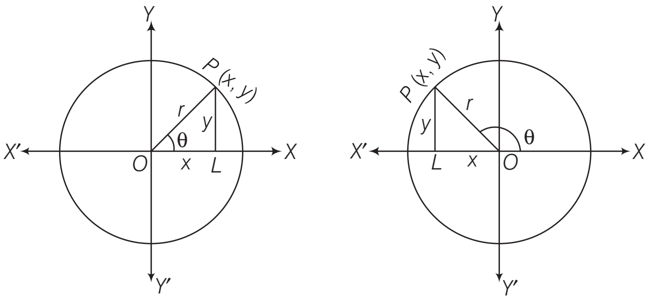
Session 4

Signs and Graph of Trigonometric Functions

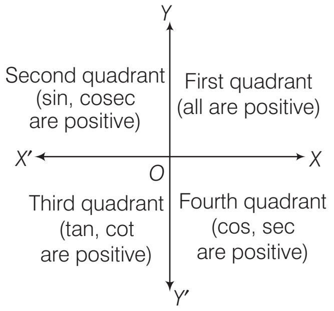
Signs of Trigonometric Functions

The signs of the trigonometric ratios of an angle depend on the quadrant in which the terminal side of the angle lies. We always take $OP = r$ to be positive (see figure). Thus the signs of all the trigonometric ratios depend on the signs of x and/or y.



An angle is said to be in that quadrant in which its terminal ray lies

For positive acute angles this definition gives the same result as in case of a right angled triangle since x and y are both positive for any point in the first quadrant and consequently they are the length of base and perpendicular of the angle θ.



1. Clearly in first quadrant $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\text{cosec}\,\theta$ are all positive as x, y are positive.
2. In second quadrant, x is negative and y is positive, therefore, only $\sin\theta$ and $\text{cosec}\,\theta$ are positive.
3. In third quadrant, x and y are both negative, therefore, only $\tan\theta$ and $\cot\theta$ are positive.
4. In fourth quadrant, x is positive and y is negative, therefore, only $\cos\theta$ and $\sec\theta$ are positive.

Quadrant →	I	II	III	IV
$\sin\theta$	+	+	–	–
$\cos\theta$	+	–	–	+
$\tan\theta$	+	–	+	–
$\text{cosec}\,\theta$	+	+	–	–
$\sec\theta$	+	–	–	+
$\cot\theta$	+	–	+	–

Variation in the Values of Trigonometric Functions in Different Quadrants

We observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, $\sin x$ increases from 0 to 1 and in the second quadrant as x increases from $\frac{\pi}{2}$ to π, $\sin x$ decreases from 1 to 0.

In the third quadrant, as x increases from π to $\frac{3\pi}{2}$, $\sin x$ decreases from 0 to -1 and finally, in the fourth quadrant, $\sin x$ increases from -1 to 0 as x increase from $\frac{3\pi}{2}$ to 2π.

Function	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	↑ from 0 to 1	↓ from 1 to 0	↓ from 0 to -1	↑ from -1 to 0
$\cos\theta$	↓ from 1 to 0	↓ from 0 to -1	↑ from -1 to 0	↑ from 0 to 1
$\tan\theta$	↑ from 0 to ∞	↑ from $-\infty$ to 0	↑ from 0 to ∞	↑ from $-\infty$ to 0
$\cot\theta$	↓ from ∞ to 0	↓ from 0 to $-\infty$	↓ from ∞ to 0	↓ from 0 to $-\infty$
$\sec\theta$	↑ from 1 to ∞	↑ from $-\infty$ to -1	↓ from -1 to $-\infty$	↓ from ∞ to 1
$\text{cosec}\,\theta$	↓ from ∞ to 1	↑ from 1 to ∞	↑ from $-\infty$ to -1	↓ from -1 to $-\infty$

Note

$+\infty$ and $-\infty$ are two symbols. These are not real numbers. When we say that $\tan\theta$ increases from 0 to ∞ as θ varies from 0 to $\frac{\pi}{2}$, it means that $\tan\theta$ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and it attains arbitrarily large positive values as θ tends to $\frac{\pi}{2}$. This rule applies to other trigonometric functions also.

Graphs of Trigonometric Functions

As in case of algebraic function, we can have some idea about the nature of a trigonometric function by its graph. Graph has many important applications in mathematical problems. We shall discuss the graphs of trigonometrical functions. We know that $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π and $\tan x$ and $\cot x$ are trigonometric functions of period π. Also if the period of function $f(x)$ is T, then period of $f(ax+b)$ is $\frac{T}{|a|}$.

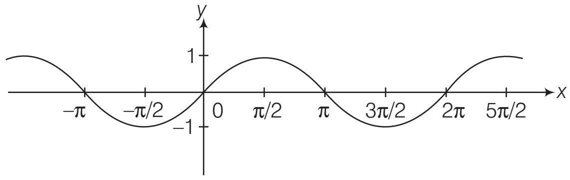
Graph and Other Useful Data of Trigonometric Functions

1. $\boldsymbol{y = f(x) = \sin x}$

   Domain $\rightarrow R$,

   Range $\rightarrow [-1, 1]$

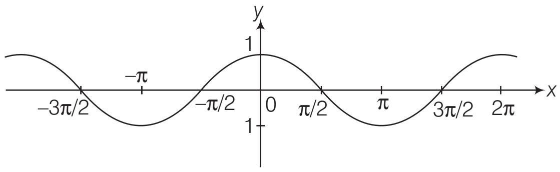
   Period $\rightarrow 2\pi$



2. $\boldsymbol{y = f(x) = \cos x}$

   Domain $\rightarrow R$, Range $\rightarrow [-1, 1]$

   Period $\rightarrow 2\pi$

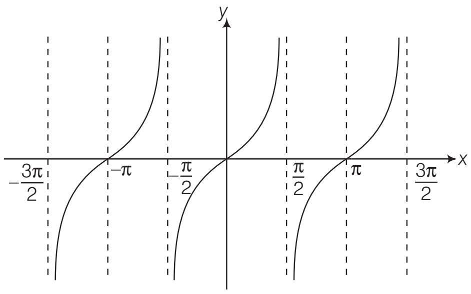


3. $\boldsymbol{y = f(x) = \tan x}$

   Domain $\rightarrow R \sim (2n+1)\frac{\pi}{2}, n \in I$

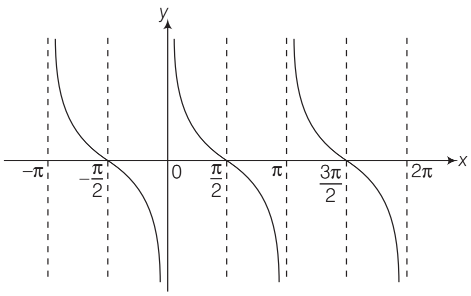
   Range $\rightarrow (-\infty, \infty)$

   Period $\rightarrow \pi$



4. $\boldsymbol{y = f(x) = \cot x}$

   Domain $\rightarrow R \sim n\pi, n \in I$; Range $\rightarrow (-\infty, \infty)$; Period $\rightarrow \pi$,

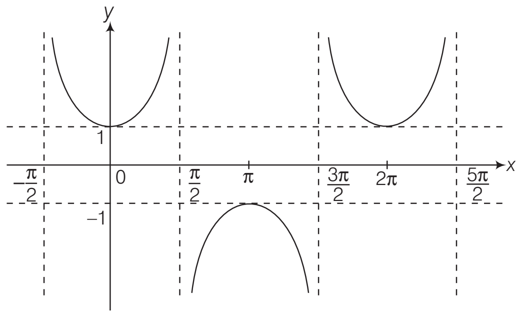


5. $\boldsymbol{y = f(x) = \sec x}$

   Domain $\rightarrow R \sim (2n+1)\frac{\pi}{2}, n \in I$

   Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

   Period $\rightarrow 2\pi$, $\sec^2 x$, $|\sec x| \in [1, \infty)$

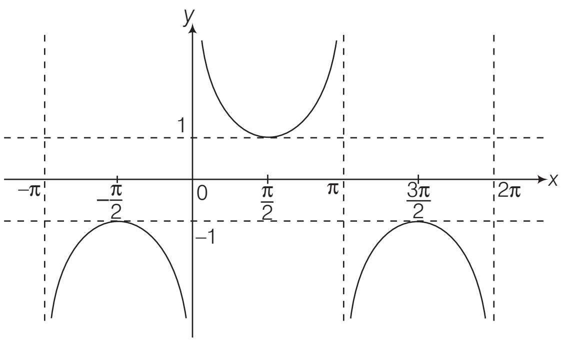


6. $\boldsymbol{y = f(x) = \operatorname{cosec} x}$

   Domain $\rightarrow R \sim n\pi, n \in I$;

   Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

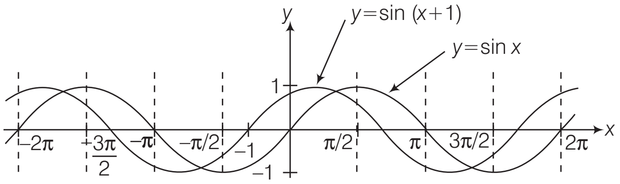
   Period $\rightarrow 2\pi$, $\operatorname{cosec}^2 x$, $|\operatorname{cosec} x| \in [1, \infty)$



Transformation of the Graphs of Trigonometric Functions

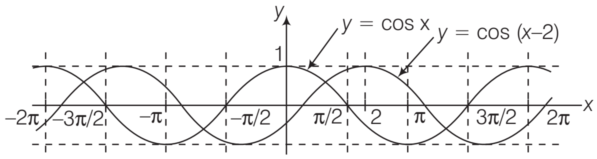
1. To draw the graph of $y = f(x+a)$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units left along the x-axis.

   Consider the following illustration.



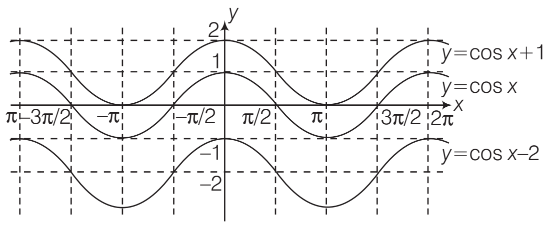
   To draw the graph of $y = f(x-a)$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units right along the x-axis.

   Consider the following illustration.

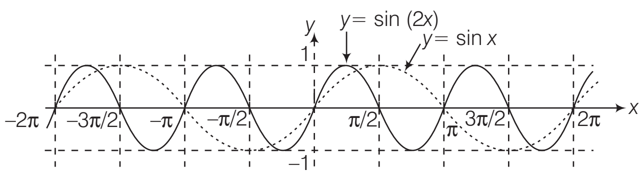


2. To draw the graph of $y = f(x) + a$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units upwards along the y-axis.

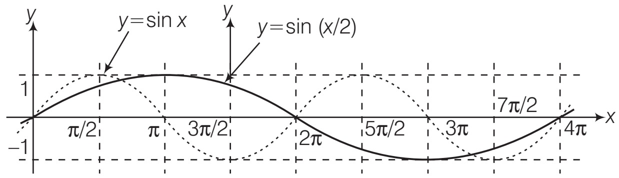
   To draw the graph of $y = f(x) - a$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units downward along the y-axis.



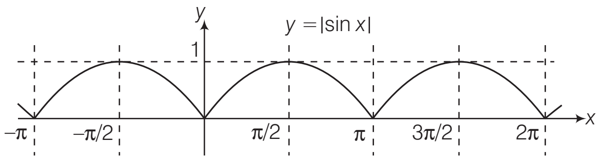
3. If $y = f(x)$ has period T, then period of $y = f(ax)$ is $\frac{T}{|a|}$.



   Period of $y = \sin(2x)$ is $\frac{2\pi}{2} = \pi$



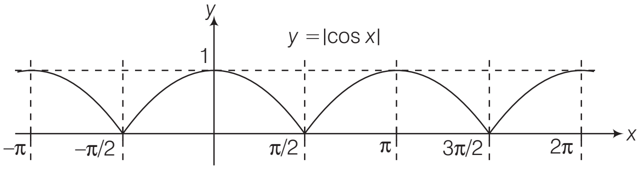
   Period of $y = \sin\left(\frac{x}{2}\right)$ is $\frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$



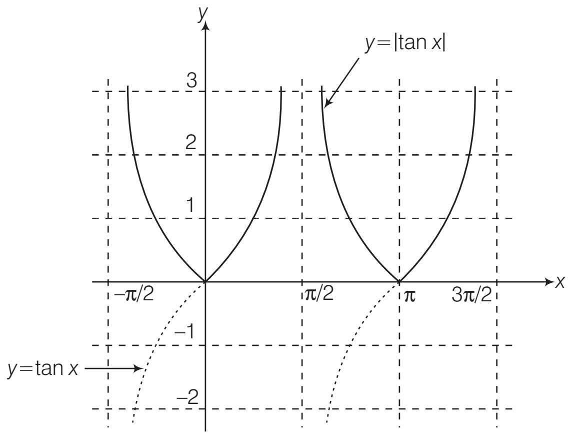
4. Since $y = |f(x)| \geq 0$, to draw the graph of $y = |f(x)|$, take the mirror of the graph of $y = f(x)$ in the x-axis for $f(x) < 0$, retaining the graph for $f(x) > 0$.

   Consider the following illustrations.

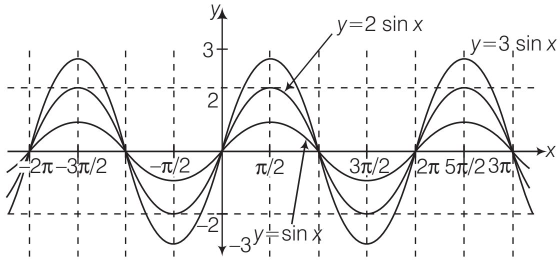
   Here, period of $f(x) = |\sin x|$ is π.

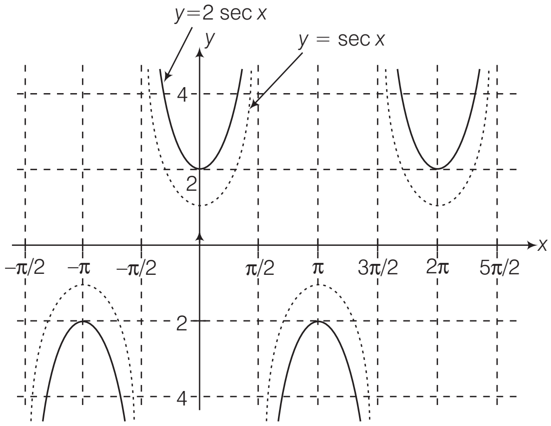


   Here, period of $f(x) = |\cos x|$ is π.



5. Graph of $y = af(x)$ from the graph of $y\ f(x)$

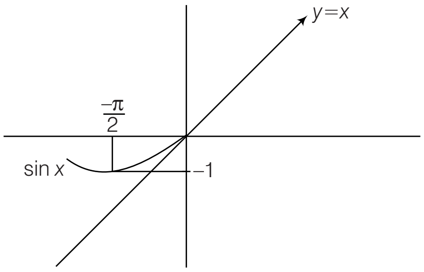




Some Important Graphical Deductions

To find relation between $\sin x$, x and $\tan x$

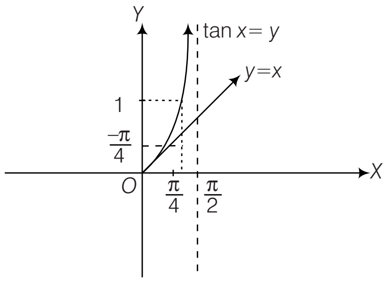
(i)



Thus, when $-\infty < x < 0$

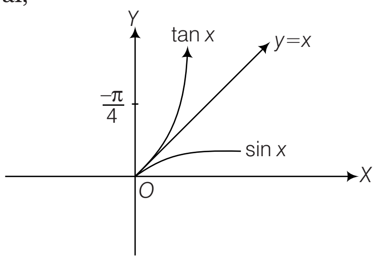
$\Rightarrow \sin x > x$

(ii)



$\therefore \quad \tan x > x$, when $0 > x > \frac{\pi}{2}$

(iii) In general,



Thus, $\tan x > x > \sin x, \forall\ x \in \left(0, \frac{\pi}{2}\right)$

and $\sin x > x > \tan x, \forall\ x \in \left(-\frac{\pi}{2}, 0\right)$.

Example 32. Find the values of the other five trigonometric functions in each of the following questions

(i) $\tan\theta = \frac{5}{12}$, where θ is in third quadrant.

(ii) $\sin\theta = \frac{3}{5}$, where θ is in second quadrant.

Sol. (i) Since θ is in third quadrant,

$\therefore$ Only $\tan\theta$ and $\cot\theta$ are positive

Now, $\quad \tan\theta = \frac{5}{12}$

Therefore, $\quad \cot\theta = \frac{12}{5}$,

$\sin\theta = -\frac{5}{13}$,

$\text{cosec} = -\frac{13}{5}$

$\cos\theta = -\frac{12}{13}$ and $\sec\theta = -\frac{13}{12}$.

(ii) Since θ is in the second quadrant,

$\therefore$ Only $\sin\theta$ and $\text{cosec}\,\theta$ will be positive.

Now, $\quad \sin\theta = \frac{3}{5}$.

Therefore,

$\text{cosec}\,\theta = \frac{5}{3}, \cos\theta = -\frac{4}{5},$

$\sec\theta = -\frac{5}{4}, \tan\theta = -\frac{3}{4}$

and $\quad \cot\theta = -\frac{4}{3}$.

Example 33. If $\sin\theta = \frac{12}{13}$ and θ lies in the second quadrant, find the value of $\sec\theta + \tan\theta$.

Sol. We have $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \quad \cos\theta = \pm\sqrt{1 - \sin^2\theta}$

In the second quadrant, $\cos\theta$ is negative

$\therefore \quad \cos\theta = -\sqrt{1 - \sin^2\theta}$

Now, $\sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta}$

$$= \frac{1 + \sin\theta}{-\sqrt{1 - \sin^2\theta}} = \frac{1 + \frac{12}{13}}{-\sqrt{1 - \left(\frac{12}{13}\right)^2}}$$

$$= \frac{\frac{25}{13}}{-\sqrt{\frac{25}{169}}} = \frac{\frac{25}{13}}{-\frac{5}{13}} = -5$$

Example 34. Draw the graph of $y = 3\sin 2x$.

Sol. $\sin x$ is a periodic function with period 2π, therefore,

$\sin 2x$ will be a periodic function of period $\frac{2\pi}{|2|} = \pi$

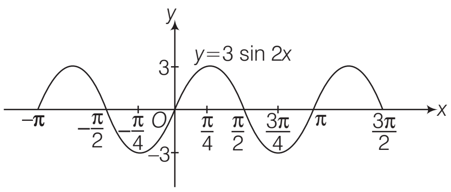
Also $\quad -1 \le \sin 2x \le 1$

$\therefore \quad -3 \le 3\sin 2x \le 3$

In order to draw the graph of $y = 3\sin 2x$, draw the graph of $y = \sin x$ and on X-axis change k to $\frac{k}{2}$, i.e. write $\frac{k}{2}$ wherever it is k. For example, write 15° in place of 30°, 45° in place of 90° etc.

On Y-axis change k to $3k$, i.e. write $3k$ wherever it is k for example, write 3 in place of 1, − 3 in place of − 1, 1.5 in place of 0.5 etc.

The graph of $y = 3\sin 2x$ will be as given in the figure.



Example 35. Draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$

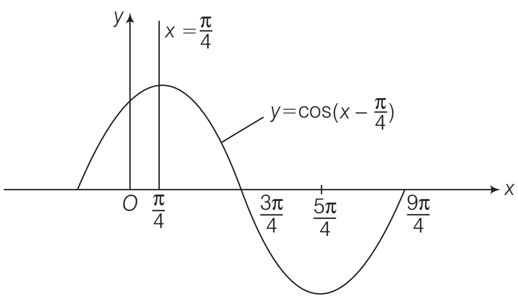
Sol. Given function is $y = \cos\left(x - \frac{\pi}{4}\right)$ ...(i)

Given function is $Y = \cos X$, where

$$X = x - \frac{\pi}{4} \text{ and } Y = y$$

or $\quad Y = 0 \Rightarrow y = 0 \text{ and } X = 0$

$\Rightarrow \quad x - \frac{\pi}{4} = 0 \Rightarrow x = \frac{\pi}{4}$



In order to draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$, we draw the graph of $y = \cos x$ and shift it on the right side through a distance of $\frac{\pi}{4}$ unit.

Example 36. Which of the following is the least?

(a) $\sin 3$ (b) $\sin 2$

(c) $\sin 1$ (d) $\sin 7$

Sol. (a) $\sin 3 = \sin[\pi - (\pi - 3)] = \sin(\pi - 3) = \sin(0.14)$

$\sin 2 = \sin[\pi - (\pi - 2)]$

$= \sin(\pi - 2) = \sin(1.14)$

$\sin 7 = \sin[2\pi + (7 - 2\pi)]$

$= \sin(7 - 2\pi) = \sin(0.72)$

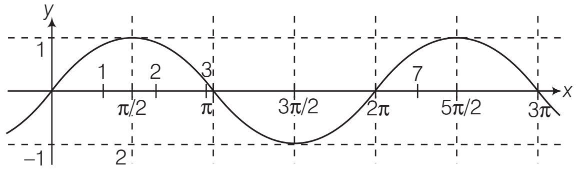
Now, $\quad 1.14 > 1 > 0.72 > 0.14$

$\Rightarrow \quad \sin(1.14) > \sin 1 > \sin(0.72) > \sin(0.14)$

[as 1.14, 0.72, 0.14 lie in the first quadrant and sine functions increase in the first quadrant]

Hence, among the given values, $\sin 3$ is the least.

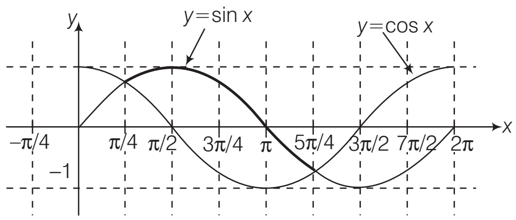
Alternate solution



From the graph, obviously $\sin 3$ is the least.

Example 37. *Find the value of* x *for which* $f(x) = \sqrt{\sin x - \cos x}$ ***is defined,*** $x \in [0, 2\pi]$.

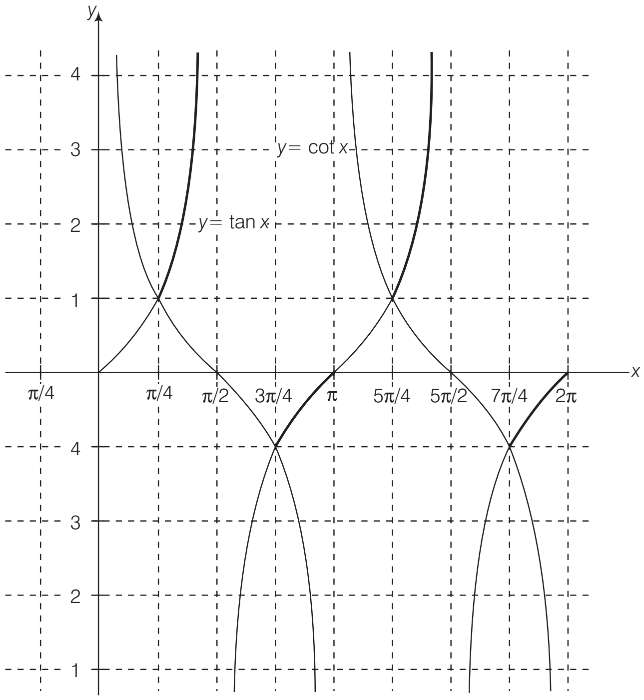
Sol. $f(x) = \sqrt{\sin x - \cos x}$ is defined if $\sin x \ge \cos x$.



From the graph, $\sin x \ge \cos x$, for $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Example 38. ***Solve*** $\tan x > \cot x$, ***where*** $x \in [0, 2\pi]$.

Sol.



We find that $\tan x \geq \cot x$. Therefore, the values of $\tan x$ are more than the value of $\cot x$.

That is, the value of x for which graph of $y = \tan x$ is above the graph of $y = \cot x$.

From the graph, it is clear that

$$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right) \cup \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right).$$

Exercise for Session 4

1. If $\tan x = -\frac{4}{3}, \frac{3\pi}{2} < x < 2\pi$, find the value of $9\sec^2 x - 4\cot x$.

2. Show that $\sin^2 x = p + \frac{1}{p}$ is impossible if x is real.

3. If $\cos x = \frac{3}{5}$ and x lies in the fourth quadrant find the values of $\operatorname{cosec} x + \cot x$.

4. Draw the graph of $y = \sin x$ and $y = \sin \frac{x}{2}$.

5. Draw the graph of $y = \sec^2 x - \tan^2 x$. Is $f(x)$ periodic? If yes, what is its fundamental period?

6. Prove that $\sin \theta < \theta < \tan \theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$.

7. Find the value of x for which $f(x) = \sqrt{\sin x - \cos x}$ is defined, $x \in [0, 2\pi]$.

8. Draw the graph of $y = \sin x$ and $y = \cos x$, $0 \leq x \leq 2\pi$.

9. Draw the graph of $y = \tan(3x)$.

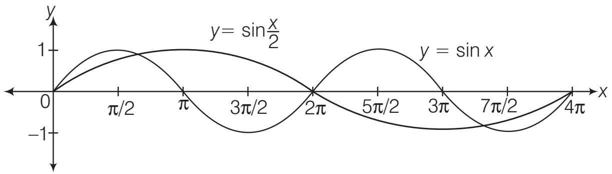
10. If $\cos x = -\frac{\sqrt{15}}{4}$ and $\frac{\pi}{2} < x < \pi$, find the value of $\sin x$.

Answers

Exercise for Session 4

1. 28 **3.** − 2

4.



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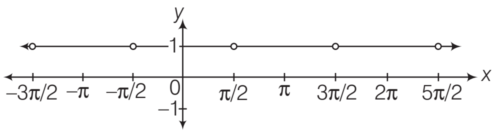
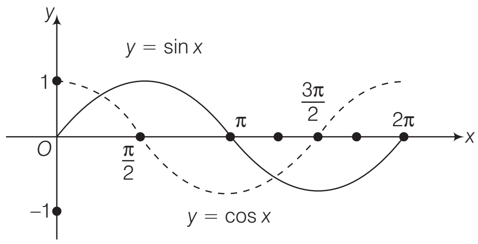


From the graph, the period of the function is π.

7. $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

8.



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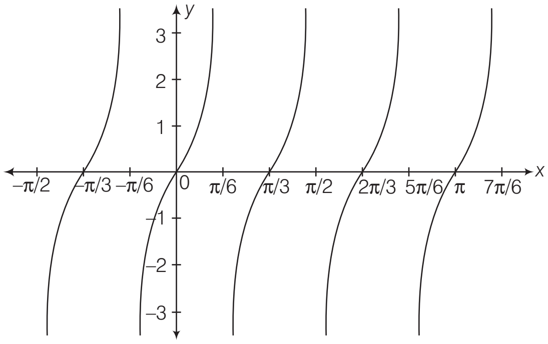

10. $\dfrac{1}{4}$