6. Conics

- A parabola is defined as the locus of a point P equidistant from a fixed point (called focus) and a fixed line (called directrix).
- The standard equation of the horizontal parabola is y2=4ax.
- The constant ratio is called the eccentricity and is denoted by *e*. When the eccentricity is unity; $e = 1$, the conic is called a Parabola.
- The line which passes through the focus and perpendicular to the directrix is called axis of the parabola.
- The vertex of a parabola is defined as the intersection point of the parabola and its axis.
- The chord passing through the focus and perpendicular to the axis is called latus rectum.
- Any chord which is perpendicular to the axis of the parabola is called double ordinate.
- The straight line passing through the vertex and perpendicular to the axis of the parabola is called tangent at vertex.
- The end points of the latus rectum are L1a, 2a and L2a,-2a.
- Equation of the Parabola in Non-standard Form

	$(y-k)^2 = 4a\,(x-h)$	$(x-h)^2 = 4b\,(y-k)$
Vertex	(h, k)	(h, k)
Focus	$(a+h, k)$	$(h, b+k)$
Equation of the Directrix	$(x-h)+a=0$	$(y-k)+b=0$
Equation of the axis	$y=k$	$x=h$
Tangent at the vertex	$x=h$	$y=k$
Equation of latus rectum	$x-a=h$	$y-k=b$
Length of latus rectum	$\lvert 4a \rvert$	$\lvert 4b \rvert$
End points of latus rectum	$L_1\,(a+h, 2a+k)$ and $L_2\,(a+h, -2a+k)$	$L_1(-2b+h, b+k)$ and $L_2\,(2b+h, b+k)$

- The parametric equation of the standard parabola y2=4ax is x=at2, y=2at.
- The general equation of second degree ax2+2hxy+by2+2gx+2fy+c=0 represents a parabola, if abc+2fgh-af2-bg2-ch2≠0 and h2=ab.
- Focal distance of a point P on parabola is defined as the distance between the point P and its focus S.
- For any point P1x1, y1 outside the parabola, we have y12-4ax1>0
- For any point P2x2, y2 inside the parabola, we have y22-4ax2<0

- An ellipse is the locus of a point which moves such that the ratio of its distance from a fixed point and a fixed line is a constant ratio that is less than one. The fixed point is called the focus; the fixed line is called directrix and the constant ratio is called the eccentricity of the ellipse.
- Standard equation of an ellipse is x2a2+y2b2=1, where b2=a21-e2 and e<1 is the eccentricity of the ellipse.
- Equation of an ellipse whose centre is (h, k) and axes are parallel to the x-axis and y-axis is x-h2a2+y-k2b2=1 .
- General equation of an ellipse whose focus is (h, k); the equation of the directrix is $ax + by + c = 0$ and the eccentricity, e, is x-h2+y-k2=e2×ax+by+c2a2+b2.
- The general equation of second degree ax2+2hxy+by2+2gx+2fy+c=0 represents an ellipse, if abc+2fgh-af2-bg2-ch2≠0 and h2<ab.
- If the equation of the ellipse is x2a2+y2b2=1, then
  - Point x1, y1 lies inside the ellipse, if x12a2+y12b2-1<0.
  - Point x1, y1 lies outside the ellipse, if x12a2+y12b2-1>0.
- The parametric equation of the ellipse x2a2+y2b2=1 is x=acosθ, y=bsinθ, 0≤θ<2π. Thus, the coordinates of any point on the ellipse can be taken as acosθ, bsinθ.
- The circle described on the major axis of an ellipse as a diameter is called the auxillary circle of the ellipse.
- A hyperbola is the locus of a point which moves such that the ratio of its distance from a fixed point called the focus and a fixed line called the directrix is a constant which is greater than unity. This constant ratio is called the eccentricity of the hyperbola.
- The standard equation of a hyperbola is x2a2-y2b2=1, where b2=a2e2-1 and e>1 is the eccentricity of the hyperbola.
- General equation of a hyperbola whose focus is (h, k) and the equation of the corresponding directrix is $ax + by + c = 0$, is x-h2+y-k2=e2×ax+by+c2a2+b2, where e is the eccentricity of the hyperbola.
- If abc+2fgh-af2-bg2-ch2≠0 and h2>ab, then the general equation of second degree **ax2+2hxy+by2+2gx+2fy+c=0** represents a hyperbola.
- The parametric equations of the hyperbola x2a2-y2b2=1 are x=asecθ, y=btanθ, where 0≤θ≤2π .
- The coordinates of any point on the hyperbola may be taken as asecθ, btanθ. The angle θ is called the eccentric angle of the point on the hyperbola.
- The point x1, y1 lies outside or inside the hyperbola if x12a2-y12b2-1<0 or x12a2-y12b2-1>0.

Equation of Tangent in Different Forms
Let us consider the parabola **y2=4ax**.

- Equation of tangent to the parabola at the point x1, y1 is $yy_1 = 2a(x + x_1)$.
- Equation of tangent to the parabola at the point at2, 2at is ty=x + at2.

  The point of intersection of the tangents at the points at12, 2at1and at22, 2at2 is at1t2, at1+t2.

- Equation of tangent in terms of the slope and condition of tangency:

  The line $y = mx + c$ is tangent to the parabola **y2=4ax** if **c=am**.
  Hence, **y=mx+am** is a tangent to the parabola **y2=4ax**. The point of contact of the tangent is a m2 , 2am.

Some Important Propositions on Parabola

- A tangent at any point P on the parabola bisects the angle between the focal chord through P and forms the perpendicular P on the directrix.
- The portion of a tangent to a parabola cut off between the directrix and the parabola subtends a right angle at the focus.
- Tangents at the extremities of any focal chord intersect at right angles on the directrix.
- Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.
- The area of triangle formed by three points on a parabola is twice the area of the triangle formed by corresponding tangents.

Equation of Normals in Different Forms

Let us consider the parabola **y2=4ax**.

- Equation of normal to the parabola at point x1, y1 is y-y1=-y12ax-x1 .
- Equation of normal to the parabola at point at2, 2at is y=-tx+2at+at3.

  Point of intersection of the normals to the parabola **y2=4ax** at points at12, 2at1 and at22, 2at2 is 2a+at12+t22+t1t2, -at1t2t1+t2.

  If a normal at point at12, 2at1 meets the parabola again at at22, 2at2, then t2=-t1-2t1.

- Equation of normal to the parabola at point am2,-2am is y=mx-2am-am3.

Equation of tangent in different forms

Let the equation of an ellipse be x2a2+y2b2=1.

- Equation of a tangent to the ellipse at the point x1, y1 is xx1a2+yy1b2=1.
- Equation of a tangent to the ellipse at the point θ, i.e. acosθ, bsinθ is xacosθ+ybsinθ=1.
- Coordinates of the point of intersection of the tangents to the ellipse at the points acosθ, bsinθ and acosφ, bsinφare acosθ+φ2cosθ-φ2, bsinθ+φ2cosθ-φ2.
- If c2=a2m2+b2, then the line $y = mx + c$ is tangent to the ellipse x2a2+y2b2=1. Putting c=±a2m2+b2 in y=mx+c, we get the equation of the tangent to the ellipse in slope form as **y=mx±a2m2+b2**.
- The coordinates of the point of contact are -a2ma2m2+b2, b2a2m2+b2 or a2ma2m2+b2, -b2a2m2+b2.
- The equation of the director circle of the ellipse x2a2+y2b2=1 is x2+y2=a2+b2.

Equation of the normal in different forms

Let the equation of an ellipse be $x2a2+y2b2=1$.

- Equation of the normal to the ellipse at point $x1, y1$ is $a2 xx1 -b2 yy1 =a2-b2$.
- Equation of the normal to the ellipse at point $acos\theta, bsin\theta$ is $axsec \theta-bycosec\theta=a2-b2$.
- Equation of the normal to the ellipse in terms of the slope *m* is $y=mx-a2-b2ma2+b2m2$.

Equation of the tangent in different forms

Let the equation of the hyperbola be $x2a2-y2b2=1$.

- Equation of the tangent to the hyperbola at the point $Px1, y1$ is $xx1a2-yy1b2=1$.
- Equation of the tangent to the hyperbola at the point $asec\theta, btan\theta$ is $xasec\theta-ybtan\theta=1$.
- If $c2=a2m2-b2$, then the straight line $y=mx+c$ is a tangent to the hyperbola $x2a2-y2b2=1$. Thus,the equation of the tangent to the hyperbola in slope form is $\mathbf{y=mx\pm a2m2-b2}$.
- The coordinates of the point of contact are $-a2ma2m2-b2, -b2a2m2-b2$ or $a2ma2m2-b2, b2a2m2-b2$.
- The equation of the director circle of the hyperbola $x2a2-y2b2=1$ is $x2+y2=a2-b2$.

Equation of the normal in different forms

Let the equation of the hyperbola be $x2a2-y2b2=1$.

- Equation of the normal to the hyperbola $x2a2-y2b2=1$ at the point $x1, y1$ is $a2 xx1+b2 yy1=a2+b2$.
- Equation of the normal to the hyperbola $x2a2-y2b2=1$ at the point $asec\theta, btan\theta$ is $ax \cos\theta+by \cot\theta=a^2+b^2$.
- Equation of the normal to the hyperbola $x2a2-y2b2=1$, having slope *m*, is $y=mx\pm ma2+b2a2-b2m2$.
- Coordinates of the points of contact are $\pm a2a2-b2m2, \mp mb2a2-b2m2$.

In general, four normals can be drawn from a point in the plane of the hyperbola. These four points are called the **co-normal points.**

Number of Tangents :

1) The equation of a tangent to a parabola y2 = 4ax with slope *m* is y = mx + am.

If this tangent passes through $P(x_1, y_1)$, then

y1 = mx1 + am⇒ m2x1 - my1 + a = 0 ...1

This is a quadratic equation in *m*. Therefore, at most, two tangents can be drawn to a parabola from a given point in its plane.

2). The equation of a tangent to an ellipse x2a2 + y2b2= 1 with slope *m* is y = mx ± a2m2 + b2.

If this tangent passes through P($x_{1,}\, y_1$), then

y1 = mx1 ± a2m2 + b2⇒ x12 - a2m2 - 2x1y1m + y12 - b2 = 0

This is a quadratic equation in *m*. Therefore, at most, two tangents can be drawn to the ellipse from a given point in its plane.

3). The equation of a tangent to a hyperbola x2a2 - y2b2= 1 with slope *m* is y = mx ± a2m2 - b2.

If this tangent passes through P($x_{1,}\, y_1$), then

y1 = mx1 ± a2m2 - b2⇒ x12 - a2m2 - 2x1y1m + y12 + b2 = 0

This is a quadratic equation in *m*. Therefore, at most, two tangents can be drawn to the hyperbola from a given point in its plane.**Locus of the point of intersection of perpendicular tangents**

1) The equation of the locus of the point of intersection of mutually perpendicular tangents to the parabola y2 = 4ax is x = -a. It is also the directrix of the parabola.

2) The equation of the locus of the point of intersection of mutually perpendicular tangents to the ellipse x2a2 + y2b2= 1 is x12 + y12 = a2 + b2. It is the circle with the centre at the origin and radius a2 + b2 and is the director circle of the given ellipse.

3) The equation of the locus of the point of intersection of mutually perpendicular tangents to the hyperbola x2a2 - y2b2= 1 is x12 + y12 = a2- b2. It is the circle with the centre at the origin and radius a2 - b2 and is the director circle of the given hyperbola.

Note:

Since the radius of the director circle is a2 - b2, the circle is real whenever b2 < a2.

If b2 = a2, the radius becomes zero and it reduces to a point circle at the origin. In this case, the centre of the hyperbola is the only point from which perpendicular tangents can be drawn to the curve.

If b2 > a2, the radius of the director circle is imaginary and no perpendicular tangents can be drawn to the hyperbola.