13. LIMITS AND DERIVATIVES

LIMITS

MEANING OF $x \to a$

If x be a variable and x takes values such as 1.99,1.999,1.9999, … . From these values it is clear that x takes these values, from left to right, the numerical difference between x and 2 gets closure and closure to 0. Similarly, if x take values 2.01,2.001,2.0001, … . Even then the numerical difference between x and 2 gets closure to 0. In such a situation, we say that x approaches to 2 and we write $x \to 2$.

In general, $x \to a$ means that the variable x and x takes values either less than or greater than that of a and the numerical difference between x and a can be made as small as we please.

Let $y = f(x)$ be a function of x and let a and k be the constant such that as $x \to a$, $f(x) = k$, the numerical value of the difference between $f(x)$ and k can be made as small as we possible by taking x is sufficiently closure to a. It can be symbolically written as: $\lim\limits_{x \to a} f(x) = k$.

Note: (i) $\mathop{Lt}\limits_{x \to a} f(x)$ is same as $\lim\limits_{x \to a} f(x)$.

(ii) $\lim\limits_{nx \to \infty} \left(Area\,of\;\; polygon\;\; of\;\; n\,sides \right) = Area\,of\;\; circle$

STANDARD RESULTS

- Limit of a constant function is a constant. i.e., If $f(x) = k$, then $\lim\limits_{x \to a} k = k$, where 'k' is any constant
- $\lim\limits_{x \to a} k.f(x) = k.\lim\limits_{x \to a} f(x)$
- $\lim\limits_{x \to a} \left[f(x) + g(x) \right] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} \left[f(x) - g(x) \right] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} \left[f(x).\, g(x) \right] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}, g(x) \neq 0$
- $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$

E.g.: i. $\lim_{x\to 1}\left(\frac{x+2}{x+5}\right)=\frac{1+2}{1+5}=\frac{3}{6}=\frac{1}{2}$

ii. $\lim_{x\to 2}\left(\frac{x^2+2x-5}{3x+5}\right)=\frac{2^2+2\times 2-5}{3\times 2+5}=\frac{4+4-5}{6+5}=\frac{3}{11}$

iii. $\lim_{x\to -2}\left(\frac{x^2+3x+2}{x+3}\right)=\frac{(-2)^2+3\times(-2)+2}{-2+3}=\frac{4-6+2}{1}=\frac{0}{1}=0$

iv. $\lim_{x\to 3}(3x+2)(x^2+2x)=(3\times 3+2)(3^2+2\times 3)=(9+2)(9+6)=11\times 15=165$

Evaluation of $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$

$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$, when $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, is known as indeterminate form.

The form $\frac{0}{0}$ is called indeterminate form.

Note: The other indeterminate forms are $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0, etc.

We cannot find the limits such functions directly. The following methods are used to find the limits:

1. **Factorization Method:**
   a) Factorize the numerator and denominator and cancel the common factors from the numerator and the denominator.
   b) Be sure that the limit of the resulting denominator is non-zero.
   c) Apply quotient rule of limit.

E.g.: Evaluate $\lim_{x\to 3}\frac{x^2-5x+6}{x-3}=\frac{0}{0}$

$=\lim_{x\to 3}\frac{(x-2)(x-3)}{x-3}=\lim_{x\to 3}(x-2)=3-2=1$

2. **Substitution Method:**

In this method, put $x=a+h$. As $x\to a$, $h\to 0$. Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{h\to 0}\frac{f(a+h)}{g(a+h)}$. It can be simplified by cancelling the powers of h and can be simplified.

E.g.: Evaluate $\lim_{x\to\pi}\frac{\sin x}{\pi-x}=\frac{\sin\pi}{\pi-\pi}=\frac{0}{0}$

put $x=\pi+h$ *as* $x\to\pi,\ h\to 0$

Now $\lim\limits_{x\to\pi}\dfrac{\sin x}{\pi-x}=\lim\limits_{x\to\pi}\dfrac{\sin(\pi+h)}{\pi-(\pi+h)}=\lim\limits_{x\to\pi}\dfrac{\sin(\pi+h)}{-h}=\lim\limits_{x\to\pi}\dfrac{-\sinh}{-h}=\lim\limits_{x\to\pi}\dfrac{\sinh}{h}=1$

3. Rationalization Method:

a) Rationalize the expression, which involve square roots.
b) Cancelling the factors from the numerator and denominator
c) Be sure that the limit of the resulting denominator is non-zero
d) Apply quotient rule of limit.

E.g. i. Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x}-1}{x}=\dfrac{\sqrt{1-0}-1}{0}=\dfrac{1-1}{0}=\dfrac{0}{0}$

$$=\lim_{x\to 0}\left[\frac{\left(\sqrt{1-x}-1\right)}{x}\cdot\frac{\left(\sqrt{1-x}+1\right)}{\left(\sqrt{1-x}+1\right)}\right]=\lim_{x\to 0}\left[\frac{1-x-1}{x\left(\sqrt{1-x}+1\right)}\right]$$

$$=\lim_{x\to 0}\left[\frac{-x}{x\left(\sqrt{1-x}+1\right)}\right]=\lim_{x\to 0}\left[\frac{-1}{\left(\sqrt{1-x}+1\right)}\right]=\frac{-1}{\left(\sqrt{1-0}+1\right)}=\frac{-1}{(1+1)}=\frac{-1}{2}=-\frac{1}{2}$$

ii) Evaluate $\lim\limits_{x\to 0}\left[\dfrac{\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\sin^{-1}x}\right]$

The expression $=\lim\limits_{x\to 0}\left[\dfrac{\left(\sqrt{1+x}-\sqrt{1-x}\right)}{\sin^{-1}x}\right]=\dfrac{1-1}{0}=\dfrac{0}{0}$

put $x=\sin\theta$. As $x\to 0,\ \sin\theta\to 0\Rightarrow\theta\to 0$

The expn. $=\lim\limits_{\theta\to 0}\left[\dfrac{\left(\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}\right)}{\theta}\right]$

$$=\lim_{\theta\to 0}\left[\frac{\left(\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}\right)}{\theta}\times\frac{\left(\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}\right)}{\left(\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}\right)}\right]$$

$$=\lim_{\theta\to 0}\left[\frac{1+\sin\theta-(1-\sin\theta)}{\theta\left(\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}\right)}\right]=\lim_{\theta\to 0}\left[\frac{2\sin\theta}{\theta\left(\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}\right)}\right]$$

$$=\lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\times\frac{2}{\left(\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}\right)}\right]=1\times\frac{2}{\left(\sqrt{1+0}+\sqrt{1-0}\right)}$$

$$=\frac{2}{1+1}=1\qquad\left[\because\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

Evaluation of $\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$ $, n, a \in R \; a > 0$

(Proof is not included. Please refer the class room note)

E.g.: i) Evaluate $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$

The expn. $= \lim_{x \to 0} \frac{(1+x)^n - 1}{x} = \lim_{x \to 0} \left[\frac{(1+x)^n - 1}{(1+x) - 1} \right]$

As $x \to 0, 1+x \to 1$

$$= \lim_{(1+x) \to 1} \left[\frac{(1+x)^n - 1^n}{(1+x) - 1} \right] = n.1^{n-1} = n \qquad (\because 1^n = 1)$$

ii) $\lim_{x \to 2} \frac{x^9 - 512}{x^4 - 16} = \lim_{x \to 2} \frac{x^9 - 2^9}{x^4 - 2^4} = \lim_{x \to 2} \left[\frac{\frac{x^9 - 2^9}{x-2}}{\frac{x^4 - 2^4}{x-2}} \right] = \frac{9 \times 2^{9-1}}{4 \times 2^{4-1}} = \frac{9 \times 2^8}{4 \times 2^3} = 9 \times 2^{8-5} = 9 \times 2^3 = 72$

If x **is measured in radians, then**

1. $\lim_{x \to 0} \sin x = 0$
2. $\lim_{x \to 0} \cos x = 1$
3. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
4. $\lim_{x \to 0} \frac{x}{\sin x} = 1$
5. $\lim_{x \to 0} \frac{\tan x}{x}$
6. $\lim_{x \to 0} \frac{x}{\tan x}$
7. $\lim_{x \to 0} \frac{\sin ax}{x} = a$
8. $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$
9. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

OTHER IMPORTANT THEOREMS

1. $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = \log e = 1$ $\left(\because \log e = \log_e^e = 1\right)$

2. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = \log e = 1$

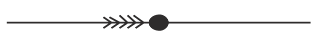
3. $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log_e^a = \log a, \ for \ a > 0$

E.g.: $lim_{x\to 0} \dfrac{e^{3x} - 1}{x} = \dfrac{e^0 - 1}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}$

$$= lim_{x\to 0} \frac{e^{3x} - 1}{3x} \times 3 = 3 \times lim_{x\to 0} \frac{e^{3x} - 1}{3x} = 3 \times 1 = 3$$

Meaning of $x \to a^-$

If the variable x takes values, which are close to a constant a and always remains on the left of a then we say that x approaches to a from left and we write as $x \to a^-$.



Meaning of $x \to a^+$

Similarly, If the variable x takes values, which are close to a constant a and always remains on the right of a then we say that x approaches to a from right and we write as $x \to a^+$.

LEFT OR LEFT HAND LIMIT AND RIGHT LIMIT OR RIGHT HAND LIMIT

If $f(x)$ be a function of x and a and l be any two constant, then $\lim\limits_{x\to a^-} f(x) = l$ is known as left hand limit or left limit of the function $f(x)$ and $\lim\limits_{x\to a^+} f(x) = l$, is known as right hand limit or right limit of the function $f(x)$.

Working rule:

1. To find $\lim\limits_{x\to a^-} f(x)$:

   put $x = a - h$

   as $x \to a^-, h \to 0^+$

   then $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0^+} f(a-h)$

2. To find $\lim_{x \to a^+} f(x)$,

put $x = a + h$

as $x \to a^+, h \to 0^+$

then $\lim_{x \to 0^+} f(x) = \lim_{h \to 0^+} f(a+h)$

EXISTENCE OF A FUNCTION

1. If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.
2. If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ exists and is equal to $\lim_{x \to a} f(x)$ is equal to $f(a)$.

E.g.: Does the function $\lim_{x \to 0} f(x)$ exists, if $f(x) = \begin{cases} \frac{x - |x|}{x}, & x \neq 0 \\ 2 & , x = 0 \end{cases}$

$$LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left[\frac{x - |x|}{x} \right]$$

Put $x = 0 - h$ *as* $x \to 0^-$, $h \to 0^+$

$$\lim_{x \to 0^-} \left[\frac{x - |x|}{x} \right] = \lim_{h \to 0^+} \left[\frac{(-h) - |-h|}{-h} \right] = \lim_{h \to 0^+} \left[\frac{(-h) - h}{-h} \right] = \lim_{h \to 0^+} \left[\frac{-2h}{-h} \right] = \lim_{h \to 0^+} 2 = 2$$

$$RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left[\frac{x - |x|}{x} \right]$$

Put $x = 0 + h$ *as* $x \to 0^+$, $h \to 0^+$

$$\lim_{x \to 0^+} \left[\frac{x - |x|}{x} \right] = \lim_{h \to 0^+} \left[\frac{h - |h|}{-h} \right] = \lim_{h \to 0^+} \left[\frac{0}{-h} \right] = \lim_{h \to 0^+} 0 = 0$$

$\therefore LHL \neq RHL$. $\therefore \lim_{x \to 0} f(x)$ does not exist.

DERIVATIVES

Derivative of a function

A function $f(x)$ is said to be derivable or differentiable if it is derivable at every points in its domain.

Suppose $f(x) = \frac{1}{x}$. Domain of the function is $R - \{0\}$

$f(x)$ is derivable at every point in R except 0.

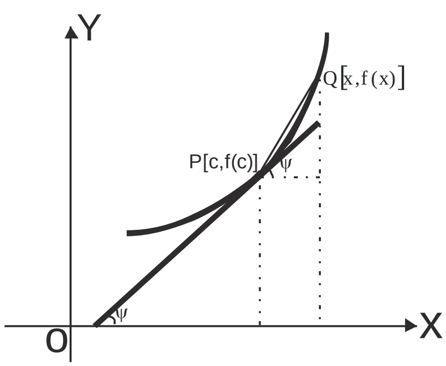
Derivability of a function on an interval

i. A function $f(x)$ is said to be a derivable function on the open interval (a,b), it is derivable at every points in the open interval (a,b).

ii. A function $f(x)$ is said to be a derivable function on the closed interval $[a,b]$,
   a. it is derivable at every points in the open interval (a,b),
   b. it is derivable at $x=a$ from right
   c. it is derivable at $x=b$ from left

Let $f(x)$ be a differentiable function on $[a,b]$. Then corresponding to each point $x\in[a,b]$, we get a unique real number equal to the derivative of $f'(x)$ and are denoted by $f'(x)$ or $\frac{dy}{dx}$ or Dy y_1 or y', etc.. i.e., $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ (or) $\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$. The process of obtaining the derivative of a function is called differentiation.

Geometrical meaning of the derivative at a point: Consider the curve $y=f(x)$. Let $f(x)$ is differentiable at $x=c$. Let $P[c, f(c)]$ be a point on the curve and let Q be a neighbouring point on the curve. Then slope of the chord $PQ=\frac{f(x)-f(c)}{x-c}$. Taking limit as $Q\to P$ i.e., $x\to c$, we get $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$. As $Q\to P$, the chord PQ becomes tangent at P.



Note: derivative of y w.r.t. x $=\frac{d}{dx}(y)=\frac{dy}{dx}$

derivative of y w.r.t. t $=\frac{d}{dt}(y)=\frac{dy}{dt}$

derivative of x w.r.t. t $=\frac{d}{dt}(x)=\frac{dx}{dt}$, etc.

Derivative of a function $y=f(x)$

Let $y=f(x)$ is a finite, single valued function of x. Let Δx be a small increment in x and Δy be the corresponding increment in y respectively.

Then $y+\Delta y = f(x+\Delta x)$

$\Delta y = f(x+\Delta x) - f(x)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

taking limits we have,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\boxed{\frac{dy}{dx} = f'(x)}$$

i.e., $\frac{d}{dx}[f(x)] = f'(x)$.This is called derivative of y w.r.t x or differential coefficient of y w.r.t x. This method is called first principles or delta (Δ *or* δ) method or differentiation by definition or ab initio.

Note: Other forms of $\frac{dy}{dx}$ **are** $f'(x),\ y',\ y_1,\ Dy,$ **etc..**

Note: If $y = f(x)$ is a real function defined at a real constant 'h', then

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(h)}{h}}$$

Find the derivative of the following functions using the first principle:

1. Let $f(x) = x^2$

$f(x+h) = (x+h)^2$

$f(x+h) - f(x) = (x+h)^2 - x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$$

$$= \lim_{h\to 0}\left(\frac{(x+h)^2 - x^2}{(x+h)-x}\right) = \lim_{(x+h)\to x}\left(\frac{(x+h)^2 - x^2}{(x+h)-x}\right) = 2x^{2-1} = 2x$$

i.e., $\frac{d}{dx}\left(x^2\right) = 2x$

2. Let $f(x) = e^x$

$f(x+h) = e^{x+h}$

$$f(x+h)-f(h)=e^{x+h}-e^x=e^xe^h-e^x=e^x\left(e^h-1\right)$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$$

$$=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}=e^x.\lim_{h\to 0}\frac{\left(e^h-1\right)}{h}$$

$$\frac{dy}{dx}=e^x\times 1=e^x \qquad \left(\because \lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=\log e=1\right)$$

$$\frac{d}{dx}\left(e^x\right)=e^x$$

3. Let $f(x)=a^x$

$$f(x+h)=a^{x+h}$$

$$f(x+h)-f(h)=a^{x+h}-a^x=a^xa^h-a^x=a^x\left(a^h-1\right)$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(h)}{h}$$

$$=\lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h}=a^x.\lim_{h\to 0}\frac{\left(a^h-1\right)}{h}$$

$$\frac{dy}{dx}=a^x\times 1=a^x \qquad \left(\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\right)$$

$$\frac{d}{dx}\left(a^x\right)=a^x$$

$$\frac{dy}{dx}=a^x\times\log a=a^x \qquad \left(\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\right)$$

$$\frac{d}{dx}\left(a^x\right)=a^x.\log a$$

4. Let $f(x)=\sqrt{x}$

$$f(x+h)=\sqrt{x+h}$$

$$f(x+h)-f(x)=\sqrt{x+h}-\sqrt{x}=\left(\sqrt{x+h}-\sqrt{x}\right)\times\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{x+\Delta x-x}{\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{h}{\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

$$= \lim_{h \to 0} \left[\frac{h}{\left(\sqrt{x+h} + \sqrt{x}\right) \times h} \right] = \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$\frac{dy}{dx} = \frac{1}{\left(\sqrt{x+0} + \sqrt{x}\right)} = \frac{1}{\left(\sqrt{x} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

5. Let $f(x) = \frac{1}{x^2} = x^{-2}$

$$f(x+h) = \frac{1}{(x+h)^2} = (x+h)^{-2}$$

$$f(x+h) - f(x) = (x+h)^{-2} - x^{-2}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^{-2} - x^{-2}}{h} = \lim_{(x+h) \to x} \frac{(x+h)^{-2} - x^{-2}}{(x+h) - x} = (-2)x^{-2-1} = (-2)x^{-3} = \frac{-2}{x^3}$$

6. Let $f(x) = \frac{1}{x^n} = x^{-n}$

$$f(x+h) = \frac{1}{(x+h)^n} = (x+h)^{-n}$$

$$f(x+h) - f(x) = (x+h)^{-n} - x^{-n}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^{-n} - x^{-n}}{h} = \lim_{(x+h) \to x} \frac{(x+h)^{-n} - x^{-n}}{(x+h) - x} = (-n)x^{-n-1} = (-n)x^{-(n+1)} = \frac{-n}{x^{n+1}}$$

7. Let $f(x) = \sin x$

$$f(x+h) = \sin(x+h)$$

$$f(x+h)-f(x)=\sin(x+h)-\sin x=2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right) \;=2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$f'(x) \quad =\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$\therefore\ f'(x) \ =\lim_{h\to 0}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right]=\lim_{h\to 0}\left[2\cos\left(\frac{2x+h}{2}\right)\times\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}\times 2}\right]$$

$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+h}{2}\right)\times\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] \qquad \left(as\ \ h\to 0,\ \frac{h}{2}\to 0\right)$$

$$=\cos x\times 1=\cos x \qquad \left(\because\ \lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

i.e., $\frac{d}{dx}(\sin x)=\cos x$

8. Let $f(x)=\cos x$

$$f(x+h)=\cos(x+h)$$

$$f(x+h)-f(x)=\cos(x+h)-\cos x=-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)=-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

$$f'(x) \quad =\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$=\lim_{h\to 0}\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}=\lim_{h\to 0}\left[-2\sin\left(\frac{2x+h}{2}\right)\times\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}\times 2}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h}{2}\right)\times\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] \qquad \left(as\ h\to 0\ ,\frac{h}{2}\to 0\right)$$

$$=\left[-\sin\left(\frac{2x+0}{2}\right)\times 1\right] \qquad \left(\because\ \lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

i.e., $\frac{dy}{dx} \ =-\sin\left(\frac{2x}{2}\right)=-\sin x$

$$\frac{d}{dx}(\cos x) = -\sin x$$

9. Let $f(x) = tan\,x$

$f(x+h) = tan(x+h)$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[tan(x+h) - tan\,x\right] = \lim_{h\to 0}\left[\frac{sin(x+h)}{cos(x+h)} - \frac{sin\,x}{cos\,x}\right]$$

$$= \frac{sin(x+h)cos\,x - cos(x+h)sin\,x}{cos(x+h)cos\,x}$$

$$= \lim_{h\to 0}\left[\frac{sin(x+h-x)}{cos(x+h)cos\,x \times h}\right] = \lim_{h\to 0}\left[\frac{1}{cos(x+h)cos\,x} \times \frac{sinh}{h}\right] \qquad \left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \left(\frac{1}{\cos(x+0)\cos x} \times 1\right)$$

$$\frac{dy}{dx} = \frac{1}{\cos x \cos x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

10. Let $f(x) = cot\,x$

$f(x+h) = cot(x+h)$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[cot(x+h) - cot\,x\right] = \lim_{h\to 0}\left[\frac{cos(x+h)}{sin(x+h)} - \frac{cos\,x}{sin\,x}\right]$$

$$= \frac{sin\,x\,cos(x+h) - cos\,x\,sin(x+h)}{sin(x+h)sin\,x}$$

$$= \lim_{h\to 0}\left[\frac{sin(x-(x+h))}{sin(x+h)sin\,x \times h}\right] = \lim_{h\to 0}\left[\frac{1}{sin(x+h)sin\,x} \times \frac{sin(-h)}{h}\right] \qquad \left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= \lim_{h\to 0}\left[\frac{-1}{sin(x+h)\,sin\,x}\times\frac{sinh}{h}\right]=\left(\frac{-1}{sin(x+0)\,sin\,x}\times 1\right)$$

$$\frac{dy}{dx} = \frac{-1}{sin\,x\,sin\,x} = -\frac{1}{sin^2 x} = -cosec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cos}ec^2 x$$

11. Let $f(x)=sec\,x$

$$f(x+h)=sec(x+h)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[sec(x+h)-sec\,x\right]=\lim_{h\to 0}\left[\frac{1}{cos(x+h)}-\frac{1}{cos\,x}\right]=\lim_{h\to 0}\left[\frac{cos\,x-cos(x+h)}{cos(x+h)cos\,x}\right]$$

$$= \lim_{h\to 0}\left\{\frac{-2\,sin\left(\frac{x+x+h}{2}\right)sin\left[\frac{x-(x-h)}{2}\right]}{cos(x+h)cos\,x\times h}\right\}=\lim_{h\to 0}\left[\frac{-2\,sin\left(\frac{2x+h}{2}\right)sin\left(\frac{-h}{2}\right)}{cos(x+h)cos\,x\times h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2\,sin\left(\frac{2x+h}{2}\right)}{cos(x+h)cos\,x}\times\frac{-sin\left(\frac{h}{2}\right)}{\frac{h}{2}\times 2}\right]=\lim_{h\to 0}\left[\frac{sin\left(\frac{2x+h}{2}\right)}{cos(x+h)cos\,x}\times\frac{sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]$$

$$= \left[\frac{\sin\left(\frac{2x+0}{2}\right)}{\cos(x+0)\cos x}\times 1\right] \quad \left(\because \lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos x.\cos x} = \frac{1}{\cos x}.\frac{\sin x}{\cos x} = \sec x.\tan x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

12. Let $f(x)=cosec\,x$

$$f(x+h)=sec(x+h)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left[cosec(x+h)-cosecx\right] = \lim_{h\to 0}\left[\frac{1}{sin(x+h)}-\frac{1}{sin x}\right] = \lim_{h\to 0}\left[\frac{sin x - sin(x+h)}{sin(x+h)sin x}\right]$$

$$= \lim_{h\to 0}\left\{\frac{2cos\left(\frac{x+x+h}{2}\right)sin\left[\frac{x-(x-h)}{2}\right]}{sin(x+h)sin x\times h}\right\} = \lim_{h\to 0}\left[\frac{2cos\left(\frac{2x+h}{2}\right)sin\left(\frac{-h}{2}\right)}{sin(x+h)sin x\,\Delta x}\right]$$

$$= \lim_{h\to 0}\left[\frac{2cos\left(\frac{2x+h}{2}\right)}{sin(x+h)sin x}\times\frac{-sin\left(\frac{h}{2}\right)}{\frac{h}{2}\times 2}\right] = \lim_{h\to 0}\left[-\frac{cos\left(\frac{2x+h}{2}\right)}{sin(x+h)sin x}\times\frac{sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]$$

$$= \lim_{h\to 0}\left[-\frac{cos\left(\frac{2x+h}{2}\right)}{sin(x+h)sin x}\times\frac{sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] = \left[-\frac{cos\left(\frac{2x+0}{2}\right)}{sin(x+0)sin x}\times 1\right] \quad \left(\because \lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$\frac{dy}{dx} = -\frac{\cos x}{\sin x.\sin x} = \frac{1}{\sin x}.\frac{\cos x}{\sin x} = -\cos ecx.\cot x$$

$$\frac{d}{dx}(\cos ecx) = -\cos ecx.\cot x$$

STANDARD RESULTS

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cos ecx$	$-\cos ecx \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\cos ec^2 x$
x^n	nx^{n-1}
e^x	e^x

e^{-x}	$-e^{x}$
a^{x}	$a^{x}.\log a$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\log x$	$\frac{1}{x}$
x	1
x^{2}	$2x$
$\boldsymbol{\frac{1}{x^{n}}}$	$\boldsymbol{-\frac{1}{x^{n+1}}}$
$\frac{1}{x}$	$-\frac{1}{x^{2}}$
$\frac{1}{x^{2}}$	$-\frac{2}{x^{3}}$
y	$\frac{dy}{dx}$
y^{2}	$2y\frac{dy}{dx}$

Note: Derivative of any trigonometric function starting with 'co' is negative.

FUNDAMENTAL RESULTS IN DIFFERENTIATION

1. **Differential coefficient of a constant is zero**. i.e., $\boldsymbol{\frac{d}{dx}(c)=0}$, where c is a constant.

   E.g.: $\frac{d}{dx}(5)=0$, $\frac{d}{dx}(-10)=0$, etc.

2. If u and v are functions of x, then $\boldsymbol{\frac{d}{dx}(u\pm v)=\frac{d}{dx}(u)\pm\frac{d}{dx}(v)}$

   $$\frac{d}{dx}(5\sin x+\log x) = \frac{d}{dx}(5\sin x)+\frac{d}{dx}(\log x)=5\frac{d}{dx}(\sin x)+\frac{d}{dx}(\log x)=5\cos x+\frac{1}{x}$$

   $$\frac{d}{dx}\left(2e^{x}-\tan x\right)=\frac{d}{dx}\left(2e^{x}\right)-\frac{d}{dx}(\tan x)=2\frac{d}{dx}\left(e^{x}\right)-\frac{d}{dx}(\tan x)=2e^{x}-\sec^{2}x$$

3. **Product rule**: If u and v are functions of x, then derivative of the product of two functions is equal to *first function x derivative of the second function + (plus) second function x derivative of the first function*.

i.e., $\boldsymbol{\frac{d}{dx}(uv) = u.\frac{d}{dx}(v) + v.\frac{d}{dx}(u)}$

E.g.: i. $y = e^{3x}\sin 4x$

$$\frac{dy}{dx} = e^{3x}\frac{d}{dx}(\sin 4x) + \sin 4x.\frac{d}{dx}\left(e^{3x}\right)$$

$$= e^{3x}.\cos 4x.\ 4 + \sin 4x.e^{3x}.3 = e^{3x}(4\cos 4x + 3\sin 4x)$$

ii. $y = x^2 \tan x$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}\left(x^2\right)$$

$$= x^2\sec^2 x + \tan x.2x = x^2\sec^2 x + 2x\tan x$$

Corollary of product rule:

If u, v and w are functions of x, then $\boldsymbol{\frac{d}{dx}(uvw) = uv.\frac{d}{dx}(w) + vw.\frac{d}{dx}(u) + uw.\frac{d}{dx}(v)}$

E.g.: $y = x^2 e^x \tan x$

$$\frac{dy}{dx} = x^2 e^x\frac{d}{dx}(\tan x) + e^x\tan x\frac{d}{dx}\left(x^2\right) + x^2\tan x\frac{d}{dx}\left(e^x\right)$$

$$= x^2 e^x\sec^2 x + e^x\tan x.2x + x^2\tan x.e^x$$

$$= xe^x\left(x\sec^2 x + 2\tan x. + x\tan x.\right) = xe^x\left(x\sec^2 x + (2+x)\tan x.\right)$$

4. **QUOTIENT FORMULA**: If u and v are any two functions of x, then quotient of two functions is equal to (2nd function x derivative of the 1st function minus 1st function x derivative of the 2nd function) divided by square of the 2nd function.

i.e., $\boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{d}{dx}(u) - u.\frac{d}{dx}(v)}{v^2}}$

E.g.: $y = \frac{Sin\,x + Cos\,x}{Sin\,x - Cos\,x}$.

$$\frac{dy}{dx} = \frac{(\sin x - \cos x).\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x).-(\sin x - \cos x) - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} = \frac{(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= \frac{\sin^2 x - 2\sin x.\cos x + \cos^2 x - (\sin^2 x + 2\sin x.\cos x + \cos^2 x)}{(\sin x - \cos x)^2}$$

$$= \frac{\sin^2 x - 2\sin x.\cos x + \cos^2 x - \sin^2 x - 2\sin x.\cos x - \cos^2 x)}{(\sin x - \cos x)^2}$$

$$= \frac{-2\sin x.\cos x - 2\sin x.\cos x}{(\sin x - \cos x)^2} = \frac{-2.2\sin x.\cos x}{(\sin x - \cos x)^2} = \frac{-2\ \sin 2x}{(\sin x - \cos x)^2}$$

FUNCTION OF A FUNCTION

Let $y = f(u)$, where $u = \phi(x)$, then y is called function of a function from.

Note:

$$\frac{d}{dx}\{f[\phi(x)]\} = f'[\phi(x)] \times \phi'(x)$$

E.g.: $f(x) = \sqrt{2x+3}$

$$f'(x) = \frac{1}{2\sqrt{2x+3}} \times \frac{d}{dx}(2x+3) = \frac{1}{2\sqrt{2x+3}} \times 2 \times 1 = \frac{1}{\sqrt{2x+3}}$$

ii. $y = e^{-ax^2}$

$$\frac{dy}{dx} = e^{-ax^2} \times \frac{d}{dx}\left(-ax^2\right) = e^{-ax^2} \times -a \times 2x = -2axe^{-ax^2}$$

iii. $y = sin^2 x$

$$\frac{dy}{dx} = 2\,sin\,x \times \frac{d}{dx}(sin\,x) = 2\,sin\,x \times cos\,x = sin\,2x$$

iv. $f(x) = sin^n x$

$$f'(x) = n\,sin^{n-1} x \times cos\,x$$