CHAPTER

5

CALCULUS

Calculus is the branch of mathematics that deals with continuous change. It helps us to understand the changes between the values which are related by a function. Calculus Math mainly focused on some important topics such as differentiation, integration, limits, functions, and so on.

LIMIT

If $f(x)$ approaches to a real number l, when x approaches to a *i.e.*, if $f(x) \to l$ when $x \to a$, then l is called the limit of the function $f(x)$. In symbolic form, it can be written as:

$$\lim_{x\to a} f(x) = l$$

LEFT HAND AND RIGHT HAND LIMIT

$$\text{LHL} = \lim_{x\to a^-} f(x) = l_1$$

$$\text{RHL} = \lim_{x\to a^+} f(x) = l_2$$

Formulae for Limits

(*a*) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$

(*b*) $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan x}$

(*c*) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin^{-1} x}$

(*d*) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan^{-1} x}$

(*e*) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a,\ a > 0$

(*f*) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(*g*) $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

(*h*) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(*i*) $\lim_{x\to 0} (1+kx)^{1/x} = e^k$, where k is any constant

(*j*) $\lim_{x\to \infty} \frac{\log_a(1+x)}{x} = \log_a e,\ a > 0 \neq 1$

(*k*) $\lim_{x\to \infty} \frac{\sin x}{x} = \lim_{x\to \infty} \frac{\cos x}{x} = 0$

(*l*) $\lim_{x\to \infty} \cos x = 1$

CONTINUITY AT A POINT

A function $f(x)$ is continuous at a point $x = a$, were $a \in$ domain of $f(x)$, if

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

where, $\lim_{x\to a^-} f(x)$ is Left Hand Limit (LHL) of $f(x)$ at $x = a$

$\lim_{x\to a^+} f(x)$ is Right Hand Limit (RHL) of $f(x)$ at $x = a$

and $f(a)$ is the value of $f(x)$ at $x = a$.

DIFFERENTIABILITY

Let $f(x)$ be defined at any point c in the interval (a, b). Then $f(x)$ is said to be differentiable at $x = c$ if the function has a derivative at this point, *i.e.*, if $f'(c)$ exists. Hence, if

$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists, then the function $f(x)$ is called differentiable at point $x = c$.

For the existence of this limit it is necessary that when $h \to 0$, the *left-hand* and *right-hand* limits both must exist and they must be equal.

The Right Hand Derivative (RHD) of $f(x)$ at the point $x = c$ is defined as

$$Rf'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}, h > 0$$

Similarly, the Left Hand Derivative (LHD) of $f(x)$ at $x = c$ is defined as

$$Lf'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}, h > 0$$

or

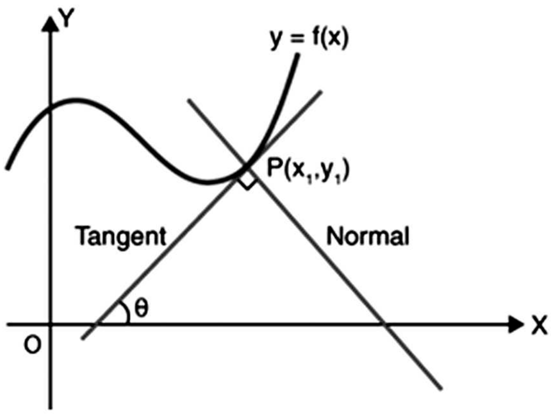
$$Lf'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}, h > 0$$

Hence, function $f(x)$ is differentiable at $x = c$ iff

$$Rf'(c) = Lf'(c).$$

TANGENT AND NORMAL LINES

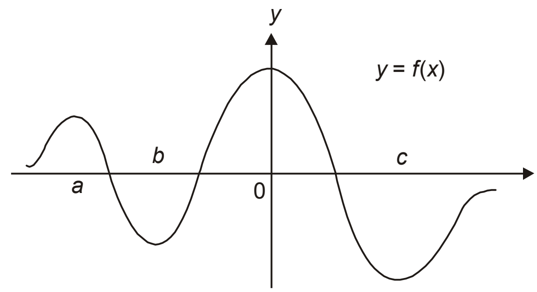
In calculus, the tangent line touches the curve at exactly one point and this line has the same slope as the curve at that point. The line which is perpendicular to the tangent line is called the normal line. These two lines can be shown as given in the below figure:



For a function $y = f(x)$, the slope of the tangent line is given by dy/dx.

MAXIMA AND MINIMA

Maxima and minima are known as the extrema of a function. Maxima and minima are the maximum or the minimum value of a function within the given set of ranges.



There are two types of maxima and minima that exist in a function, which are:

- Local Maxima and Minima
- Absolute or Global Maxima and Minima

A second-order derivative test for maxima and minima tests whether the slope is equal to 0 at the critical point $x = c (f'(c) = 0)$, at which point we find the second derivative of the function. Within the given range, if the second derivative of the function is present:

- **Local maxima :** If $f''(c) < 0$
- **Local Minima :** If $f''(c) > 0$
- **Test fails :** If $f''(c) = 0$

INTEGRATION

INTEGRATION BY SUBSTITUTION

Let f be a continuous function of u and u be a continuously differentiable function of x on $[a, b]$. Then

$$\int f(u)du = \int f(u(x))u'(x)\,dx$$

Integration by parts is the inverse of the product rule.

INTEGRATION BY PARTS

Let f and g be continuously differentiable functions on $[a, b]$. Then

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

This rule is often written with differentials in the form $\int u\,dv = uv - \int v\,du.$

Standard Formulas for Indefinite Integration

(a) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

(b) $\int a^x dx = \frac{a^x}{lna} + c$

(c) $\int e^x dx = e^x + c$

(d) $\int \frac{1}{x} dx = lnx + c$

(e) $\int \sin x \, dx = -\cos x + c$

(f) $\int \cos x \, dx = \sin x + c$

(g) $\int \sec^2 x \, dx = \tan x + c$

(h) $\int \text{cosec}^2 x \, dx = -\cot x + c$

(i) $\int \sec x \tan x \, dx = \sec x + c$

(j) $\int \text{cosec } x \cot x \, dx = -\text{cosec } x + c$

(k) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

(l) $\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + c$

(m) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

(n) $\int \frac{1}{1+x^2} dx = -\cot^{-1} x + c$

(o) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

(p) $\int \frac{1}{x\sqrt{x^2-1}} dx = -\text{cosec}^{-1} x + c$

DIFFERENTIAL EQUATIONS WITH VARIABLE SEPARABLE

We know that the first order, first degree differential equation is of the form:

$$\frac{dy}{dx} = F(x, y) \quad ...(i)$$

If $F(x, y)$ is expressed as the product of $g(x)\, h(y)$, where $g(x)$ is the function of x and $h(y)$ is the function of y, then the differential equation is said to be of variable separable type. Thus, the differential equation (*i*) takes the form:

$$\frac{dy}{dx} = g(x).h(y) \quad ...(ii)$$

If $h(y) \neq 0$, and separating the variables, equation (*ii*) becomes

$$(1/h(y))\; dy = g(x)\; dx \quad ...(iii)$$

Now, integrate the equation (*iii*) on both sides, we get

$$\int (1/h(y))dy = \int g(x)dx \quad ...(iv)$$

Hence, equation (*iv*) provides the solution for the differential equation in the form:

$$H(y) = G(x) + C$$

Where $H(y)$ and $G(x)$ are the antiderivatives of $1/h(y)$ and $g(x)$, respectively and C is called the arbitrary constant.

MULTIPLE CHOICE QUESTIONS

1. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is:

A. 10π B. 12π
C. 8π D. 11π

2. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is:

A. 0 B. 1
C. 3 D. $\frac{1}{3}$

3. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}, 0 \le x \le 1$ is:

A. $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ B. $\frac{1}{2}$
C. 1 D. 0

4. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?

A. $(-1, 1)$ B. $(0, 1)$
C. $\left(\frac{\pi}{2}, \pi\right)$ D. $\left(0, \frac{\pi}{2}\right)$

5. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of:

A. 1 m^3/h B. 0.1 m^3/h
C. 1.1 m^3/h D. 0.5 m^3/h

6. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is:

A. $\frac{22}{7}$ B. $\frac{6}{7}$
C. $\frac{7}{6}$ D. $\frac{-6}{7}$

7. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is:

A. 1 B. 2
C. 3 D. $\frac{1}{2}$

8. The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is:

A. $x + y = 0$ B. $x - y = 0$
C. $x + y + 1 = 0$ D. $x - y + 1 = 0$

9. The function $f : R \to R$ given by $f(x) = -|x - 1|$ is:

A. continuous as well as differentiable at $x = 1$
B. not continuous but differentiable at $x = 1$
C. continuous but not differentiable at $x = 1$
D. neither continuous nor differentiable at $x = 1$

10. The function $f(x) = \cot x$ is discontinuous on the set:

A. $\{x = n\pi : n \in Z\}$
B. $\{x = 2n\pi : n \in Z\}$
C. $\left\{x = (2n+1)\frac{\pi}{2};\ n \in Z\right\}$
D. $\left\{x = \frac{n\pi}{2};\ n \in Z\right\}$

11. The function $f(x) = \frac{x-1}{x(x^2-1)}$ is discontinuous at:

A. exactly one point
B. exactly two points
C. exactly three points
D. no point

12. If $f(x) = x^2 \sin\frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is:

A. 0 B. –1
C. 1 D. None of these

13. The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $\left[0, \sqrt{3}\right]$ is:

A. 1 B. –1
C. $\frac{3}{2}$ D. $\frac{1}{3}$

14. If $y = \log\sqrt{\tan x}$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is:

A. 0 B. 1
C. $\frac{1}{2}$ D. ∞

15. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to:

A. $\frac{\cos x}{2y-1}$ B. $\frac{\cos x}{1-2y}$
C. $\frac{\sin x}{1-2y}$ D. $\frac{\sin x}{2y-1}$

16. What is the value of the given limit, $\lim_{x\to 0}\frac{2x}{x}$?

A. 2 B. 0
C. $\frac{1}{2}$ D. $\frac{2}{3}$

17. The value of $\lim_{x\to 0}\frac{8^x - 4^x}{4x}$ is:

A. $\frac{1}{2}\ln 2$ B. $\frac{1}{4}\ln 2$
C. 0 D. ln 2

18. $\lim_{x\to\frac{x}{2}}\left[x\tan x - \frac{\pi}{2}\sec x\right] =$

A. 0 B. 1
C. –1 D. $\frac{\pi}{2}$

19. If $y = Ae^{5x} + Be^{-5x}$, then $\frac{d^2y}{dx^2}$ is equal to:

A. $25y$ B. $5y$
C. $-25y$ D. $15y$

20. $x = at^2$, $y = 2at$, then $\frac{d^2y}{dx^2}$ at $t = 2$ is:

A. $\frac{1}{4}$
B. $\frac{-1}{4}$
C. $\frac{1}{16a}$
D. $\frac{-1}{16a}$

ANSWERS

1	2	3	4	5	6	7	8	9	10
B	D	C	D	A	B	A	B	C	A
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
C	A	A	B	A	A	B	C	A	D

EXPLANATORY ANSWERS

1. Area of circle (A) = πr^2

The rate of change of area w.r.t. its radius $r = \frac{dA}{dr}$

Now, A = πr^2

$$\therefore \quad \frac{dA}{dr} = 2\pi \times 6$$

$$= 12\pi \text{ cm}^2\text{/radius.}$$

2. Let $y = \frac{1-x+x^2}{1+x+x^2}$

$$\frac{dy}{dx} = \frac{(-1+2x)(1+x+x^2)-(1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

Numerator of $\frac{dy}{dx} = 2(x-1)(x+1)$,

$\therefore \frac{dy}{dx} = \frac{2(x-1)(x+1)}{(x^2+x+1)^2}, \frac{dy}{dx} = 0$ at $x = 1, -1$

At $x = 1$, $\frac{dy}{dx}$ changes sign from –ve to +ve

$\therefore$ y is minimum at $x = 1$

Minimum value of $\frac{1-x+x^2}{1+x+x^2} = \frac{1-1+1}{1+1+1} = \frac{1}{3}$.

3. Let $y = [x(x-1)+1]^{1/3}$

$\therefore \frac{dy}{dx} = \frac{(2x-1)}{3[x(x-1)+1]^{2/3}}, \frac{dy}{dx} = 0$ at $x = \frac{1}{2}$

$\frac{dy}{dx}$ changes sign from –ve to +ve at $x = \frac{1}{2}$

$\therefore$ y is minimum at $x = \frac{1}{2}$

Value of y at $x = 0$, $(0+1)^{1/3} = 1^{1/3} = 1$

Value of y at $x = 1$, $(0+1)^{1/3} = 1^{1/3} = 1$

$\therefore$ The maximum value of y is 1.

4. Let $f(x) = x^{100} + \sin x - 1$

$\therefore f(x) = 100\,x^{99} + \cos x$

(A) for $(-1, 1)$ *i.e.*, $-1 < x < 1$, $-1 < x^{99} < 1$

$\Rightarrow -100 < 100\,x^{99} < 100$

Also $0 < \cos x < 1$

$\Rightarrow f'(x)$ can either be +ve or –ve on $(-1, 1)$

$\therefore f(x)$ is neither increasing nor decreasing on $(-1, 1)$.

(B) For $(0, 1)$ *i.e.*, $0 < x < 1$, x^{99} and $\cos x$ are both +ve

$\therefore f(x) > 0$

$\Rightarrow f(x)$ is increasing on $(0, 1)$.

(C) For $\left(\frac{\pi}{2}, \pi\right)$ *i.e.*, $\frac{\pi}{2} < x < \pi$, x^{99} is +ve and $-1 < \cos x < 0$

$\therefore f'(x) > 0$

$\Rightarrow f(x)$ is increasing on $\left(\frac{\pi}{2}, \pi\right)$

(D) For $\left(0, \frac{\pi}{2}\right)$, *i.e.*, $0 < x < \frac{\pi}{2}$, x^{99} and $\cos x$ are both –ve

$\therefore f'(x) < 0$

$\Rightarrow f(x)$ is decreasing on $\left(0, \frac{\pi}{2}\right)$.

5. Let h be the height of the cylindrical tank at any instant.

Volume of cylindrical tank = $\pi r^2 h = \pi(10)^2 h$,

$\therefore v = 100\,\pi h$

Rate of change of volume

$$\therefore \frac{dv}{dt} = 100\pi\frac{dh}{dt} \quad \text{...}(i)$$

The tank is filled at the rate of 314 cubic metre per hour *i.e.*,

$\frac{dv}{dt} = 314$ from (*i*) $314 = 100\pi\frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = \frac{314}{100\pi} = \frac{314}{100 \times 3.14} = 1$$

Hence, the depth of the tank changes at 1 cubic m/h.

6. The curve is $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ ...(*i*)

Put $x = 2$,

$\Rightarrow \quad (t+5)(t-2) = 0$

Put $t = 2$ in $y = 2t^2 - 2t - 5 = 8 - 4 - 5 = -1$

At $x = 2$, $y = -1$, $t = 2$

Differentiating (*i*)

$$\frac{dx}{dt} = 2t + 3$$

$$\frac{dy}{dx} = 4t - 2$$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{4t-2}{2t+3}$

At $t = 2$, $\quad \frac{dy}{dx} = \frac{4 \times 2 - 2}{2 \times 2 + 3}$

$= \frac{6}{7}$.

7. The equation of the curve is $y^2 = 4x$

Differentiating w.r.t. x

$\therefore \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$

Slope of tangent $= \frac{2}{y} = m$

$\therefore \quad y = \frac{2}{m}$ ...(*i*)

(x_1, y_1) lies on $y^2 = 4x$, $y_1^2 = 4x_1$ ...(*ii*)

Equation of tangent at (x_1, y_1)

or $\quad y = mx + y_1 - mx_1$ ...(*iii*)

$y = mx + 1$ ...(*iv*)

Comparing (*iii*) & (*iv*)

$y_1 - mx_1 = 1$ ...(*v*)

from (*i*) & (*ii*) $\quad m = \frac{2}{y_1}$

$x_1 = \frac{y_1^2}{4}$

$\therefore$ Put these values in (*v*)

or $\quad y_1 - \frac{y_1}{2} = \frac{y_1}{2} = 1$

$\therefore y_1 = 2$

$\therefore m = \frac{2}{y_1} = \frac{2}{2} = 1$

8. The equation of the curve $2y + x^2 = 3$

Differentiating,

$\therefore \quad \frac{dy}{dx} = -x$

$\frac{dy}{dx}$ at $(1, 1) = -1 =$ slope of tangent

Since slope of normal $= \frac{-1}{\text{Slope of tangent}} = 1$

$\therefore$ The equation of the normal is

$y - y_1 =$ (slope of normal) $(x - x_1)$

$y - 1 = 1\ (x - 1)$

or $x - y = 0$.

9. We have,

$$f(x) = -|x-1| = \begin{cases} x-1, & \text{if } x \le 1 \\ -(x-1), & \text{if } x > 1 \end{cases}$$

At $x = 1$

$\text{LHL} = \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} (1-h) - 1$

$= 0$

$\text{RHL} = \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} -(1+h-1)$

$= 0$

$f(1) = 1 - 1 = 0$

$\therefore$ LHL = RHL = $f(0)$

$\Rightarrow f(x)$ is continuous everywhere

Now, at $x = 1$

$\text{LHD} = \frac{d}{dx}(x-1) = 1$

$\text{RHD} = \frac{d}{dx}\{-(x-1)\} = -1$

LHD $\neq$ RHD

$\therefore f(x)$ is not differentiable of $x = 1$.

10. We know that, $f(x) = \cot x$ is continuous in $\mathrm{R} - \{n\pi : n \in \mathrm{Z}\}$

Since, $\quad f(x) = \cot x = \frac{\cos x}{\sin x}$

[Since, $\sin x = 0$ at $\{n\pi, n \in \mathrm{Z}\}$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in \mathrm{Z}\}$.

11. We have,

$$f(x) = \frac{x-1}{x(x^2-1)}$$

$\therefore f(x)$ is discontinuous when $x(x^2 - 1) = 0$

$\Rightarrow \quad x = 0, x = \pm 1$

$\therefore f(x)$ is discontinuous

at $\quad x = 0, -1, 1$

i.e., exactly at three points.

12. $\because \quad f(x) = x^2 \sin\left(\frac{1}{x}\right)$,

where $x \neq 0$

$\because \lim_{x \to 0} f(x) = 0$

Hence, value of the function f at $x = 0$, so that it is continuous at $x = 0$ is 0.

13. $\because \quad f'(c) = 0 \quad [\because f'(x) = 3x^2 - 3]$

$\Rightarrow \quad 3c^2 - 3 = 0$

$$\Rightarrow \quad c^2 = \frac{3}{3} = 1$$

$$\Rightarrow \quad c = \pm 1$$

where $1 \in \left(0, \sqrt{3}\right)$

$$\Rightarrow \quad c = 1.$$

14. We have, $y = \log\sqrt{\tan x}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{\tan x}} \times \frac{1}{2\sqrt{\tan x}} \times \sec^2 x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sec^2 x}{2\tan x}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{at } x=\frac{\pi}{4}} = \frac{\left(\sqrt{2}\right)^2}{2\times 1} = \frac{2}{2} = 1.$$

15. $\because \quad y = (\sin x + y)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(\sin x + y)^{-1/2}.\frac{d}{dx}(\sin x + y)$$

$$= \frac{1}{2}.\frac{1}{(\sin x + y)^{1/2}}.\left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}\left(\cos x + \frac{dy}{dx}\right) \quad [\because (\sin x + y)^{1/2} = y]$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1 - \frac{1}{2y}\right) = \frac{\cos x}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y}.\frac{2y}{2y-1}$$

$$= \frac{\cos x}{2y-1}.$$

16. Given $\lim\limits_{x\to 0} \frac{2x}{x}$

Using L'Hospital's Rule, by differentiating both the numerator and denominator with respect to x,

$$\lim_{x\to 0} \frac{2}{1} = 2.$$

17. $\lim\limits_{x\to 0} \frac{8^x - 4^x}{4x} \left[\frac{0}{0} \text{ form}\right]$

Applying L'Hospital's Rule,

$$= \lim_{x\to 0} \frac{8^x \ln 8 - 4^x \ln 4}{4}$$

$$= \frac{1}{4}(\ln 8 - \ln 4) = \frac{1}{4}\ln\frac{8}{4}$$

$$= \frac{1}{4}\ln 2.$$

18. $\lim\limits_{x\to \frac{\pi}{2}} \left[x\tan x - \frac{\pi}{2}\sec x\right]$

$$= \lim_{x\to\frac{\pi}{2}} \left[\frac{x\sin x - \frac{\pi}{2}}{\cos x}\right] \quad \left(\frac{0}{0} \text{ form}\right)$$

By L'Hospital's Rule,

$$\lim_{x\to\frac{\pi}{2}} \left[\frac{x\cos x + \sin x}{-\sin x}\right] = -1$$

19. $y = Ae^{5x} + Be^{-5x}$

$$\Rightarrow \quad \frac{dy}{dx} = 5Ae^{5x} - 5Be^{-5x}$$

$$\Rightarrow \quad \frac{d^2y}{d^2x} = 25Ae^{5x} + 25Be^{-5x}$$

$$= 25y.$$

20. $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{t^2} \times \frac{dt}{dx}$$

$$= \frac{-1}{t^2} \times \frac{1}{2at}$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=2} = \frac{-1}{2a\times 2^3}$$

$$= \frac{-1}{16a}.$$