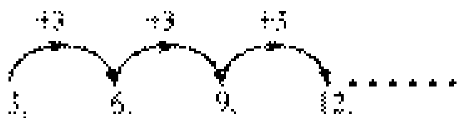
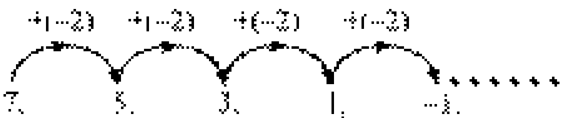
Progressions

- **The Concept of Arithmetic Progression**
  - An arithmetic progression is a list of numbers in which the difference between any two consecutive terms is equal.
  - In an AP, each term, except the first term, is obtained by adding a fixed number called common difference to the preceding term.
  - The common difference of an AP can be positive, negative or zero.

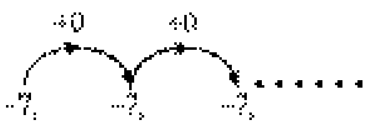
Example 1:

1. 

is an AP whose first term and common difference are 3 and 3 respectively.

2. 

is an AP whose first term and common difference are 7 and –2 respectively.

3. 

is an AP whose first term and common difference are –7 and 0 respectively.

  - The general form of an AP can be written as a, $a + d$, $a + 2d$, $a + 3d$ …, where a is the first term and d is the common difference.
  - A given list of numbers i.e., a_1, a_2, a_3 … forms an AP if $a_{k+1} - a_k$ is the same for all values of k.

Example 2:

Which of the following lists of numbers forms an AP? If it forms an AP, then write its next three terms.

(a) –4, 0, 4, 8, …

(b) 2, 4, 8, 16, …

Solution:

(a) –4, 0, 4, 8, …

$a_2 - a_1 = 0 - (-4) = 4$
$a_3 - a_2 = 4 - 0 = 4$
$a_4 - a_3 = 8 - 4 = 4$
$a_{n+1} - a_n = 4$; for all values of n

Therefore, the given list of numbers forms an AP with 4 being its common difference.
The next three terms of the AP are 8 + 4 = 12, 12 + 4 = 16, 16 + 4 = 20
Hence, AP: –4, 0, 4, 8, 12, 16, 20 …

(b) 2, 4, 8, 16, …

$a_2 - a_1 = 4 - 2 = 2$
$a_3 - a_2 = 8 - 4 = 4$
$a_3 - a_2 \neq a_2 - a_1$

Therefore, the given list of numbers does not form an AP.

- **The terminology related to arithmetic progression**
  - An arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.
  - The fixed number is called the common difference (d) of the A.P. The common difference can be either positive or negative or zero.
- **The general form of an A.P.**
  - a, $(a + d)$, $(a + 2d)$, $(a + 3d)$, ... , $[a + (n - 1)d]$, ... where *a is the* first term and *d is* common difference
- **Type of AP**
  - Finite AP: The APs have finite number of terms.
  - Infinite AP: The APs have not finite number of terms.

- **n^{th} term of an AP**

The n^{th} term (a_n) of an AP with first term a and common difference d is given by $a_n = a + (n - 1)\, d$.
Here, a_n is called the general term of the AP.

- **n^{th} term from the end of an AP**

The n^{th} term from the end of an AP with last term l and common difference d is given by $l - (n - 1) d$.

Example:

Find the 12^{th} term of the AP 5, 9, 13 …

Solution:

Here, $a = 5$, $d = 9 - 5 = 4$, $n = 12$

$$a_{12} = a + (n - 1) d$$
$$= 5 + (12 - 1) 4$$
$$= 5 + 11 \times 4$$
$$= 5 + 44$$
$$= 49$$

- **Sum of *n* terms of an AP**
  - The sum of the first n terms of an AP is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$ Sn=n22a+n-1d , where a is the first term and d is the common difference.
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  - If there are only n terms in an AP, then $S_n = \frac{n}{2}[a + l]$ Sn=n2a+l, where $l = a_n$ is the last term.

Example :

Find the value of 2 + 10 + 18 + …. + 802.

Solution:

2, 10, 18… 802 is an AP where $a = 2$, $d = 8$, and $l = 802$.

Let there be n terms in the series. Then,

$a_n = 802$

$\Rightarrow a + (n - 1) d = 802$

$\Rightarrow 2 + (n - 1) 8 = 802$

$\Rightarrow 8(n - 1) = 800$

$\Rightarrow n - 1 = 100$

$\Rightarrow n = 101$

Thus, required sum =
$\frac{n}{2}(a + l) = \frac{101}{2}(2 + 802) = 40602$ n2a+l = 10122+802 = 40602

- **Properties of an Arithmetic progression**
  - If a constant is added or subtracted or multiplied to each term of an A.P. then the resulting sequence is also an A.P.
  - If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

- **Arithmetic mean**
  - For any two numbers *a* and *b*, we can insert a number A between them such that *a*, A, *b* is an A.P. Such a number i.e., A is called the arithmetic mean (A.M) of numbers *a* and *b* and it is given by $A = \frac{a+b}{2}$.
  - For any two given numbers *a* and *b*, we can insert as many numbers between them as we want such that the resulting sequence becomes an A.P.

Let $A_1, A_2 \ldots A_n$ be *n* numbers between *a* and *b* such that $a, A_1, A_2 \ldots A_n, b$ is an A.P.

Here, common difference (*d*) is given by $\frac{b-a}{n+1}$.

Example:
Insert three numbers between –2 and 18 such that the resulting sequence is an A.P.
Solution:
Let A_1, A_2, and A_3 be three numbers between – 2 and 18 such that – 2, A_1, A_2, A_3, 18 are in an A.P.
Here, $a = -2$, $b = 18$, $n = 5$
$\therefore 18 = -2 + (5 - 1)\, d$
$\Rightarrow 20 = 4\, d$
$\Rightarrow d = 5$
Thus, $A_1 = a + d = -2 + 5 = 3$
$A_2 = a + 2d = -2 + 10 = 8$
$A_3 = a + 3d = -2 + 15 = 13$
Hence, the required three numbers between –2 and 18 are 3, 8, and 13.

- **Geometric Progression:** A sequence is said to be a geometric progression (G.P.) if the ratio of any term to its preceding term is the same throughout. This constant factor is called the common ratio and it is denoted by *r*.

- In standard form, the G.P. is written as a, ar, ar^2 ... where, a is the first term and r is the common ratio.

- **General Term of a G.P.:** The n^{th} term (or general term) of a G.P. is given by $a_n = ar^{n-1}$

Example: Find the number of terms in G.P. 5, 20, 80 ... 5120.

Solution: Let the number of terms be n.

Here $a = 5$, $r = 4$ and $t_n = 5120$

n^{th} term of G.P. = ar^{n-1}

$$\therefore 5(4)^{n-1} = 5120$$
$$\Rightarrow 4^{n-1} = \frac{5120}{5} = 1024$$
$$\Rightarrow (2)^{2n-2} = (2)^{10}$$
$$\Rightarrow 2n - 2 = 10$$
$$\Rightarrow 2n = 12$$
$$\therefore n = 6$$

- **Sum of n Term of a G.P.:** The sum of n terms (S_n) of a G.P. is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & \text{if } r < 1 \quad \text{or} \quad \frac{a(r^n-1)}{r-1}, \text{ if } r > 1 \\ na, & \text{if } r = 1 \end{cases}$$

Example: Find the sum of the series 1 + 3 + 9 + 27 + ... to 10 terms.

Solution: The sequence 1, 3, 9, 27, ... is a G.P.

Here, $a = 1$, $r = 3$.

Sum of n terms of G.P. = $\frac{a(r^n-1)}{r-1}$ $[r>1]$

$S_{10} = 1 + 3 + 9 + 27 + \ldots$ to 10 terms

$= \frac{1\times[(3)^{10}-1]}{(3-1)}$

$= \frac{59049-1}{2}$

$= \frac{59048}{2}$

$= 29524$

- Three consecutive terms can be taken as $\frac{a}{r}$, a, ar ar, a, ar. Here, common ratio is r.
- Four consecutive terms can be taken as $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3 ar3, ar, ar, ar3. Here, common ratio is r^2 r2.

- **Geometric Mean:** For any two positive numbers a and b, we can insert a number G between them such that a, G, b is a G.P. Such a number i.e., G is called a geometric mean (G.M.) and is given by $G = \sqrt{ab}$

In general, if $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then $G_1, G_2, \ldots, G_n$ are given by

$G_1 = ar$, $G_2 = ar^2, \ldots, G_n = ar^n$

Where, r is calculated from the relation $b = ar^{n+1}$, that is $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

Example: Insert three geometric means between 2 and 162.
Solution:

Let G_1, G_2, G_3 be 3 G.M.'s between 2 and 162.

Therefor 2, G_1, G_2, G_3, 162 are in G.P.

Let r be the common ratio of G.P.

Here, $a = 2$, $b = 162$ and $n = 3$

$$r = \left(\frac{162}{2}\right)^{\frac{1}{3+1}} = (81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$$

$G_1 = ar = 2 \times 3 = 6$
$G_2 = ar^2 = 2 \times (3)^2 = 2 \times 9 = 18$
$G_3 = ar^3 = 2 \times (3)^3 = 2 \times 27 = 54$

Thus, the required three geometric means between 2 and 162 are 6, 18, and 54.

- **Relation between A.M. and G.M.:** Let A and G be the respective A.M. and G.M. of two given positive real numbers a and b. Accordingly, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$.

Then, we will always have the following relationship between the A.M. and G.M.: $A \geq G$