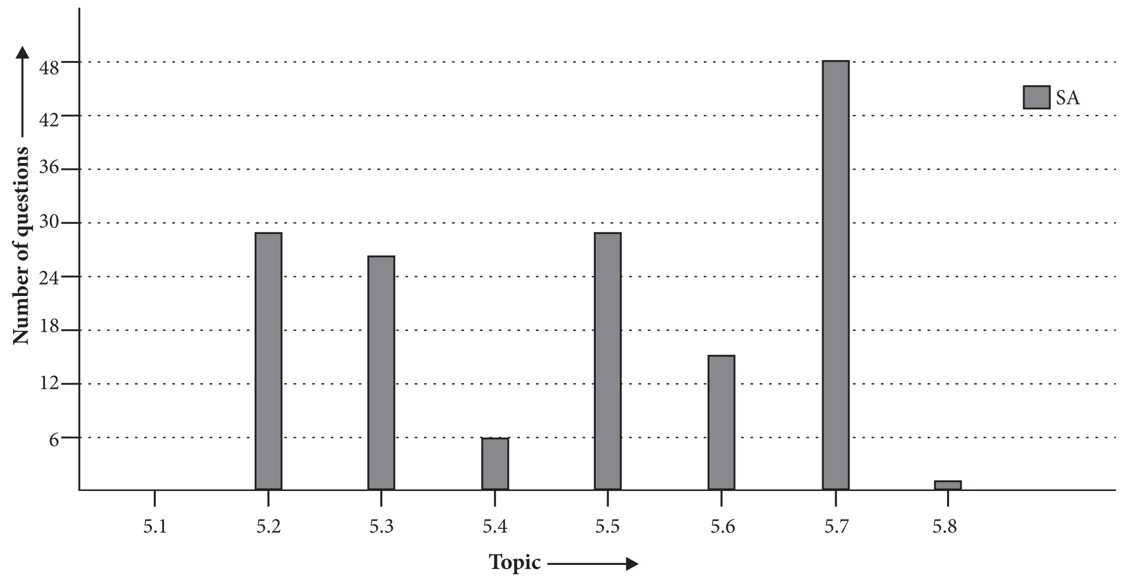
05 Continuity and Differentiability

Topicwise Analysis of Last 10 Years' CBSE Board Questions



- Maximum weightage is of *Second Order Derivative*
- Only SA type questions were asked till now
- No VBQ type questions were asked till now

QUICK RECAP

CONTINUITY

- A real valued function f is said to be continuous at a point $x = c$, if the function is defined at $x = c$ and $\lim_{x\to c} f(x) = f(c)$ or we say f is continuous at $x = c$ iff $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$

- **Discontinuity of a Function :** A real function f is said to be discontinuous at $x = c$, if it is not continuous at $x = c$.
  i.e., f is discontinuous if any of the following reasons arise:
  (i) $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ or both does not exist.

(ii) $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

(iii) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \neq f(c)$

- A function f is said to be continuous in an interval (a, b) iff f is continuous at every point in the interval (a, b) ; and f is said to be continuous in the interval $[a, b]$ iff f is continuous in the interval (a, b) and it is continuous at a from the right and at b from the left.
- A function f is said to be discontinuous in the interval (a, b) if it is not continuous at atleast one point in the given interval.
- **Algebra of Continuous Functions :** If f and g be two real valued functions, continuous at $x = c$, then
  (i) $f + g$ is continuous at $x = c$.
  (ii) $f - g$ is continuous at $x = c$.
  (iii) $f \cdot g$ is continuous at $x = c$.
  (iv) $\left(\frac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).
- Composition of two continuous functions is continuous *i.e.,* if f and g are two real valued functions and g is continuous at c and f is continuous at $g(c)$, then fog is continuous at c.
- The following functions are continuous everywhere.
  (i) Constant function
  (ii) Identity function
  (iii) Polynomial function
  (iv) Modulus function
  (v) Sine and cosine functions
  (vi) Exponential function

DIFFERENTIABILITY

Let $f(x)$ be a real function and a be any real number. Then, we define

(i) **Right-hand derivative :**

$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$ and is denoted by $Rf'(a)$.

(ii) **Left-hand derivative :**

$\lim_{h\to 0^-} \frac{f(a-h)-(fa)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$ and is denoted by $Lf'(a)$.

A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.

The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and it known as the derivative of $f(x)$ at $x = a$. If, however, $Rf'(a) \neq Lf'(a)$ we say that $f(x)$ is not differentiable at $x = a$.

- A function is said to be differentiable in (a, b), if it is differentiable at every point of (a, b).
- Every differentiable function is continuous but the converse is not necessarily true.

SOME GENERAL DERIVATIVES

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\text{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}};\ x\in(-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}};\ x\in(-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2};\ x\in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2};\ x\in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x\in R-[-1, 1]$	$\text{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x\in R-[-1,1]$
$\log_e x$	$\frac{1}{x};\ x>0$	a^x	$a^x \log_e a;\ a>0$	$\log_a x$	$\frac{1}{x\log_e a};\ x>0$ and $a>0$

EXPONENTIAL FUNCTION

- If a is any positive real number, then the function f defined by $f(x) = a^x$ is called the exponential function.

LOGARITHMIC FUNCTION

- Let $a > 1$ be a real number. The logarithmic function of x to the base a is the function $y = f(x) = \log_a x$ *i.e.*, $\log_a x = b$, if $x = a^b$
- The logarithm function, with base $a = 10$, is called common logarithm and with base $a = e$, is called natural logarithm.
- The function $\log_a x$ $(a > 0, \neq 1)$ has the following properties :
  - (i) $\log_a(mn) = \log_a m + \log_a n$; $m, n > 0$
  - (ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$; $m, n > 0$
  - (iii) $\log_a m^n = n\log_a m$; $m > 0$
  - (iv) $\log_a x = \frac{\log x}{\log a}$; $x > 0$
  - (v) $\log_a a = 1$, $\log_a 1 = 0$

SOME PROPERTIES OF DERIVATIVES

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.
6.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
7.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
8.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$

ROLLE'S THEOREM

- Let $f : [a, b] \to R$ be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$

MEAN VALUE THEOREM

- Let $f : [a, b] \to R$ be a continuous function on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Previous Years' CBSE Board Questions

5.2 Continuity

SA (4 marks)

1. Find the values of p and q, for which

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\pi/2\\ p, & \text{if } x=\pi/2\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\pi/2\end{cases}$$

is continuous at $x = \pi/2$.

(Delhi 2016, Foreign 2008)

2. Find the value of the constant k so that the function f, defined below, is continuous at $x= 0$, where

$$f(x)=\begin{cases}\left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

(AI 2014C)

3. Find the value of k, for which

$$f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x<1\end{cases}$$

is continuous at $x = 0$. *(AI 2013)*

4. If $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{when } x<0\\ a, & \text{when } x=0\\ \dfrac{\sqrt{x}}{\left(\sqrt{16+\sqrt{x}}\right)-4}, & \text{when } x>0\end{cases}$

and f is continuous at $x = 0$, find the value of a.

(Delhi 2013C, AI 2012C, 2010C)

5. If $f(x)=\begin{cases}1, & \text{if } x\le 3\\ ax+b, & \text{if } 3<x<5,\\ 7, & \text{if } x\ge 5\end{cases}$

find the values of a and b so that $f(x)$ is a continuous function.

(AI 2013C, Delhi 2012C)

6. Find the value of k so that the following function is continuous at $x = 2$.

$$f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}; & x\neq 2\\ k; & x=2\end{cases}$$

(Delhi 2012C)

7. Find the value of k so that the following function is continuous at $x=\dfrac{\pi}{2}$:

$$f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 5, & \text{if } x=\dfrac{\pi}{2}\end{cases}$$

(Delhi 2012C)

8. If the function $f(x)$ given by

$$f(x)=\begin{cases}3ax+b, & \text{if } x>1\\ 11, & \text{if } x=1\\ 5ax-2b, & \text{if } x<1\end{cases}$$

is continuous at $x = 1$, find the values of a and b.

(Delhi 2012C, 2011, AI 2010C)

9. Find the values of a and b such that the following function $f(x)$ is a continuous function :

$$f(x)=\begin{cases}5, & x\le 1\\ ax+b, & 2<x<10\\ 21, & x\ge 10\end{cases}$$

(Delhi 2011)

10. For what value of a is the function f defined by

$$f(x)=\begin{cases}a\sin\dfrac{\pi}{2}(x+1), & x\le 0\\ \dfrac{\tan x-\sin x}{x^3}, & x>0\end{cases}$$

continuous at $x = 0$? *(Delhi 2011)*

11. Find the relationship between a and b so that the function 'f' defined by

$$f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+3, & \text{if } x>3\end{cases}$$

is continuous at $x = 3$. *(AI 2011)*

12. Discuss the continuity of the function $f(x)$ at $x = \frac{1}{2}$, when $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2} + x, & 0 \le x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ \frac{3}{2} + x, & \frac{1}{2} < x \le 1 \end{cases}$$ *(Delhi 2011C)*

13. Find the value of 'a' if the function $f(x)$ defined by

$$f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$$ is continuous at $x = 2$.

(AI 2011C)

14. Find all points of discontinuity of f, where f is defined as follows :

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$ *(Delhi 2010)*

15. For what value of k is the function defined by

$$f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \le 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? Also, write whether the function is continuous at $x = 1$. *(Delhi 2010C)*

16. Find the values of a and b such that the function defined as follows is continuous :

$$f(x) = \begin{cases} x + 2, & x \le 2 \\ ax + b, & 2 < x < 5 \\ 3x - 2, & x \ge 5 \end{cases}$$ *(Delhi 2010C)*

17. Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & x > 0 \\ 2, & x = 0 \\ \frac{4(1 - \sqrt{1 - x})}{x}, & x < 0 \end{cases}$$

is continuous at $x = 0$. *(AI 2009)*

18. If the function defined by

$$f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$$

is continuous at $x = 2$, find the value of a. Also, discuss the continuity of $f(x)$ at $x = 3$.

(Delhi 2009C)

19. For what value of k is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x + 1, & x < 2 \\ k, & x = 2 \\ 3x - 1, & x > 2 \end{cases}$$ *(Delhi 2008)*

20. If $f(x)$ defined by the following is continuous at $x = 0$, find the value of a, b and c.

$$f(x) = \begin{cases} \frac{\sin(a + 1)x + \sin x}{x}, & \text{if } x < 0 \\ c, & \text{if } x = 0 \\ \frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{if } x > 0 \end{cases}$$

(AI 2008)

21. If the following function $f(x)$ is continuous at $x = 0$, find the value of k.

$$f(x) = \begin{cases} \frac{1 - \cos 2x}{2x^2}, & x \ne 0 \\ k, & x = 0 \end{cases}$$ *(Delhi 2008C)*

22. Find the value of k if the function

$$f(x) = \begin{cases} kx^2, & x \ge 1 \\ 4, & x < 1 \end{cases}$$ is continuous at $x = 1$.

(Delhi 2007)

23. If $f(x) = \begin{cases} \frac{x^2 - 25}{x - 5}, & \text{if } x \ne 5 \\ k, & \text{if } x = 5 \end{cases}$

is continuous at $x = 5$, find the value of k.

(AI 2007)

5.3 Differentiability

SA (4 marks)

24. Find the values of a and b, if the function f defined by $f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$

is differentiable at $x = 1$. *(Foreign 2016)*

25. If $y = \tan^{-1}\left(\frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}\right)$, $x^2 \le 1$

then find $\frac{dy}{dx}$. *(Delhi 2015)*

26. If $f(x)=\sqrt{x^2+1}; g(x)=\dfrac{x+1}{x^2+1}$ and $h(x)=2x-3$, then find $f'[h'(g'(x))]$. *(AI 2015)*

27. Show that the function $f(x) = |x - 1| + |x + 1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$. *(AI 2015)*

28. Find whether the following function is differentiable at $x = 1$ and $x = 2$ or not.

$$f(x)=\begin{cases} x, & x<1 \\ 2-x, & 1\le x\le 2 \\ -2+3x-x^2, & x>2 \end{cases}$$

(Foreign 2015)

29. For what value λ of the function defined by $f(x)=\begin{bmatrix} \lambda(x^2+2), & \text{if } x\le 0 \\ 4x+6, & \text{if } x>0 \end{bmatrix}$ is continuous at $x = 0$? Hence check the differentiability of $f(x)$ at $x = 0$. *(AI 2015C)*

30. If $\cos y = x\cos(a + y)$, where $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx}=\dfrac{\cos^2(a+y)}{\sin a}$. *(Foreign 2014)*

31. If $y=\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$ and $0 < x < 1$, then find $\dfrac{dy}{dx}$. *(AI 2014C, Delhi 2010)*

32. Show that the function $f(x) = |x - 3|, x \in R$, is continuous but not differentiable at $x = 3$. *(Delhi 2013, AI 2012C)*

33. If $\sin y = x \sin(a + y)$, then prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$. *(Delhi 2012, 2011C, AI 2009)*

34. Differentiate $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x. *(AI 2012)*

35. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ for $x \neq y$. Prove the following :

$\dfrac{dy}{dx}=\dfrac{-1}{(1+x)^2}$. *(Delhi 2011C, AI 2008C)*

36. If $y = a \sin x + b \cos x$, prove that

$y^2+\left(\dfrac{dy}{dx}\right)^2=a^2+b^2$. *(AI 2011C)*

37. Show that the function defined as follows, is continuous at $x = 2$, but not differentiable.

$$f(x)=\begin{cases} 3x-2, & 0<x<1 \\ 2x^2-x, & 1<x\le 2 \\ 5x-4, & x>2 \end{cases}$$

(Delhi 2010)

38. If $y=\cos^{-1}\left[\dfrac{3x+4\sqrt{1-x^2}}{5}\right]$, find $\dfrac{dy}{dx}$. *(AI 2010)*

39. If $y=\cos^{-1}\left[\dfrac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right]$, find $\dfrac{dy}{dx}$. *(AI 2010C)*

40. Find $\dfrac{dy}{dx}$, if $(x^2 + y^2)^2 = xy$. *(Delhi 2009)*

41. If $y=\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, find $\dfrac{dy}{dx}$. *(Delhi 2008)*

42. Differentiate $\tan^{-1}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$ w.r.t. x. *(Delhi 2008)*

43. If $xy + y^2 = \tan x + y$, find $\dfrac{dy}{dx}$. *(AI 2008)*

44. Differentiate $\sin^{-1}\left[\dfrac{5x+12\sqrt{1-x^2}}{13}\right]$ w.r.t. x. *(AI 2008)*

5.4 Exponential and Logarithmic Functions

SA (4 marks)

45. If $y=\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log\sqrt{1-x^2}$, then prove that $\dfrac{dy}{dx}=\dfrac{\cos^{-1}x}{(1-x^2)^{3/2}}$. *(Delhi 2015C)*

46. If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} + e^{y-x} = 0$ *(Foreign 2014)*

47. If $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx} = \frac{2a^3}{x^4 - a^4}$. *(AI 2014C)*

48. If $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$, show that $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$ *(AI 2011C)*

49. If $y = \sqrt{x^2+1} - \log\left[\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right]$, find $\frac{dy}{dx}$. *(Delhi 2008)*

50. If $y = \log\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$, then show that $\frac{dy}{dx} = 2\operatorname{cosec} 2x$. *(AI 2007C)*

5.5 Logarithmic Differentiation

SA (4 marks)

51. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x. *(AI 2016, Delhi 2009)*

52. If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$. *(Delhi 2015C, 2013C, 2009, AI 2009C)*

53. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$. *(Foreign 2014)*

54. If $(x-y)\cdot e^{\frac{x}{x-y}} = a$, prove that $y\frac{dy}{dx} + x = 2y$. *(Delhi 2014C)*

55. If $(\tan^{-1}x)^y + y^{\cot x} = 1$, then find $\frac{dy}{dx}$. *(AI 2014C)*

56. Differentiate the following function with respect to x : $(\log x)^x + x^{\log x}$. *(Delhi 2013)*

57. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$. *(AI 2013)*

58. Differentiate the following with respect to x : $\sin^{-1}\left(\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right)$ *(AI 2013)*

59. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$. *(AI 2013, Delhi 2010C)*

60. Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$. *(AI 2013C)*

61. If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$. *(Delhi 2012, AI 2009)*

62. If $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$, find $\frac{dy}{dx}$. *(Delhi 2012C)*

63. Find $\frac{dy}{dx}$ when $y = x^{\cot x} + \frac{2x^2-3}{x^2+x+2}$. *(AI 2012C)*

64. Differentiate $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. x. *(Delhi 2011)*

65. If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}$. *(AI 2011)*

66. Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$. *(Delhi 2010)*

67. If $y = (\sin x - \cos x)^{\sin x - \cos x}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$, then find $\frac{dy}{dx}$. *(AI 2010C)*

68. Differentiate the following with respect to x. $(x)^{\cos x} + (\sin x)^{\tan x}$ *(Delhi 2009)*

69. If $y = (\log x)^x + (x)^{\cos x}$, find $\frac{dy}{dx}$. *(Delhi 2009C)*

70. If $y = (x)^{\sin x} + (\log x)^x$, find $\frac{dy}{dx}$. *(Delhi 2009 C)*

71. If $y = x^x - (\sin x)^x$, find $\frac{dy}{dx}$. *(AI 2009C)*

72. If $y = (\log x)^{\cos x} + \dfrac{x^2+1}{x^2-1}$, find $\dfrac{dy}{dx}$. *(Delhi 2008C)*

73. Differentiate $(\sin x)^{\tan x} + (\cos x)^{\sec x}$ w.r.t. x. *(Delhi 2007C)*

5.6 Derivatives of Functions in Parametric Forms

SA (4 marks)

74. If $x = a \sin 2t(1 + \cos 2t)$ and $y = b \cos 2t(1 - \cos 2t)$, find the values of $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{4}$ and $t = \dfrac{\pi}{3}$. *(Delhi 2016, AI 2016)*

75. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\dfrac{2x}{1+x^2}$, if $x \in (-1, 1)$ *(Foreign 2016, Delhi 2014)*

76. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, prove that $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$. *(AI 2015C)*

77. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, when $x \neq 0$. *(Delhi 2014)*

78. Differentiate $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. *(Delhi 2014)*

79. Find the value of $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{4}$, if $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$. *(AI 2014)*

80. If $x = a \sin 2t\,(1 + \cos 2t)$ and $y = b \cos 2t(1 - \cos 2t)$, show that at $t = \dfrac{\pi}{4}, \left(\dfrac{dy}{dx}\right) = \dfrac{b}{a}$. *(AI 2014)*

81. If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, find the value of $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{4}$. *(AI 2014)*

82. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, then prove that $\dfrac{dy}{dx} = \tan\left(\dfrac{3\theta}{2}\right)$. *(Delhi 2013C)*

83. If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$, show that $\dfrac{dy}{dx} = -\dfrac{y}{x}$. *(AI 2012)*

84. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{3}$. *(Delhi 2011C)*

85. If $x = a(\cos t + \log\tan\dfrac{t}{2})$ and $y = a \sin t$, find $\dfrac{dy}{dx}$. *(Delhi 2011C)*

86. If $x = a\left(\cos\theta + \log\tan\dfrac{\theta}{2}\right)$ and $y = a\sin\theta$, find the value of $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{4}$. *(AI 2008)*

5.7 Second Order Derivative

SA (4 marks)

87. If $y = x^x$, prove that $\dfrac{d^2y}{dx^2} - \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{y}{x} = 0$. *(Delhi 2016, 2014)*

88. If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ *(AI 2016)*

89. If $x = \sin t$ and $y = \sin pt$. Prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$ *(Foreign 2016)*

90. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that $y^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$ *(Delhi 2015, Foreign 2014, AI 2013C)*

91. If $y = e^{m\sin^{-1}x}, -1 \le x \le 1$, then show that $(1-x^2)\dfrac{d^2y}{dx^2} - \dfrac{xdy}{dx} - m^2y = 0$. *(AI 2015, 2010)*

92. If $y = (x + \sqrt{1+x^2})^n$, then show that $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = n^2y$. *(Foreign 2015, Delhi 2013C)*

93. If $x = a\sec^3\theta$, $y = a\tan^3\theta$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$.
(Delhi 2015C)

94. If $y = Ae^{mx} + Be^{nx}$, show that
$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$$
(AI 2015C, 2014, 2009C, 2007)

95. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.
(Delhi 2014C)

96. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$, evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. *(Delhi 2014C)*

97. If $x = a\sin t$ and $y = a(\cos t + \log\tan(t/2))$, find $\frac{d^2y}{dx^2}$. *(Delhi 2013)*

98. If $y = \log\left[x + \sqrt{x^2 + a^2}\right]$, show that
$$(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$ *(Delhi 2013, 2013C)*

99. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$. *(AI 2013)*

100. If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that
$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$ *(Delhi 2013C)*

101. If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that
$$(1+x^2)\frac{d^2y}{dx^2} + (2x - a)\frac{dy}{dx} = 0.$$ *(AI 2013C, 2011)*

102. If $x = \cos\theta$ and $y = \sin^3\theta$, then prove that
$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta(5\cos^2\theta - 1).$$
(AI 2013C)

103. If $y = \sin^{-1}x$, show that
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$ *(Delhi 2012)*

104. If $y = (\tan^{-1}x)^2$, show that
$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$
(Delhi 2012, AI 2012)

105. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$
(Delhi 2012, 2009, 2009C)

106. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.
(AI 2012, 2011C, Delhi 2012C)

107. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$, find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$. *(AI 2012)*

108. If $x = \cos t + \log\tan\frac{t}{2}$, $y = \sin t$, then find the value of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. *(AI 2012C)*

109. If $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$.
(Delhi 2011)

110. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$. *(AI 2011C)*

111. If $y = \text{cosec}^{-1}x$, $x > 1$, then show that
$$x(x^2 - 1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0.$$ *(AI 2010)*

112. If $y = (\cot^{-1}x)^2$, then show that
$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$
(Delhi 2010C)

113. If $y = 3e^{2x} + 2e^{3x}$, then prove that
$$\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 0.$$ *(AI 2009, 2007)*

114. If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, then show that

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0.$$ *(AI 2009)*

115. If $y = e^x(\sin x + \cos x)$, then prove that

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y = 0.$$ *(AI 2009)*

116. If $y = e^x \sin x$, then prove that

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y = 0.$$ *(AI 2009C)*

117. If $y = \sin(\log x)$, then prove that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$ *(Delhi 2007)*

118. If $y = x + \tan x$, then prove that

$$\cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0.$$ *(AI 2007)*

5.8 Mean Value Theorem

SA (4 marks)

119. Verify Rolle's theorem for the function $f(x) = x^2 - 4x + 3$ on $[1, 3]$. *(AI 2007)*

Detailed Solutions

1. $\because$ $f(x)$ is continuous at $\pi/2$.

$\therefore \quad \lim_{x\to\pi/2^-} f(x) = \lim_{x\to\pi/2^+} f(x) = f(\pi/2)$ ...(1)

Now, $\lim_{x\to\pi/2^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$

$$= \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$= \lim_{h\to 0}\frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$$

$$= \lim_{h\to 0}\frac{(1+\cos^2 h+\cos h)}{3(1+\cos h)} = \frac{1+1+1}{3(1+1)} = \frac{1}{2}$$

and $\lim_{x\to\pi/2^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$

$$= \lim_{h\to 0}\frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2} = \lim_{h\to 0}\frac{q(1-\cos h)}{4h^2}$$

$$= \frac{q}{4}\times\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4} = \frac{q}{4}\times\frac{2}{4} = \frac{q}{8}$$

and $f(\pi/2) = p \;\therefore\; \frac{1}{2} = \frac{q}{8} = p$

$\Rightarrow \quad p = \frac{1}{2}$ and $q = 4$ [From (1)]

2. $\because$ $f(x)$ is continuous at $x = 0$.

$\therefore \quad f(0) = k$

and $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos 4x}{8x^2}$

$$= \lim_{x\to 0}\frac{2\cdot\sin^2 2x}{8x^2} = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$$

$\because \quad f$ is continuous at $x = 0$

$\therefore \quad f(0) = \lim_{x\to 0} f(x) \Rightarrow k = 1$

3. $\because$ $f(x)$ is continuous at $x = 0$

$\therefore \quad \lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$ ...(1)

Now $f(0) = \frac{2\times 0+1}{0-1} = -1$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{2+h+1}{h-1} = -1$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{\sqrt{1+kh}-\sqrt{1-kh}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1+kh}-\sqrt{1-kh}}{h}\times\frac{\sqrt{1+kh}+\sqrt{1-kh}}{\sqrt{1+kh}+\sqrt{1-kh}}$$

$$= \lim_{h\to 0}\frac{(1+kh)-(1-kh)}{h[\sqrt{1+kh}+\sqrt{1-kh}]}$$

$$= \lim_{h\to 0}\frac{2k}{\sqrt{1+kh}+\sqrt{1-kh}} = \frac{2k}{2} = k$$

$\therefore \quad$ From (1), we get $k = -1$

4. $\because$ $f(x)$ is continuous at $x = 0$.

$\therefore \quad \lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$ ...(1)

Now $f(0) = a$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}$$

$$= \lim_{h\to 0}\frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}\times\frac{\sqrt{16+\sqrt{h}}+4}{\sqrt{16+\sqrt{h}}+4}$$

$$= \lim_{h\to 0}\frac{\sqrt{h}\left(\sqrt{16+\sqrt{h}}+4\right)}{16+\sqrt{h}-4^2} = \lim_{h\to 0}\sqrt{16+\sqrt{h}}+4 = 8$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{1-\cos 4(-h)}{(-h)^2}$$

$$= \lim_{h\to 0}\frac{1-\cos 4h}{h^2} = \lim_{h\to 0}\frac{2\sin^2 2h}{h^2}$$

$$= 8.\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2 = 8$$

$\therefore \quad$ From (1), we get $a = 8$

5. Continuity at $x = 3$

$\because \quad f(x)$ is continuous at $x = 3$

$\therefore \quad f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ ...(1)

Now, $f(3) = 1$

$$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0} 1 = 1$$

$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h)$$

$$= \lim_{h\to 0}[a(3+h)+b] = 3a+b$$

$\therefore$ From (1), $3a + b = 1$ ...(2)

Continuity at $x = 5$

$\because$ $f(x)$ is continuous at $x = 5$

$$\therefore\ f(5) = \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) \quad ...(3)$$

Now $f(5) = 7$

$$\lim_{x\to 5^+} f(x) = \lim_{h\to 0} f(5+h) = \lim_{h\to 0} 7 = 7$$

$$\lim_{x\to 5^-} f(x) = \lim_{h\to 0} f(5-h)$$

$$= \lim_{h\to 0}[a(5-h)+b] = 5a+b$$

$\therefore$ From (3), $5a + b = 7$ ...(4)

Solving (2) and (4) we get

$a = 3, b = -8$.

6. $\because$ $f(x)$ is continuous at $x = 2$

$$\therefore\ \lim_{x\to 2} f(x) = f(2) \quad ...(1)$$

Here $f(2) = k$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3+x^2-16x+20}{(x-2)^2}$$

$$= \lim_{x\to 2} \frac{(x-2)^2(x+5)}{(x-2)^2} = \lim_{x\to 2}(x+5) = 7$$

$\therefore$ From (1), we get $k = 7$

7. $\because$ $f(x)$ is continuous at $x = \frac{\pi}{2}$,

$$\therefore\ f\left(\frac{\pi}{2}\right) = \lim_{x\to \frac{\pi}{2}} f(x) \quad ...(1)$$

Here $f\left(\frac{\pi}{2}\right) = 5$

$$\lim_{x\to \frac{\pi}{2}} f(x) = \lim_{x\to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)}$$

$$= \lim_{h\to 0} \frac{-k\sin h}{-2h} = \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{k}{2}.1 = \frac{k}{2}$$

$\therefore$ From (1), we get $k = 10$

8. $\because$ $f(x)$ is continuous at $x = 1$

$$\therefore\ \lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x) \quad ...(1)$$

Here $f(1) = 11$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}[5a(1-h)-2b] = 5a-2b$$

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}[3a(1+h)+b] = 3a+b$$

$\therefore$ From (1), we get

$5a - 2b = 11$ and $3a + b = 11$

Solving these, we get

$a = 3, b = 2$.

9. *Refer to answer 5.*

10. $\because$ $f(x)$ is continuous at $x = 0$,

$$\therefore\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \quad ...(1)$$

Here, $f(0) = a\sin\frac{\pi}{2} = a$

$$\therefore\ \lim_{x\to 0^-} f(x) = \lim_{h\to 0} a\sin\left(\frac{\pi}{2}(-h+1)\right)$$

$$= \lim_{h\to 0} a\sin\left(\frac{\pi}{2} - h\frac{\pi}{2}\right) = \lim_{h\to 0} a\cos\left(\frac{-\pi}{2}h\right) = a$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h\to 0}\frac{\frac{\sin h}{\cos h} - \sin h}{h^3} = \lim_{h\to 0}\frac{\sin h\left(\frac{1}{\cos h}-1\right)}{h^3}$$

$$= \lim_{h\to 0}\frac{\sin h}{h} \times \lim_{h\to 0}\frac{1-\cos h}{\cos h \cdot h^2}$$

$$= 1\times \lim_{h\to 0}\frac{1}{\cos h} \times \lim_{x\to 0}\frac{2\sin^2\left(\frac{h}{2}\right)}{4\times\left(\frac{h}{2}\right)^2} = 1\times\frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow\ a = \frac{1}{2}$$

Hence, $f(x)$ is continuous at $x = 0$, if $a = \frac{1}{2}$.

11. $\because$ $f(x)$ is continuous at $x = 3$

$$\therefore\ \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \quad ... (1)$$

$$\lim_{h\to 3^-} f(x) = \lim_{h\to 0}(a(3-h)+1) = 3a+1$$

$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0}(b(3+h)+3) = 3b+3$$

Also, $f(3) = 3a + 1$

From (1), $3a + 1 = 3b + 3 \Rightarrow a - b = \frac{2}{3}$,

which is the required relation between a and b.

12. Here, $f\left(\frac{1}{2}\right)=1.$

$$\lim_{x\to\left(\frac{1}{2}\right)^-} f(x)=\lim_{h\to0} f\left(\frac{1}{2}-h\right)=\lim_{h\to0}\left[\frac{1}{2}+\left(\frac{1}{2}-h\right)\right]=1$$

$$\lim_{x\to\left(\frac{1}{2}\right)^+} f(x)=\lim_{h\to0} f\left(\frac{1}{2}+h\right)=\lim_{h\to0}\left[\frac{3}{2}+\left(\frac{1}{2}+h\right)\right]=2$$

Since $\lim_{x\to\left(\frac{1}{2}\right)^+} f(x)\neq \lim_{x\to\left(\frac{1}{2}\right)^-} f(x)$

$\Rightarrow$ f is not continuous at $x=\frac{1}{2}$.

13. For f to be continuous at $x = 2$, we must have

$\lim_{x\to2^-} f(x) = f(2) = \lim_{x\to2^+} f(x)$ ... (1)

Now $f(2) = a$

$\lim_{x\to2^-} f(x) = \lim_{h\to0} f(2-h) = \lim_{h\to0}[2(2-h)-1] = 3$

$\lim_{x\to2^+} f(x) = \lim_{h\to0} f(2+h) = \lim_{h\to0}[(2+h)+1] = 3$

$\therefore$ From (1), we get $a = 3$

14. Continuity at $x = -3$:

$\lim_{x\to-3^-} f(x)= \lim_{x\to-3^-} |x|+3= \lim_{x\to-3} (-x+3)=3+3=6$

$\lim_{x\to-3^+} f(x)= \lim_{x\to-3} (-2x)=6$

$f(-3) = |-3| + 3 = 3 + 3 = 6$

Thus, $\lim_{x\to-3^-} f(x)= \lim_{x\to-3^+} f(x)=f(-3)$

$\therefore$ f is continuous at $x = -3$.

Continuity at $x = 3$:

$\lim_{x\to3^-} f(x)= \lim_{x\to3} (-2x)=-6$

$\lim_{x\to3^+} f(x)= \lim_{x\to3} (6x+2)=6(3)+2=20$

Thus, $\lim_{x\to3^-} f(x)\neq \lim_{x\to3^+} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x = 3$.

So, the only point of discontinuity of f is $x = 3$.

15. We have, $\lim_{x\to0^-} f(x) = \lim_{h\to0} k(h^2+2) = 2k$

$\lim_{x\to0^+} f(x) = \lim_{h\to0} (3h+1) = 1$

and $f(0) = 2k$

As $f(x)$ is continuous at $x = 0$

$\therefore$ $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$

$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$

$\therefore$ $f(x)$ is continuous at $x = 0$, if $k=\frac{1}{2}$.

Also, $f(x)$ is continuous at $x = 1$ as $f(x) = 3x + 1$ is a polynomial function.

16. $\because$ $f(x)$ is continuous at $x = 2$ and $x = 5$

$\therefore$ $\lim_{x\to2^+} f(x) = f(2)$ and $\lim_{x\to5^-} f(x) = f(5)$

$\therefore$ $\lim_{x\to2}(ax+b) = 4$ and $\lim_{x\to5}(ax+b) = 13$

$\Rightarrow$ $2a + b = 4$ ...(1) and $5a + b = 13$ ...(2)

Solving these equations, we get $a = 3$ and $b = -2$

17. $\lim_{x\to0^-} f(x) = \lim_{h\to0} f(0-h)$

$$= \lim_{h\to0}\frac{4[1-\sqrt{1-(0-h)}]}{0-h} = \lim_{h\to0}\frac{4[1-\sqrt{1+h}]}{-h}$$

$$= \lim_{h\to0}\frac{4[1-\sqrt{1+h}]}{-h}\times\frac{1+\sqrt{1+h}}{1+\sqrt{1+h}}$$

$$= \lim_{h\to0}\frac{4[1-(1+h)]}{-h[1+\sqrt{1+h}]} = \lim_{h\to0}\frac{4(-h)}{-h[1+\sqrt{1+h}]}$$

$$= \lim_{h\to0}\frac{4}{1+\sqrt{1+h}} = \frac{4}{1+1} = 2$$

$\lim_{x\to0^+} f(x)= \lim_{h\to0} f(0+h)$

$$= \lim_{h\to0}\left[\frac{\sin(0+h)}{0+h}+\cos(0+h)\right] = \lim_{h\to0}\left[\frac{\sin h}{h}+\cos h\right]$$

$$= \lim_{h\to0}\frac{\sin h}{h} + \lim_{h\to0}\cos h = 1+1 = 2$$

and $f(0) = 2$

$\Rightarrow$ $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$

18. *Refer to answer 13.*

Also $f(x)$ is continuous at $x = 3$ as $f(x) = x + 1$ is a polynomial function.

19. *Refer to answer 13.*

20. For $f(x)$ to be continuous at $x = 0$, we must have

$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$

Now,

$$\lim_{x\to0^-} f(x)= \lim_{h\to0}\frac{\sin(a+1)(0-h)+\sin(0-h)}{0-h}$$

$$=\lim_{h\to0}\frac{-h\sin(a+1)-\sin h}{-h}$$

$= \lim_{h\to 0}\frac{\sin(a+1)h}{(a+1)h}\times(a+1)+\lim_{h\to 0}\frac{\sin h}{h}$

$= (a + 1) + 1 = a + 2$ ...(1)

And, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{\sqrt{0+h+b(0+h)^2}-\sqrt{0+h}}{b(0+h)^{3/2}}$

$=\lim_{h\to 0}\frac{\sqrt{h+bh^2}-\sqrt{h}}{bh^{3/2}}=\lim_{h\to 0}\frac{\left(\sqrt{1+bh}-1\right)\sqrt{h}}{bh^{3/2}}$

$=\lim_{h\to 0}\frac{\sqrt{1+bh}-1}{bh}\times\frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}$

$=\lim_{h\to 0}\frac{1+bh-1}{bh(\sqrt{1+bh}+1)}=\lim_{h\to 0}\frac{bh}{bh(\sqrt{1+bh}+1)}=\frac{1}{2}$ ...(2)

Also, $f(0) = c$ ...(3)

$\therefore$ From (1), (2) and (3), we get

$a+2=\frac{1}{2}=c\Rightarrow a=\frac{-3}{2}$ and $c=\frac{1}{2}$

and b can be any real number.

21. *Refer to answer 2.*

22. $\because$ $f(x)$ is continuous at $x = 1$

$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$

$\Rightarrow 4=k(1)^2\Rightarrow k=4$

$\therefore$ $f(x)$ is continuous at $x = 1$, if $k = 4$.

23. $\because$ $f(x)$ is continuous at $x = 5$

$\therefore \lim_{x\to 5} f(x)=f(5)=k$

$\Rightarrow \lim_{x\to 5}\frac{x^2-25}{x-5}=k \Rightarrow \lim_{x\to 5}\frac{(x+5)(x-5)}{x-5}=k$

$\Rightarrow \lim_{x\to 5}(x+5)=k \Rightarrow k=10$

$\therefore$ $f(x)$ is continuous at $x = 5$, if $k = 10$.

24. Given that $f(x)$ is differentiable at $x = 1$. Therefore, $f(x)$ is continuous at $x = 1$

$\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$

$\Rightarrow \lim_{x\to 1}(x^2+3x+a)=\lim_{x\to 1}(bx+2)=1+3+a$

$\Rightarrow 1 + 3 + a = b + 2$

$\Rightarrow a - b + 2 = 0$ ...(1)

Again, $f(x)$ is differentiable at $x = 1$. So,

(L.H.D. at $x = 1$) = (R.H.D. at $x = 1$)

$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$

$\Rightarrow \lim_{x\to 1}\frac{(x^2+3x+a)-(4+a)}{x-1}=\lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$

$\Rightarrow \lim_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{bx-2-a}{x-1}$

$\Rightarrow \lim_{x\to 1}\frac{(x+4)(x-1)}{x-1}=\lim_{x\to 1}\frac{bx-b}{x-1}$ [From (1)]

$\Rightarrow \lim_{x\to 1}(x+4)=\lim_{x\to 1}\frac{b(x-1)}{x-1}\Rightarrow 5=b$

Putting $b = 5$ in (1), we get $a = 3$

Hence, $a = 3$ and $b = 5$

25. We have,

$y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$

Putting $x^2 = \cos\theta \Rightarrow \theta = \cos^{-1}(x^2)$ we get

$y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$

$=\tan^{-1}\left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)=\tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)=\frac{\pi}{4}+\frac{\theta}{2}$

$\Rightarrow y=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$

Differentiating w.r.t x on both sides, we get

$\frac{dy}{dx}=-\frac{1\times 2x}{2\sqrt{1-x^4}}=\frac{-x}{\sqrt{1-x^4}}$

26. Here $f(x)=\sqrt{x^2+1}=(x^2+1)^{\frac{1}{2}}$

$\Rightarrow f'(x)=\frac{1}{2}\cdot(x^2+1)^{-\frac{1}{2}}\cdot 2x=\frac{x}{\sqrt{x^2+1}},$ ...(1)

$g(x)=\frac{x+1}{x^2+1}$

$\Rightarrow g'(x)=\frac{(x^2+1)\cdot 1-(x+1)\cdot 2x}{(x^2+1)^2}=\frac{-x^2-2x+1}{(x^2+1)^2}$ ...(2)

and $h(x) = 2x - 3$

$\Rightarrow h'(x) = 2$ ...(3)

$\therefore f'[h'(g'(x))]=f'\left[h'\left(\frac{-x^2-2x+1}{(x^2+1)^2}\right)\right]$ [Using (2)]

$=f'(2)$ [Using (3)]

$=\frac{2}{\sqrt{2^2+1}}=\frac{2}{\sqrt{5}}$ [Using (1)]

27. The given function is $f(x) = |x-1| + |x+1|$

$$= \begin{cases} -(x-1)-(x+1), & x<-1 \\ -(x-1)+x+1, & -1\le x\le 1 \\ x-1+x+1, & x>1 \end{cases} = \begin{cases} -2x, & x<-1 \\ 2, & -1\le x\le 1 \\ 2x, & x>1 \end{cases}$$

At $x = 1$,

$$f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{2-2}{-h} = 0$$

$$f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{2(1+h)-2}{h} = \lim_{h\to 0}\frac{2h}{h} = 2$$

$\therefore\quad f'(1^-) \ne f'(1^+)$

$\Rightarrow\quad f$ is not differentiable at $x = 1$.

At $x = -1$,

$$f'(-1^-) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h}$$

$$= \lim_{h\to 0}\frac{-2(-1-h)-(2)}{-h} = \lim_{h\to 0}\frac{2h}{-h} = -2$$

$$f'(-1^+) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{2-2}{h} = 0$$

$\therefore\quad f'(-1^-) \ne f'(-1^+)$

$\Rightarrow\quad f$ is not differentiable at $x = -1$.

28. At $x = 1$:

$$f'(1^-) = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x-1}{x-1} = 1$$

$$f'(1^+) = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1}\frac{2-x-1}{x-1} = \lim_{x\to 1}\frac{1-x}{x-1} = -1$$

Since, $f'(1^-) \ne f'(1^+)$

$\therefore\quad f(x)$ is not differentiable at $x = 1$.

At $x = 2$:

$$f'(2^-) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{2-x-0}{x-2} = -1$$

$$f'(2^+) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2}\frac{-2+3x-x^2-0}{x-2} = \lim_{x\to 2}\frac{(1-x)(x-2)}{x-2} = -1$$

Since, $f'(2^-) = f'(2^+)$

$\therefore\quad f(x)$ is differentiable at $x = 2$.

29. Here $f(x) = \begin{cases} \lambda(x^2+2), & \text{if } x\le 0 \\ 4x+6, & \text{if } x>0 \end{cases}$

At $x = 0$, $f(0) = \lambda(0^2 + 2) = 2\lambda$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0}\lambda[(0-h)^2+2] = 2\lambda$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0}[4(0+h)+6] = 6$$

$\therefore\quad$ For f to be continuous at $x = 0$

$2\lambda = 6 \Rightarrow \lambda = 3$.

Hence the function becomes

$$f(x) = \begin{cases} 3(x^2+2), & \text{if } x\le 0 \\ 4x+6 & \text{if } x>0 \end{cases}$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h}$$

$$= \lim_{h\to 0}\frac{3(h^2+2)-6}{-h} = \lim_{h\to 0}(-3h) = 0$$

and $f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h} = \lim_{h\to 0}\frac{4h+6-6}{h} = 4$

$\Rightarrow\quad f'(0^-) \ne f'(0^+)$

$\therefore\quad f$ is not differentiable at $x = 0$.

30. We have $\cos y = x\cos(a + y)$

$$\Rightarrow\quad x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y on both sides, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\left(\frac{d}{dy}\cos y\right) - \cos y\left(\frac{d}{dy}\cos(a+y)\right)}{\cos^2(a+y)}$$

$$\Rightarrow\quad \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y)+\cos y\sin(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow\quad \frac{dx}{dy} = \frac{\cos y\sin(a+y)-\cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin[(a+y)-y]}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore\quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

31. We have, $y = \sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$

$\Rightarrow\ y = \sin^{-1}\left(x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right)$

$\Rightarrow\ y = \sin^{-1}x - \sin^{-1}\sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{x-x^2}}$$

32. $f(x) = |x-3| = \begin{cases} x-3, & \text{if } x \ge 3 \\ -(x-3), & \text{if } x < 3 \end{cases}$

We have $f(3) = |3-3| = 0$

$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0}(3+h)-3 = \lim_{h\to 0} h = 0$

$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0}[-(3-h-3)] = \lim_{h\to 0} h = 0$

$\because\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3) = 0$

So, $f(x)$ is continuous at $x = 3$.

Now, $Rf'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$

$= \lim_{h\to 0}\frac{(3+h-3)-0}{h} = \lim_{h\to 0}\frac{h}{h} = 1$

And $Lf'(3) = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}$

$= \lim_{h\to 0}\frac{[-(3-h-3)]-0}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$

Thus, $Rf'(3) \ne Lf'(3)$

$\therefore$ $f(x)$ is not differentiable at $x = 3$.

33. *Refer to answer 30.*

34. Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\therefore\ y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$

$\Rightarrow\ y = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$\Rightarrow\ y = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x$

$\therefore\ \frac{dy}{dx} = \frac{1}{2}\cdot\left(\frac{1}{1+x^2}\right)$

35. We have, $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow\ x\sqrt{1+y} = -y\sqrt{1+x} \Rightarrow x^2(1+y) = y^2(1+x)$

$\Rightarrow\ (x^2-y^2) + xy(x-y) = 0 \Rightarrow y = \frac{-x}{x+1}$

$\therefore\ \frac{dy}{dx} = \frac{(x+1)(-1)-(-x)(1)}{(x+1)^2} = \frac{-x-1+x}{(x+1)^2} = \frac{-1}{(x+1)^2}$

36. Here, $y = a\sin x + b\cos x$

$\Rightarrow\ \frac{dy}{dx} = a\cos x - b\sin x$

Now, L.H.S. $= y^2 + \left(\frac{dy}{dx}\right)^2$

$= (a\sin x + b\cos x)^2 + (a\cos x - b\sin x)^2$

$= a^2\sin^2 x + b^2\cos^2 x + 2ab\sin x\cos x + a^2\cos^2 x + b^2\sin^2 x - 2ab\sin x\cos x$

$= a^2(\sin^2 x + \cos^2 x) + b^2(\cos^2 x + \sin^2 x)$

$= a^2 + b^2 =$ R.H.S.

37. L.H.L. $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2}(2x^2 - x) = 6$

R.H.L. $= \lim_{x\to 2^+} f(x) = \lim_{x\to 2}(5x-4) = 6$

Also, $f(2) = 2(2)^2 - 2 = 6$

As, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$.

Test of differentiability :

We have, $Lf'(2) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$= \lim_{h\to 0}\frac{2(2-h)^2-(2-h)-6}{-h}$

$= \lim_{h\to 0}\frac{8+2h^2-8h+h-8}{-h}$

$= \lim_{h\to 0}\frac{2h^2-7h}{-h} = \lim_{h\to 0}\frac{-h(-2h+7)}{-h} = 7$

$Rf'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$

$= \lim_{h\to 0}\frac{5(2+h)-4-6}{h}$

$= \lim_{h\to 0}\frac{10+5h-10}{h} = \lim_{h\to 0}\frac{5h}{h} = 5$

$\because\ Lf'(2) \ne Rf'(2)$

Hence, $f(x)$ is not differentiable at $x = 2$.

38. We have, $y=\cos^{-1}\left[\dfrac{3x+4\sqrt{1-x^2}}{5}\right]$

Putting $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$, we get

$$y=\cos^{-1}\left[\frac{3\sin\theta+4\cos\theta}{5}\right]$$

$$\Rightarrow\ y=\cos^{-1}\left[\frac{3}{5}\sin\theta+\frac{4}{5}\cos\theta\right]$$

Let $\dfrac{3}{5}=\sin\alpha$ and $\dfrac{4}{5}=\cos\alpha$

$\Rightarrow\ y = \cos^{-1}[\sin\alpha\sin\theta + \cos\alpha\cos\theta]$

$\Rightarrow\ y = \cos^{-1}[\cos(\alpha - \theta)] \Rightarrow y = \alpha - \theta$

$\Rightarrow\ y = \alpha - \sin^{-1}x$

$\therefore\ \dfrac{dy}{dx}=\dfrac{-1}{\sqrt{1-x^2}}$

39. *Refer to answer 38.*

40. We have, $(x^2 + y^2)^2 = xy$

Differentiating w.r.t. x, we get

$$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow\ \frac{dy}{dx}(4y(x^2+y^2)-x)=y-4x(x^2+y^2)$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$$

41. We have,

$$y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$$

$$\Rightarrow\ y=\cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$\Rightarrow\ y=\cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right]$$

$$\Rightarrow\ y=\cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]\Rightarrow y=\cot^{-1}\left[\cot\frac{x}{2}\right]$$

$$\Rightarrow\ y=\frac{x}{2}$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{2}$$

42. Let $y=\tan^{-1}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

Putting $x = \cos 2\theta \Rightarrow \theta=\dfrac{1}{2}\cos^{-1}x$, we get

$$y=\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$\Rightarrow\ y=\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$\Rightarrow\ y=\tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]=\tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$\Rightarrow\ y=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$

$$\Rightarrow\ y=\frac{\pi}{4}-\theta\ \Rightarrow\ y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$$

43. Given, $xy + y^2 = \tan x + y$

Differentiating w.r.t. x, we get

$$x\frac{dy}{dx}+y+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

$$\Rightarrow\ \frac{dy}{dx}(x+2y-1)=\sec^2 x-y$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{\sec^2 x-y}{x+2y-1}$$

44. *Refer to answer 38.*

45. Here $y=\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log\sqrt{1-x^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{\sqrt{1-x^2}\cdot\frac{d}{dx}(x\cos^{-1}x)-x\cos^{-1}x\cdot\frac{d}{dx}\sqrt{1-x^2}}{1-x^2}-\frac{1}{\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$=\frac{\sqrt{1-x^2}\cdot\left(1\cdot\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}\right)-x\cos^{-1}x\cdot\frac{-x}{\sqrt{1-x^2}}}{1-x^2}-\frac{1}{\sqrt{1-x^2}}\cdot\frac{-x}{\sqrt{1-x^2}}$$

$$=\frac{\sqrt{1-x^2}\cos^{-1}x-x+\dfrac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{1-x^2}+\frac{x}{1-x^2}$$

$$=\frac{(1-x^2)\cos^{-1}x+x^2\cos^{-1}x}{(1-x^2)\sqrt{1-x^2}}=\frac{\cos^{-1}x}{(1-x^2)^{3/2}}$$

46. Given $e^x + e^y = e^{x+y} \Rightarrow 1 + e^{y-x} = e^y$ ...(1)

Differentiating (1) w.r.t. x, we get

$$e^{y-x}\cdot\frac{d}{dx}(y-x)=e^y\frac{dy}{dx}$$

$$\Rightarrow\ e^{y-x}\left(\frac{dy}{dx}-1\right)=e^y\frac{dy}{dx}\Rightarrow\ \frac{dy}{dx}(e^{y-x}-e^y)=e^{y-x}$$

$$\Rightarrow\ \frac{dy}{dx}(-1)=e^{y-x}\qquad\text{[Using (1)]}$$

$$\Rightarrow\ \frac{dy}{dx}+e^{y-x}=0$$

47. Here, $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}\log\left(\frac{x-a}{x+a}\right)$$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}[\log(x-a)-\log(x+a)]$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx}=\frac{1}{1+\dfrac{a^2}{x^2}}\cdot\frac{d}{dx}\left(\frac{a}{x}\right)+\frac{1}{2}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$$=\frac{x^2}{x^2+a^2}\cdot a\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{2}\left[\frac{(x+a)-(x-a)}{x^2-a^2}\right]$$

$$=\frac{-a}{x^2+a^2}+\frac{a}{x^2-a^2}=\frac{-a(x^2-a^2)+a(x^2+a^2)}{x^4-a^4}$$

$$=\frac{2a^3}{x^4-a^4}$$

48. We have, $\log(\sqrt{1+x^2}-x)=y\sqrt{1+x^2}$

Differentiating w.r.t. x, we get

$$\frac{1}{\sqrt{1+x^2}-x}\cdot\left[\frac{1}{\sqrt{1+x^2}}\cdot x-1\right]=\frac{dy}{dx}\sqrt{1+x^2}+y\cdot\frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow\ \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}-x}\cdot\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}=(1+x^2)\frac{dy}{dx}+xy$$

$$\Rightarrow\ (1+x^2)\frac{dy}{dx}+xy+1=0$$

49. We have, $y=\sqrt{x^2+1}-\log\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)$

$$\Rightarrow\ y=\sqrt{x^2+1}-\log\left(\frac{1+\sqrt{x^2+1}}{x}\right)$$

$$\Rightarrow\ y=\sqrt{x^2+1}-\log\left(1+\sqrt{x^2+1}\right)+\log x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{2x}{2\sqrt{x^2+1}}-\left(\frac{1}{1+\sqrt{x^2+1}}\right)\left(\frac{2x}{2\sqrt{x^2+1}}\right)+\frac{1}{x}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{x}{\sqrt{x^2+1}}-\frac{x}{\sqrt{x^2+1}\left(1+\sqrt{x^2+1}\right)}+\frac{1}{x}$$

$$=\frac{x\left(1+\sqrt{x^2+1}-1\right)}{\sqrt{x^2+1}\left(1+\sqrt{x^2+1}\right)}+\frac{1}{x}=\frac{x}{1+\sqrt{x^2+1}}+\frac{1}{x}$$

$$=\frac{x^2+1+\sqrt{x^2+1}}{x\left(1+\sqrt{x^2+1}\right)}=\frac{\sqrt{x^2+1}\left(\sqrt{x^2+1}+1\right)}{x\left(1+\sqrt{x^2+1}\right)}$$

$$=\frac{\sqrt{x^2+1}}{x}$$

50. We have, $y=\log\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$

$$\Rightarrow\ y=\log\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}=\log(\tan x)$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\tan x}\times\sec^2 x$$

$$=\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x}=\frac{2}{2\sin x\cos x}=\frac{2}{\sin 2x}=2\operatorname{cosec}2x$$

51. Let $y = x^{\sin x} + (\sin x)^{\cos x}$

$\Rightarrow\ y = e^{\sin x\cdot\log x} + e^{\cos x\cdot\log\sin x}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=e^{\sin x\cdot\log x}\left(\sin x\cdot\frac{1}{x}+\log x\cdot\cos x\right)$$

$$+e^{\cos x\cdot\log\sin x}\cdot\left(\cos x\cdot\frac{\cos x}{\sin x}+\log\sin x\cdot(-\sin x)\right)$$

$$\Rightarrow\ \frac{dy}{dx}=x^{\sin x}\left(\frac{\sin x}{x}+\log x\cdot\cos x\right)$$

$$+(\sin x)^{\cos x}\cdot\left(\frac{\cos^2 x}{\sin x}-\sin x\cdot\log\sin x\right)$$

52. Here, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$\Rightarrow \quad y = e^{x \cdot \log \sin x} + \sin^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = e^{x\cdot\log\sin x}\left[\frac{x\cdot\cos x}{\sin x} + \log\sin x\right] + \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^x(\log\sin x + x\cot x) + \frac{1}{2\sqrt{x-x^2}}$$

53. Given $x^m y^n = (x+y)^{m+n}$

Taking log on both the sides, we get

$\log x^m + \log y^n = (m+n)\log(x+y)$

$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y)$

Differentiating w.r.t. x, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\cdot\frac{dy}{dx} = (m+n)\cdot\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right) = \frac{nx-my}{x(x+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}.$$

54. Here, $(x-y)\cdot e^{\frac{x}{x-y}} = a$

Taking log on both sides, we get

$$\Rightarrow \quad \log\left\{(x-y)\cdot e^{\frac{x}{x-y}}\right\} = \log a$$

$$\Rightarrow \quad \log(x-y) + \frac{x}{x-y} = \log a$$

Differentiating w.r.t. x, we get

$$\frac{1}{x-y}\cdot\left(1-\frac{dy}{dx}\right) + \frac{(x-y)\cdot 1 - x\left(1-\frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$\Rightarrow \quad (x-y)\left(1-\frac{dy}{dx}\right) + x - y - x\left(1-\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad -y\left(1-\frac{dy}{dx}\right) + x - y = 0 \Rightarrow y\frac{dy}{dx} + x = 2y$$

55. Here, $(\tan^{-1}x)^y + y^{\cot x} = 1$

$\Rightarrow \quad u + v = 1$ where $u = (\tan^{-1}x)^y$ and $v = y^{\cot x}$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \text{...(1)}$$

Now, $u = (\tan^{-1}x)^y$

$\Rightarrow \quad \log u = y\log(\tan^{-1}x)$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \frac{dy}{dx}\log(\tan^{-1}x) + y\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \quad \frac{du}{dx} = (\tan^{-1}x)^y \times \left[\frac{dy}{dx}\log(\tan^{-1}x) + \frac{y}{(1+x^2)\tan^{-1}x}\right] \quad \text{...(2)}$$

And $v = y^{\cot x}$

$\Rightarrow \quad \log v = \cot x \cdot \log y$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cot x\cdot\frac{1}{y}\cdot\frac{dy}{dx} - \operatorname{cosec}^2 x\cdot\log y$$

$$\Rightarrow \quad \frac{dv}{dx} = y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx} - \operatorname{cosec}^2 x\cdot\log y\right] \quad \text{...(3)}$$

From (1), (2) and (3), we get

$$(\tan^{-1}x)^y\left[\frac{dy}{dx}\log(\tan^{-1}x) + \frac{y}{(1+x^2)\tan^{-1}x}\right] + y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx} - \operatorname{cosec}^2 x\cdot\log y\right] = 0$$

$$\Rightarrow \quad \frac{dy}{dx}[(\tan^{-1}x)^y\cdot\log(\tan^{-1}x) + y^{\cot x - 1}\cdot\cot x]$$

$$= y^{\cot x}\cdot\operatorname{cosec}^2 x\cdot\log y - (\tan^{-1}x)^{y-1}\cdot\frac{y}{(1+x^2)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y^{\cot x}\cdot\operatorname{cosec}^2 x\cdot\log y - (\tan^{-1}x)^{y-1}\cdot\frac{y}{(1+x^2)}}{(\tan^{-1}x)^y.\log(\tan^{-1})x + y^{\cot x-1}.\cot x}$$

56. Let $y = (\log x)^x + x^{\log x}$

$\therefore \quad y = e^{x\log(\log x)} + e^{(\log x)^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = (\log x)^x\frac{d}{dx}\{x\log(\log x)\} + x^{\log x}\frac{d}{dx}\{(\log x)^2\}$$

$$= (\log x)^x\left\{x\left(\frac{1}{\log x}\right)\frac{1}{x} + \log(\log x)\right\} + x^{\log x}\left(2(\log x)\frac{1}{x}\right)$$

$$= (\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\} + 2\left(\frac{\log x}{x}\right)x^{\log x}$$

57. Here $y^x = e^{y-x}$

Taking log on both sides, we get

$x\log y = (y - x)\log e = y - x$

$\Rightarrow \; x(1 + \log y) = y$

$$\Rightarrow \; x = \frac{y}{1+\log y}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{(1+\log y)\cdot 1 - y\cdot \frac{1}{y}}{(1+\log y)^2} = \frac{\log y}{(1+\log y)^2}$$

$$\Rightarrow \; \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

58. Let $y = \sin^{-1}\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$

$$\Rightarrow \; y = \sin^{-1}\left[\frac{2\cdot 2^x\cdot 3^x}{1+(36)^x}\right] = \sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right]$$

Put $6^x = \tan\theta \Rightarrow \theta = \tan^{-1}6^x$

$$\therefore \; y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$\Rightarrow \; y = 2\theta = 2\tan^{-1}(6^x)$

Now differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\cdot\frac{1}{1+(6^x)^2}\cdot\frac{d}{dx}(6^x)$$

$$= \frac{2}{1+(36)^x}\cdot 6^x\log 6 = \frac{2\log 6\cdot 6^x}{1+(36)^x}$$

59. We have, $x^y = e^{x-y}$

Taking log on both sides, we get

$y\log x = (x - y)\log e = x - y$

$$\Rightarrow \; y(1+\log x) = x \;\; \Rightarrow \; y = \frac{x}{(1+\log x)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)1 - x\left(\frac{1}{x}\right)}{(1+\log x)^2} = \frac{1+\log x - 1}{(1+\log x)^2}$$

$$\therefore \; \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

60. *Refer to answer 58.*

61. We have, $(\cos x)^y = (\cos y)^x$

Taking log on both sides, we get

$y\log(\cos x) = x\log(\cos y)$ ... (i)

Differentiating w.r.t. x, we get

$$y\cdot\frac{1}{\cos x}\cdot(-\sin x) + \log(\cos x)\cdot\frac{dy}{dx}$$

$$= x\cdot\frac{1}{\cos y}\cdot(-\sin y)\cdot\frac{dy}{dx} + \log(\cos y)\cdot 1$$

$$\Rightarrow \; \frac{dy}{dx}(x\tan y + \log(\cos x)) = y\tan x + \log(\cos y)$$

$$\Rightarrow \; \frac{dy}{dx} = \frac{y\tan x + \log(\cos y)}{x\tan y + \log(\cos x)}$$

62. Here $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$

Let $u = x^{\sin x - \cos x}$ and $v = \frac{x^2-1}{x^2+1}$

$\therefore \; y = u + v$

$$\Rightarrow \; \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Now, $u = x^{\sin x - \cos x}$

$\Rightarrow \; \log u = (\sin x - \cos x)\log x$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = (\sin x - \cos x)\cdot\frac{1}{x} + (\cos x + \sin x)\cdot\log x$$

$$\Rightarrow \; \frac{du}{dx} = x^{\sin x - \cos x} \times\left[\frac{\sin x - \cos x}{x} + (\cos x + \sin x)\cdot\log x\right] \quad ...(2)$$

$$\text{Now, } v = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$$

$$\Rightarrow \; \frac{dv}{dx} = 0 - 2\cdot(-1)(x^2+1)^{-2}\cdot 2x = \frac{4x}{(x^2+1)^2} \quad ...(3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = x^{\sin x - \cos x}\cdot\left[\frac{\sin x - \cos x}{x} + (\sin x + \cos x)\cdot\log x\right] + \frac{4x}{(x^2+1)^2}$$

63. Here, $y = x^{\cot x} + \frac{2x^2-3}{x^2+x+2}$

Let $u = x^{\cot x}, v = \frac{2x^2-3}{x^2+x+2}$

$$\Rightarrow \; y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Now, $u = x^{\cot x}$

$\Rightarrow \; \log u = \cot x\cdot\log x$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\cot x\cdot\frac{1}{x}-\operatorname{cosec}^2 x\cdot\log x$$

$$\therefore\quad \frac{du}{dx}=x^{\cot x}\left(\frac{\cot x}{x}-\operatorname{cosec}^2 x\cdot\log x\right)\quad ...(2)$$

Also, $v=\dfrac{2x^2-3}{x^2+x+2}$

$$\Rightarrow\quad \frac{dv}{dx}=\frac{(x^2+x+2)\cdot 4x-(2x^2-3)\cdot(2x+1)}{(x^2+x+2)^2}$$

$$\Rightarrow\quad \frac{dv}{dx}=\frac{(4x^3+4x^2+8x)-(4x^3+2x^2-6x-3)}{(x^2+x+2)^2}$$

$$\Rightarrow\quad \frac{dv}{dx}=\frac{2x^2+14x+3}{(x^2+x+2)^2}\quad ...(3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx}=x^{\cot x}\left(\frac{\cot x}{x}-\operatorname{cosec}^2 x\cdot\log x\right)+\frac{2x^2+14x+3}{(x^2+x+2)^2}$$

64. Let $y=x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$

Let $u=x^{x\cos x}$ and $v=\dfrac{x^2+1}{x^2-1}$

$\therefore\quad y=u+v$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\quad ...(1)$$

Now, $u=x^{x\cos x}$

Taking log on both sides, we get

$\log u = x\cos x\cdot\log x$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=x\cos x\frac{d}{dx}(\log x)+x\log x\frac{d}{dx}(\cos x)+\cos x\log x\frac{d}{dx}(x)$$

$$\Rightarrow\quad \frac{1}{u}\cdot\frac{du}{dx}=x\cos x\cdot\frac{1}{x}+x\log x(-\sin x)+\cos x\log x$$

$$=\cos x-x\sin x\log x+\cos x\log x$$

$$\therefore\quad \frac{du}{dx}=x^{x\cos x}[\cos x-x\sin x\log x+\cos x\log x]\quad ...(2)$$

Also, $v=\dfrac{x^2+1}{x^2-1}$

Differentiating w.r.t. x, we get

$$\frac{dv}{dx}=\frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$=\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$=\frac{2x[x^2-1-x^2-1]}{(x^2-1)^2}=\frac{-4x}{(x^2-1)^2}\quad ...(3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx}=x^{x\cos x}[\cos x-x\sin x\log x+\cos x\log x]-\frac{4x}{(x^2-1)^2}$$

$$=x^{x\cos x}[\cos x(1+\log x)-x\sin x\log x]-\frac{4x}{(x^2-1)^2}$$

65. *Refer to answer 59.*

We have,

$$\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}=\frac{\log x}{(\log e+x)^2}=\frac{\log x}{(\log(xe))^2}$$

66. We have, $y=(\cos x)^x+(\sin x)^{1/x}$

$$\Rightarrow\quad y=e^{x\log\cos x}+e^{\frac{\log\sin x}{x}}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{x\log\cos x}\left[x\times\frac{1}{\cos x}(-\sin x)+\log\cos x\right]+e^{\frac{\log\sin x}{x}}\left[\frac{1}{x}\cdot\frac{1}{\sin x}(\cos x)+\log\sin x\left(\frac{-1}{x^2}\right)\right]$$

$$=(\cos x)^x[\log\cos x-x\tan x]+(\sin x)^{\frac{1}{x}}\left[\frac{\cot x}{x}-\frac{\log\sin x}{x^2}\right]$$

67. We have, $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking log on both sides, we get

$\log y=(\sin x-\cos x)\log(\sin x-\cos x)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=(\sin x-\cos x)\frac{(\cos x+\sin x)}{(\sin x-\cos x)}+\log(\sin x-\cos x)\cdot(\cos x+\sin x)$$

$$\Rightarrow\quad \frac{1}{y}\cdot\frac{dy}{dx}=(\cos x+\sin x)[1+\log(\sin x-\cos x)]$$

$$\therefore\quad \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\times[(\cos x+\sin x)(1+\log(\sin x-\cos x))]$$

68. Let $y = (x)^{\cos x} + (\sin x)^{\tan x}$

$\Rightarrow \quad y = e^{\cos x \log x} + e^{\tan x \log \sin x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{\cos x \log x}\left[\frac{\cos x}{x} - \sin x \log x\right] + e^{\tan x \log \sin x}\left[\frac{\tan x \cos x}{\sin x} + \sec^2 x \log \sin x\right]$$

$$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x \log x\right] + (\sin x)^{\tan x}\left[1 + \sec^2 x \log \sin x\right]$$

69. We have, $y = (\log x)^x + (x)^{\cos x}$

$\Rightarrow \quad y = e^{x \log(\log x)} + e^{\cos x \cdot \log x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{x \log(\log x)}\left[\frac{x}{x \log x} + \log(\log x)\right] + e^{\cos x \cdot \log x}\left[\frac{\cos x}{x} + \log x(-\sin x)\right]$$

$$= (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\cos x}\left[\frac{\cos x}{x} - \sin x \cdot \log x\right]$$

70. *Refer to answer 69.*

71. We have, $y = x^x - (\sin x)^x$

$\Rightarrow \quad y = e^{x \log x} - e^{x \log \sin x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{x \log x}\left[\frac{x}{x} + \log x\right] - e^{x \log \sin x}\left[\frac{x}{\sin x}(\cos x) + \log \sin x\right]$$

$= (x)^x(1 + \log x) - (\sin x)^x (x\cot x + \log \sin x)$

72. We have, $y = (\log x)^{\cos x} + \dfrac{x^2+1}{x^2-1}$

$$\Rightarrow \quad y = e^{\cos x \log(\log x)} + \frac{x^2+1}{x^2-1}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} + \log(\log x)\cdot(-\sin x)\right] + \left[\frac{(x^2-1)2x - (x^2+1)(2x)}{(x^2-1)^2}\right]$$

$$= (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right] - \left[\frac{4x}{(x^2-1)^2}\right]$$

73. Let $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$

$\Rightarrow \quad y = e^{\tan x \cdot \log \sin x} + e^{\sec x \cdot \log \cos x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = (\sin x)^{\tan x}\left\{\sec^2 x \cdot \log \sin x + \tan x \cdot \frac{1}{\sin x}\cos x\right\} + (\cos x)^{\sec x} \times \left\{\sec x \tan x \cdot \log \cos x + \sec x\left(\frac{1}{\cos x}\right)(-\sin x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\tan x}\left\{\sec^2 x \log \sin x + 1\right\} + (\cos x)^{\sec x}\{\sec x \tan x \cdot \log \cos x - \sec x \tan x\}$$

74. $x = a \sin 2t\,(1 + \cos 2t)$, $y = b \cos 2t\,(1 - \cos 2t)$

Now, $\dfrac{dx}{dt} = 2a\cos 2t(1 + \cos 2t) + a \sin 2t(-2\sin 2t)$

$= 2a \cos 2t + 2a[\cos^2 2t - \sin^2 2t]$

$= 2a \cos 2t + 2a \cos 4t$

Also, $\dfrac{dy}{dt} = -2b \sin 2t(1 - \cos 2t) + b\cos 2t(2\sin 2t)$

$= -2b \sin 2t + 4b(\sin 2t \cos 2t)$

$= -2b \sin 2t + 2b \sin 4t$

So, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2b(\sin 4t - \sin 2t)}{2a(\cos 4t + \cos 2t)}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } t=\pi/4} = \frac{b}{a}\left[\frac{\sin \pi - \sin(\pi/2)}{\cos \pi + \cos(\pi/2)}\right] = \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}$$

$$\left(\frac{dy}{dx}\right)_{\text{at } t=\pi/3} = \frac{b}{a}\left[\frac{\sin(4\pi/3) - \sin(2\pi/3)}{\cos(4\pi/3) + \cos(2\pi/3)}\right] = \frac{b}{a}\left[\frac{\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\frac{-1}{2} - \frac{1}{2}}\right] = \frac{\sqrt{3}b}{a}$$

75. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore \quad u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow \quad u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$\Rightarrow\ u=\tan^{-1}\left(\dfrac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\Rightarrow u=\tan^{-1}\left(\tan\dfrac{\theta}{2}\right)$

$\therefore\ u=\dfrac{\theta}{2}\ \Rightarrow\ u=\dfrac{1}{2}\tan^{-1}x$

Differentiating w.r.t. x, we get

$\dfrac{du}{dx}=\dfrac{1}{2(1+x^2)}$

Also, let $v=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)\ \Rightarrow\ v=2\tan^{-1}x$

Differentiating w.r.t. x, we get

$\dfrac{dv}{dx}=\dfrac{2}{1+x^2}$

$\therefore\ \dfrac{du}{dv}=\dfrac{\frac{du}{dx}}{\frac{dv}{dx}}=\dfrac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}}\Rightarrow\dfrac{du}{dv}=\dfrac{1}{4}$

76. We have $x=ae^t(\sin t+\cos t)$

$\Rightarrow\ \dfrac{dx}{dt}=ae^t(\sin t+\cos t)+ae^t(\cos t-\sin t)=2ae^t\cos t$

and $y=ae^t(\sin t-\cos t)$

$\Rightarrow\ \dfrac{dy}{dt}=ae^t(\sin t-\cos t)+ae^t(\cos t+\sin t)=2ae^t\sin t$

$\therefore$ L.H.S. $=\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{2ae^t\sin t}{2ae^t\cos t}=\tan t$

Also, R.H.S. $=\dfrac{x+y}{x-y}$

$=\dfrac{ae^t(\sin t+\cos t)+ae^t(\sin t-\cos t)}{ae^t(\sin t+\cos t)-ae^t(\sin t-\cos t)}$

$=\dfrac{2ae^t\sin t}{2ae^t\cos t}=\tan t=\text{L.H.S.}$

77. Let $u=\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

Put $x=\cos\theta$

$\therefore\ u=\tan^{-1}\left[\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right]=\tan^{-1}\left(\dfrac{\sin\theta}{\cos\theta}\right)$

$=\tan^{-1}(\tan\theta)=\theta$

$\Rightarrow\ \dfrac{du}{d\theta}=1$

Also let,

$v=\cos^{-1}(2x\sqrt{1-x^2})\Rightarrow v=\cos^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$

$=\cos^{-1}(2\cos\theta\sin\theta)=\cos^{-1}(\sin 2\theta)$

$=\cos^{-1}\left(\cos\left(\dfrac{\pi}{2}-2\theta\right)\right)=\dfrac{\pi}{2}-2\theta$

$\Rightarrow\ \dfrac{dv}{d\theta}=-2$

Now $\dfrac{du}{dv}=\dfrac{du/d\theta}{dv/d\theta}=\dfrac{-1}{2}$

78. *Refer to answer 77.*

79. *Refer to answer 76.*

$\therefore\ \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{2ae^\theta\cos\theta}{2ae^\theta\sin\theta}=\cot\theta$

$\Rightarrow\ \left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}=\cot\dfrac{\pi}{4}=1$

80. *Refer to answer 74.*

81. Here, $x=\cos t(3-2\cos^2 t)$, $y=\sin t(3-2\sin^2 t)$

$\Rightarrow\ \dfrac{dx}{dt}=-\sin t(3-2\cos^2 t)+\cos t\,[2\cdot 2\cos t\sin t]$

$=-3\sin t+6\cos^2 t\sin t$

and $\dfrac{dy}{dt}=\cos t(3-2\sin^2 t)+\sin t(-2\cdot 2\sin t\cos t)$

$=-3\cos t+6\sin^2 t\cos t$

$\therefore\ \dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{3\cos t-6\sin^2 t\cos t}{-3\sin t+6\cos^2 t\sin t}$

$=\dfrac{3\cos t\cdot\cos 2t}{3\sin t\cdot\cos 2t}=\cot t$

$\Rightarrow\ \left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{4}}=\cot\dfrac{\pi}{4}=1$

82. Here, $x=2\cos\theta-\cos 2\theta$, $y=2\sin\theta-\sin 2\theta$

$\Rightarrow\ \dfrac{dx}{d\theta}=-2\sin\theta+2\sin 2\theta$ and $\dfrac{dy}{d\theta}=2\cos\theta-2\cos 2\theta$

$\therefore\ \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{2(\cos\theta-\cos 2\theta)}{2(\sin 2\theta-\sin\theta)}$

$=\dfrac{2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}=\tan\left(\dfrac{3\theta}{2}\right).$

83. Here $x=\sqrt{a^{\sin^{-1}t}}$

$\Rightarrow \log x=\frac{1}{2}\sin^{-1}t\cdot\log a$

Differentiating w.r.t. t, we get

$$\frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}} \quad ...(1)$$

Also, $y=\sqrt{a^{\cos^{-1}t}}$

$\Rightarrow \log y=\frac{1}{2}\cos^{-1}t\cdot\log a$

Differentiating w.r.t. t, we get

$$\frac{1}{y}\cdot\frac{dy}{dt}=\frac{1}{2}\log a\cdot\frac{-1}{\sqrt{1-t^2}} \quad ...(2)$$

Now dividing (2) by (1), we get

$$\frac{\frac{1}{y}\cdot\frac{dy}{dt}}{\frac{1}{x}\cdot\frac{dx}{dt}}=-1\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$$

84. Here , $x = a(\theta - \sin\theta) \Rightarrow \frac{dx}{d\theta}=a(1-\cos\theta)$

and $y=a(1+\cos\theta) \Rightarrow \frac{dy}{d\theta}=a(-\sin\theta)$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-\sin\theta}{(1-\cos\theta)}$$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}=\frac{-\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}}=\frac{-\sqrt{3}/2}{1-(1/2)}=-\sqrt{3}$$

85. Here, $x=a\,(\cos t+\log\tan\frac{t}{2})$

$$\Rightarrow \frac{dx}{dt}=a\left(-\sin t+\frac{1}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}\right)$$

$$=a\left(-\sin t+\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\cdot\frac{1}{2\cos^2\frac{t}{2}}\right)=a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$=a\frac{(-\sin^2 t+1)}{\sin t}=\frac{a\cos^2 t}{\sin t}$$

Also, $y = a \sin t$

$$\Rightarrow \frac{dy}{dt}=a\cos t$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=a\cos t\cdot\frac{\sin t}{a\cos^2 t}=\tan t.$$

86. *Refer to answer 85.*

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}=\tan\frac{\pi}{4}=1$$

87. We have, $y = x^x$

$\Rightarrow y=e^{x\log x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{x\log x}\left(x\times\frac{1}{x}+\log x\right)$$

$$\Rightarrow \frac{dy}{dx}=x^x(1+\log x)\Rightarrow\frac{dy}{dx}=y(1+\log x) \quad ...(1)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=(1+\log x)\cdot\frac{dy}{dx}+y\times\frac{1}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{y}\left(\frac{dy}{dx}\right)^2+\frac{y}{x} \quad \text{[From (1)]}$$

$$\Rightarrow \frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$$

88. We have, $y = 2\cos(\log x) + 3\sin(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=-2\sin(\log x)\times\frac{1}{x}+3\cos(\log x)\times\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx}=-2\sin(\log x)+3\cos(\log x) \quad ...(1)$$

Again differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-2\cos(\log x)\times\frac{1}{x}-3\sin(\log x)\times\frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-[2\cos(\log x)+3\sin(\log x)]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y \Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

89. We have, $x = \sin t$ and $y = \sin pt$

$$\frac{dx}{dt}=\cos t \text{ and } \frac{dy}{dt}=p\cos pt$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{p\cos pt}{\cos t}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{-p^2\sin pt\cos t+p\cos pt\sin t}{\cos^2 t}\times\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{-p^2\sin pt\cos t+p\cos pt\sin t}{\cos^3 t}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{p^2 \sin pt \cos t}{\cos^3 t} + \frac{p \cos pt \sin t}{\cos^3 t}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-p^2 y}{\cos^2 t} + \frac{x\frac{dy}{dx}}{\cos^2 t}$

$\Rightarrow \quad \cos^2 t \frac{d^2y}{dx^2} = -p^2 y + x\frac{dy}{dx}$

$\Rightarrow \quad (1-\sin^2 t)\frac{d^2y}{dx^2} = -p^2 y + x\frac{dy}{dx}$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$

90. Given, $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$

$\Rightarrow \quad x^2 = a^2\cos^2\theta + b^2\sin^2\theta + 2ab\cos\theta\sin\theta$

and $y^2 = a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta$

Adding (1) and (2), we get

$x^2 + y^2 = a^2 + b^2$

Differentiating w.r.t x, we get

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow x + y\frac{dy}{dx} = 0$ ...(1)

Again differentiating w.r.t. x, we get

$\Rightarrow \quad 1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

Multiplying by y on both sides, we get

$\Rightarrow \quad y^2\frac{d^2y}{dx^2} + \left(y\cdot\frac{dy}{dx}\right)\frac{dy}{dx} + y = 0$

$\Rightarrow \quad y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ [From (1)]

91. We have, $y = e^{m\sin^{-1}x}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = e^{m\sin^{-1}x}\left(\frac{m}{\sqrt{1-x^2}}\right) = \frac{my}{\sqrt{1-x^2}}$ ...(1)

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = m\left[\frac{\sqrt{1-x^2}\frac{dy}{dx} - \frac{y}{2\sqrt{1-x^2}}\cdot(-2x)}{(1-x^2)}\right]$$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = m\left[my + \frac{xy}{\sqrt{1-x^2}}\right]$ [From (1)]

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = m\left[my + x\cdot\left(\frac{1}{m}\cdot\frac{dy}{dx}\right)\right]$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = m^2y + x\frac{dy}{dx}$

$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

92. We have, $y = (x + \sqrt{1+x^2})^n$

Differentiating w.r.t. x, we get

$\Rightarrow \quad \frac{dy}{dx} = n(x+\sqrt{1+x^2})^{n-1}\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$

$\Rightarrow \quad \frac{dy}{dx} = n(x+\sqrt{1+x^2})^{n-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}} = \frac{ny}{\sqrt{1+x^2}}$ ...(1)

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = n\left[\frac{\sqrt{1+x^2}\cdot\frac{dy}{dx} - \frac{2(xy)}{2\sqrt{1+x^2}}}{1+x^2}\right]$$

$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} = n\left[\sqrt{1+x^2}\times\frac{ny}{\sqrt{1+x^2}} - \frac{xy}{\sqrt{1+x^2}}\right]$

$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} = n^2y - \frac{nxy}{\sqrt{1+x^2}}$

$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} = n^2y - x\frac{dy}{dx}$ [From (1)]

$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$

93. Here $x = a\sec^3\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = a\cdot 3\sec^2\theta\cdot\sec\theta\tan\theta = 3a\sec^3\theta\tan\theta$

and $y = a\tan^3\theta$

$\Rightarrow \quad \frac{dy}{d\theta} = a\cdot 3\tan^2\theta\cdot\sec^2\theta$

$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$

On differentiating w.r.t. x, we get

$\frac{d^2y}{dx^2} = \cos\theta\cdot\frac{d\theta}{dx} = \frac{\cos\theta}{3a\sec^3\theta\tan\theta} = \frac{1}{3a}\cos^4\theta\cdot\cot\theta$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}} = \frac{1}{3a}\cos^4\frac{\pi}{4}\cdot\cot\frac{\pi}{4} = \frac{1}{3a}\cdot\left(\frac{1}{\sqrt{2}}\right)^4\cdot 1$$

$$= \frac{1}{3a}\cdot\frac{1}{4} = \frac{1}{12a}$$

94. Given $y = Ae^{mx} + Be^{nx}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{mx}\cdot m + Be^{nx}\cdot n$$

$$\Rightarrow \frac{d^2y}{dx^2} = m^2Ae^{mx} + n^2Be^{nx}$$

Now, $\text{L.H.S.} = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

$= m^2Ae^{mx} + n^2Be^{nx} - (m+n)(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$

$= Ae^{mx}[m^2 - (m+n)m + mn] + Be^{nx}[n^2 - (m+n)n + mn]$

$= Ae^{mx} \times 0 + Be^{nx} \times 0 = 0 = \text{R.H.S.}$

95. Here, $x = a(\cos t + t\sin t)$

$$\Rightarrow \frac{dx}{dt} = a[-\sin t + 1\cdot\sin t + t\cos t] = at\cos t \quad ...(1)$$

and $y = a(\sin t - t\cos t)$

$$\Rightarrow \frac{dy}{dt} = a[\cos t - (1\cdot\cos t - t\sin t)] = a\,t\sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\,t\sin t}{a\,t\cos t} = \tan t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \frac{\sec^2 t}{a\,t\cos t} \quad \text{[Using (1)]}$$

$$= \frac{1}{a}\cdot\frac{1}{t\cos^3 t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = \frac{1}{a}\cdot\frac{1}{\frac{\pi}{4}\cos^3\frac{\pi}{4}} = \frac{4}{\pi a}\cdot(\sqrt{2})^3 = \frac{8\sqrt{2}}{\pi a}$$

96. *Refer to answer 85.*

We get $\frac{dy}{dx} = \tan t$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t\cdot\frac{dt}{dx} = \frac{\sec^2 t}{a\cos^2 t/\sin t} = \frac{\sin t}{a\cos^4 t}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{a\cos^4\frac{\pi}{3}} = \frac{\sqrt{3}/2}{a(1/2)^4} = \frac{8\sqrt{3}}{a}$$

97. *Refer to answer 96.*

98. Given that $y = \log(x + \sqrt{x^2 + a^2})$ ...(1)

Differentiating (1) w.r.t. 'x' on both sides, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}}\cdot\frac{d}{dx}(x + \sqrt{x^2 + a^2})$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}}\left(1 + \frac{1}{2\sqrt{x^2 + a^2}}\cdot 2x\right)$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}}\cdot\frac{(\sqrt{x^2 + a^2} + x)}{\sqrt{x^2 + a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}} \Rightarrow \sqrt{x^2 + a^2}\,\frac{dy}{dx} = 1 \quad ...(2)$$

Again differentiating (2) on both sides w.r.t. x, we get

$$\sqrt{x^2 + a^2}\,\frac{d^2y}{dx^2} + \frac{2x}{2\sqrt{x^2 + a^2}}\,\frac{dy}{dx} = 0$$

$$\Rightarrow (x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

99. Here $x = a\cos^3\theta$, $y = a\sin^3\theta$

$$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta\cdot(-\sin\theta) \text{ and } \frac{dy}{d\theta} = 3a\sin^2\theta\cdot\cos\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\sec^2\theta\frac{d\theta}{dx} = -\frac{\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{1}{3a}\cdot\frac{1}{\cos^4\theta\cdot\sin\theta}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{6}} = \frac{1}{3a}\cdot\frac{1}{\cos^4\frac{\pi}{6}.\sin\frac{\pi}{6}}$$

$$= \frac{1}{3a}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^4\cdot\frac{1}{2}} = \frac{32}{27a}$$

100. Here, $y = x\log\left(\frac{x}{a+bx}\right)$ ... (1)

$\Rightarrow y = x[\log x - \log(a+bx)] = x\log x - x\log(a+bx)$

$$\Rightarrow \frac{dy}{dx} = x\cdot\frac{1}{x} + 1\cdot\log x - \left[1\cdot\log(a+bx) + x\cdot\frac{1}{a+bx}\cdot b\right]$$

$$= 1 - \frac{bx}{a+bx} + \log x - \log(a+bx)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right) \quad ...(2)$$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{a}{a+bx} + \dfrac{y}{x}$ [Using (1)]

Again differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = a\cdot(-1)(a+bx)^{-2}\cdot b + \frac{x\dfrac{dy}{dx} - y}{x^2}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a}{x(a+bx)} = \frac{-abx + a(a+bx)}{x(a+bx)^2} = \frac{a^2}{x(a+bx)^2}$$

$$\text{Now, R.H.S.} = \left(x\frac{dy}{dx} - y\right)^2$$

$$= \left\{x\cdot\left[\frac{a}{a+bx} + \frac{y}{x}\right] - y\right\}^2 = \left(\frac{ax}{a+bx}\right)^2$$

$$\text{and L.H.S.} = x^3\frac{d^2y}{dx^2}$$

$$= \frac{a^2x^2}{(a+bx)^2} = \left[\frac{ax}{a+bx}\right]^2 = \text{R.H.S.}$$

101. We have, $x = \tan\left(\dfrac{1}{a}\log y\right)$

$\Rightarrow \quad \dfrac{1}{a}\log y = \tan^{-1} x$

Differentiating w.r.t. x, we get

$$\frac{1}{a}\cdot\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{1+x^2} \Rightarrow (1+x^2)\frac{dy}{dx} = ay$$

Again differentiating w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\cdot\frac{dy}{dx} = a\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$$

102. Given $x = \cos\theta$ and $y = \sin^3\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = -\sin\theta \text{ and } \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\sin^2\theta\cos\theta}{-\sin\theta} = -3\sin\theta\cos\theta$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = [-3\cos^2\theta - 3\sin\theta(-\sin\theta)]\frac{d\theta}{dx}$$

$$= (-3\cos^2\theta + 3\sin^2\theta)\cdot\frac{-1}{\sin\theta}$$

$$\text{Now, L.H.S.} = y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$$

$$= \sin^3\theta\left(\frac{3\cos^2\theta - 3\sin^2\theta}{\sin\theta}\right) + (-3\sin\theta\cos\theta)^2$$

$= \sin^2\theta(3\cos^2\theta - 3\sin^2\theta) + 9\sin^2\theta\cos^2\theta$

$= 3\sin^2\theta(\cos^2\theta - \sin^2\theta + 3\cos^2\theta)$

$= 3\sin^2\theta(4\cos^2\theta - \sin^2\theta)$

$= 3\sin^2\theta(4\cos^2\theta - 1 + \cos^2\theta)$

$= 3\sin^2\theta(5\cos^2\theta - 1) =$ R.H.S.

103. Given $y = \sin^{-1} x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\sqrt{1-x^2}\cdot 0 - 1\cdot\dfrac{1(-2x)}{2\sqrt{1-x^2}}}{1-x^2}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\cdot\frac{1}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\cdot\frac{dy}{dx} = 0$$

104. $y = (\tan^{-1}x)^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\cdot\tan^{-1}x\cdot\frac{d}{dx}(\tan^{-1}x) = 2\tan^{-1}x\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} = 2\tan^{-1}x$$

Again differentiating. w.r.t. x, we get

$$(x^2+1)\frac{d^2y}{dx^2} + 2x\cdot\frac{dy}{dx} = 2\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \quad (x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$$

105. *Refer to answer 88.*

106. *Refer to answer 95.*

Since, we have $\dfrac{dx}{dt} = at\cos t$

$$\therefore \quad \frac{d^2x}{dt^2} = a[t(-\sin t) + \cos t] = -at\sin t + a\cos t$$

$$\text{and } \frac{dy}{dt} = at\sin t \Rightarrow \frac{d^2y}{dt^2} = a[t\cos t + \sin t]$$

107. *Refer to answer 96.*

$\because \quad y = a \sin t$

$\therefore \quad \frac{dy}{dt} = a\cos t$

Again differentiating w.r.t. t, we get

$\frac{d^2y}{dt^2} = -a \sin t$

108. *Refer to answer 96.*

109. *Refer to answer 84.*

We have, $\frac{dy}{dx} = \frac{-\sin\theta}{(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$\Rightarrow \frac{dy}{dx} = -\cot\frac{\theta}{2} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\text{cosec}^2\frac{\theta}{2}\times\frac{d\theta}{dx}$

$= \frac{1}{2}\times\text{cosec}^2\frac{\theta}{2}\times\frac{1}{2a\sin^2\frac{\theta}{2}} = \frac{\text{cosec}^4\frac{\theta}{2}}{4a}$

110. *Refer to answer 109.*

111. We have, $y = \text{cosec}^{-1} x$

$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}} \Rightarrow x\sqrt{x^2-1}\,\frac{dy}{dx} = -1$

Differentiating w.r.t. x, we get

$x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \left[x\times\frac{(2x)}{2\sqrt{x^2-1}} + \sqrt{x^2-1}\right]\frac{dy}{dx} = 0$

$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2} + \left(\frac{x^2+x^2-1}{\sqrt{x^2-1}}\right)\frac{dy}{dx} = 0$

$\Rightarrow x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$

112. We have, $y = (\cot^{-1}x)^2$

$\Rightarrow \frac{dy}{dx} = 2(\cot^{-1}x)\times\frac{-1}{1+x^2}$

$\Rightarrow (x^2+1)\frac{dy}{dx} = -2\cot^{-1}x$

$\Rightarrow (x^2+1)^2\left(\frac{dy}{dx}\right)^2 = 4(\cot^{-1}x)^2$

$\Rightarrow (x^2+1)^2\left(\frac{dy}{dx}\right)^2 = 4y$

Differentiating w.r.t. x, we get

$(x^2+1)^2\left(2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\right) + 2(x^2+1)(2x)\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$

$\therefore \quad (x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$

113. *Refer to answer 94.*

114. We have, $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\cdot y = \sin^{-1}x$

Differentiating w.r.t. x, we get

$\sqrt{1-x^2}\,\frac{dy}{dx} - \frac{x}{\sqrt{1-x^2}}\,y = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\frac{dy}{dx} - xy = 1$

Again differentiating w.r.t. x, we get

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$

115. $y = e^x(\sin x + \cos x)$

$\Rightarrow \frac{dy}{dx} = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x\cos x$

Again differentiating w.r.t. x, we get

$\frac{d^2y}{dx^2} = 2e^x\cos x - 2e^x\sin x$

$\Rightarrow \frac{d^2y}{dx^2} = 2e^x(\cos x - \sin x)$

$\text{L.H.S.} = \frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y$

$= 2e^x(\cos x - \sin x) - 4e^x\cos x + 2[e^x(\sin x + \cos x)]$

$= 2e^x[\cos x - \sin x - 2\cos x + \sin x + \cos x]$

$= 2e^x \times 0 = 0 = \text{R.H.S.}$

116. We have, $y = e^x \sin x$

$\Rightarrow \frac{dy}{dx} = e^x\cos x + e^x\sin x$

$\Rightarrow \frac{dy}{dx} = e^x\cos x + y \qquad ...(1)$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x \cos x - e^x \sin x + \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx} - y\right) - y + \frac{dy}{dx} \quad \text{[From (1)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y = 0$$

117. We have, $y = \sin(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\Rightarrow x\frac{dy}{dx} = \cos(\log x)$$

Again differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-\sin(\log x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

118. We have, $y = x + \tan x$

$$\Rightarrow \frac{dy}{dx} = 1 + \sec^2 x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\sec x \cdot \sec x \tan x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\tan x}{\cos^2 x} \Rightarrow \cos^2 x \cdot \frac{d^2y}{dx^2} = 2(y - x)$$

$$\Rightarrow \cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$$

119. We have, $f(x) = x^2 - 4x + 3$

(i) $f(x)$ being a polynomial function is continuous in $[1,3]$

(ii) $f(x)$ being a polynomial function is differentiable in $(1, 3)$

(iii) $f(3) = 3^2 - 4(3) + 3 = 0$
and $f(1) = 1^2 - 4(1) + 3 = 0$. Thus $f(1) = f(3)$

Thus, all the conditions of Rolle's theorem are satisfied, so there exists atleast one point $c \in (1, 3)$ such that $f'(c) = 0$

$f'(x) = 2x - 4 \Rightarrow f'(c) = 2c - 4$

$\therefore \quad f'(c) = 2c - 4 = 0 \Rightarrow c = 2 \in (1, 3)$

Hence, the Rolle's theorem is verified.