# CHAPTER 31

# MATRICES AND DETERMINANTS

## 31.1 MATRIX

A rectangular array of (m × n) objects arranged along m-horizontal lines (called rows) and along n-vertical lines (called columns) as shown below:

$$\begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{ln} \\ \vdots & \vdots & \vdots & \vdots \\ am_1 & am_2 & am_3 & a_{mn} \end{bmatrix}$$

Here, $a_{ij}$ = elements in $i^{th}$ row and $j^{th}$ column. The matrix as shown here, is denoted by $[a_{ij}]_{m\times n}$.

**Order of Matrix:** Matrix having m-rows and n–columns is said to have order m × n.

**Real Matrix:** A matrix having all real elements.

**Complex Matrix:** A matrix having atleast one imaginary element.

**Complex Conjugate of a Matrix:** A matrix obtained by replacing the elements of a complex matrix $A = [a_{ij}]_{m\times n}$ by their conjugate is called complex conjugate of matrix A, and it is denoted by $\bar{A} = \left[\bar{a}_{ij}\right]_{m\times n}$.

**Rectangular Matrix:** A matrix of order m × n; where m, n ∈ ℕ and m ≠ n. These are of two types:

**(a) Horizontal Matrix:** A matrix of order m × n; where n > m, i.e., number of columns is greater than number of rows.

**(b) Vertical Matrix:** A matrix of order m × n; where m > n, i.e., number of rows is greater than number of columns.

**Row Matrix:** A matrix of order 1 × n, that is, a matrix having one row only.

**Column Matrix:** A matrix of order n × 1, that is, a matrix having one column only.

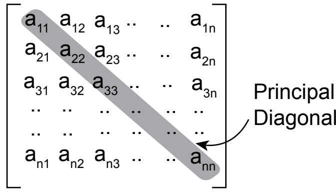
#### Remark:

*Clearly, row matrix is horizontal, whereas column matrix is vertical.*

**Square Matrix:** Matrix of order m × n; that is, a matrix having equal number of rows and columns. Such a matrix is called m–rowed square matrix.

**Principal (Leading) Diagonal and Off-diagonal of Square Matrix**
Diagonal along which the elements $a_{11}$, $a_{22}$, $a_{33}$,…, $a_{nn}$ lie, is called principal diagonal, or simply diagonal, when there is number chance of confusion. The other diagonal is called off-diagonal.

The elements lying diagonal are called **diagonal elements**.




**Trace of a Square Matrix:** The sum of diagonal elements

i.e., $\sum_{i=1}^{n} a_{ii} = (a_{11} + a_{22} + a_{33} + .... + a_{nn}) = \underbrace{Tr(A)}_{\text{(notation)}}$ .

**Diagonal Matrix:** A square matrix having all non-diagonal elements zeros, i.e.,

$$\begin{bmatrix} a_{11} & 0 & 0 & ... & ... & 0 \\ 0 & a_{22} & 0 & ... & ... & 0 \\ 0 & 0 & a_{33} & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & a_{nn} \end{bmatrix} = \text{diagonal } [a_{11}, a_{22,} a_{33}, \ldots, a_{nn}] \text{ or } \underbrace{\text{diagonal}.(a_{11}\ a_{22}\ a_{33}\ .....,a_{nn})}_{\text{(Notation)}}$$

**Scalar Matrix:** A diagonal matrix having all diagonal elements equal, i.e.,

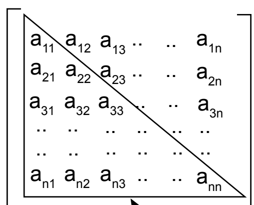
$$\begin{bmatrix} k & 0 & 0 & ... & ... & 0 \\ 0 & k & 0 & ... & ... & 0 \\ 0 & 0 & k & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & k \end{bmatrix} = \text{diagonal}(k\ k\ k\ ....k)$$

**Unit Matrix (Identity Matrix):** A scalar matrix having each diagonal element unit, i.e., 1.

$$\begin{bmatrix} 1 & 0 & 0 & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & 0 \\ 0 & 0 & 1 & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & 1 \end{bmatrix}$$

**Remark:**

*$I_1 = [1]$; $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are called unit matrix of order 1, order 2 and order 3 and so on, respectively.*



Lower triangle containing non-zero elements, non-null matrix

**Null Matrix (Zero Matrix):** A matrix having its all elements zero.

**Triangular Matrix:** A square matrix, in which, all the elements above the principal diagonal, or below the principal diagonal, are zero, is called triangular matrix.

**Lower Triangular Matrix:** A square matrix having its all elements above diagonal zeros, i.e., having non-zero elements (if not, only null) on principal diagonal or below it, i.e., $a_{ij} = 0$ for $a_{11}$ $i < j$.

**Upper Triangular Matrix:** Square matrix having its all elements below diagonal zero, i.e., having non-zero elements (if non-null) on or above the principal diagonal, i.e., $a_{ij} = 0$ for all $i > j$

**Example:**
Upper triangle contains non-zero elements if non-null matrix.

Upper triangle containing non-zero elements, if non-null matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & .. & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & .. & a_{3n} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & a_{n3} & .. & .. & a_{nn} \end{bmatrix}$$

**Remarks:**

*(i)* *Null square matrix is simultaneously both upper as well as lower triangular matrix.*

*(ii)* *Minimum number of zeros in a triangular matrix of order* $n = \frac{n(n-1)}{2}$.

*(iii)* *Maximum number of non-zero entries in a triangular matrix of order* $n = \frac{n(n+1)}{2}$.

*(iv)* *Diagonal matrix is simultaneously both upper as well as lower triangular matrix.*

*(v)* *Minimum number of zero entries in a diagonal matrix* $= (n^2 - n) = n\,(n-1)$.

*(vi)* *Maximum number of non-zero entries in a diagonal matrix of order* $n = n$.

*(vii)* *Maximum number of zero entries in a diagonal matrix of order* $n = n^2$ *(when its is null).*

*(viii)* *Maximum number of different elements in a triangular matrix of order* $n = \frac{n^2 + n + 2}{2}$.

*(ix)* *Minimum number of different elements in a non-null diagonal matrix of order* $n = 2$.

*(x)* *Minimum number of different elements in a non-null triangular matrix* $= 2$.

*(xi)* *Minimum number of zeros in a scalar matrix* $= (n^2 - n)$.

*(xii)* *Number of zeros in a non-null scalar matrix* $= (n^2 - n)$.

*(xiii)* *Number of different entries in a non null scalar matrix* $= 2$.

*(xiv)* *A triangle matrix is called strictly triangular iff* $a_{ii} = 0$, *for all* $i$; $1 \le i \le n$.

## 31.2 SUB MATRIX

Matrix obtained by leaving some rows, or columns, or both of a matrix A, is called a sub-matrix of matrix A.

For example, $\begin{bmatrix} 2 & 5 \\ 7 & 9 \end{bmatrix}$ is a sub-matrix of matrix $\begin{bmatrix} 2 & 5 & 8 \\ 7 & 9 & 4 \\ 1 & 3 & 5 \end{bmatrix}$.

### 31.2.1 Equal Matrices

Two matrices are said to be equal iff they are of same order and the elements on their corresponding positions are same, i.e., $A = [a_{ij}]_{m \times n} = B\,[b_{ij}]_{r \times p} \Leftrightarrow m = r, n = p$ and $a_{ij} = b_{ij}$.

#### *31.2.1.1 Addition of matrices*

Two matrices $A = [a_{ij}]$ and $B = [B_{ij}]$ are said to be conformable for addition iff they are of same order. Further, $A + B = [a_{ij} + b_{ij}]_{m \times n}$; where $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$.

**Properties of Matrix Addition:**

1. Matrix addition is commutative: A + B = B + A.
   Matrix addition is associative: A + (B + C) = (A + B) + C.
2. Null matrix of order m × n additive identity in the set of matrices of order m × n: If $[a_{ij}]_{m\times n} = 0$ and $B = [b_{ij}]_{m\times n}$, then $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [b_{ij}]_{m\times n} = [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$; where $a_{ij} = O$ for all i and j.
3. $-A = [-a_{ij}]_{m\times n}$ is additive inverse of $A = [a_{ij}]_{m\times n}$.
4. Left cancellation law: $A + B = A + C \Rightarrow B = C$.
   Right cancellation law: $A + B = C + B \Rightarrow A = C$.
5. A + X = O has a unique solution X = –A; of order m × n, and $X = [-a_{ij}]_{m\times n}$ if $A = [a_{ij}]_{m\times n}$

**Subtraction of Matrices:** If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, i.e., A and B are of same order (conformable for subtraction), then $A - B = [a_{ij} - b_{ij}]_{m\times n}$.

**Properties of Subtraction of Matrices:**

1. Neither commutative nor associative.
2. Follows left concellation and right concllation.
3. Left cancellation law: $A - B = A - C \Rightarrow B = C$.
4. Right cancellation law: $A - B = C - B \Rightarrow A = C$.
5. Equation A – X = O; where O is a null matrix of order m × n and A and X are matrices of order (m × n), has a unique solution X = A.

**Multiplication of Matrix by a Scalar:**

$\lambda.A = \lambda[a_{ij}]_{m\times n} = [\lambda a_{ij}]_{m\times n}$, i.e., scalar multiplication of a matrix. A gives a new matrix of same order whose elements are scalar (λ) times the corresponding elements of matrix A.

**Scalar Multiplication is Commutative and Distributive:**

(i) Matrix addition is commutative and associative.
(ii) Follows cancellation and right cancellation law.

## 31.3 MULTIPLICATION OF MATRIX

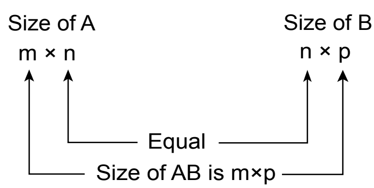
Two matrices, A and B, are said to be conformable for the product AB, if $A = (a_{ij})$ is of the order m × n and $B = (b_{ij})$ is of the order n × p, the resulting matrix is of the order m × p, and $AB = (C_{ij})$; where $(C_{ij}) = \sum_{k=1}^{n} a_{ik} b_{kj}$ $= a_{i1} b_{1j} + a_{i2} b_{2j} + .......a_{in} b_{nj}$ for i = 1, 2, 3,......,m and j = 1, 2, 3,.......,p.

As an aid to memory, denote the rows of matrix A by $R_1$, $R_2$, $R_3$ and columns of B by $C_1$, $C_2$ and $C_3$.

$$\text{Also, } A\times B = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \times (C_1.C_2.C_3) = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix};$$

where $R_i C_j$ is the scalar product of $R_i$ and $C_j$.

The diagrammatical working of product of two matrices is shown as in the figure.



**Remarks:**

(i) *In the product AB, A is called post-multiplied by B and B is called P multiplied by A.*

(II) $A = [a_1, a_2, \ldots\ldots, a_n]$ *and* $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$; $\Rightarrow AB = [a_1 b_1 + a_2 b_2 + \ldots.. + a_n b_n]$.

## 31.3.1 Properties of Multiplication of Matrices

1. AB and BA both may be defined yet $AB \neq BA$.
2. AB and BA both may be defined and $AB = BA$.
3. One of the products AB or BA may not be defined.
4. If A be a square matrix of the same order as I, then $IA = A$, $I = A$ and $OA = AO = O$, where O is a null matrix, i.e., multiplication by identity and null matrix is commutative.
5. AB may be a zero matrix, and BA may be a non-zero matrix, or vice versa, when $A \neq O$, $B \neq O$.
6. AB and BA both may be a zero matrix, when $A \neq 0$, $B \neq 0$.
7. Multiplication of matrices is associative and distributive over addition.
8. The matrix AB is the matrix B pre-multiplied by A and the matrix BA is the B post multiplied by A.
9. If A, B are suitable matrices and λ is a scalar, then $\lambda(AB) = (\lambda A)B = A(\lambda B)$.
10. **Existence of multiplicative Identity:** If $A = [a_{ij}]$ is an $m \times n$ matrix, then $I_m A = A = AI_n$.
11. The product of any matrix, and null matrix of a suitable order is a null matrix.
    If $A = [a_{ij}]$ is an $m \times n$ matrix, then $O_{p \times m} A = O_{p \times n}$ and $AO_{n \times q} = O_{m \times q}$.
12. **Powers of a square matrix:** Let A be a square matrix of order n, then AA makes sense, and it is also a square matrix of order n. We define:
    $A^1 = A$; $A^2 = AA, \ldots\ldots, A^m = A^{m-1}A = AA^{m-1}$ for all positive integers m.

## 31.3.2 Transpose of a Matrix

A matrix obtained by interchanging rows and columns of a matrix A is called the transpose of a matrix.

If A is a matrix, then its transpose must be denoted as A′ or $A^T$, e.g., if $A = \begin{bmatrix} 2 & 3 & 5 \\ 5 & 6 & 8 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 5 & 8 \end{bmatrix}$.

**Properties of Transpose of a Matrix**

(i) $(A^T)^T = A$, i.e., the transpose of the transpose of a matrix is the matrix itself.
(ii) $(A + B)^T = A^T + B^T$, i.e., the transpose of the sum of two matrices is the sum of their transpose.
(iii) $(kA)^T = kA^T$ (where k is a scalar).
(iv) $(AB)^T = B^T A^T$, i.e., the transpose of the product of two matrice is the product in reverse order of their transpose.
(v) $(-A)^T = ((-1)A)^T = (-1)A^T = -A^T$.
(vi) $(A - B)^T = (A + (-B))^T = A^T + (-B)^T = A^T + (-B^T) = A^T - B^T)$.
(vii) If A is $m \times n$ matrix, then $A^T$ is $n \times m$ matrix.

## 31.3.3 Symmetric Matrix

A square matrix will be called symmetric, if the elements across principal diagonal are symmetrically equal.

**Skew Symmetric Matrix:** A square matrix $A = [a_{ij}]_{m\times n}$ is said to be skew symmetric, iff $a_{ij} = -a_{ij}$ $\forall$ i and j $\Rightarrow a_{ii} = 0$ $\forall$ i; i.e., the diagonal elements are zeros.

### *31.3.3.1 Properties of symmetric/skew-symmetric matrix*

1. A symmetric/skew-symmetric matrix is necessarily a square matrix.
2. Symmetric matrix does not change by interchanging the rows and columns.
   i.e., symmetric matrices are transpose of themselves.
3. A is symmetric, if $A^T = A$ and A is skew-symmetric if $A^T = -A$.
4. $A + A^T$ is a symmetric matrix, and $A - A^T$ is a skew-symmetric matrix.
   Consider $(A + A^T) = A^T + (A^T)^T = A^T + A = A + A^T = A + A^T$ is symmetric.
   Similarly, we can prove that $A - A^T$ is skew-symmetric.
5. The sum of two symmetric matrix is a symmetric matrix and the sum of two skew-symmetric matrix, is a skew symmetric matrix.
6. If A and B are symmetric matrices, then AB + BA is a symmetric matrix and AB – BA is a skew symmetric matrix.
7. Every square matrix can be uniquely expressed as the sum of symmetric and skew-symmetric matrix.
8. Maximum number of distinct entries in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.
9. Maximum number of distinct elements in a skew symmetric matrix of order $n = n^2 - n + 1$.
10. Maximum number of distinct non–zero elements in a skew-symmetric matrix of order $n = (n^2 - n) = n(n - 1)$.
11. Maximum number of elements with distinct magnitude in a skew-symmetric matrix $= \left(\frac{n^2 - n}{2}\right) + 1$.
12. The matrix (B') AB is symmetric or skew-symmetric, according as A is symmetric or non-symmetric, respectively.
13. The determinant of a skew-symmetric matrix with real entries and odd order always vanishes.
14. The determinant of a skew-symmetric matrix with even real entries order is always a perfect square.

### *31.3.3.2 Properties of trace of a matrices*

(i) $tr(\lambda A) = \lambda tr(A)$
(ii) $tr(A \pm B) = tr(A) \pm tr(B)$
(iii) $tr(AB) = tr(BA)$
(iv) tr (skew-symmetric matrix) = 0
(v) tr(A) = na; where A is a scalar matrix of order n and with diagonal elements a.
(vi) tr [diagonal (a, b, c), diagonal (d, e, f)] = tr [diagonal (ad, be, cf)] = (ad + be + cf)
(vii) $tr(\bar{A}) = \overline{tr(A)}$; $\bar{A}$ = conjugate matrix of A.
(viii) $tr(A') = tr(A)$; $A'$ = transpose of matrix A.

## 31.4 HERMITIAN MATRIX

If A = $[a_{ij}]_{m\times n}$ is such that $a_{ij} = \overline{aji}$; i.e., $\left(\overline{A}'\right) = A$, i.e., $A^\theta = A$; where $A^\theta = \left(\overline{A}'\right) = \left(\overline{A}\right)'$

$$\text{e.g., } A\begin{bmatrix} 2 & 3+2i \\ 3-2i & 7 \end{bmatrix} \Rightarrow A'\begin{bmatrix} 2 & 3-2i \\ 3+2i & 7 \end{bmatrix} \Rightarrow \left(\overline{A}'\right) = \begin{bmatrix} 2 & 3+2i \\ 3-2i & 7 \end{bmatrix} = A.$$

### 31.4.1 Properties of Hermitian Matrices

1. Diagonal elements are purely real, $a_{ii} = \overline{a}_{ii} \Rightarrow a_{ii} - \overline{a}_{ii} = 0 \;\Rightarrow 2I_m(a_{ii}) = 0$.
2. Every symmetric matrix with real number as elements is hermitian, e.g., $a_{ij} = a_{ij} = \overline{a}_{ji} \rightarrow A$ is hermitian.

### 31.4.2 Skew-Hermitian Matrix

If A = $[a_{ij}]_{m\times n}$ is such that $-a_{ij} = \overline{a_{ij}}$, i.e., $(\overline{A}') = -A$, i.e., $A^\theta = -A$, e.g., $A = \begin{bmatrix} 3i & 1-3i & 2 \\ -1-3i & 0 & 4+i \\ -2 & -4+i & 2i \end{bmatrix}$.

#### *31.4.2.1 Properties of hermitian/skew–hermitian matrix*

1. Elements on principal diagonal are either purely imaginary or zero, e.g., for i = j

   $a_{ii} = -\overline{a_{ii}} \;\Rightarrow \mathbb{R}(a_{ii}) = 0 \Rightarrow a_{ii}$, is purly imaginary.
2. Every skew-symmetric matrix with real numbers as elements is skew-Hermitian.
3. Every square matrix can be uniquely represented as the sum of a hermitian and skew-Hermitian matrices.
4. If A is any matrix, then $A = \frac{1}{2}\{A + A^\theta\} + \frac{1}{2}\{A - A^\theta\}$ = Hermitian + skew-Hermitian.

### 31.4.3 Orthogonal Matrix

A square matrix A is called an orthogonal matrix, if the product of the matrix A and its transpose A' is an identity matrix, i.e., AA' = A'A = I.

#### *31.4.3.1 Properties of Orthogonal Matrix*

(i) If $AA' = I$ then $A^{-1} = A'$ $\because$ $AA' = I \Rightarrow A^{-1}(AA') = A^{-1}.I = A^{-1} \Rightarrow A' = A^{-1}$

(ii) If A and B are orthogonal, then AB is also orthogonal.

$\because$ $(AB)(AB) = (AB)(B'A') = A(BB')A' = AIA' = AA' = I$; similarly $(AB')(AB) = I$.

(iii) Value of corresponding determinant of orthogonal matrix is $\pm1$.

### 31.4.4 Idempotent Matrix

A square matrix A is called idempotent, provided that it satisfies the relation $A^2 = A$.

**Properties:**

(i) If A and B are idempotent matrices, then AB is as idempotent matrix, if AB = BA.

(ii) If A and B are idempotent matrices, then A + B is an idempotent if AB + BA = O.

(iii) A is idempotent and A + B = I, then B is also idempotent and AB = BA = O.

## 31.4.5 Periodic Matrix

A square matrix A is called periodic, if $A^{k+1} = A$; where k is a positive integer. If k is the least positive integer, for which $A^{k+1} = A$, then k is said to be period of A. For k = 1, we get $A^2 = A$, and we called it to be an idempotent matrix.

## 31.4.6 Nilpotent Matrix

A square matrix A is called Nilpotent matrix of order k, provided that it satisfies the relation $A^k = O$ and $A^{k-1} \neq A$; where k is positive integer and O is null matrix, and k is the order of the nilpotent matrix A.

## 31.4.7 Involutory Matrix

A square matrix A is called involutory matrix, provided that it satisfies the relation $A^2 = I$; where I is identity matrix, e.g., $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

**Properties:**

(i) A is involutory iff $(A + I)(A - I) = O$.

(ii) Identity matrix is a trivial example of involutory matrix.

## 31.4.8 Unitary Matrix

A square matrix A is called a unitary matrix if $A.\ A^{\theta} = I$, where I is an identity matrix and $A^{\theta}$ is the transpose conjugate of A.

### *31.4.8.1 Properties of Unitary Matrix*

(i) If A is unitary matrix, then $A'$ is also unitary.

(ii) If A is unitary matrix, then $A^{-1}$ is also unitary.

(iii) If A and B are unitary matrices, then AB is also unitary.

### *31.4.8.2 Determinant of a square matrix*

A number associate with every square matrix A is called its determinant and denoted by $|A|$ or det (A).

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$.

**Evaluation of Determinant of Order 3:**

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

**Singular Matrix:** Square matrix having its determinant = 0.

## 31.4.9 Non-singular Matrix

Square matrix A, for which $|A| \neq 0$.

### *31.4.9.1 Minor of elements of a square matrix*

The determinant obtained by deleting the $i^{th}$ row and $j^{th}$ column, passing through the $a_{ij}$ element, is called minor of element $a_{ij}$, and is denoted by $M_{ij}$, e.g., $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ = $(a_{11}.a_{32} - a_{31}a_{12})$ = minor element $a_{23}$; where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$.

**Co-factors of Element of Square Matrix:** The determinant obtained by deleting $i^{th}$ row and $j^{th}$ column when multiplied by $(-1)^{i+j}$ gives us the co-factors of element $a_{ij}$ and is denoted by $A_{ij}$ or $C_{ij}$. In other words, $C_{ij} = (-1)^{i+j} M_{ij}$, i.e., $(-1)^{i+j}$ times the minor of element $a_{ij}$.

e.g., $C_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}.a_{32} - a_{31}.a_{12})$ = co-factor of element $a_{23}$.

**Remarks:**

*(i)* $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$; *in general,* $|A| = \sum_{k=1}^{3} a_{ik}C_{ik}$; *i = 1 or 2 or 3 (expansion along rows), or*

$|A| = \sum_{k=1}^{3} a_{kj}.C_{kj}$; *j = 1 or 2 or 3 (expansion along columns)*

*(ii)* $\sum_{k=1}^{3} a_{ik}C_{jk} = \sum_{k=1}^{3} a_{ki}C_{kj} = 0$ *for* $i \neq j$

## 31.5 ADJOINT OF A SQUARE MATRIX

The transpose of the matrix containing co-factors of elements of square matrix A. It is denoted by Adj(A)

i.e., $\text{Adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$; where $C_{ij}$ = co-factors of $a_{ij}$ $\Rightarrow$ $\text{Adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

### 31.5.1 Properties of Adjoint of Square Matrix A

(i) A(adj A) = (adj A) (A) = $|A| I_n$; where A is a square matrix of order n.
(ii) If A is a singular matrix, then A(Adj A) = (Adj A). A = 0 ($\because$ $|A| = 0$).
(iii) $|\text{Adj } A| = |A|^{n-1}$.
(iv) Adj (AB) = (Adj B). (Adj A); provided that A, B are non-singular square matrices of order n.
(v) Adj $(A^T)$ = $(\text{Adj } A)^T$.
(vi) Adj.( Adj A) = $|A|^{n-2}$ .A.
(vii) $|\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2}$.
(viii) Adjoint of a diagonal matrix is a diagonal matrix.
(ix) adj($\lambda$.A) = $\lambda^{n-1}$. (Adj A); where $\lambda$ is a scalar and $(A)_{n\times n}$.

### 31.5.2 Inverse of Non-singular Square Matrix

A square matrix B of order n, is called inverse of non-singular square matrix.
A of order n iff A.B. = B.A = $I_n$.

Let $B = \frac{\text{Adj} A}{|A|}; |A| \neq 0 \Rightarrow A.B = \frac{A.(\text{Adj} A)}{|A|} = \frac{|A|.I_n}{|A|} = I_n$; similarly, $B.A = \frac{(\text{Adj} A)}{|A|}.A = \frac{|A|.I_n}{|A|} = I_n$.

Thus, A.B. = B.A. = $I_n \Rightarrow B = A^{-1}$. Thus, $A^{-1} = \frac{\text{Adj} A}{|A|}$; provided that $|A| \neq 0$.

**Invertible Matrix:** A square matrix iff it is non-singular, i.e., $|A| \neq 0$.

### *31.5.2.1 Properties of inverse of square matrix*

1. Every invertible matrix possesses a unique inverse.
2. A square matrix is invertible, if and only if, it is non-singular.
3. If A, B be two non-singular matrices of the same order, then AB is also non-singular and $(AB)^{-1} = B^{-1} A^{-1}$ (reversal law of inverse).
4. (i) $AB = AC \Rightarrow B = C$ (ii) $BA = CA \Rightarrow B = C$
5. Since, we already know that $(AB)^{-1} = B^{-1} A^{-1}$, therefore, in general, we can say that $(ABC,.......,Z)^{-1} = Z^{-1} Y^{-1} ....B^{-1} A^{-1}$.
6. If A is an invertible square matrix, then adj $(A') = (\text{adj } A)'$.
7. $(A^T)^{-1} = (A^{-1})^T$
8. $(\overline{A}^T)^{-1} = (\overline{A}^{-1})^T$
9. $AA^{-1} = A^{-1}A = I$
10. $(A^{-1})^{-1} = A$

## 31.6 MATRIX POLYNOMIAL

Let $f(x) = a_0 x^m + a_1 x^{m-1} + .... + a_{m-1} x + a_m$ be a polynomial in x and A be a square matrix of order n, then $f(A) = a_0 A^m + a_1 A^{m-1} + .... + a_{m-1} A + a_m I_n$ is called a matrix polynomial in A. Thus, to obtain f (A), replace x by A in f(x), and the constant term is multiplied by the identity matrix of the order equal to that of A.

The polynomial equation f (x) = 0 is said to be satisfied by the matrix A iff f(A) = O.

e.g., if $f(x) = 2x^2 - 3x + 7$ and A is a square matrix of order 3 then $f(A) = 2A^2 - 3A + 7I_3$.

The polynomial $| A - x I_n|$, is called characteristic polynomial of square matrix A.

The equation $| A - x I_n| = O$, is called characteristic equation of matrix A.

### 31.6.1 Cayley Hamilton Theorem

Every matrix satisfies its characteristics equation $|A - xI| = 0$ because $|A - AI| = |A - A| = 0$.

So, $a_0 A^n + a_1 A^{n-1} + .... + a_2 A^{n-2} + ..... + a_n I = O \Rightarrow A^{-1} = -\left[\frac{a_0}{a_n} A^{n-1} + \frac{a_1}{a_n} A^{n-2} + ......\right]$

### 31.6.2 Elementry Transformation

- **Interchange of any two rows or columns:** Denotion by $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$.
- **Multiplication by non-zero scalar:** Denotion $R_i \leftrightarrow kR_i$ or $C_j \leftrightarrow kC_j$.
- Replacing the $i^{th}$ row (or column) by the sum of its elements and scalar multiplication of corresponding elements of any other row (or column).

  **Denotion:** $R_i \to R_i + kR_j$ or $C_i \to C_i + k C_j$.

- Transformed matrix using sequence of elementary transformations (one or more) is known as equivalent matrix of A.

## 31.6.3 Elementary Matrix

Elementary matrix obtained from identities matrix by single elementary transformation:

$$\text{e.g., } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

## 31.6.4 Equivalent Matrices

Two matrices A and B are equivalent, if one can be obtained from the other by a sequence of elementary transformations denoted by $A \sim B$.

### *31.6.4.1 Inverse of a matrix A by using elementary row operations*

**Step 1:** Write $A = I_n \cdot A$; i.e., $\begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 1 \end{bmatrix}.A\cdot$

**Step 2:** Now, applying the sequence of elementary row operation on matrix A and matrix $I_n$, simultaneously till matrix A on L.H.S of the above equation get converted to identity matrix $I_n$.

**Step 3:** After (Step 2) reaching at $I_n = B.A \Rightarrow B = A^{-1}$.

### *31.6.4.2 Inverse of matrix A by using elementary column operations*

**Step 1:** Write $A = A\,I_n$.

**Step 2:** Now, apply as above sequence of elementary column operations on matrix A on the left hand side, and same sequence of elementary column operations on identity matrix. $I_n$ on the right hand side of the above equation till matrix A on the left hand side gets converted to $I_n$.

**Step 3:** After (Step 2) reaching at $I_n = A.B \Rightarrow B = A^{-1}$.

### *31.6.4.3 System of simultaneous equations*

The system of n equations in n-unknown, given by: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$
$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$; where $b_1, b_2, b_3, \ldots\ldots, b_n$ are not all zeros, is called non-homogenous system of equations.

This system of equation can be written in matrix form, as: $\begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

AX = B. Here, A is a square matrix. A system is said to be consistent if it has atleast one set of solution; otherwise, known as inconsistent equation.

### *31.6.4.4 Solutions of non-homogenous systems of equation*

There are three methods of solving non-homogenous equations in three variables

(i) Matrix method
(ii) Determinant method (Cramer's rule)
(iii) By using elementary row and column operations

### *31.6.4.5 Matrix method of solving non-homogeneous system of equations*

Let the given system of equation be $AX = B \Rightarrow X = A^{-1}B$, gives us:

1. Unique solution of system of non-homogenous equations, provided $|A| \neq 0$.
2. No solution, if $|A| = 0$ and (adj A). $B \neq 0$ (null matrix).
3. Infinitely many solutions, if $|A| = 0$ and (adj A). $B = 0$. For getting infinitely many solutions, take any $(n-1)$ equations. Take any one variable, say $x_n = k$, and solve these $(n-1)$ equations for $x_1, x_2, x_3, \ldots, x_{n-1}$ in terms of k.
   The infinitely many solutions are given by $x_1 = f_1(k), x_2 = f_2(k), \ldots, x_{n-1} = f_{n-1}(k); x_n = k$ and $k \in \mathbb{R}$.

## 31.7 DETERMINANT METHOD (CRAMER'S RULE) FOR SOLVING NON-HOMOGENOUS EQUATIONS

### 31.7.1 For Two Variables

Let $a_1x + b_1y = C_1$ and $a_2x + b_2y = C_2$, then take $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}; \Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}; \Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

i.e., $\Delta$ is determinant formed by coefficient of x and y.

$\Delta_1$ is determinant formed by replacing elements of first column of $\Delta$ by $C_1$ and $C_2$ and $\Delta_2$ is determinant formed by replacing elements of second column of $\Delta$ by $C_1$ and $C_2$.

**Case (i):** If $\Delta \neq 0$; then system of equation has a unique solution given by $x = \frac{\Delta_1}{\Delta}; y = \frac{\Delta_2}{\Delta}$.

**Case (ii):** If $\Delta = 0$;

(a) If $\Delta_1, \Delta_2$ both are not zeros, i.e., atleast one of $\Delta_1$ and $\Delta_2$ is non zero, then there is no solution.

(b) If $\Delta_1 = \Delta_2 = 0$, then the system of equation has infinitely many solution. Take x or y say $y = k$

$\Rightarrow \; x = \frac{C_1 - b_1k}{a_1}$. Thus, $x = \frac{C_1 - b_1k}{a_1}; y = k; k \in \mathbb{R}$ gives infinitely many solutions.

### 31.7.2 For Three Variables

$a_1x + b_1y + c_1z = d_1;\; a_2x + b_2y + c_2z = d_2;\; a_3x + b_3y + c_3z = d_3$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}; \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

**Case (i):** For $\Delta \neq 0$, there will be unique solutions $x = \frac{\Delta_1}{\Delta}; y = \frac{\Delta_2}{\Delta}; z = \frac{\Delta_3}{\Delta}$.

**Case (ii):** For $\Delta = 0$

(a) If atleast one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero, there is no solution. i.e., system of equations is consistent.

(b) If $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then there will be infinitely many solutions. For these infinitely many solutions take any two equations, say (i) and (ii) and put $z = k$, to obtain $a_1x + b_1y = d_1 - c_1k$ and $a_2x + b_2y = d_2 - c_2k$. Solving, we get x and y in term of k (say) $x = f_1(k)$ and $y = f_2(k)$.
Thus $x = f_1(k)$; $y = f_2(k)$; $z = k$; $k \in \mathbb{R}$ gives us infinitely many solutions.

## 31.8 SOLUTION OF NON-HOMOGENEOUS LINEAR EQUATIONS BY ELEMENTARY ROW OR COLUMN OPERATIONS

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be such that AX = B, i.e., $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(by using elementary row operations)

Apply elementary row operations on matrix A, and same operations simultaneously on B, to reduce it into $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ \mu \\ \alpha \end{bmatrix}$

$\Rightarrow \quad ax + by + cz = \lambda$ ...(i) $\qquad dy + ez = \mu$ ....(ii) $\qquad fz + \alpha$ ....(iii)

from equation (iii), we get $z = \alpha/f$; from equation (ii), we get $y = \dfrac{\mu - e\left(\frac{\alpha}{f}\right)}{d}$.

And from equation (i), we get $x = \dfrac{\lambda - d\left(\dfrac{\mu - e\left(\frac{\alpha}{f}\right)}{d}\right) - C\left(\frac{\alpha}{f}\right)}{a}$ (By using elementary column operations).

Now applying elementary column operations to A′ and simultaneously same elementary column operation's to B′, to get:

$$\Rightarrow \quad [x_1\ x_2\ x_3]\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix} = [\lambda\ \mu\ \alpha]$$

$\Rightarrow \quad ax_1 + bx_2 + cx_3 = \lambda$ ......(i)

$dx_2 + ex_3 = \mu$ ......(ii)

$fx_3 = \alpha$ .....(iii)

$\therefore$ From (iii) $x_3 = \dfrac{\alpha}{f}$; from (ii) $x_2 = \dfrac{\mu - e\left(\frac{\alpha}{f}\right)}{d}$; from (iii) $x_1 = \dfrac{\lambda - d\left(\dfrac{\mu - e\left(\frac{\alpha}{f}\right)}{d}\right) - C\left(\frac{\alpha}{f}\right)}{a}$.

### 31.8.1 Solutions of Homogenous System of Equation

Consider the following system of homogenous linear equation in n unknowns $x_1, x_2, \ldots, x_n$

$$a_{11}x_1 + a_{22}x_2 + \ldots + a_{1n}x_m = 0$$
$$a_{22}x_1 + a_{22}x_2 + \ldots + a_{2n}x_m = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0$$

This system of equation can be written in matrix form, as follows:

$$\begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \ldots\ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \quad AX = O$$

(i) If $|A| \neq 0$, the system of equations has only trivial solution and that will be the only solution.
(ii) If $|A| = 0$, the system of equations has non-trivial solution and it has infinite solutions.
(iii) If number of equations < Number of unknowns, then it has non-trivial solution.

**Remark:**

*If numbers of equations < number of unknown variables, then either the system of equations have no solutions or infinitely many solutions.*

## 31.9 ELIMINANT

Eliminant of a given number of equation in some variables is an expression which is obtained by eliminating the variables out of these equations.

### 31.9.1 Linear Transformation

The transformation in which the straight line remains straight and origin does not change its position. We represent point (x, y) by column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ and transformation mapping is denoted by a matrix operation which transform $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} X \\ Y \end{bmatrix}$.

**Definition:** Any transformation of $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} X \\ Y \end{bmatrix}$ that can be expressed by the linear equation $a_1x + b_1y = X$ and $a_2x + b_2y = Y$ is called linear transformation.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \text{ operator } M = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \text{ is matrix of transformation.}$$

Origin remains invariant of such transformation. Some common linear transformations are:

1. Drag by a factor k along x-axis
2. Enlargment or reduction
3. Reflection in any line through origin
4. Rotation through any angle about origin
5. Shearing parallel to x-axis/y-axis

## 31.9.2 Compound Transformation

When a transformation (2) is carried out after (1) the compound transformation is denoted by a matrix operator $M_2$ o $M_1 = M_2 M_1$, where $M_2$ and $M_1$ are respective matrix operators for (i) and (ii) operation. $M_2$ o $M_1$ is known as composition of $M_2$ with $M_1$ (order of performance of operations must be mentioned).

**Matrix representing reflection in x-axis**

If P(x,y) be any point and P′ (X,Y) is its reflection on x –axis, then X = 1(x) + 0(y) and Y = 0(x) + (–1)y

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \text{ Thus } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{described reflection of point P(x,y) on x-axis}$$

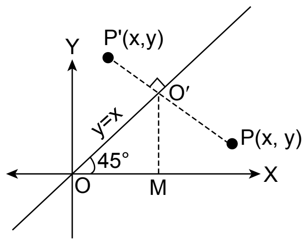
**Matrix representing reflection in y-axis**

Here, X = (–1)x + 0(y) and Y = (0) x + 1(y).

**Matrix representing reflection through the origin**

If P(x,y) is any point then P′ (X,Y), i.e., reflection of P(x,y) on origin is given by X = –1(x) + 0(y) and

$$Y = 0(x) + (-1)y \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

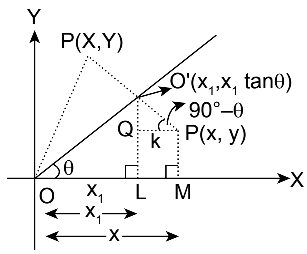



**Matrix representing reflection in the line y = x**

Let, P(x,y) be any point, and (X,Y) be its reflection on line y = x

Here, X = y and Y = x

$$\Rightarrow \quad X = 0(x) + 1(y) \text{ and } Y = 1(x) + 0(y) \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

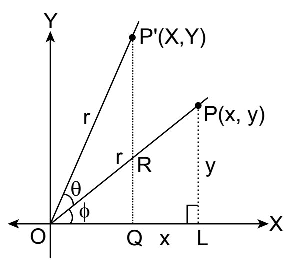



**Matrix representing reflection in the line y = x tanθ**

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

**Matrix representing rotation through an angle θ**

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$




**Expansion of determinant using co-factor (Laplace method)**

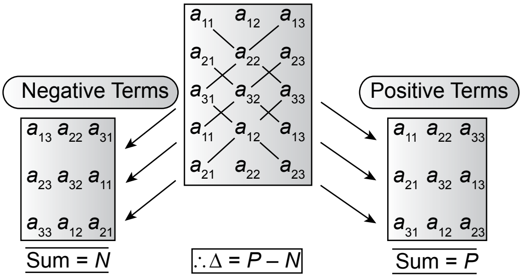
Let, $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be determinant or order 3 × 3, then

$\Delta = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3} + C_{i3} = \sum_{k=1}^{3} a_{ik}C_{ik}$ = expansion of Δ along $i^{th}$ rows and $\Delta = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j} + C_{3j} = \sum_{k=1}^{3} a_{kj}C_{kj}$ = expansion of Δ along $j^{th}$ column.

**Sarrus rule of expanding a determinant of third order**

Sarrus gave a rule for evaluating a determinant of the order three mentioned as follows:

**Rule:** Write down the three rows of the determinant, and rewrite the first two rows just below them. The three diagonals sloping down to the right give the three positive terms and the three diagonals sloping down to the left give the three negative terms. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then




## 31.9.3 Application of Determinant

Out of wide applications of determinants, a few are given below:

- **Area of Δ with vertices** $A(x_1, y_1), B(x_2, y_2), C(c_3, y_3)$

$$\Rightarrow \quad \Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$ ; where $|x|$ denotes absolute value of x.

- **Cross product of vectors** $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}; \vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\Rightarrow \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

It is also used to find the scalar triple product of three vector $\vec{a}.(\vec{b} \times \vec{c})$ is S.T.P. of $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$.

## 31.9.4 Properties of Determinants

**Property 1.** The value of determinant remains unaltered, if the rows are changed into columns and columns into rows. For example, if $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ $(a_1 b_2 - b_1 a_2)$ and $\Delta' = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = (a_1b_2 - a_2b_1) \Rightarrow \Delta = \Delta'$.

**Property 2.** If all the elements of a row/column are zero, then the value of determinant will be zero.

**Property 3. Reduction and increase of order of determinant**

(a) If all the elements in a row (or a column) except one element, are zeros the determinant reduces to a determinant of an order less by one.

(b) A determinant can be replaced by a determinant of a higher order by one as per the requirment.

**Property 4.** If any two rows or two columns of a determinant are interchanged, the determinant retains its absolute value, but changes its sign and symbolically the interchange of $i^{th}$ and $j^{th}$ rows or $i^{th}$ and $j^{th}$ columns is written as $\Delta = -\Delta_{R_i \rightleftarrows R_j}$ (or $-\Delta_{C_i \rightleftarrows C_j}$).

**Property 5.** The value of a determinant is zero, if any two rows or columns are identical. Symbolically, it is written as $\Delta_{R_i \equiv R_j} = 0$ or $\Delta_{C_i \equiv C_j} = 0$.

**Property 6.** (a) If every element of a given row of matrix A is multiplied by a number $\lambda$, the matrix thus obtained has determinant equal to $\lambda$ (det A). As a consequence, if every element in a row of a determinant has the same factor this can be factored out of the determinant. Symbolically, it is written as $\Delta = m.\Delta_{R_i \to \frac{1}{m} R_i}$.

(b) If all the elements of a row (column) of a determinant are multiplied by a constant (k), then the determinant gets multiplied by that constant.

**Property 7.** The value of the determinant corresponding to a triangular determinant is equal to product of its principal diagonal elements.

**Property 8.** If any row or column of a determinant be passed over n rows or columns, the resulting determinant will be $(-1)^n$ times the original determinant.

**Property 9.** (a) If every element of a column or (row) is the sum (difference) of two terms, then the determinant is equal to the sum (difference) of two determinants of same order; one containing only the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as in the given determinant.

(b) A determinant having two or more terms in the elements of a row (or column) can be written as the sum of two or more determinants.

**Property 10.** The value $\Delta$ of a determinant A remains unchanged, if all the elements of one row (column) are multiplied by a scalar and added or subtracted to the corresponding elements of another row (column). Symbolically, it is written as $\Delta = \Delta_{R_i \to R_i + mR_j}$ (or $\Delta_{C_j \to C_j + mC_i}$) and operation is, also symbolically written as $R_i \to R_i + mR_j$ or $C_j = C_j + mC_i$.

**Property 11.** (a) The sum of the products of elements of a row (or column) with their corresponding co-factors is equal to the value of the determinant. For example, $a_{11}C_{11} + a_{12} C_{12} + a_{13} C_{13} = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} = \Delta$

(b) Sum of the products of elements of any row (or column) with the co-factors of the corres sponding elements of a parallel row (or column) is always zero. For example, $a_{11} C_{21} + a_{12} C_{22} + a_{13}C_{23} = 0$

**Property 12.** If the elements of a determinant $\Delta$ involve x, i.e., the determinant is a polynomial in x and if it vanishes for x = a, then (x – a) must be a factor of $\Delta$. In other words, if two rows (or two column) become identical for x = a then (x – a) is a factor of $\Delta$. Generalizing this result, we can say, if r rows (or r columns) become identical when a is substituted for x, then $(x - a)^{r-1}$ should be a factor of $\Delta$.

For example, if $\Delta = \begin{vmatrix} x & 5 & 2 \\ x^2 & 9 & 4 \\ x^3 & 16 & 8 \end{vmatrix}$ at x = 2, $\Delta = 0$ ($\because C_1$ and $C_2$ become identical at x = 2).

## 31.9.5 Caution

While applying all the above properties from property 1 to property 10, atleast one row (or column) must remain unchanged.

## 31.10 SPECIAL DETERMINANT

### 31.10.1 Symmetric Determinant

Symmetric determinant is a determinant in which the elements situated at equal distance (symmetrically) from the principle diagonal are equal both in magnitude and sign, i.e., (i, j)$^{th}$ element ($a_{ij}$) = (j, i)$^{th}$ element ($a_{ji}$); e.g., $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} abc + 2fgh - af^2 - bg^2 - ch^2$.

### 31.10.2 Skew-Symmetric Determinant

All the diagonal elements are zero and the elements situated at equal distance from the diagonal are equal in magnitude but opposite in sign, i.e., (i, j)$^{th}$ element = –(j, i)$^{th}$ element, i.e., $a_{ij} = -a_{ji}$. The value of a skew-symmetric determinant of odd order is zero, e.g., $\Delta = \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$.

### 31.10.3 Cyclic Determinants

Determinants in which if a is replaced by b, b by c and c by a, then value of determinants remains unchanged are called cyclic determinants.

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ (Already proved in previous article)

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$ (can be proved using factorization)

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$ (can be proved using factorization)

### 31.10.4 Circulants

Circulants are those determinants in which the elements of rows (or columns) are cyclic arrangements of letters

(i) $\begin{vmatrix} x+a & x+b & x+c \\ x+b & x+c & x+a \\ x+c & x+a & x+b \end{vmatrix}$

(ii) $\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix}$, e.g., $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c-3abc)$

(iii) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

(iv) $\begin{vmatrix} a^2 & b^2 & c^2 \\ b^2 & c^2 & a^2 \\ a^2 & b^2 & c^2 \end{vmatrix}; \begin{vmatrix} x+a & y+b & z+c \\ y+b & z+c & x+a \\ z+c & x+a & y+b \end{vmatrix}$

**Remarks:**

1. *An expression is called cyclic in x, y, z iff cyclic replacement of variables does not change the expression. e.g., $x + y + z$, $xy + yz + zx$ etc. Such expression can be abbreviated by cyclic sigma notation as below:* $\sum x^2 = x^2 + y^2 + z^2, \sum xy = xy + yz + zx, \sum (x - y) = 0 \;=\; x + y + z + x^2 + y^2 + z^2 = \sum x + \sum x^2$
2. *An expression is called symmetric in variable x and y iff interchanging x and y does not change the expression.* $x^2 + y^2$, $x^2 + y^2 - xy$; $x^3 + y^3 + x^2y + y^2x$ . $x^3 - y^3$ *is not symmetric.*

## 31.10.5 Product of Two Determinant

Two determinants are conformable to multiply iff they are of same size. Since, $|A|\,|B| = |AB| = |A^TB^T| = |A^T B| = |A|\,B^T|$. There are four method of taking product of two determinant.

Let, $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$ and $\Delta = [P_{ij}]_{3\times3}$.

**Method 1:** Method of Multiplication (Row by column); $\Delta = |AB|$

$$\Delta = \Delta_1.\Delta_2 = \begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 & a_1l_3 + b_1m_3 + c_1n_3 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 & a_2l_3 + b_2m_3 + c_2n_3 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 & a_3l_3 + b_3m_3 + c_3n_3 \end{vmatrix};$$

$p_{ij}$ = scalar product of $i^{th}$ row vector and $j^{th}$ column vectors of $\Delta_1$ and $\Delta_2$ respectively.

**Method 2:** Method of multiplication (Row by Row); $\Delta = |AB^T|$

**Method 3:** Method of multiplication (Column by Row); $\Delta = |A^TB^T|$

**Method 4:** Method of multiplication (Column by Column); $\Delta = |A^TB|$

**Remark:**

*Since $|AB| = |A|\,|B| = |B|\,|A| = |BA| = |B^TA| = |BA^T| = |B^TA^T|$, thus $|AB|$ can also be obtained by row-column, row-row, column-row or column-column multiplication of B and A. Thus there are eight ways of obtaining ($\Delta_1$. $\Delta_2$).*

## 31.10.6 Adjoint or Adjugate of Determinant

If $\Delta = |a_{ij}|_{n\times n}$ is a determinant of order $n \times n$; then, $\Delta' = |C_{ij}|_{3\times3}$; where $C_{ij}$ is co-factor of element $a_{ij}$ is called Adjoint or Adjugate of determinant.

### *31.10.6.1 Jacobi's theorm*

Its states that $\Delta' = \Delta^{n-1}$; $\Delta \neq 0$; where $\Delta'$ = adjoint of $\Delta$ = determinant $|C_{ij}|$; $C_{ij}$ = co-factor of $a_{ij}$.

### *31.10.6.2 Reciprocal determinant*

If $\Delta = |a_{ij}| \neq 0$, then $\Delta'' = \left|\dfrac{C_{ij}}{\Delta}\right|$; where $C_{ij}$ is the cofactor of $a_{ij}$ is called the reciprocal determinant of $\Delta$.

$$\Delta'' = \left|\frac{C_{ij}}{\Delta}\right| = \frac{1}{\Delta^n}|C_{ij}| = \frac{\Delta'}{\Delta^n} = \frac{\Delta^{n-1}}{\Delta^n} = \frac{1}{\Delta}$$

### 31.10.6.3 Method to break a determinant as the product of two determinants

(a) Observe the diagonal symmetry of the elements and apply the following facts:
- ❑ The determinant of skew symmetric determinant with odd order always vanishes. Therefore, any odd order skew symmetric determinant can be broken into product of two matrices of which atleast one is singular.
- ❑ The determinant of skew symmetric determinant with even order is a perfect square. Therefore, an even ordered skew symmetric determinant can be written as a square of a determinant having symmetrical elements.

(b) Observe the symmetry of the elements and make sure whether $(i, j)^{th}$ element of the given determinant can be written as $R_i \cdot C_j$; where $R_i$ is the $i^{th}$ row of the first factor (determinant) and $C_j$ is the $j^{th}$ column of the second factor (determinant).

(c) While applying the approach (b), it is advised to choose the $(i, j)^{th}$ element to be diagonal elements.

## 31.11 DIFFERENTIATION OF DETERMINANTS

The differentiation of a determinant can be obtained as the sum of as many determinants as the order. The process can be carried out along the row/column by differentiating one row/column at a time and retaining the others as they are:

$\therefore$ If $\Delta = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$ of order 2, which is a function of x, then

$$\frac{d\Delta}{dx} = \frac{d}{dx}\begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix} = \frac{d}{dx}(f_1(x)g_2(x) - g_1(x)f_2(x))$$

$$= (f_1(x)g_2'(x) - g_2(x)f_1'(x) - g_1(x)f_2'(x) - f_2(x)g_1'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$$

**Note:**

*In order to find out the coefficient of $x^r$ in any polynomial f(x), differentiate the given polynomial f(x) r times successively and then substitute x = 0.*

*i.e., the coefficient of* $x^r = \left[\frac{f^r(0)}{r!}\right]$; *where* $f^r(0) = \left(\frac{d^r f(x)}{dx^r}\right)$ *at x = 0.*

### 31.11.1 Integration of a Determinant

Integration of a determinant: As determinant is a numerical value, so it can always be integrated by expanding but the integration of the determinant can be done without expansion, if it has only one variable row/column.

Given a determinant Δ (x) =
(where a, b, c, l, m and n are constants) as a function of x.

$$\text{So, } \int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ a & b & c \\ l & m & n \end{vmatrix}.$$