Differential Equations

Question 1.
Find the order and degree of the following differential equations.

(i) $\frac{dy}{dx}+2y=x^3$

(ii) $\frac{d^3y}{dx^3}+3\left(\frac{dy}{dx}\right)^3+2\frac{dy}{dx}=0$

(iii) $\frac{d^2y}{dx^2}=\sqrt{y-\frac{dy}{dx}}$

(iv) $\frac{d^3y}{dx^3}=0$

(v) $\frac{d^2y}{dx^2}+y+\left(\frac{dy}{dx}-\frac{d^3y}{dx^3}\right)^{\frac{3}{2}}=0$

(vi) $(2-y'')^2=y''^2+2y'$

(vii) $\left(\frac{dy}{dx}\right)^3+y=x-\frac{dx}{dy}$

Solution:

(i) $\frac{dy}{dx}+2y=x^3$

Highest order derivative is $\frac{dy}{dx}$

Power of $\frac{dy}{dx}$ is 1

$\therefore$ order = 1 Degree =1

(ii) $\frac{d^3y}{dx^3}+3\left(\frac{dy}{dx}\right)^3+2\frac{dy}{dx}=0$

Highest order derivative is $\frac{d^3y}{dx^3}$

Power of $\frac{d^3y}{dx^3}$ is 1

$\therefore$ order = 3 Degree =1

(iii) $\frac{d^2y}{dx^2}=\sqrt{y-\frac{dy}{dx}}$

Here we eliminate the radical sign. Squaring both sides we get,

$$\left(\frac{d^2y}{dx^2}\right)^2=y-\frac{dy}{dx}$$

$\therefore$ order = 2, Degree =2

(iv) $\frac{d^3y}{dx^3} = 0$

order = 3, Degree = 1

(v) $\frac{d^2y}{dx^2} + y + \left(\frac{dy}{dx} - \frac{d^3y}{dx^3}\right)^{\frac{3}{2}} = 0$

Here we eliminate the radical sign

For this write the equation as $\frac{d^2y}{dx^2} + y = -\left(\frac{dy}{dx} - \frac{d^3y}{dx^3}\right)^{\frac{3}{2}}$

Squaring both the sides, we get $\left(\frac{d^2y}{dx^2} + y\right)^2 = \left(\frac{dy}{dx} - \frac{d^3y}{dx^3}\right)^3$

$\therefore$ Order = 3, Degree = 3

(vi) $(2 - y'')^2 = y''^2 + 2y'$

$4 - 4y'' + (y'')^2 = (y'')^2 + 2y'$

$\Rightarrow 4 - 4y'' = 2y'$

$\therefore$ order = 2, Degree =1

(vii) $\left(\frac{dy}{dx}\right)^3 + y = x - \frac{dx}{dy}$

multiplying the equation by $\frac{dy}{dx}$

$\left(\frac{dy}{dx}\right)^4 + y\frac{dy}{dx} = x\frac{dy}{dx} - 1$

$\therefore$ Order = 1, Degree = 4

Question 2.
Find the differential equation of the following
(i) $y = cx + c - c^3$
(ii) $y = c(x - c)^2$
(iii) $xy = c^2$
(iv) $x^2 + y^2 = a^2$
Solution:
(i) $y = cx + c - c^3$ ……. (1)
Here c is a constant which has to be eliminated

Differentiating w.r.t x, $\frac{dy}{dx}$ = c ...... (2)

Using (2) in (1) we get,

$y = \left(\frac{dy}{dx}\right)x + \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3$ which is the required differential equation.

(ii) $y = c(x - c)^2$ ...... (1)

We have to eliminate c

Differentiating w.r.t x ,we get, $\frac{dy}{dx}$ = 2c(x – c) ..... (2)

Dividing (2) by (1) we get

$$\frac{\frac{dy}{dx}}{y} = \frac{2c(x-c)}{c(x-c)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y}{x-c}$$

$$(x-c)\frac{dy}{dx} = 2y$$

$$\frac{x\,dy}{dx} - \frac{c\,dy}{dx} = 2y$$

$$\frac{x\,dy}{dx} - 2y = \frac{c\,dy}{dx}$$

$$\Rightarrow \quad c = \frac{\frac{x\,dy}{dx} - 2y}{\frac{dy}{dx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Using (3) in (1) we get

$$y = \left[\frac{\frac{x\,dy}{dx} - 2y}{\frac{dy}{dx}}\right]\left[x - \left(\frac{\frac{x\,dy}{dx} - 2y}{\frac{dy}{dx}}\right)\right]^2$$

$$y = \frac{\left(\frac{x\,dy}{dx} - 2y\right)}{\frac{dy}{dx}}\left[\frac{\frac{x\,dy}{dx} - \frac{x\,dy}{dx} + 2y}{\left(\frac{dy}{dx}\right)}\right]^2$$

$$y = \frac{\left(\frac{x\,dy}{dx} - 2y\right)}{\frac{dy}{dx}} \left[\frac{\frac{x\,dy}{dx} - \frac{x\,dy}{dx} + 2y}{\left(\frac{dy}{dx}\right)} \right]^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3 y = \left(\frac{x\,dy}{dx} - 2y\right)(4y^2)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3 = 4xy\frac{dy}{dx} + 8y^2$$

(or) $\left(\frac{dy}{dx}\right)^3 - 4xy\frac{dy}{dx} + 8y^2 = 0$ is the required differential equation

(*iii*) $xy = c^2$

Differentiating w.r.t x ,we get,

$\frac{x\,dy}{dx} + y = 0$ is the required differential equation

(*iv*) $x^2 + y^2 = a^2$

Differentiating w.r.t x ,we get,

$$2x + 2y\frac{dy}{dx} = 0$$

(or) $x + y\frac{dy}{dx} = 0$ is the required differential equation

Question 3.

Form the differential equation by eliminating α and β from $(x - \alpha)^2 + (y - \beta)^2 = r^2$

Solution:

$(x - \alpha)^2 + (y - \beta)^2 = r^2$ ....(1)

Differentiating *w.r.t* x we get,

$$2(x - \alpha) + 2(y - \beta)\frac{dy}{dx} = 0$$

(or) $(x - \alpha) + (y - \beta)\frac{dy}{dx} = 0$ ....(2)

Again differentiating *w.r.t* x,

$$1+(y-\beta)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0$$

$$(y-\beta)\frac{d^2y}{dx^2} = -1-\left(\frac{dy}{dx}\right)^2$$

For convenience we use $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$

The above equation becomes,

$$(y-\beta)y'' = -(1+y'^2)$$

$$y-\beta = -\frac{(1+y'^2)}{y''} \quad \text{.............(3)}$$

Using (3) in (2) we get

$$(x-\alpha)-\frac{(1+y'^2)}{y''}y' = 0$$

$$\text{(or)}(x-\alpha) = \frac{(1+y'^2)}{y''}y' \quad \text{...........(4)}$$

Now using (3) and (4) in (1) we get

$$\left[\frac{(1+y'^2)}{y''}y'\right]^2+\left[\frac{-(1+y'^2)}{y''}\right]^2 = r^2$$

$$\text{(or)} \quad \frac{(1+y'^2)^2}{(y'')^2}(y'^2+1) = r^2$$

$$(1+y'^2)^3 = r^2(y'')^2$$

$$\Rightarrow \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2 \text{ is the required differential equation}$$

Question 4.

Find the differential equation of the family of all straight lines passing through the origin.

Solution:

The general equation for a family of lines passing through the origin is

y = mx ……. (1)

Differentiating w.r.t x,

$\frac{dy}{dx}$ = m ……. (2)

Using (2) in (1)

$y = \left(\frac{dy}{dx}\right) x$ is the required differential equation

Question 5.
Form the differential equation that represents all parabolas each of which has a latus rectum 4a and whose axes are parallel to the x-axis.
Solution:
Equation of parabola whose axis is parallel to the x-axis with latus rectum 4a is
$(y - \beta)^2 = 4a(x - \alpha)$ ....(1)
Here (α, β) is the vertex of the parabola.
Differentiating (1) w.r.t x, we get

$$2(y - \beta) \frac{dy}{dx} = 4a \text{ .... } (2)$$

Again, differentiating (2) w.r.t x, we get

$$2\left[(y-\beta)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \qquad .....(3)$$

From (2) we have,

$$(y - \beta)\frac{dy}{dx} = 2a$$

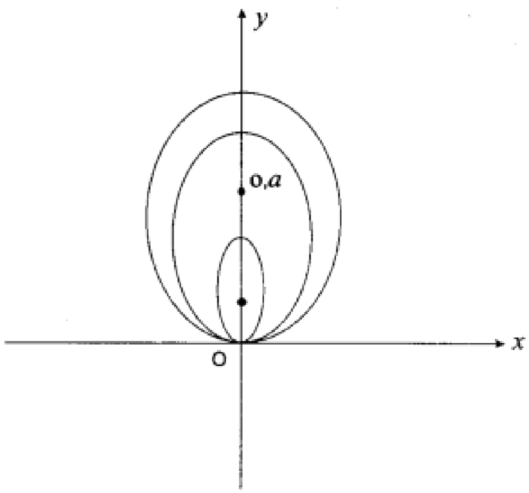
$$y - \beta = \frac{2a}{\frac{dy}{dx}}$$

Using this in (3) we get

$$\frac{2a}{\left(\frac{dy}{dx}\right)}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

(or) $2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ is the required differential equation

Question 6.
Find the differential equation of all circles passing through the origin and having their centers on the y-axis.
Solution:



The circles pass through the origin. They have their centres at (o, a)
The circles have radius a. so the equation of the family of circles is given by
$x^2 + (y - a)^2 = a^2$
$x^2 + y^2 - 2ay + a^2 = a^2$
$x^2 + y^2 = 2ay$ ……. (1)
Differentiating w.r.t x,

$$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$$

$$\Rightarrow \quad a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Using (2) in (1)

$$x^2 + y^2 = 2y\left[\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right]$$

$$x^2\frac{dy}{dx} + y^2\frac{dy}{dx} = 2xy + 2y^2\frac{dy}{dx}$$

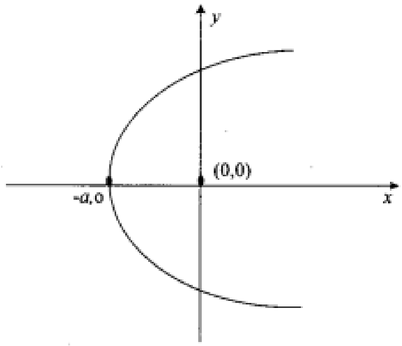
$$\Rightarrow (x^2 - y^2)\frac{dy}{dx} = 2xy$$

$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$ is the required differential equation of all circles passing through origin and having

their centres on the y-axis.

Question 7.
Find the differential equation of the family of a parabola with foci at the origin and axis along the x-axis.

Solution:
The given family of parabolas have foci at the origin and axis along the x-axis.



The equation of such family of parabolas is given by
$y^2 = 4a(x + a)$ …… (1)
Differentiating w.r.t x,

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad a = \frac{y}{2}\frac{dy}{dx} \quad \text{.........................(2)}$$

Using (2) in (1) gives

$$y^2 = 2y\frac{dy}{dx}\left[x + \frac{y}{2}\frac{dy}{dx}\right]$$

$$\text{(or)} \quad y = 2\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$$

$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$ is the required differential equation.

Question 8.
Solve: $(1 - x)\, dy - (1 + y)\, dx = 0$
Solution:
$(1 - x)\, dy - (1 + y)\, dx = 0$
Separating the variables,
$(1 - x)dy = (1 + y)\, dx$

$$\frac{dy}{1+y} = \frac{dx}{1-x}$$

Integrating, we get

$$\int \frac{1}{1+y}\, dy = \int \frac{1}{1-x}\, dx$$

log(y + 1) = -log(1 – x) + log c
log(y + 1) + log(1 – x) = log c
(y + 1) (1 – x) = c

Question 9.
Solve:

(i) $\frac{dy}{dx}$ = y sin 2x
(ii) log($\frac{dy}{dx}$) = ax + by

Solution:

(i) $\frac{dy}{dx}$ = y sin 2x

Separating the variables, $\frac{dy}{y}$ = sin 2x dx

Integrating, we get

$\int \frac{1}{y}$ dy = $\int$sin 2x dx
log y = $-\frac{\cos 2x}{2} + c$ is the required solution

(ii) $\log\left(\frac{dy}{dx}\right) = ax + by$

$$\Rightarrow \quad \frac{dy}{dx} = e^{ax+by}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{ax} e^{by}$$

$$\frac{dy}{e^{by}} = e^{ax}\, dx$$

Integrating, we get

$$\int \frac{1}{e^{by}}\, dy = \int e^{ax}\, dx$$

$$\int e^{-by}\, dy = \int e^{ax}\, dx$$

$$\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + c$$

(or) $$\frac{e^{ax}}{a} = -\frac{e^{-by}}{b} + c$$

Question 10.

Find the curve whose gradient at any point P(x, y) on it is $\frac{x-a}{y-b}$ and which passes through the origin.

Solution:

The gradient at any point P (x, y) on the curve is given by $\frac{dy}{dx}$

According to the problem

$$\frac{dy}{dx} = \frac{x-a}{y-b}$$

Separating the variables,
(y – b) dy = (x – a) dx
Integrating,
∫(y – b) dy = ∫(x – a) dx

$$\frac{(y-b)^2}{2} = \frac{(x-a)^2}{2} + c$$

The curve passes through the origin, $y = 0$, $x = 0$

So $$\frac{b^2}{2} = \frac{a^2}{2} + c$$

$$\Rightarrow \quad c = \frac{b^2 - a^2}{2}$$

Thus the curve becomes

$$\frac{(y-b)^2}{2} = \frac{(x-a)^2}{2} + \frac{b^2 - a^2}{2}$$

(or) $(y - b)^2 = (x - a)^2 + b^2 - a^2$ is the required equation of the curve.