Trigonometry

* Trigonometric Ratios: sine, cosine, tangent, cotangent, secant, cosecant.

* Results:

1. $\cos n\pi = (-1)^n$, $\sin n\pi = 0$, $n \in I$.
2. $\cos \frac{n\pi}{2} = 0$, $\sin \frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, n is odd Integer.
3. $\cos(n\pi + \theta) = (-1)^n \cos\theta$, $n \in I$.
   $\sin(n\pi + \theta) = (-1)^n \sin\theta$, $n \in I$
4. $\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$, n is odd integer
   $\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$, n is odd integer.

* $f(\theta) = a\cos\theta + b\sin\theta$.

$f_{max} = \sqrt{a^2+b^2}$ $\quad f_{min} = -\sqrt{a^2+b^2}$.

* $\sin 18° = \frac{\sqrt{5}-1}{4}$, $\sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4}$

$\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$, $\cos 36° = \frac{\sqrt{5}+1}{4}$

$\tan 15° = 2-\sqrt{3}$, $\tan 22.5° = \sqrt{2}-1$, $\tan 67.5° = \sqrt{2}+1$,

$\tan 75° = 2+\sqrt{3}$.

* For any three angles A, B, C

1. $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B + \sin A\sin B\sin C$.
2. $\cos(A+B+C) = \cos A\cos B\cos C + (-\cos A\sin B\sin C) - \cos B\sin C\sin A - \cos C\sin A\sin B$.
3. $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C - \tan A\tan B\tan C}{1-\tan A\tan B - \tan B\tan C - \tan C\tan A}$
4. $\cot(A+B+C) = -\dfrac{\cot A\cot B\cot C - (\cot A+\cot B+\cot C)}{\cot A\cot B + \cot B\cot C + \cot C\cot A - 1}$

* Identities: If A, B, C are angles of a triangle,

1. $\sin(B+C) = \sin A$, $\cos B = -\cos(C+A)$

2. $\cos(A+B) = -\cos C$, $\sin C = \sin(A+B)$.

3. $\tan(C+A) = -\tan B$, $\cot A = -\cot(B+C)$.

4. $\cos\frac{A+B}{2} = \sin\frac{C}{2}$, $\cos\frac{C}{2} = \sin\frac{A+B}{2}$

5. $\sin\frac{C+A}{2} = \cos\frac{B}{2}$, $\sin\frac{A}{2} = \cos\frac{B+C}{2}$

6. $\tan\frac{B+C}{2} = \cot\frac{A}{2}$, $\tan\frac{B}{2} = \cot\frac{C+A}{2}$

7. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

8. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$.

9. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

10. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

11. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

12. $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$.

13. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$.

14. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

15. $\sin 2mA + \sin 2mB + \sin 2mC = (-1)^{m+1} \cdot 4\sin mA \sin mB \cdot \sin mC$

16. $\cos mA + \cos mB + \cos mC = 1 \pm 4\sin\frac{mA}{2}\sin\frac{mB}{2}\sin\frac{mC}{2}$.

according as m is of the form $4n+1$ or $4n+3$.

17. $\cos A + \cos B + \cos C + \cos(A+B+C) =$

$$4\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{C+A}{2}\right).$$

18. $\sin A + \sin B + \sin C - \sin(A+B+C) =$

$$4\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{C+A}{2}\right).$$

Trigonometry.

* Trig. Series:

- $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin\{\alpha+(n-1)\beta\} = \dfrac{\sin\left\{\dfrac{2\alpha+(n-1)\beta}{2}\right\}\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}.$

- $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots + \cos\{\alpha+(n-1)\beta\}. = \dfrac{\cos\left\{\dfrac{2\alpha+(n-1)\beta}{2}\right\}\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}.$

* Trig. Formulae:

1. $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
2. $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B.$
3. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
4. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B.$
5. $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}.$
6. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}.$
7. $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

   $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}.$
8. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}.$
9. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
10. $\cot(A \pm B) = \dfrac{\cot A\cot B \mp 1}{\cot A \pm \cot B},$

11. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$,

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

12. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}.$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}.$$

$$\tan A + \tan B = \frac{\sin(A+B)}{\cos A\cos B}$$

$$\tan A - \tan B = \frac{\sin(A-B)}{\cos A\cos B}.$$

13. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B).$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B).$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B).$$

* Trig. Equ^{n}s :

1. If $\sin\theta = \sin\alpha$ or $\csc\theta = \csc\alpha$ then

$$\theta = n\pi + (-1)^n\alpha$$

$\cos\theta = \cos\alpha$ or $\sec\theta = \sec\alpha$ then

$$\theta = 2n\pi \pm \alpha$$

$\cot\theta = \cot\alpha$ or $\tan\theta = \tan\alpha$ then

$$\theta = n\pi + \alpha$$

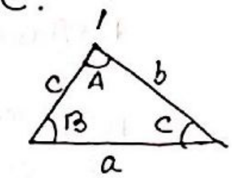
2. If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$ then $\theta = n\pi \pm \alpha$

3. Equ^n of the type $a\cos\theta + b\sin\theta = c$ then put $a = r\cos\alpha$, $b = r\sin\alpha \Rightarrow r = \sqrt{a^2+b^2}$.
& $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

$$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = c.$$
$$\Rightarrow r\cos(\theta-\alpha) = c.$$
$$\Rightarrow \theta - \alpha = 2n\pi \pm \cos^{-1}\left(\frac{c}{r}\right).$$
$$\Rightarrow \theta = \alpha + 2n\pi \pm \cos^{-1}\left(\frac{c}{r}\right).$$

• Solutions of Triangle | Properties of Triangle:

(*) In any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
(sine formula)



$$\cos A = \frac{b^2+c^2-a^2}{2bc} \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}.$$
$$\cos B = \frac{c^2+a^2-b^2}{2ca}$$
(Cosine formula).

$a = b\cos C + c\cos B$ $\qquad c = a\cos B + b\cos A$.
$b = c\cos A + a\cos C$. (Projection formula).

(*) Napier's analogy:
$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$
$$\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\left(\frac{B}{2}\right)$$
$$\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right).$$

* Mollweide's Rule: In ΔABC, $\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$, $\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}$

* $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ $\quad$ $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$.

$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ $\quad$ $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$.

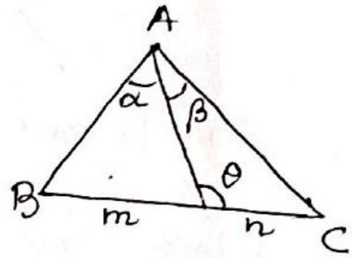
$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ & so on

* m-n Rule: In any triangle,

$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$.

$= n\cot B - m\cot C$.



* Area of $\Delta ABC = \frac{1}{2} ab\sin C = \frac{1}{2} bc\sin A = \frac{1}{2} ca\sin B$.

* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$. $\left[R = \frac{abc}{4\Delta}\right.$, radius of circumcircle$\left.\right]$.

* Radius of Incircle 'r':

1. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$.
2. $r = \frac{a\sin(B/2)\sin(C/2)}{\cos(A/2)}$ & so on
3. $r = \frac{\Delta}{s} = 4R\sin(A/2)\sin(B/2)\sin(C/2)$.

* Radius of ex-circles r_1, r_2, r_3:

1. $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$.
2. $r_1 = s\tan\left(\frac{A}{2}\right)$, $r_2 = s\tan\left(\frac{B}{2}\right)$, $r_3 = s\tan\left(\frac{C}{2}\right)$.
3. $r_1 = 4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$ & so on
4. $r_1 = a\frac{\cos(B/2)\cos(C/2)}{\cos(A/2)}$ & so on.

* $\Delta = 2R^2 \sin A \sin B \sin C = \frac{abc}{4R}$.

* Length of angle bisector & medians:

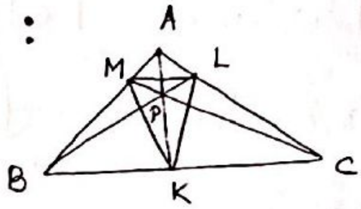
If m_a & β_a are the lengths of a median & an angle bisector from the angle A, then

$$m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2} \qquad \beta_a = \frac{2bc\cos(A/2)}{b+c}.$$

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2+b^2+c^2\right).$$

* Orthocentre & Pedal Triangle:

$\Delta KLM \rightarrow$ Pedal Triangle.



1. $PA = 2R\cos A$, $PB = 2R\cos B$, $PC = 2R\cos C$.

2. $PK = 2R\cos B\cos C$, $PL = 2R\cos C\cos A$, $PM = 2R\cos A\cos B$

3. Sides of ΔKLM − $a\cos A\ (= R\sin 2A)$, $b\cos B$ $(= R\sin 2B)$, $c\cos C\ (= R\sin 2C)$.

Angles of ΔKLM − $\pi - 2A$, $\pi - 2B$, $\pi - 2C$.

4. Circumradii of ΔPBC, ΔPCA, ΔPAB, ΔABC are equal.

* Excentral Triangle: Triangle joining three excentres.

1. Incentre of ΔABC is the orthocentre of the excentric triangle.

2. ΔABC is pedal triangle of $\Delta I_1 I_2 I_3$.

3. Sides of excentral triangle − $4R\cos(A/2)$, $4R\cos(B/2)$, $4R\cos(C/2)$.

Angles - $\left(\frac{\pi}{2}-\frac{A}{2}\right), \left(\frac{\pi}{2}-\frac{B}{2}\right), \left(\frac{\pi}{2}-\frac{C}{2}\right)$.

1. Distances between incentre of ΔABC & the angular points of excentric triangles -

$4R\sin\left(\frac{A}{2}\right)$, $4R\sin\left(\frac{B}{2}\right)$, $4R\sin\left(\frac{C}{2}\right)$.

* Distance between circumcentre & orthocentre $= R\sqrt{1-8\cos A\cos B\cos C}$.

* Distance between circumcentre & incentre $= \sqrt{R^2-2Rr}$.

* Distance between incentre & orthocentre =

$$\sqrt{2r^2-4R^2\cos A\cos B\cos C}.$$

* Perimeter (P) & Area (A) of regular polygon of n sides inscribed in a circle of radius r are given by

$$P = 2nr\sin\left(\frac{\pi}{n}\right)$$

$$A = \frac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right).$$

* Perimeter (P) & Area (A) of a regular polygon of n sides circumscribed about a given circle of radius r.

$$P = 2nr\tan\left(\frac{\pi}{n}\right); \quad A = nr^2\tan\left(\frac{\pi}{n}\right).$$

M

Solution of Triangles.

1. If a, b, c are given, then $\cos A = \frac{b^2+c^2-a^2}{2bc}$ (Similarly B & C.).

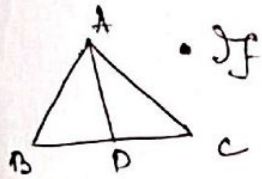
2. If b, c and A given, then

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2} \quad \& \quad \frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}.$$

3. If b, c & B given, then $\sin C = \frac{c}{b}\sin B$.

$$A = \pi - (B+C). \qquad a = \frac{b\sin A}{\sin B} \text{ [ambiguous case].}$$

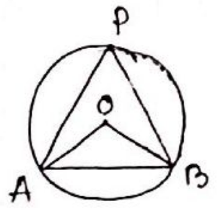
- Heights & Distance:
- Angle of elevation & depression



- If AD is median,

$$AB^2 + AC^2 = 2(AD^2 + BD^2).$$

- If AD is angle bisector, $\frac{BD}{DC} = \frac{AB}{AC}$.



$$\angle AOB = 2\angle APB.$$