$

$\Rightarrow \quad \cos\theta \cdot \sin\theta = y\cos\theta$

$\Rightarrow \quad y = \sin\theta$

$\therefore x^2 + y^2 = \cos^2\theta + \sin^2\theta$

$= 1$

Hence proved.

4. If $a \cos\theta - b \sin\theta = c$, then prove that $(a \sin\theta + b \cos\theta) = \pm\sqrt{a^2 + b^2 - c^2}$

Solution :

Given $a \cos\theta - b \sin\theta = c$

Squaring on both sides

$(a \cos\theta - b \sin\theta)^2 = c^2$

$a^2 \cos^2\theta + b^2 \sin^2\theta - 2ab \cos\theta \sin\theta = c^2$

$\Rightarrow a^2 (1 - \sin^2\theta) + b^2 (1 - \cos^2\theta) - 2ab \cos\theta \sin\theta = c^2$

$\Rightarrow a^2 - a^2 \sin^2\theta + b^2 - b^2 \cos^2\theta - 2ab \cos\theta \sin\theta = c^2$

$\Rightarrow a^2 \sin^2\theta + b^2 \cos^2\theta + 2ab \cos\theta \sin\theta + a^2 + b^2 - c^2$

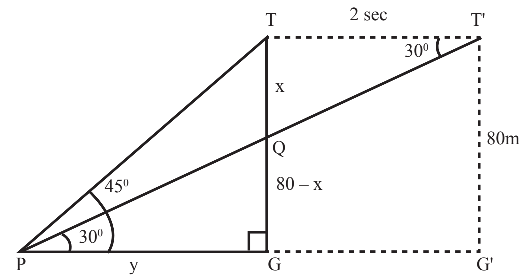
$(a \sin\theta + b \cos\theta)^2 = a^2 + b^2 - c^2$

$\therefore a \sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

Hence proved.

5. A bird is sitting on the top of a 80 m high tree. From a point on the ground, the angle of elevation of the bird is 45°. The bird flies away horizontally in such away that it remained at a constant height from the ground. After 2 seconds, the angle of elevation of the bird from the same point is 30°. Determine the speed at which the bird flies. ($\sqrt{3}$ = 1.732)

Solution :



P → Point of observation

T, T ' → Initial & final positions of the bird

TG = T'G' = 80m = height at which the bird is on the tree, from the ground.

$\angle QTG = \angle QT'T = 30^0$

In ΔPTG,

$$\tan 45^0 = \frac{TG}{PG}$$

$$\Rightarrow 1 = \frac{80}{y}$$

$$\Rightarrow y = 80 \quad ....(1)$$

In ΔPQG,

$$\tan 30^0 = \frac{80\text{-}x}{y}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{80\text{-}x}{80} \quad \text{(from (1))}$$

$$\Rightarrow 80 = 80\sqrt{3} - \sqrt{3}x$$

$$\Rightarrow \sqrt{3}x = 80\left(\sqrt{3} - 1\right)$$

$$\therefore \quad x = \frac{80\left(\sqrt{3}-1\right)}{\sqrt{3}} \quad ....(2)$$

In Δ TQT'

$$\tan 30^0 = \frac{TQ}{TT'}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{x}{TT'}$$

$$\Rightarrow TT' = \sqrt{3}x$$

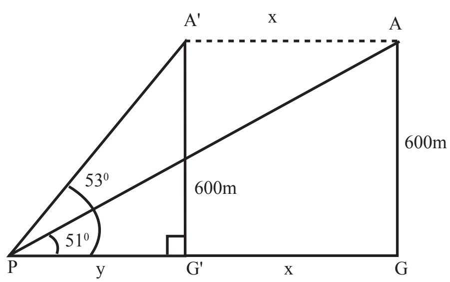
$$\Rightarrow \quad = 80\left(\sqrt{3} - 1\right) \quad \text{(from (2))}$$

Given, time taken by the bird from T to reach T' = 2 sec.

$$\therefore \text{ Speed of the bird} = \frac{\text{Distance}}{\text{Time}}$$
$$= \frac{80\left(\sqrt{3}-1\right)}{2}$$
$$= 40\left(\sqrt{3}-1\right)$$
$$= 40 \times 1.732$$
$$= 29.28 \text{ m/sec.}$$

6. An aeroplane is flying parallel to the Earth's surface at a speed of 175 m/sec and at a height of 600 m. The angle of elevation of the aeroplane from a point on the Earth's surface is 37° at a given point. After what period of time does the angle of elevation increase to 53° ? (tan 53° = 1.3270, tan 37° = 0.7536)

Solution :



P → Point of observation.

A, A' → Initial & final positions of the plane.

Speed of the plane = 175 m/sec.

AG = A'G' = 600 m

= height at which the plane is flying.

In ΔPAG,

$$\tan 37^0 = \frac{AG}{PG}$$
$$\Rightarrow \tan 37^0 = \frac{600}{y+x}$$
$$\Rightarrow y + x = \frac{600}{\tan 37^0}$$
$$\Rightarrow y = \frac{600}{\tan 37^0} - x \qquad \text{..........(1)}$$

In ΔPA' G',

$$\tan 53^0 = \frac{A'G'}{PG'}$$
$$\Rightarrow \tan 53^0 = \frac{600}{y}$$
$$\Rightarrow y = \frac{600}{\tan 53^0} \qquad \text{..........(2)}$$

∴ From (1) & (2)

$$\frac{600}{\tan 53^0} = \frac{600}{\tan 37^0} - x$$
$$\Rightarrow x = 600\left(\frac{1}{\tan 37^0} - \frac{1}{\tan 53^0}\right)$$
$$\Rightarrow x = 600\left(\frac{\tan 53^0 - \tan 37^0}{\tan 53^0 \tan 37^0}\right)$$
$$= 600\left(\frac{\tan 53^0 - \tan 37^0}{\tan 53^0 \tan 37^0}\right) \qquad (\because \tan 53^0 = \cot 53^0 = 1)$$
$$= 600\left(\frac{\tan 53^0 - \tan 37^0}{1}\right)$$
$$= 600\,(1.3270 - 0.7536)$$
$$= 600 \times 0.5734$$
$$= 344.04 \text{ m}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$
$$= \frac{344.04}{175}$$
$$= 1.9659$$
$$\simeq 1.97 \text{ sec.}$$

$\therefore$ After 1.97 sec (approx), angle of elevation changes from 37^0 to 53^0.

7. A bird is flying from A towards B at an angle of 35° , a point 30 km away from A. At B it changes its course of flight and heads towards C on a bearing of 48° and distance 32 km away.

(i) How far is B to the North of A?

(ii) How far is B to the West of A?

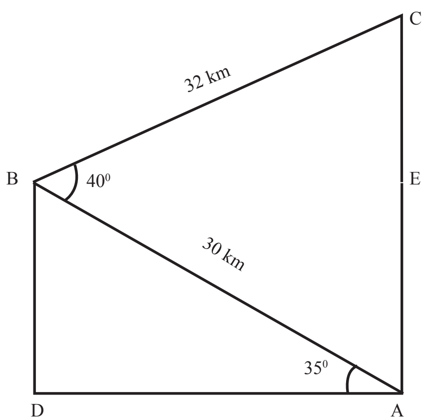
(iii) How far is C to the North of B?

(iv) How far is C to the East of B?

(sin 55° = 0.8192, cos 55° = 0.5736,

sin 42° = 0.6691, cos 42° = 0.7431)

Solution :



i) The distance of B to the north of A [BE]

In ΔBCE,

$$\Rightarrow \cos 48^0 = \frac{BE}{BC}$$

$$\Rightarrow \sin 42^0 = \frac{BE}{32}$$

$$\Rightarrow \quad BE = 32\,(0.6691)$$

$$= 21.4112$$

$$\simeq 21.41 \text{ km.}$$

ii) The distance of B to the west of A [BD]

In ΔBAD,

$$\Rightarrow \sin 35^0 = \frac{BD}{30}$$

$$\Rightarrow \cos 55^0 = \frac{BD}{30} \quad (\because \sin(90-\theta) = \cos\theta)$$

$$\Rightarrow \quad BD = 30^0 \,.\, \cos 55^0$$

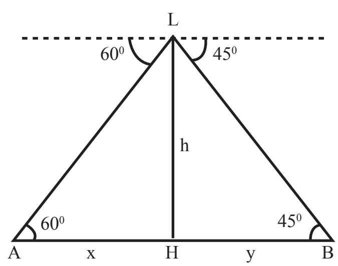
$$= 30\,(0.5736)$$

$$= 17.208$$

$$\simeq 17.21\, km.$$

8. Two ships are sailing in the sea on either side of the lighthouse. The angles of depression of two ships as observed from the top of the lighthouse are 60° and 45° respectively. If the distance between the ships is $200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$ metres, find the height of the lighthouse.

Solution :



LH = h m = height of the light house

AB = Dist. between 2 ships =

$$x + y = 200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$$

In ΔLBH,

$$\tan 45^0 = \frac{h}{y}$$

$$\Rightarrow \quad 1 = \frac{h}{y}$$

$$\Rightarrow \quad y = h \qquad \text{..........} (1)$$

In ΔLAH,

$$\tan 60^0 = \frac{h}{x}$$

$$\Rightarrow \quad \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow \quad x = \frac{h}{\sqrt{3}} \quad ....(2)$$

∴ Adding (1) & (2)

$$x + y = h + \frac{h}{\sqrt{3}}$$

$$\Rightarrow \quad 200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) = h\left(1 + \frac{1}{\sqrt{3}}\right)$$

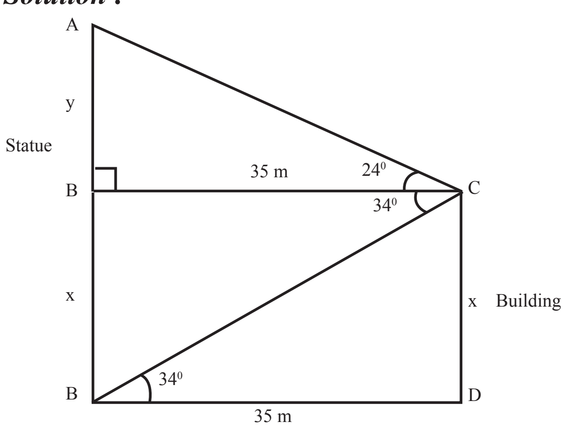
$$\Rightarrow \quad \frac{200(\sqrt{3}+1)}{\sqrt{3}} = h\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$$

$$\therefore \; h = 200$$

∴ Height of the light house = 200 m.

9. A building and a statue are in opposite side of a street from each other 35 m apart. From a point on the roof of building the angle of elevation of the top of statue is 24° and the angle of depression of base of the statue is 34° . Find the height of the statue.
(tan24° = 0.4452, tan 34° = 0.6745)

Solution :



Let CD = x m = height of the building = EB

AB = x + y m = height of the statue

In ΔCBD,

$$\tan 34^0 = \frac{CD}{BD}$$

$$\Rightarrow \quad \tan 34^0 = \frac{x}{35}$$

$$x = 35 \tan 34^0 \quad ....(1)$$

In ΔACE,

$$\tan 24^0 = \frac{AE}{EC}$$

$$\Rightarrow \quad \tan 24^0 = \frac{y}{35}$$

$$y = 35 \tan 24^0 \quad ....(2)$$

Height of the statue = x + y

= 35 ($\tan 34^0 + \tan 24^0$)

= 35 (0.6745 + 0.4452)

= 35 (1.1197)

= 39.1895

≃ 39.19 m

PROBLEMS FOR PRACTICE

1. If $7 \sin^2\theta + 3 \cos^2\theta = 4$, show that $\tan \theta = {}^1\!/\!{}_{\sqrt{3}}$.
2. Prove that $2(\sin^6 \theta + \cos^6 \theta) - 3 (\sin^4 \theta + \cos^4 \theta = 0$
3. Prove that $\frac{\cot A + \text{cosec} A - 1}{\cot A - \text{cosec} A + 1} = \frac{1 + \cos A}{\sin A}$
4. If $\tan \theta + \sin \theta = p$, $\tan \theta - \sin \theta = q$, prove that $p^2 - q^2 = 4\sqrt{pq}$
5. Prove that $\cot \theta - \tan \theta = \frac{2\cos^2\theta - 1}{\sin\theta \,.\, \cos\theta}$
6. If x = secA + sinA, y = secA – sinA, prove that $\left(\frac{2}{x+y}\right)^2 + \left(\frac{x-y}{2}\right)^2 = 1$

7. If $\sin\theta = \frac{15}{17}$, find the value of $\frac{3-4\sin^2\theta}{4\cos^2\theta-3}$ $\left(\text{Ans}: \frac{33}{611}\right)$

8. If a = sinθ + cosθ, b = $\sin^3\theta + \cos^3\theta$, then show that (3a – 2b) = a^3

9. If $\sec\theta = x + \frac{1}{4x}$ prove that secθ + tanθ= 2x

10. Prove that

$$\frac{\sin\theta}{\cot\theta + \operatorname{cosec}\theta} = 2 + \frac{\sin\theta}{\cot\theta - \operatorname{cosec}\theta}$$

11. Prove that $(\tan\alpha + \operatorname{cosec}\beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha \cdot \cot\beta(\operatorname{cosec}\alpha + \sec\beta)$.

12. If $\sin\theta + \sin^2\theta = 1$, prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta + 2\cos^4\theta + 2\cos^2\theta - 2 = 0$

13. Two vertical lamp - posts of equal height stand on either side of a roadway which is 50m wide. At a point in the road between lamp posts, elevations of the tops of the lamp posts are 60^0 and 30^0. Find the height of each lamp post. **(Ans : 21.65 m)**

14. The angle of elevation of a jet plane from a point P on the ground is 60^0. After 15 seconds the angle of elevation changes to 30^0. If the jet is flying at a speed of 720 km/h, find the height at which the jet is flying.

(Ans : 2.6 km)

15. From an aeroplane flying horizontally above a straight road, the angles of depression of two consecutive kilometer stones on the road are 45^0 and 30^0 respectively. Find the height of the aeroplane above the road when the km stones are i) on the same side of the vertical through aeroplane

ii) on the opposite sides.

(Ans : i) 1.366 kmii) 0.366 km)

Objective Type Qustions :

1. If $\cos\theta = a/b$, then cosec θ is equal to

a) b/a b) $\frac{b}{\sqrt{b^2-a^2}}$

c) $\frac{\sqrt{b^2-a^2}}{b}$ d) $\frac{a}{\sqrt{b^2-a^2}}$

(Ans : (b))

2. If sin θ – cos θ = 0, then the value of $\sin^4\theta + \cos^4\theta$ is

a) $1/2$ b) $1/4$ c) $3/4$ d) 1

(Ans : (d))

3. The value of $\frac{11}{\cot^2\theta} - \frac{11}{\cos^2\theta}$ is

a) 11 b) 0 c) $1/11$ d) –11

(Ans : (d))

4. If tanθ + cotθ = 5, then the value of $\tan^2\theta + \cot^2\theta$ is

a) 23 b) 25 c) 27 d) 15

(Ans : (a))

5. The value of $(1 + \cot^2\theta) \cdot (1 + \cos\theta)(1 - \cos\theta)$ is

a) $\sin^2\theta$ b) $\operatorname{cosec}^2\theta$ c) 1

d) $\sec^2\theta$ **(Ans : (c))**

6. The value of $\tan 10^0 \cdot \tan 15^0 \cdot \tan 75^0 \tan 80^0$ is

a) –1 b) 0 c) 1 d) 4

(Ans : (c))

7. A pole 6m high cases a shadow $2\sqrt{3}$ m long on the ground, then the sun's elevation is

a) 60^0 b) 45^0 c) 30^0 d) 90^0

(Ans : (a))

8. The angle of elevation of the top of a tower from a point situated at a distance of 100m from the base of a tower is 30^0. The height of the tower is

a) $\frac{100}{\sqrt{3}}$m b) $100\sqrt{3}$m

c) $\frac{50}{\sqrt{3}}$m d) $50\sqrt{3}$m

(Ans : (a))

9. The angle of depression of a point on the horizontal from the top of a hill is 60^0. If one has towalk 300m to reach the top from this point, then the distance of this point from the base of the hill is

a) $300\sqrt{3}$m b) 150 m

c) $150\sqrt{3}$m d) $\frac{150}{\sqrt{3}}$ m

(Ans : (b))

10. The value of $\sin\theta \cdot \operatorname{cosec}\theta + \cos\theta \sec\theta$ is

a) 1 *b*) 2 *c*) 0 *d*) $\frac{1}{2}$

(Ans : (b))

11. If $5 \cos \theta = 7\sin \theta$, then the value of $\frac{7\sin\theta + 5\cos\theta}{5\sin\theta + 7\cos\theta}$ is

a) $\frac{37}{35}$ *b*) 1 *c*) $\frac{5}{7}$ *d*) $\frac{35}{37}$

(Ans : (d))

12. If $\sin A = \frac{1}{\sqrt{5}}$, then sec A is

a) $\frac{1}{\sqrt{5}}$ *b*) $\frac{2}{\sqrt{5}}$ *c*) $\frac{\sqrt{5}}{2}$ *d*) $\sqrt{5}$

(Ans : (c))

13. The acute angle 'θ' when $\sec^2 \theta + \tan^2 \theta = 3$ is

a) 30^0 b) 45^0 c) 60^0 d) 90^0

(Ans : (b))

14. If $\tan (20^0 - 3\alpha) = \cot (5\alpha - 20^0)$, then α is

a) 45^0 b) 30^0 c) 90^0

d) none of these **(Ans: (a))**

15. If $\tan \alpha = \sqrt{3}$, $\tan \beta = \frac{1}{\sqrt{3}}$, then $\cot (\alpha + \beta)$ is

a) $\sqrt{3}$ *b*) 0 *c*) $\frac{1}{\sqrt{3}}$ *d*) 1

(Ans : (b))