Inverse Trigonometric Function

Q.1. Evaluate the following :

i. $\sin(\cot^{-1} x)$
ii. $\sin(2 \sin^{-1} 0.8)$
iii. $\tan[2 \tan^{-1}(1/5) - \pi / 4]$
iv. $\sin \cot^{-1} \cos \tan -1 x$.
v. $\cos^{-1} x + \cos^{-1}[x/2 + \{\sqrt{(3 - 3x^2)/2}\}]$, $1/2 \le x \le 1$.

Solution : 1

i. $\sin(\cot^{-1}) = \sin[\sin^{-1} 1/\sqrt{(1 + x^2)}] = 1/\sqrt{(1 + x^2)}$.

ii. As $2 \sin^{-1} x = \sin^{-1} 2x \sqrt{(1 - x^2)}$;

Therefore, $\sin(2 \sin^{-1} 0.8)$

$= \sin[\sin^{-1}\{2 \times 8/10 \times \sqrt{(1 - 64/100)}\}]$

$= \sin(\sin -1\ 0.96) = 0.96$.

iii. As $2 \tan^{-1} x = \tan^{-1}\{2x / (1 - x^2)\}$;

Therefore, $\tan[2 \tan^{-1}(1/5) - \pi / 4]$

$= \tan[\tan^{-1}\{2 \times (1/5) / (1 - 1/25)\} - \tan^{-1} 1]$

$= \tan[\tan^{-1}(5/12) - \tan^{-1} 1]$

$= \tan[\tan^{-1}\{(5/12^{-1}) / (1 + 5/12 \times 1)\}]$

$= \tan \tan^{-1}(-7/17) = -7/17$.

iv. We have to evaluate , $\sin \cot^{-1} \cos \tan^{-1} x$.

Let $\tan^{-1} x = \theta$. Then $\cos \tan^{-1} x = \cos\theta = 1/\sqrt{(1 + x^2)} = z$ (say) .

Let $\cot^{-1} z = \varphi$. Then $\cot\varphi = z \Rightarrow \sin\varphi = 1/\sqrt{(1 + z^2)} = \sqrt{(1 + x^2)}/\sqrt{(2 + x^2)}$.

v. We have to evaluate , $\cos^{-1} x + \cos^{-1} [x/2 + \sqrt{(3 - 3x^2)}/2]$.

Let $\cos^{-1} x = \theta$. Then $\cos \theta = x$.

Now , $1/2 \le x \le 1 \Rightarrow \cos^{-1} 1 \le \cos^{-1} x \le \cos^{-1} 1/2 \Rightarrow 0 \le \theta \le \pi/3$.

Therefore , given expression = $\cos^{-1} x + \cos^{-1} [x . 1/2 + \sqrt{3}/2 . \sqrt{(1 - x^2)}]$

$= \theta + \cos -1 [\cos \theta . \cos \pi/3 + \sin \pi/3 . \sin \theta]$

$= \theta + \cos^{-1} \cos (\pi/3 - \theta) = \theta + \pi/3 - \theta = \pi/3$.

Q.2. Show that : $\tan^{-1} 1/2 + \tan^{-1} 1/3 = \pi/4$.

Solution : 2

We know that : $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\{(x + y)/(1 - xy)\}$, if $xy < 1$.

[Here $x = 1/2$, $y = 1/3$, $xy = 1/6 < 1$].

Hence, $\tan^{-1} 1/2 + \tan^{-1} 1/3$

$= \tan^{-1} [(1/2 + 1/3)/(1 - 1/2 \times 1/3)]$

$= \tan^{-1} [(5/6)/(5/6)] = \tan^{-1} 1 = \pi/4$. **[Proved.]**

Q.3. Prove that : $\tan^{-1} x + \cot^{-1} (x + 1) = \tan^{-1} (x^2 + x + 1)$

Solution : 3

L. H. S. = $\tan^{-1} x + \cot^{-1} (x + 1)$

$= \tan^{-1} x + \tan^{-1} \{1/(x + 1)\}$

$= \tan^{-1} [\{x + 1/(x + 1)\}/\{1 - x/(x + 1)\}]$

[As, $x \times\{1/ (x + 1)\} = x/(x + 1) < 1$]

$= \tan^{-1}(x^2 + x + 1)$ = R. H. S. **[Proved.]**

Q.4. Prove that : $\sin^{-1}x/\sqrt{(1 + x^2)} + \cos^{-1}(x + 1)/\sqrt{(x^2 + 2x + 2)} = \tan^{-1}(x^2 + x + 1)$.

Solution : 4

We know that $\sin^{-1} x = \tan^{-1}\{x/\sqrt{(1 - x^2)}\}$

Therefore, $\sin^{-1}\{x/\sqrt{(1 + x^2)}\} = \tan^{-1}[\{x/\sqrt{(1 + x^2)}\}/\sqrt{\{1 - x^2/(1 + x^2)\}}]$

$= \tan^{-1} x$..

Also, $\cos^{-1} x = \tan^{-1}\{\sqrt{(1 - x^2)}/x\}$

Therefore, $\cos^{-1}\{(x + 1)/\sqrt{(x^2 + 2x + 2)}\}$

$= \tan^{-1}[\sqrt{\{1 - (x + 1)2/(x^2 + 2x + 2)\}}/\{(x + 1)/\sqrt{(x^2 + 2x + 2)}\}]$

$= \tan^{-1}\{1/(x + 1)\} = \cot -1 (x + 1)$.

Now it is the question no. 3 above.

Q.5. Solve for x : $\tan^{-1}(2 + x) + \tan^{-1}(2 - x) = \tan^{-1}(2/3)$.

Solution : 5

We have, $\tan^{-1}(2 + x) + \tan^{-1}(2 - x) = \tan -1 (2/3)$

Or, $\tan^{-1}[\{(2 + x) + (2 - x)\}/\{1 - (2 + x)(2 - x)\}] = \tan^{-1}(2/3)$ $[(2 + x)(2 - x) < 1]$

Or, $\tan^{-1}[4/(x^2 - 3)] = \tan^{-1}(2/3)$ [x2 > 3]

Or, $4/(x^2 - 3) = 2/3$, $[| x | > \sqrt{3}]$

Or, $x^2 - 3 = 6$ $[| x | > \sqrt{3}]$

Or, $x^2 = 9$ $[| x | > \sqrt{3}]$

Or, x = 3 or –3, these values satisfy $| x | > \sqrt{3}$.

Q.6. Solve for x : $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$.

Solution : 6

We have, $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$

Or, $\tan^{-1} \{(2x + 3x)/(1 - 2x.3x)\} = \pi/4$ $2x.3x < 1$

Or, $5x/(1 - 6x2) = \tan \pi/4$, $x2 < 1/6$

Or, $5x/(1 - 6x2) = 1$, $| x | < 1/\sqrt{6}$

Or, $6x^2 + 5x - 1 = 0$, $- 1/\sqrt{6} < x < 1/\sqrt{6}$

Or, $(6x^{-1})(x + 1) = 0$, $- 1/\sqrt{6} < x < 1/\sqrt{6}$ Or, $x = 1/6$ or -1 , $- 1/\sqrt{6} < x < 1/\sqrt{6}$

But only $x = 1/6$ satisfy the condition $- 1/\sqrt{6} < x < 1/\sqrt{6}$,

Hence $x = 1/6$ is the solution of the given equation.

Q.7. Prove that : $\tan^{-1} 1/4 + \tan^{-1} 2/9 = \cos^{-1} 2/\sqrt{5}$.

Solution : 7

$\tan^{-1} 1/4 + \tan^{-1} 2/9$

$= \tan^{-1} \{(1/4 + 2/9)/(1 - 1/4 .2/9)$

$= \tan^{-1} \{(17/36)/(17/18)\} = \tan^{-1} 1/2$ --------------- (1)

we know that, $\cos^{-1} x = \tan -1 \{\sqrt{(1 - x^2)}/x\}$

Therefore, $\cos^{-1} 2/\sqrt{5} = \tan -1 [\{\sqrt{(1 - 4/5)}\}/(2/\sqrt{5})]$

$= \tan^{-1} 1/2$ ------------------------- (2)

From (1) and (2) we get , $\tan^{-1} 1/4 + \tan -1\ 2/9 = \cos^{-1} 2/\sqrt{5}$. **[Proved.]**

Q.8. Prove that : $4 \tan^{-1} 1/5 - \tan^{-1} 1/70 + \tan^{-1} 1/99 = \pi/4$.

Solution : 8

L. H. S. = 2 (2 tan $^{-1}$ 1/5) – (tan $^{-1}$ 1/70 – tan $^{-1}$ 1/99)

= 2 tan^{-1} [(2/5)/{1 – (1/5)2}] – tan $^{-1}$ [(1/70 – 1/99)/(1 + 1/70.1/99)]

= 2 tan $^{-1}$ 5/12 – tan $^{-1}$ 29/6931 = 2 tan $^{-1}$ 5/12 – tan $^{-1}$ 1/239

= tan $^{-1}$ [(2 × 5/12)/{1 – (5/12)2}] $^{-1}$ –1 1/239 = tan $^{-1}$ 120/119 – tan $^{-1}$ 1/239

= tan $^{-1}$ [(120/119 – 1/239)/(1 + 120/119 × 1/239)]

= tan $^{-1}$ (28561/28561) = tan $^{-1}$ 1 = π/4 = R. H. S. **[Proved.]**

Q.9. Prove that : 2 tan $^{-1}$ (1/3) + cot $^{-1}$ (4) = tan $^{-1}$ (16/13) .

Solution : 9

. H. S. = 2 tan $^{-1}$ (1/3) + cot $^{-1}$ (4)

= 2 tan $^{-1}$ (1/3) + tan $^{-1}$ (1/4)

= tan^{-1} [(2/3)/{1 – (1/3)2}] + tan $^{-1}$ (1/4)

= tan $^{-1}$ [(2/3)/(8/9)] + tan $^{-1}$ (1/4)

= tan $^{-1}$ (3/4) + tan $^{-1}$ (1/4)

= tan $^{-1}$ [(3/4 + 1/4)/(1 – 3/16)]

= tan $^{-1}$ [1/(13/16)] = tan $^{-1}$ (16/13) = R. H. S. **[Proved.]**

Q.10. Show that : sin $^{-1}$ (1/√17) + cos $^{-1}$ (9/√85) = tan $^{-1}$ (1/2) .

Solution : 10

Let sin $^{-1}$ (1/√17) = θ , then sin θ = 1/√17 ,

Therefore, tan θ = sin θ/cos θ = sin θ/√(1 – sin2 θ) = (1/√17)/√{1 – (1/√17)2}

Or, tan θ = 1/4 => θ = tan^{-1} (1/4) ;

Again , let $\cos^{-1}(9/\sqrt{85}) = \varphi$, then $\cos\varphi = 9/\sqrt{85}$,

Therefore , $\tan\varphi = \sin\varphi/\cos\varphi = \sqrt{(1 - \cos2\ \varphi)}/\cos\varphi = \sqrt{\{1 - (9/\sqrt{85})2\}}/9/\sqrt{85}$

Or, $\tan\varphi = 2/9 \Rightarrow \varphi = \tan^{-1}(2/9)$;

Now, L.H.S. = $\sin^{-1}(1/\sqrt{17}) + \cos -1\ (9/\sqrt{85})$

$= \tan^{-1}(1/4) + \tan^{-1}(2/9)$

$= \tan^{-1}[(1/4 + 2/9)/\{1 - (1/4)(2/9)\}]$

$= \tan^{-1}[\{(9 + 8)/36\}/\{(36 - 2)/36\}] = \tan^{-1}(17/34) = \tan -1\ (1/2)$ **[Proved.]**

Q.11. Prove that : $\cot(\pi/4 - 2\cot^{-1} 3) = 7$.

Solution : 11

L.H.S. = $\cot(\pi/4 - 2\cot^{-1} 3)$

$= \cot\{\pi/4 - 2\tan^{-1}(1/3)\}$ [As, $\cot^{-1} x = \tan^{-1}(1/x)$]

$= \cot[\tan^{-1}(1) - \tan^{-1}\{(2/3)/(1 - 1/9)\}]$

$= \cot[\tan^{-1}(1) - \tan^{-1}\{(2/3)/(8/9)\}]$

$= \cot[\tan^{-1}(1) - \tan^{-1}(3/4)]$

$= \cot[\tan^{-1}\{(1 - 3/4)/(1 + 3/4)\}]$

$= \cot[\tan^{-1}(1/7)] = \cot[\cot^{-1}(7)] = 7$ R.H.S. **[Proved.]**

Q.12. Prove that : $2(\tan^{-1} 1 + \tan^{-1} 1/2 + \tan^{-1} 1/3) = \pi$.

Solution : 12

L.H.S. = $2(\tan^{-1} 1 + \tan^{-1} 1/2 + \tan^{-1} 1/3)$

$= 2[\tan^{-1}(1 + 1/2)/(1 - 1/2) + \tan^{-1} 1/3]$

$= 2[\tan^{-1}\{(3/2)/(1/2)\} + \tan^{-1} 1/3]$

$= 2\ [\tan^{-1} 3 + \tan^{-1} 1/3]$

$= 2\ [\tan^{-1} \{(3 + 1/3)/(1 - 3 \times 1/3)\}]$

$= 2\ [\tan^{-1} \{(10/3)/0\}]$

$= 2 \tan^{-1} (\infty) = 2 \times \pi/2 = \pi$ = R.H.S. **[Proved.]**

Q.13. Show that : $\sin^{-1} 4/5 + \cos^{-1} 2/\sqrt{5} = \cot^{-1} 2/11$.

Solution : 13

We know that : $\sin^{-1} 4/5 = \tan^{-1} 4/3$ and $\cos^{-1} 2/\sqrt{5} = \tan^{-1} 1/2$.

L.H.S. = $\sin^{-1} 4/5 + \cos^{-1} 2/\sqrt{5}$

$= \tan^{-1} 4/3 + \tan^{-1} 1/2$

$= \tan^{-1} [(4/3 + 1/2)/\{1 - (4/3 \times 1/2)\}]$

$= \tan^{-1} [(11/6)/(6 - 4)/6] = \tan^{-1} [(11/6)/(2/6)]$

$= \tan^{-1} 11/2 = \cot^{-1} 2/11$ = R.H.S. **[Proved.]**

Q.14. Show that : $\sin^{-1} \sqrt{3}/2 + 2 \tan^{-1} 1/\sqrt{3} = 2\pi/3$.

Solution : 14

L.H.S. = $\sin^{-1} (\sqrt{3}/2) + 2 \tan^{-1} (1/\sqrt{3})$

$= \sin^{-1} (\sin 60^\circ) + 2 \tan^{-1} (\tan 30^\circ)$

$= 60^\circ + 2 \times 30^\circ = 60^\circ + 60^\circ = 120^\circ = 2\pi/3$ = R.H.S. **[Proved.]**

Q.15. Prove that : $\sin^{-1} x/\sqrt{(1+ x^2)} + \cos^{-1} (x+1)/\sqrt{(x^2 + 2x + 2)} = \tan^{-1} (x^2 + x + 1)$.

Solution : 15

Let $\sin^{-1} x/\sqrt{(1 + x^2)} = \theta \Rightarrow \sin\theta = x/\sqrt{(1 + x^2)}$

Therefore, $\tan\theta = \sin\theta/\sqrt{(1 - \sin 2\theta)}$

$= \{x/\sqrt{(1 + x^2)}\}/\sqrt{[1 - \{x/\sqrt{(1 + x^2)}\}2]}$

$= \{x/\sqrt{(1 + x^2)}\}/\{1/\sqrt{(1 + x^2)}\}$

$= x/1 \Rightarrow \theta = \tan^{-1} x$.

Again let $\cos^{-1} (x + 1)/\sqrt{(x^2 +2x + 2)} = \varphi \Rightarrow \cos\varphi = (x+ 1)/\sqrt{(x^2 + 2x + 2)}$

Therefore, $\tan\varphi = \sqrt{(1 - \cos 2\varphi)}/\cos\varphi = \sqrt{[1 - \{(x+1)/\sqrt{(x^2+2x +2)}\}2]}/(x+1)\sqrt{(x^2+2x+2)}$

$= 1/(x + 1) \Rightarrow \varphi = \tan^{-1} 1/(x+1)$

L.H.S. $= \tan^{-1} x + \tan^{-1} 1/(x + 1)$

$= \tan^{-1} [\{x + 1/(x+1)\}/\{1 - x/(x+1)\}] = \tan^{-1} (x^2 + x + 1) =$ R.H.S. **[Proved.]**

Q.16. Solve for x : $\tan^{-1} (x^{-1}) + \tan^{-1} x + \tan^{-1} (x+1) = \tan^{-1} 3x$.

Solution : 16

$\tan^{-1} (x - 1) + \tan -1\, x + \tan^{-1} (x+1) = \tan^{-1} 3x$

Or, $\tan^{-1} (x - 1) + \tan^{-1} (x+1) = \tan^{-1} 3x - \tan^{-1} x$

Or, $\tan^{-1} [(x - 1+ x + 1)/\{1 - (x - 1)(x + 1)\}] = \tan^{-1} [(3x - x)/(1 + 3x.x)]$

Or, $\tan^{-1} 2x/(2 - x^2) = \tan^{-1} 2x/(1 + 3x^2)$

Or, $2x/(2 - x^2) = 2x/(1 + 3x^2)$

Or, $x[1 + 3x^2 - 2 + x^2] = 0$

Or, $x = 0$ or $4x2 - 1 = 0$

Or, $x = \pm 1/2$.

Q.17. Solve the equation : $\tan^{-1} (2 + x) + \tan^{-1} (2 - x) = \tan^{-1} 2/3$.

Solution : 17

$\tan^{-1}(2 + x) + \tan^{-1}(2 - x) = \tan^{-1} 2/3$

Or, $\tan^{-1} [\{(2 + x) + (2 + x)\}/\{1 - (2 + x)(2 - x)\}] = \tan^{-1} 2/3$

Or, $\tan^{-1} [4/\{1 - (4 - x 2)\}] = \tan^{-1} 2/3$ Or, $\tan^{-1} [4/(x2 - 3)] = \tan^{-1} 2/3$

Or, $4/(x2 - 3) = 2/3$ Or, $2 (x2 - 3) = 12$

Or, $x2 - 3 = 6 \Rightarrow x2 = 9 \Rightarrow x \pm 3$.

Q.18. Solve the equation : $\sin^{-1}(6x) + \sin^{-1}(6\sqrt{3} x) = - \pi/2$.

Solution : 18

$\sin^{-1}(6x) + \sin^{-1}(6\sqrt{3} x) = - \pi/2$

Or, $\sin^{-1} [6x \sqrt{\{1 - (6\sqrt{3} x)2 \}} + 6\sqrt{3} x \sqrt{\{1 - (6x)^2\}}] = \sin^{-1}(-1)$

Or, $\sin^{-1} [6x\sqrt{(1 - 108 x^2)} + 6\sqrt{3} x \sqrt{(1 - 36 x^2)}] = \sin^{-1}(- 1)$

Or, $6x \sqrt{(1 - 108 x^2)} + 6\sqrt{3} x \sqrt{(1 - 36 x^2)} = - 1$

Or, $6x \sqrt{(1 - 108 x^2)} = - [1 + 6\sqrt{3} x \sqrt{(1 - 36 x^2)}]$

Squiring both sides we get

$36 x^2 (1 - 108 x^2) = 1 + 108 x^2 - 36 \times 108 x4 + 12\sqrt{3} x \sqrt{(1 - 36 x^2)}$

Or, $36 x^2 - 36 \times 108 x4 = 1 + 108 x^2 - 36 \times 108 x4 + 12\sqrt{3} x \sqrt{(1 - 36 x^2)}$

Or, $72 x^2 + 12\sqrt{3} x \sqrt{(1 - 36 x^2)} + 1 = 0$

Or, $12\sqrt{3} x \sqrt{(1 - 36 x^2)} = - (1 + 72 x^2)$

Squaring both sides again we get

$432x^2(1 - 36 x^2) = 1 + 72 \times 72 x^2 + 144 x^2$

Or, $288 \times 72 x4 - 288 x^2 + 1 = 0$

Or, $x^2 = [288 \pm \sqrt{\{(288)^2 - 288 \times 288\}}]/(2 \times 288 \times 72) = 1/144$

Or, $x = \pm 1/12 \Rightarrow x = 1/12$ or $-1/12$.

Q.19. Solve the equation : $\sin^{-1} 2a/(1 + a^2) + \sin^{-1} 2b/(1 + b^2) = 2 \tan^{-1} x$.

Solution : 19

Let $\tan \theta = a$, then $\sin^{-1} 2a/(1 + a^2) = \sin^{-1} [2\tan \theta/(1 + \tan 2\theta)]$

$= \sin^{-1} (\sin 2\theta)$

$= 2 \theta = 2 \tan^{-1} a$.

Similarily, $\sin^{-1} 2b/(1 + b^2) = 2\tan^{-1} b$.

Therefore, $\sin^{-1} 2a/(1 + a^2) + \sin^{-1} 2b/(1 + b^2)$

$= 2\tan^{-1} a + 2\tan -1\ b = 2 \tan^{-1} (a + b)/(1 - ab)$, provided $ab < 1$.

Therefore, $2 \tan^{-1} (a + b)/(1 - ab) = 2 \tan^{-1} x$

Or, $x = (a + b)/(1 - a b)$.

Q.20. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, prove that : $x^2 - y^2 - z^2 + 2yz\sqrt{(1 - x^2)} = 0$.

Solution : 20

$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

Or, $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$

Or, $\sin^{-1} [x \sqrt{(1 - y^2)} + y \sqrt{(1 - x^2)}] = \pi - \sin^{-1} z$

Or, $x \sqrt{(1 - y^2)} + y \sqrt{(1 - x^2)} = \sin(\pi - \sin -1\ z)$

Or, $x \sqrt{(1 - y^2)} + y \sqrt{(1 - x^2)} = z$

Or, $x \sqrt{(1 - y^2)} = z - y\sqrt{(1 - x^2)}$

Squaring we get,

$x^2(1 - y^2) = z^2 + y^2(1 - x^2) - 2yz\sqrt{(1 - x^2)}$

Or, $x^2 - y^2 - z^2 + 2yz\sqrt{(1 - x^2)} = 0$ **[Proved.]**