UNIT - 4 : MATRICES [JEE – MAIN CRASH COURSE]

Definition

A rectangular array of symbols (which could be real or complex numbers) along rows and columns is called a matrix.

Thus, a system of $m \times n$ symbols arranged in a rectangular formation along m rows and n columns and bonded by the brackets $[\cdot]$ is called an m by n matrix (which is written as $m \times n$ matrix). Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ is a matrix of order $m \times n$.

In a compact form the above matrix is represented by $A = [a_{ij}]$, $1 \le i \le m$, $1 \le j \le n$ or simply $[a_{ij}]_{m\times n}$. The numbers a_{11}, a_{12}, ..., etc. of this rectangular array are called the elements of the matrix. The element a_{ij} belongs to the ith row and jth column and is called the (i, j)th element of a matrix.

Determinant of Square Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$.

Notes:

- If $A_1, A_2, \ldots, A_n$ are square matrices of the same order then $|A_1A_2 \cdots A_n| = |A_1|\,|A_2| \cdots |A_n|$.
- If k is scalar then $|kA| = k^n|A|$, where n is order of the square matrix A.

Classification of Matrices

Equal matrices

Two matrices are said to be equal if they have the same order and each element of one is equal to the corresponding element of the other.

Row matrix

A matrix having a single row is called a row matrix, e.g., $[1 \quad 3 \quad 5 \quad 7]$.

Column matrix

A matrix having a single column is called a column matrix, e.g., $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$.

Square matrix

An $m \times n$ matrix A is said to be a square matrix if $m = n$, i.e., number of rows = number of columns.

The diagonal from the left-hand side upper corner to the right-hand side lower corner is known as leading diagonal or principal diagonal.

Diagonal matrix

A square matrix all of whose elements, except those in the leading diagonal, are zero is called a diagonal matrix. For a square matrix $A = [a_{ij}]_{n\times n}$ to be a diagonal matrix, $a_{ij} = 0$, whenever $i \ne j$.

A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots, d_n$ as diagonal elements is denoted by diag $[d_1, d_2, \ldots, d_n]$.

Scalar matrix

A diagonal matrix whose all the leading diagonal elements are equal is called a scalar matrix.

For a square matrix $A = [a_{ij}]_{n \times n}$ to be a scalar matrix

$$a_{ij} = \begin{cases} 0, & i \neq j \\ m, & i = j \end{cases}, \text{ where } m \neq 0.$$

Unit matrix or identity matrix

A diagonal matrix of order n, which has unity for all its diagonal elements, is called a unit matrix of order n and is denoted by I_n.

Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Triangular matrix

A square matrix in which all the elements below the diagonal are zero is called upper triangular matrix and a square matrix in which all the elements above diagonal are zero is called lower triangular matrix.

Given a square matrix $A = [a_{ij}]_{n \times n}$; for upper triangular matrix, $a_{ij} = 0$, $i > j$; and for lower triangular matrix, $a_{ij} = 0$, $i < j$.

Null matrix

If all the elements of a matrix (square or rectangular) are zero, it is called a null or zero matrix.

For $A = [a_{ij}]$ to be null matrix, $a_{ij} = 0 \ \forall \ i, j$.

Singular and non-singular matrix

A square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$.

Trace of Matrix

The sum of the elements of a square matrix A lying along the principal diagonal is called the trace of A, i.e., tr(A)

Thus if $A = [a_{ij}]_{n \times n}$, then

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

Properties of trace of a matrix

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ and λ be a scalar, then

- $\text{tr}(\lambda A) = \lambda \text{tr}(A)$
- $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- $\text{tr}(AB) = \text{tr}(BA)$

Algebra of Matrices

Addition and subtraction of matrices

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order, their sum $A + B$ is defined to be the matrix of order $m \times n$ such that $(A + B)_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

Notes:

- Only matrices of the same order can be added or subtracted.
- Addition of matrices is commutative as well as associative.
- Cancellation laws hold well in case of addition.
- That is , $A + B = A + C \Rightarrow B = C$

Scalar multiplication

The matrix obtained by multiplying every element of a matrix A by a scalar λ is called the scalar multiple of A by λ and is denoted by λA, i.e., if $A = [a_{ij}]$ then $\lambda A = [\lambda a_{ij}]$.

Multiplication of matrices

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post-multiplier). Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$$= [a_{i1} \ a_{i2} \ \cdots \ a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$= (i\text{th row of } A)(j\text{th column of } B) \quad (1)$$

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$

Notes:

- Commutative law does not necessarily hold for matrices.
- If $AB = BA$ then matrices A and B are called commutative matrices.
- If $AB = -BA$ then matrices A and B are called anti-commutative matrices.
- Matrix multiplication is associative $A(BC) = (AB)C$.
- Matrix multiplication is distributive with respect to addition, i.e., $A(B + C) = AB + AC$.

- The matrices possess divisors of zero, i.e., if the product $AB = O$, it is not necessary that at least one of the matrices should be zero matrix. For example, if $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ while neither A nor B is the null matrix.
- Cancellation law does not necessarily hold, i.e., if $AB = AC$ then in general $B \neq C$, even if $A \neq O$.
- Matrix multiplication $A \cdot A$ is represented as A^2. Thus $A^n = A \cdot A \cdots n$ times.
- If $A = \text{diag.}(a_1, a_2, a_3, \ldots, a_n)$ and $B = \text{diag.}(b_1, b_2, b_3, \ldots, b_n)$, then $A \cdot B = \text{diag.}(a_1b_1, a_2b_2, \ldots, a_nb_n)$. Thus $A^n = \text{dign.}(a_1{}^n, a_2{}^n, a_3{}^n, \ldots, a_n{}^n)$
- If A and B are diagonal matrices of the same order then $AB = BA$ or diagonal matrices are commutative.
- If A and B are commutative then

$$(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$$

Similarly, $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

In general, $(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + \cdots + {}^nC_nB^n$.

Matrices A and I are always commutative. Hence, $(I + A)^n = {}^nC_0 + {}^nC_1A + {}^nC_1A^2 + \cdots + {}^nC_nA^n$.

Transpose of Matrix

The matrix obtained from any given matrix A, by interchanging rows and columns, is called the transpose of A and is denoted by A^T.

If $A = [a_{ij}]_{m \times n}$ and $A^T = [b_{ij}]_{n \times m}$ then $b_{ij} = a_{ji}$, $\forall\ i, j$.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}_{3\times 2}$, then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}_{2\times 3}$.

Properties of transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$, A and B being conformable matrices
- $(\alpha A)^T = \alpha A^T$, α being scalar
- $(AB)^T = B^TA^T$, A and B being conformable for multiplication **(reversal law)**

Special Matrices

Symmetric matrix

A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all i, j.

For example, the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$ is symmetric, because $a_{12} = -1 = a_{21}$, $a_{13} = 1 = a_{31}$, $a_{23} = 5 = a_{32}$.

For symmetric matrix $A^T = A$.

Skew-symmetric matrix

A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j.

For example, the matrix $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$ is skew-symmetric.

For skew-symmetric matrix $A^T = -A$.

Adjoint of square matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

Inverse of matrix

A non-singular square matrix of order n is invertible if there exists a square matrix B of the same order such that $AB = I_n = BA$. In such a case, we say that the inverse of A is B and we write, $A^{-1} = B$. Also from $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$, we can conclude that $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$.

Properties of adjoint and inverse of a matrix

1. Let A be a square matrix of order n. Then $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$.
2. Every invertible matrix possesses a unique inverse.
3. **Reversal law:** If A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
   In general, if $A, B, C, \ldots$, are invertible matrices then $(ABC \cdots)^{-1} = \cdots C^{-1}B^{-1}A^{-1}$.
4. If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
5. If A is a non-singular square matrix of order n. Then $|\text{adj } A| = |A|^{n-1}$.
6. **Reversal law for adjoint:** If A and B are non-singular square matrices of the same order, then $\text{adj} \cdot (AB) = (\text{adj } B)\,(\text{adj } A)$.
7. If A is an invertible square matrix, then $\text{adj}(A^T) = (\text{adj } A)^T$.

8. If A is a non-singular square matrix, then adj(adj A) $= |A|^{n-2} A$.
9. If A is a non-singular matrix, then prove that $|A^{-1}| = |A|^{-1}$, i.e., $|A^{-1}| = \frac{1}{|A|}$.
10. The inverse of the kth power of A is the kth power of the inverse of A.

Consider the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad (1)$$

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then (1) is equivalent to the matrix equation

$$AX = D \qquad (2)$$

Multiplying both sides of (2) by the reciprocal matrix A^{-1}, we get

$$A^{-1}(AX) = A^{-1}D \Rightarrow IX = A^{-1}D \qquad [\because A^{-1}A = 1]$$

$$\Rightarrow \quad X = A^{-1}D.$$

1. If A is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$
2. If A is singular matrix, and (*ajd* A) $D = O$, then the system of the equations given by $AX = D$ is consistent with infinitely many solutions.
3. If A is singular matrix, and (adj A) $D \neq O$, then the system of equation given by $AX = D$ is inconsistent.

UNIT - 5 : DETERMINANT [JEE – MAIN CRASH COURSE]

Definition

Let a, b, c, and d be any four numbers, real or complex, the symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ denotes $ad - bc$ and is called a determinant of second order; a, b, c, and d are called elements of the determinant; and $ad - bc$ is called its value.

Let $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ be any nine numbers, then the symbol $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is another way of denoting.

That is

$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Minors and Cofactors

In the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1)$$

if we leave the row and the column passing through the element a_{ij} then the second-order determinant thus obtained is called the minor of a_{ij} and it is denoted by M_{ij}. Thus, we can get nine minors corresponding to the nine elements.

For example, in determinant (1), the minor of the element

$$a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$$

In terms of the notation of minors if we expand the determinant along the first row, then

$$\Delta = (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} + (-1)^{1+3}a_{13}M_{13}$$
$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

Similarly, expanding Δ along the second column, we have

$$\Delta = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij}. If we denote the cofactor of the element a_{ij}, by C_{ij}, then the cofactor of a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$.

Cofactor of the element a_{21} is

$$C_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

In terms of the notation of the cofactors, we have

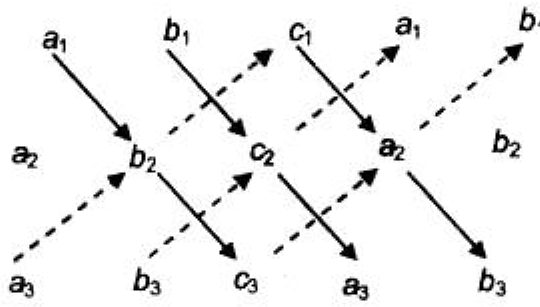
$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

Also, $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$, etc.

Sarrus Rule for Expansion

Sarrus gave a rule for a determinant of order 3.

Rule The three diagonals sloping down to the right give the three positive terms and the three diagonals sloping down to the left the three negative terms.



$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$$

Some Operations

The first, second, and third rows of a determinant are denoted by R_1, R_2, and R_3 respectively and the first, second, and third columns by C_1, C_2, and C_3 respectively.

1. The interchange of its ith row and jth row is denoted by $R_i \leftrightarrow R_j$.
2. The interchange of ith column and jth column is denoted by $C_i \leftrightarrow C_j$.
3. The addition of m times the elements of jth row of the corresponding elements of ith row is denoted by $R_i \to R_i + mR_j$.
4. The addition of m times the elements of jth column to the corresponding elements of ith column is denoted by $C_i \to C_i + mC_j$.
5. The addition of m times the elements of jth row to n–times the elements of ith row is denoted by $R_i \to nR_i + mR_j$.

Properties of Determinants

1. The value of the determinant is not changed when rows are changed into corresponding columns.
   Naturally when rows are changed into corresponding columns, then columns will be changed into corresponding rows. That is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. If any two rows or columns of a determinant are interchanged, the sign of the determinant is changed, but its magnitude remains the same.
3. The value of a determinant is zero if any two rows of columns are identical.
4. A common factor of all elements of any row (or of any column) may be taken outside the sign of the determinant. In other words, if all the elements of the same row (or the same column) are multiplied by a certain number, then the determinant gets multiplied by that number.

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

i.e.,

$$\begin{vmatrix} 32 & 24 & 16 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix} = 8\begin{vmatrix} 4 & 3 & 2 \\ 8 & 3 & 5 \\ 4 & 5 & 3 \end{vmatrix}$$

[taking 8 common from the first row]

$$= 8 \times 4 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 5 \\ 1 & 5 & 3 \end{vmatrix}$$

[taking 4 common from the first column]

5. If every element of some column or (row) is the sum of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as in the given determinant. That is,

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

6. The value of a determinant does not change when any row or column is multiplied by a number or an expression and is then added to or subtracted from any other row or column.
   Here it should be noted that if the row or column which is changed is multiplied by a number, then the determinant will have to be divided by that number. That is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix}$$

7. If $\Delta_r = \begin{vmatrix} f_1(r) & f_2(r) & f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$

   where $f_1(r)$, $f_2(r)$, and $f_3(r)$ are functions of r and a, b, c, d, e, and f are constants. Then

$$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=}^{n} f_1(r) & \sum_{r=1}^{n} f_2(r) & \sum_{r=1}^{n} f_3(r) \\ a & b & c \\ d & e & f \end{vmatrix}$$

Also for

$$\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a & b & c \\ d & e & f \end{vmatrix}$$

where $f_1(x), f_2(x)$, and $f_3(x)$ are functions of x and a, b, c, d, e, and f are constants. We have

$$\int_p^q \Delta(x)dx = \begin{vmatrix} \int_p^q f_1(x)dx & \int_p^q f_2(x)dx & \int_p^q f_3(x)dx \\ a & b & c \\ d & e & f \end{vmatrix}$$

Some important determinants

1. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$

2. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x - y)(y - z)(z - x)(x + y + z)$

3. $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x - y)(y - z)(z - x)(xy + yz + zx).$

Product of two determinants

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_2 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then row by row multiplication of Δ_1 and Δ_2 is given by

$$\Delta_1 \times \Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication can also be performed row by column; column by row or column by column as required in the problem.

To express a determinant as product of two determinants, one requires a lots of practice and this can be done only by inspection and trial.

Property If $A_1, B_1, C_1, \ldots$, are respectively the cofactors of the elements $a_1, b_1, c_1, \ldots$, of the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta \neq 0, \text{ then } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$$

Differentiation of a determinant

1. Let $\Delta(x)$ be a determinant of order 2. If we write $\Delta(x) = [C_1; C_2]$, where C_1 and C_2 denote the first and second columns then $\Delta'(x) = [C_1' ; C_2] + [C_1; C_2']$, where C_i' denotes the column which contains the derivative of all the functions in the ith column C_i. In a similar fashion, if we write

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$$

For example, let $\Delta(x) = \begin{vmatrix} \sin x & \log x \\ e & 1/x \end{vmatrix}, x > 0$, then

$$\Delta'(x) = \begin{vmatrix} \cos x & \log x \\ 0 & 1/x \end{vmatrix} + \begin{vmatrix} \sin x & 1/x \\ e & -1/x^2 \end{vmatrix}$$

2. Let $\Delta(x)$ be of order 3. If we write $\Delta(x) = [C_1; C_2; C_3]$, then $\Delta'(x) = [C_1'; C_2; C_3] + [C_1; C_2'; C_3] + [C_1; C_2; C_3']$ and similarly if we consider

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix}$$

3. If only one row (column) consists functions of x and other rows are constants, viz., let

$$\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

then

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and in general

$$\Delta^n(x) = \begin{vmatrix} f_a^n(x) & f_2^n(x) & f_3^n(x) \\ b_2 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where n is any positive integer and $f^n(x)$ denotes the nth derivative of $f(x)$.

System of Linear Equations

System of consistent linear equations System of (linear) equations is said to be consistent if it has at least one solution. For example,

1. System of equations $\left.\begin{array}{r} x+y=2 \\ 2x+2y=5 \end{array}\right\}$ is inconsistent because it has no solution, i.e., there is no value of x and y which satisfy both the equations. Here, two straight lines are parallel.
2. System of equations $\left.\begin{array}{r} x+y=2 \\ x-y=0 \end{array}\right\}$ is consistent because it has a solution $x = 1$, $y = 1$. Here lines intersect at one point.

Cramer's Rule

Use of determinants in solving linear equations with the help of Cramer's rule:

System of linear equations in two variables Let the given system of equations be

$$\left.\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{array}\right\} \qquad (2)$$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

Solving by cross-multiplication, we have

$$\frac{x}{b_1c_2-b_2c_1}=\frac{-y}{a_1c_2-a_2c_1}=\frac{1}{a_1b_2-a_2b_1}$$

or $$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

System of linear equations in three variables Let the given system of linear equations in three variables x, y, and z be

$$\begin{array}{l} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{array} \qquad (3)$$

Let

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Solutions under different conditions

1. If $\Delta \neq 0$, then given system of equations is consistent and it has unique (one) solution which is given by
$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, \text{ and } z = \frac{\Delta_3}{\Delta}$$
2. If $\Delta = 0$ and at least one of Δ_1, Δ_2, and Δ_3 is non-zero, then given system of equations is inconsistent and it will have no solution.
3. If all of Δ, Δ_1, Δ_2, and Δ_3 are zero, then given system of equations is consistent and has infinitely many solutions.

Conditions for consistency of three linear equations in two unknowns

System of three linear equations in x and y

$$\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \\ a_3x+b_3y+c_3=0 \end{array}$$

will be consistent if the values of x and y obtained from any two equations satisfy the third equation,

or $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

This is the required condition for consistency of three linear equations in two unknowns. If such system of equations is consistent, then the number of solutions is one.

System of Homogeneous Linear Equations

A system of linear equations is said to be homogeneous if the sum of powers of variable in each term is one.

Let the three homogeneous linear equations in three unknown, x, y, and z be

$$a_1x+b_1y+c_1z=0 \qquad (4)$$
$$a_2x+b_2y+c_2z=0 \qquad (5)$$
$$a_3x+b_3y+c_3z=0 \qquad (6)$$

Clearly, $x = 0$, $y = 0$, and $z = 0$ is a solution of system of equations (4), (5), and (6). This solution is called a trivial solution. Any other solution is called a non-trivial solution. Now consider

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Solutions under different conditions

1. If $\Delta \neq 0$, then the given system of equations has only trivial solution and the number of solutions in this case is one.
2. If $\Delta = 0$, then the given system of equations has non-trivial solution as well as trivial solution and number of solutions in this case is infinite.