02

Inverse Trigonometric Functions

Quick Revision

Trigonometric functions are not one-one on their natural domains, so their inverse does not exist in all values but their inverse may exists in some interval of their restricted domains and codomains. Thus, we can say that, inverse of trigonometric functions are defined within restricted domains of corresponding trigonometric functions. Inverse of f is denoted by f^{-1}.

Domain and Principle Value Branch (Range)
of Inverse Trigonometric Functions

Function	**Domain**	**Principle value branch** (Range)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	R	$(0, \pi)$

Note $\sin^{-1} x \neq (\sin x)^{-1}$, $\sin^{-1} x \neq \sin^{-1}\left(\frac{1}{x}\right)$, $\sin^{-1} x \neq \frac{1}{\sin x}$

Objective Questions

Multiple Choice Questions

1. The inverse of cosine function is defined in the intervals

(a) $[-\pi, 0]$ (b) $\left[\frac{-\pi}{2}, 0\right]$

(c) $\left[0, \frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{2}, \pi\right]$

2. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(a) $[1, 2]$ (b) $[2, 1]$

(c) $[-1, 1]$ (d) $[0, 1]$

3. The principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $-\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

4. The value of $\sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{2\pi}{3}$ (d) $-\frac{\pi}{6}$

5. The value of $\sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$ is

(a) 1 (b) 0

(c) 2 (d) 4

6. The value of $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$

(c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{3}$

7. The value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) 0 (d) $\frac{2\pi}{3}$

8. The value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$ is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{-\pi}{3}$ (d) $\frac{\pi}{6}$

9. If $\sin^{-1} x = y$, then $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(a) True (b) False

(c) Can't say (d) Partially true

10. For the value $x = \sqrt{3}$, $\sin^{-1}\left(\frac{1}{2}\right) = \tan^{-1} x$.

(a) True (b) False

(c) Can't say (d) Partially false

11. If $\tan^{-1}\left(\frac{3}{4}\right) = A$, then $\cos A = \frac{3}{5}$.

(a) True (b) False

(c) Can't say (d) Partially true

12. The set of values of $\sec^{-1}\frac{1}{2}$ is ....

(a) R (b) $\{0, 1\}$

(c) $R-\{0\}$ (d) ϕ

13. The domain of the function $\cos^{-1}(2x-1)$ is ....

(a) $[0, 1]$ (b) $[-1, 1]$

(c) $(-1, 1)$ (d) $[0, \pi]$

Assertion-Reasoning MCQs

Directions (Q. Nos. 14-15) Each of these questions contains two statements : Assertion (A) and Reason (R). Each of these questions also has four alternative choices, any one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.

(a) A is true, R is true; R is a correct explanation for A.

(b) A is true, R is true; R is not a correct explanation for A.

(c) A is true; R is False.

(d) A is false; R is true.

14. Assertion (A) We can write $\sin^{-1} x = (\sin x)^{-1}$.

Reason (R) Any value in the range of principal value branch is called principal value of that inverse trigonometric function.

15. Assertion (A) The inverse of sine function is define in the interval $[-\pi, 0]$, $[0, \pi]$ etc.

Reason (R) The inverse of sine function is denoted by $\sin^{-1}$.

Case Based MCQs

16. Ritika, a student of class XII is teaching a new topic on existence of a function in their domain. For six inverse trigonometric functions, she is teaching about their domain and range.

Based on domain and range of the inverse trigonometric functions, answer the following questions.

(i) The value of $\sin^{-1} x$ at $x = \frac{1}{2}$ is

(a) does not exist

(b) a finite value

(c) $\frac{\pi}{2}$

(d) $\frac{\pi}{4}$

(ii) The value of $\cos^{-1} x$ at $x = 2$ is

(a) does not exist (b) a finite value

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

(iii) The value of $\tan^{-1} x$ at $x = 6$ is

(a) does not exist (b) a finite value

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

(iv) The value of $\sec^{-1}\left(\frac{1}{3}\right)$ is

(a) does not exist (b) a finite value

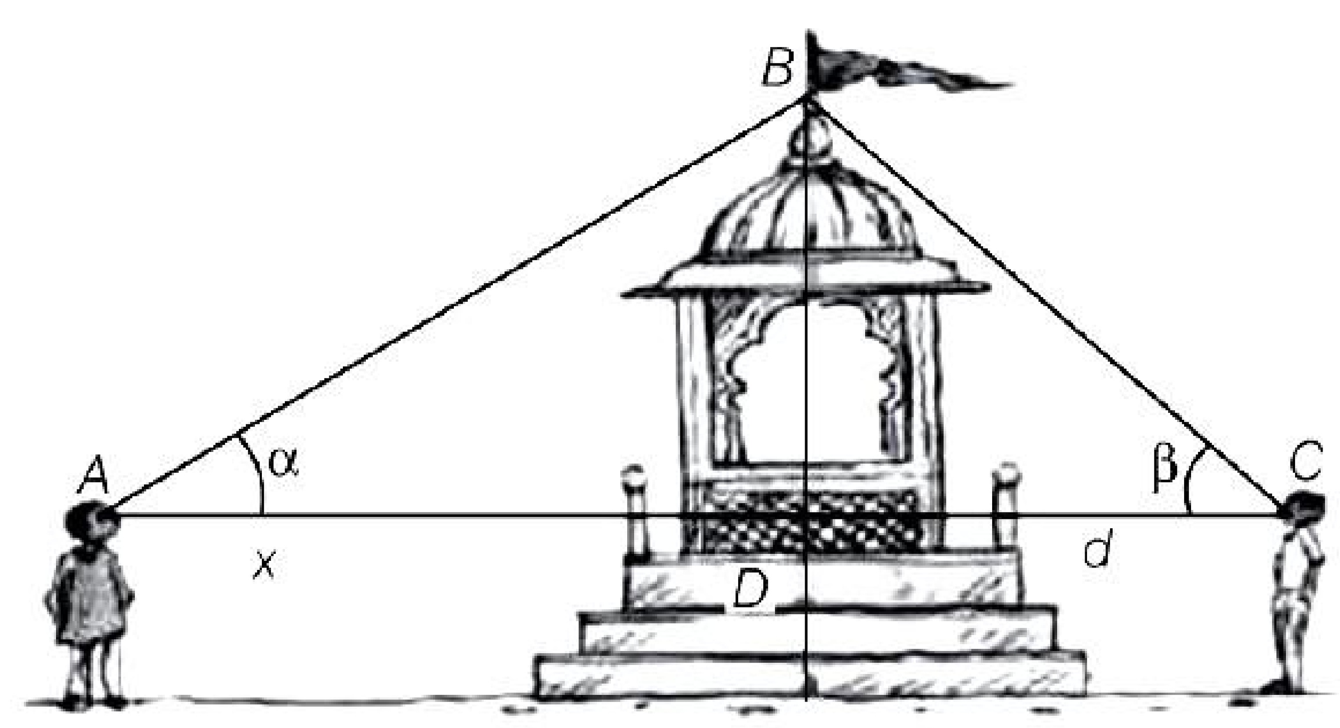
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

(v) The value of $\text{cosec}^{-1}(\sqrt{2})$ is

(a) does not exist (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

17. Two men on either side of a temple of 30 m high observe its top at the angles of elevation α and β respectively. (as shown in the figure below).



The distance between two men is $40\sqrt{3}$ m and the distance between the first person A and the temple is $30\sqrt{3}$ m. Based on the above information answer the following questions.

[CBSE Question Bank]

(i) $\angle CAB = \alpha$ is

(a) $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\sin^{-1}\left(\frac{1}{2}\right)$

(c) $\sin^{-1}(2)$ (d) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(ii) $\angle CAB = \alpha$ is

(a) $\cos^{-1}\left(\frac{1}{5}\right)$ (b) $\cos^{-1}\left(\frac{2}{5}\right)$

(c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\cos^{-1}\left(\frac{4}{5}\right)$

(iii) $\angle BCA = \beta$ is

(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}(2)$

(c) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\tan^{-1}(\sqrt{3})$

(iv) $\angle ABC$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

(v) Domain and Range of $\cos^{-1} x$ is

(a) $(-1, 1), (0, \pi)$ (b) $[-1, 1], (0, \pi)$

(c) $[-1, 1], [0, \pi]$ (d) $(-1, 1), \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

18. As the trigonometric functions are periodic functions, so these functions are many-one. Trigonometric functions are not one-one and onto over their natural domain and range, so their inverse do not exist. But if we restrict their domain and range, then their inverse may exists.

Based on the above information, answer the following questions.

(i) The value of $\tan^{-1}(-1)$ in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$

(c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$

(ii) $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{1}{2}\right)\right)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) 1

(iii) The value of $\tan^{-1}(\sqrt{3}) + \cot^{-1}(\sqrt{3}) + \tan^{-1}(\cos(0))$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$

(c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{3}$

(iv) The value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$ is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $-\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

(v) The principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

ANSWERS

Multiple Choice Questions

1. (a) 2. (a) 3. (b) 4. (b) 5. (a) 6. (c) 7. (d) 8. (a) 9. (a) 10. (b)
11. (b) 12. (d) 13. (a)

Assertion-Reasoning MCQs

14. (d) 15. (d)

Case Based MCQs

16. (i) - (b); (ii) - (a); (iii) - (b); (iv) - (a); (v) - (c) 17. (i) - (b); (ii) - (c); (iii) - (d); (iv) - (c); (v) - (b)
18. (i) - (b); (ii) - (a); (iii) - (c); (iv) - (a); (v) - (a)

SOLUTIONS

1. Cosine function restricted to any of the intervals $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$ etc., is bijective with range as $[-1, 1]$. We can, therefore, define the inverse of cosine function in each of these intervals.

2. Given function, $f(x) = \sin^{-1}\sqrt{x-1}$

For domain of $f(x)$, $0 \le \sqrt{x-1} \le 1$

$\Rightarrow \quad 0 \le x-1 \le 1 \Rightarrow 1 \le x \le 2$

$\therefore \quad x \in [1, 2]$

3. Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta \Rightarrow \sec\theta = \frac{2}{\sqrt{3}}$

We know that, the range of principal value branch of $\sec^{-1}\theta$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\therefore \sec\theta = \frac{2}{\sqrt{3}} = \sec\frac{\pi}{6}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$, where $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\Rightarrow \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

4. Given, $\sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$$= \frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

5. $\sin\left(\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right)$

$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

6. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right)$

$$= \frac{\pi}{6} + \frac{\pi}{3} + \tan^{-1}(1)$$

$$\left[\because \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}, \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}, \sin\left(\frac{\pi}{2}\right) = 1\right]$$

$$= \frac{\pi}{6} + \frac{\pi}{3} + \frac{\pi}{4} = \frac{3\pi}{4}$$

7. Let $\cos^{-1}\left(\frac{1}{2}\right) = x$

$\Rightarrow \quad \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow \quad x = \frac{\pi}{3} \in [0, \pi]$

[$\because$ principal value branch of $\cos^{-1}$ is $[0, \pi]$]

Again, let $\sin^{-1}\left(\frac{1}{2}\right) = y$

$\Rightarrow \quad \sin y = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow \quad y = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\left[\because \text{principal value branch of } \sin^{-1} \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2\cdot\frac{\pi}{6}$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

8. Given, $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$$= \tan^{-1}\left[2\sin\left(2\times\frac{\pi}{6}\right)\right]$$
$$= \tan^{-1}\left(2\sin\frac{\pi}{3}\right)$$
$$= \tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

9. For domain $x \in [-1, 1]$

Range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

10. $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (We know)

when $x = \sqrt{3}$, $\tan^{-1}(\sqrt{3})$ is $\frac{\pi}{3}$.

$\therefore \quad \frac{\pi}{6} \neq \frac{\pi}{3}$

11. Given, $\tan^{-1}\left(\frac{3}{4}\right) = A$

$\Rightarrow \quad \frac{3}{4} = \tan A$

$\Rightarrow \quad \tan A = \frac{3}{4} = \frac{P}{B}$

$\therefore \quad \cos A = \frac{B}{H} = \frac{B}{\sqrt{B^2 + P^2}}$

$$= \frac{4}{\sqrt{4^2 + 3^2}} = \frac{4}{5}$$

12. Since, domain of $\sec^{-1} x$ is $R - (-1, 1)$.

$\Rightarrow \quad (-\infty, -1] \cup [1, \infty)$

So, there is no set of values exist for $\sec^{-1}\frac{1}{2}$.

So, ϕ is the answer.

13. We have, $f(x) = \cos^{-1}(2x - 1)$

$\therefore \quad -1 \le 2x - 1 \le 1$

$\Rightarrow \quad 0 \le 2x \le 2$

$\Rightarrow \quad 0 \le x \le 1$

$\therefore \quad x \in [0, 1]$

14. **Assertion** $\sin^{-1} x$ should not be confused with $(\sin x)^{-1}$. Infact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.

Reason The value of an inverse trigonometric function which lies in the range of principal branch, is called the principal value of that inverse trigonometric function.

Hence, we can say that Assertion is false and Reason is true.

15. **Assertion** Sine function is one-one and onto in the interval $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc; and its range is $[-1, 1]$.

So, inverse of sine function is define in each of these intervals.

Reason We denote the inverse of sine function by $\sin^{-1}$ (arc sine function).

Hence, we can say that the Reason is true and Assertion is false.

16. (i) We know that,

The domain of $\sin^{-1} x$ is $[-1, 1]$

At $x = \frac{1}{2} \in [-1, 1]$;

$\sin^{-1}\left(\frac{1}{2}\right)$ exists and it is a finite value which is $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

(ii) We know that,

The domain of $\cos^{-1} x$ is $[-1, 1]$

$$x = 2 \notin [-1, 1]$$

Hence, $\cos^{-1}(2)$ does not exist.

(iii) We know that,

The domain of $\tan^{-1} x$ is $(-\infty, \infty)$.

It means $\tan^{-1} x$ exists for all real numbers.

At $x = 6$, value of $\tan^{-1}(6)$ is a finite value but not $\frac{\pi}{4}$ or $\frac{\pi}{3}$.

(iv) We know that,

The domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$

$\because \frac{1}{3} \notin$ domain of $\sec^{-1} x$.

$\therefore \sec^{-1}\left(\frac{1}{3}\right)$ does not exist.

(v) We know that,

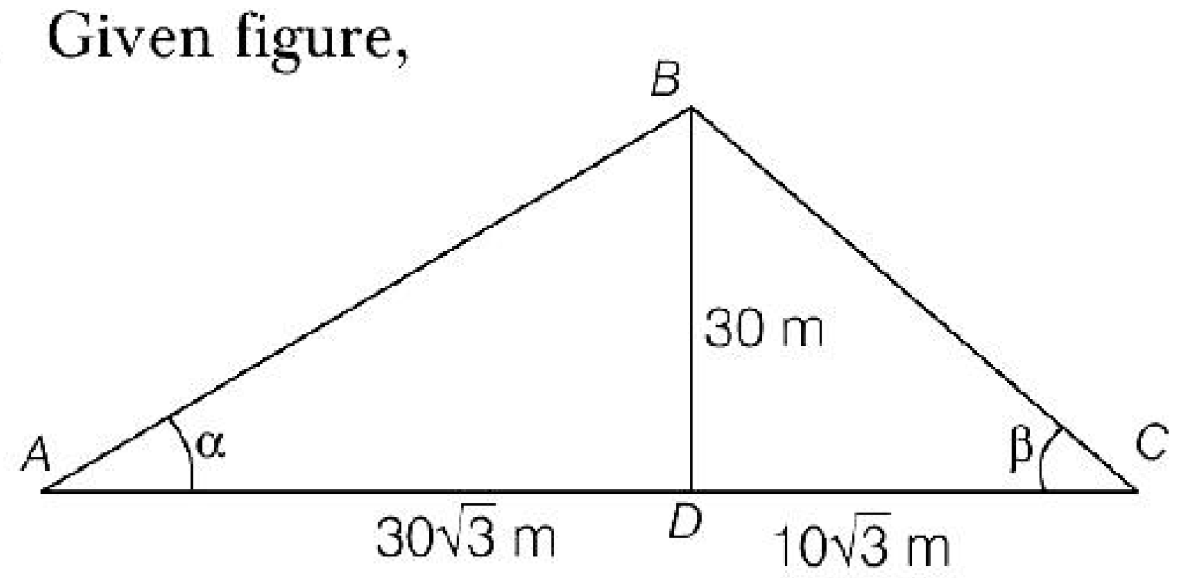
The domain of $\text{cosec}^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$.

$\because \sqrt{2} \in$ domain of $\text{cosec}^{-1} x$.

$\therefore \text{cosec}^{-1} x$ exists at $x = \sqrt{2}$

and $\text{cosec}^{-1}(\sqrt{2}) = \frac{\pi}{4}$

17. Given figure,



(i) In ΔABD, $\tan\alpha = \dfrac{BD}{AD} = \dfrac{30}{30\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \qquad \alpha = 30°$

$\therefore \quad \sin\alpha = \sin 30° = \dfrac{1}{2} \Rightarrow \alpha = \sin^{-1}\left(\dfrac{1}{2}\right)$

(ii) In ΔABD,

$\cos\alpha = \cos 30° = \dfrac{\sqrt{3}}{2} \rightarrow \alpha = \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

(iii) In ΔBDC, $\tan\beta = \dfrac{BD}{CD} = \dfrac{30}{10\sqrt{3}} = \sqrt{3}$

$\Rightarrow \qquad \beta = \tan^{-1}(\sqrt{3})$

(iv) Since, $\alpha = \sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$,

$\beta = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$

$\therefore \angle ABC = \pi - \left(\dfrac{\pi}{6} + \dfrac{\pi}{3}\right)$

$= \pi - \dfrac{\pi}{2} = \dfrac{\pi}{2}$

(v) We know that, domain and range of $\cos^{-1} x$ are $[-1, 1]$ and $[0, \pi]$ respectively.

18. (i) Let $\quad y = \tan^{-1}(-1)$

$\Rightarrow \tan y = -1$

$\Rightarrow \tan y = \tan\dfrac{3\pi}{4} \qquad \left[\because \dfrac{3\pi}{4} \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)\right]$

$\Rightarrow \qquad y = \dfrac{3\pi}{4}$

(ii) $\sin\left[\dfrac{\pi}{3} - \sin^{-1}\left(\dfrac{1}{2}\right)\right]$

$= \sin\left[\dfrac{\pi}{3} - \dfrac{\pi}{6}\right] \quad \left[\because \sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}\right]$

$= \sin\dfrac{\pi}{6} = \dfrac{1}{2}$

(iii) $\tan^{-1}(\sqrt{3}) + \cot^{-1}(\sqrt{3}) + \tan^{-1}(\cos(0))$

$= \dfrac{\pi}{3} + \dfrac{\pi}{6} + \tan^{-1}(1)$

$= \dfrac{\pi}{2} + \dfrac{\pi}{4} = \dfrac{2\pi + \pi}{4} = \dfrac{3\pi}{4}$

(iv) Given, $\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)\right)\right]$

$= \tan^{-1}\left[2\sin\left(2 \times \dfrac{\pi}{6}\right)\right] = \tan^{-1}\left(2\sin\dfrac{\pi}{3}\right)$

$= \tan^{-1}\left(2 \times \dfrac{\sqrt{3}}{2}\right) = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

(v) Let $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \theta \Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2}$

We know that, principal value branch of $\cos^{-1}$ is $[0, \pi]$.

$\therefore \quad \cos\theta = \dfrac{\sqrt{3}}{2} = \cos\dfrac{\pi}{6} \Rightarrow \theta = \dfrac{\pi}{6}$

Hence, $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6} \in [0, \pi]$