Applications of Derivatives

EXERCISE 4.1 [PAGE 105]

Exercise 4.1 | Q 1.1 | Page 105

Find the equation of tangent and normal to the curve at the given points on it.

$y = 3x^2 - x + 1$ at (1, 3)

Solution: Equation of the curve is $y = 3x^2 - x + 1$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x - 1$$

Slope of the tangent at (1, 3) is

$$\left(\frac{dy}{dx}\right)_{(1,3)} = 6(1) - 1 = 5$$

∴ Equation of tangent at (a, b) is

$$y - b = \left(\frac{dy}{dx}\right)_{(a,\ b)} (x - a)$$

Here, (a, b) ≡ (1, 3)

∴ Equation of the tangent at (1, 3) is

(y - 3) = 5(x - 1)

∴ y - 3 = 5x - 5

∴ 5x - y - 2 = 0

Slope of the normal at (1, 3) is $\dfrac{-1}{\left(\frac{dy}{dx}\right)_{(1,3)}} = \dfrac{-1}{5}$

∴ Equation of normal at (a, b) is

$$y - b = \frac{-1}{\left(\frac{dy}{dx}\right)_{(a,b)}} (x - a)$$

∴ Equation of the normal at (1, 3) is

$$(y - 3) = \frac{-1}{5}(x - 1)$$

∴ 5y - 15 = - x + 1

∴ x + 5y - 16 = 0

Exercise 4.1 | Q 1.2 | Page 105

Find the equation of tangent and normal to the curve at the given points on it.

$2x^2 + 3y^2 = 5$ at (1, 1)

Solution: Equation of the curve is $2x^2 + 3y^2 = 5$

Differentiating w.r.t. x, we get

$$4x + 6y \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-4x}{6y}$$

∴ Slope of the tangent at (1, 1) is

$$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-4(1)}{6(1)} = \frac{-2}{3}$$

∴ Equation of tangent at (a, b) is

$$y - b = \left(\frac{dy}{dx}\right)_{(a,\ b)} (x - a)$$

Here, (a, b) ≡ (1, 1)

∴ Equation of the tangent at (1, 1) is

$(y - 1) = \frac{-2}{3}(x - 1)$

$\therefore 3(y - 1) = - 2(x - 1)$

$\therefore 3y - 3 = - 2x + 2$

$\therefore 3y - 3 = - 2x + 2$

$\therefore 2x + 3y - 5 = 0$

Slope of the normal at (1, 1) is $\frac{-1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = \frac{3}{2}$

$\therefore$ Equation of normal at (a, b) is

$y - b = \frac{-1}{\left(\frac{dy}{dx}\right)_{(a,b)}}(x - a)$

$\therefore$ Equation of the normal at (1, 1) is

$(y - 1) = \frac{3}{2}(x - 1)$

$\therefore 2y - 2 = 3x - 3$

$\therefore 3x - 2y - 1 = 0$

Exercise 4.1 | Q 1.3 | Page 105

Find the equation of tangent and normal to the curve at the given points on it.

$x^2 + y^2 + xy = 3$ at (1, 1)

Solution: Equation of the curve is $x^2 + xy + y^2 = 3$

Differentiating w.r.t. x, we get

$2x + x \cdot \frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$

$\therefore (2x + y) + (x + 2y)\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-(2x + y)}{x + 2y}$

$\therefore$ Slope of the tangent at (1, 1) is

$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-(2 + 1)}{1 + 2} = -1$

$\therefore$ Equation of tangent at (a, b) is

$y - b = \left(\frac{dy}{dx}\right)_{(a,\ b)} (x - a)$

Here, (a, b) ≡ (1, 1)

$\therefore$ Equation of the tangent at (1, 1) is

$(y - 1) = -1 (x - 1)$

$\therefore (y - 1) = - x + 1$

$\therefore x + y - 2 = 0$

Slope of the normal at (1, 1) is $\frac{-1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = 1$

∴ Equation of normal at (a, b) is

$$y - b = \frac{-1}{\left(\frac{dy}{dx}\right)_{(a,b)}}(x - a)$$

∴ Equation of the normal at (1, 1) is

(y - 1) = 1 (x - 1)

∴ x - y = 0

Exercise 4.1 | Q 2 | Page 105

Find the equations of tangent and normal to the curve $y = x^2 + 5$ where the tangent is parallel to the line 4x − y + 1 = 0.

Solution: Equation of the curve is $y = x^2 + 5$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2x$$

Slope of the tangent at $P(x_1, y_1)$ is

$$\left(\frac{dy}{dx}\right)_{(x_1,x_2)} = 2x_1$$

According to the given condition, the tangent is parallel to 4x – y + 1 = 0

Now, slope of the line 4x – y + 1 = 0 is 4.

∴ Slope of the tangent = $\frac{dy}{dx} = 4$

∴ $2x_1 = 4$

∴ $x_1 = 2$

$P(x_1, y_1)$ lies on the curve $y = x^2 + 5$

∴ $y_1 = (2)^2 + 5$

∴ $y_1 = 9$

∴ The point on the curve is (2, 9).

∴ Equation of the tangent at (2, 9) is

∴ (y – 9) = 4(x – 2)

∴ y – 9 = 4x – 8

∴ 4x - y + 1 = 0

Slope of the normal at (2, 9) is $\frac{1}{\left(\frac{dy}{dx}\right)_{(2,9)}} = \frac{-1}{4}$

∴ Equation of the normal of (2, 9) is

$(y - 9) = \frac{-1}{4}(x - 2)$

∴ 4y - 36 = - x + 2

∴ x + 4y - 38 = 0

Exercise 4.1 | Q 3 | Page 105

Find the equations of tangent and normal to the curve $y = 3x^2 - 3x - 5$ where the tangent is parallel to the line 3x − y + 1 = 0.

Solution: Equation of the curve is $y = 3x^2 - 3x - 5$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = 6x - 3$

Slope of the tangent at $P(x_1, y_1)$ is

$$\left(\frac{dy}{dx}\right)_{(x_1, x_2)} = 6x_1 - 3$$

According to the given condition, the tangent is parallel to $3x - y + 1 = 0$

Now, slope of the line $3x - y + 1 = 0$ is 3.

$\therefore$ Slope of the tangent = $\frac{dy}{dx} = 3$

$\therefore 6x_1 - 3 = 3$

$\therefore x_1 = 1$

$P(x_1, y_1)$ lies on the curve $y = 3x^2 - 3x - 5$

$\therefore y_1 = 3(1)^2 - 3(1) - 5$

$\therefore y_1 = -5$

$\therefore$ The point on the curve is (1, -5).

$\therefore$ Equation of the tangent at (1, -5) is

$\therefore (y + 5) = 3(x - 1)$

$\therefore y + 5 = 3x - 3$

$\therefore 3x - y - 8 = 0$

Slope of the normal at (1, -5) is $\frac{-1}{\left(\frac{dy}{dx}\right)_{(1,-5)}} = \frac{-1}{3}$

$\therefore$ Equation of the normal of (1, -5) is

$(y + 5) = \frac{-1}{3}(x - 1)$

$\therefore 3y + 15 = -x + 1$

$\therefore x + 3y + 14 = 0$

EXERCISE 4.2 [PAGE 106]

Exercise 4.2 | Q 1.1 | Page 106

Test whether the following functions are increasing or decreasing : $f(x) = x^3 - 6x^2 + 12x - 16$, $x \in R$.

Solution: $f(x) = x^3 - 6x^2 + 12x - 16$

$$\therefore f'(x) = \frac{d}{dx}\left(x^3 - 6x^2 + 12x - 16\right)$$

$$= 3x^2 - 6 \times 2x + 12 \times 1 - 0$$

$$= 3x^2 - 12x + 12$$

$$= 3(x^2 - 4x + 4)$$

$$= 3(x - 2)^2 \geq 0 \text{ for all } x \in R$$

$$\therefore f'(x) \geq 0 \text{ for all } x \in R$$

$\therefore$ f is increasing for all $x \in R$.

Exercise 4.2 | Q 1.2 | Page 106

Test whether the function is increasing or decreasing.

$f(x) = x - 1/x$, $x \in R$, $x \neq 0$,

Solution:

$$f(x) = x - \frac{1}{x}, x \in R$$

$$\therefore f\prime(x) = 1 - \left(-\frac{1}{x^2}\right) = 1 + \frac{1}{x^2}$$

$$\because x \neq 0, \text{ for all values of x, } x^2 > 0$$

$$\therefore \frac{1}{x^2} > 0, 1 + \frac{1}{x^2} \text{ is always positive}$$

thus f'(x)>o , for all $x \in R$

Hence f(x) is increasing function.

Exercise 4.2 | Q 1.3 | Page 106

Test whether the following function is increasing or decreasing.

$f(x) = 7/x - 3, x \in R, x \neq 0$

Solution:

$$f'(x) = \frac{7}{x} - 3, x \in R, x \neq 0$$

$$\therefore f'(x) = \frac{-7}{x^2}$$

$$x \neq 0, x^2 > 0, \text{ i.e., } \frac{1}{x^2} > 0, \text{ i.e., } -\frac{7}{x^2} < 0$$

$\therefore f'(x) < 0$ for all $x \in R, x \neq 0$

Hence, f(x) is a decreasing function, for all $x \in R, x \neq 0$.

Exercise 4.2 | Q 2.1 | Page 106

Find the value of x, such that f(x) is increasing function.

$f(x) = 2x^3 - 15x^2 + 36x + 1$

Solution: $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\therefore f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x - 3)(x - 2)$

f(x) is an increasing function, if $f'(x) > 0$

$\therefore 6(x - 3)(x - 2) > 0$

$\therefore (x - 3)(x - 2) > 0$

$ab > 0 \Leftrightarrow a > 0$ and $b > 0$ or $a < 0$ or $b < 0$

$\therefore$ Either $(x – 3) > 0$ and $(x – 2) > 0$ or
$(x – 3) < 0$ and $(x – 2) < 0$

Case 1: $x – 3 > 0$ and $x – 2 > 0$

∴ x > 3 and x > 2

∴ x > 3

Case 2: x – 3 < 0 and x – 2 < 0

∴ x < 3 and x < 2

∴ x < 2

Thus, f(x) is an increasing function for x < 2 or x > 3, i.e., (- ∞, 2) ∪ (3, ∞)

Exercise 4.2 | Q 2.2 | Page 106

Find the value of x, such that f(x) is increasing function.

f(x) = x^2 + 2x - 5

Solution: f(x) = x^2 + 2x - 5

∴ f'(x) = 2x + 2

f(x) is an increasing function, if f'(x) > 0

∴ 2x + 2 > 0

∴ 2x > -2

∴ x > - 1

Thus, f(x) is an increasing function for x > -1, i.e., (-1, ∞)

Exercise 4.2 | Q 2.3 | Page 106

Find the value of x, such that f(x) is increasing function.

f(x) = $2x^3 - 15x^2$ - 144x - 7

Solution: f(x) = $2x^3 - 15x^2$ - 144x - 7

∴ f'(x) = $6x^2$ - 30x - 144

f(x) is an increasing function, if f'(x) > 0

∴ 6(x^2 - 5x - 24) > 0

∴ 6(x + 3)(x - 8) > 0

∴ (x + 3)(x - 8) > 0

ab > 0 ⇔ a > 0 and b > 0 or a < 0 or b < 0

∴ Either (x + 3) > 0 and (x – 8) > 0 or

(x + 3) < 0 and (x – 8) < 0

Case 1: $x + 3 > 0$ and $x - 8 > 0$

∴ $x > -3$ and $x > 8$

∴ $x > 8$

Case 2: $x + 3 < 0$ and $x - 8 < 0$

∴ $x < -3$ or $x < 8$

∴ $x < -3$

Thus, f(x) is an increasing function for $x < -3$, or $x > 8$ i.e., $(-\infty, -3) \cup (8, \infty)$.

Exercise 4.2 | Q 3.1 | Page 106

Find the value of x, such that f(x) is decreasing function.

$f(x) = 2x^3 - 15x^2 - 144x - 7$

Solution: $f(x) = 2x^3 - 15x^2 - 144x - 7$

∴ $f'(x) = 6x^2 - 30x - 144$

f(x) is an decreasing function, if $f'(x) < 0$

∴ $6(x^2 - 5x - 24) < 0$

∴ $6(x + 3)(x - 8) < 0$

∴ $(x + 3)(x - 8) < 0$

$ab < 0 \Leftrightarrow a > 0$ and $b < 0$ or $a < 0$ or $b > 0$

∴ Either $(x + 3) > 0$ and $(x - 8) < 0$ or

$(x + 3) < 0$ and $(x - 8) > 0$

Case 1: $x + 3 > 0$ and $x - 8 < 0$

∴ $x > -3$ and $x < 8$

Case 2: $x + 3 < 0$ and $x - 8 > 0$

∴ $x < -3$ or $x > 8$, which is not possible.

Thus, f(x) is an decreasing function for $-3 < x < 8$ i.e., $(-3, 8)$.

Exercise 4.2 | Q 3.2 | Page 106

Find the value of x such that f(x) is decreasing function.

$f(x) = x^4 - 2x^3 + 1$

Solution: $f(x) = x^4 - 2x^3 + 1$

∴ $f'(x) = 4x^3 - 6x^2 = 2x^2 (2x - 3)$

f(x) is a decreasing function, if f'(x) < 0

∴ $2x^2 (2x - 3) < 0$

As x^2 is always positive,

(2x - 3) < 0

∴ 2x < 3

∴ $x < \frac{3}{2}$

Thus, f(x) is a decreasing function for $x < \frac{3}{2}$, i.e. $\left(-\infty, \frac{3}{2}\right)$.

Exercise 4.2 | Q 3.3 | Page 106

Find the value of x, such that f(x) is decreasing function.

$f(x) = 2x^3 - 15x^2 - 84x - 7$

Solution: $f(x) = 2x^3 - 15x^2 - 84x - 7$

∴ $f'(x) = 6x^2 - 30x - 84$

$= 6(x^2 - 5x - 14)$

$= 6(x^2 - 7x + 2x - 14)$

= 6(x - 7)(x + 2)

f(x) is an decreasing function, if f'(x) < 0

∴ 6(x - 7)(x + 2) < 0

∴ (x - 7)(x + 2) < 0

ab < 0 ⇔ a > 0 and b < 0 or a < 0 or b > 0

∴ Either (x - 7) > 0 and (x + 2) < 0 or

(x - 7) < 0 and (x + 2) > 0

Case 1: x - 7 > 0 and x + 2 < 0

∴ x > 7 and x < -2, which is not possible.

Case 2: x - 7 < 0 and x + 2 > 0

∴ x < 7 and x > -2

Thus, f(x) is an decreasing function for -2 < x < 7 i.e., (-2, 7).

EXERCISE 4.3 [PAGE 109]

Exercise 4.3 | Q 1.1 | Page 109

Determine the maximum and minimum value of the following function.

$f(x) = 2x^3 - 21x^2 + 36x - 20$

Solution: $f(x) = 2x^3 - 21x^2 + 36x - 20$

$\therefore f'(x) = 6x^2 - 42x + 36$ and $f''(x) = 12x - 42$

Consider, $f'(x) = 0$

$\therefore 6x^2 - 42x + 36 = 0$

$\therefore 6(x^2 - 7x + 6) = 0$

$\therefore 6(x - 1)(x - 6) = 0$

$\therefore (x - 1)(x - 6) = 0$

$\therefore x = 1$ or $x = 6$

For $x = 1$,

$f''(1) = 12(1) - 42 = 12 - 42 = -30 < 0$

$\therefore f(x)$ attains maximum value at $x = 1$.

$\therefore$ Maximum value $= f(1)$

$= 2(1)^3 - 21(1)^2 + 36(1) - 20$

$= 2 - 21 + 36 - 20$

$= -19 - 20 + 36$

$= -39 + 36$

$= -3$

$\therefore$ The function $f(x)$ has maximum value -3 at $x = 1$.

For $x = 6$,

$f''(6) = 12(6) - 42 = 72 - 42 = 30 > 0$

$\therefore f(x)$ attains minimum value at $x = 6$.

$\therefore$ Minimum value $= f(6)$

$= 2(6)^3 - 21(6)^2 + 36(6) - 20$

$= 432 - 756 + 216 - 20$

$= -128$

$\therefore$ The function $f(x)$ has minimum value -128 at $x = 6$.

Exercise 4.3 | Q 1.2 | Page 109

Determine the maximum and minimum value of the following function.

f(x) = x log x

Solution: f(x) = x log x

$\therefore$ f'(x) = $x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$

$= x \times \frac{1}{x} + \log x \times 1 = 1 + \log x$

and f''(x) = $0 + \frac{1}{x} = \frac{1}{x}$

Consider, f'(x) = 0

$\therefore$ 1 + log x = 0

$\therefore$ log x = - 1

$\therefore$ log x = - log e = log e^{-1} = log $\left(\frac{1}{e}\right)$

$\therefore$ x = $\frac{1}{e}$

For x = $\frac{1}{e}$

$$f''\left(\frac{1}{e}\right) = \frac{1}{\frac{1}{e}} = e > 0$$

∴ f(x) attains minimum value at x = $\frac{1}{e}$.

∴ Minimum value = $f\left(\frac{1}{e}\right) = \frac{1}{e}\log\left(\frac{1}{e}\right) = \frac{1}{e}\log e^{-1}$

$= \left(\frac{-1}{e}\right)(1) = \left(\frac{-1}{e}\right)$

∴ The function f(x) has minimum value $\frac{-1}{e}$ at x = $\frac{1}{e}$.

Exercise 4.3 | Q 1.3 | Page 109

Determine the maximum and minimum value of the following function.

f(x) = $x^2 + \frac{16}{x}$

Solution:

f(x) = $x^2 + \frac{16}{x}$

∴ f'(x) = $2x - \frac{16}{x^2}$

and f''(x) = $2 + \frac{32}{x^2}$

Consider, f'(x) = 0

∴ $2x - \frac{16}{x^2} = 0$

∴ 2x = $\frac{16}{x^2}$

∴ x^3 = 8

∴ x = 2

For x = 2

$$f''(2) = 2 + \frac{32}{2^3} = 2 + \frac{32}{8} = 2 + 4 = 6 > 0$$

∴ f(x) attains minimum value at x = 2.

∴ Minimum value = $f(2) = (2)^2 + \frac{16}{2} = 4 + 8 = 12$

∴ The function f(x) has minimum value 12 at x = 2.

Exercise 4.3 | Q 2 | Page 109

Divide the number 20 into two parts such that their product is maximum.

Solution: The given number is 20.

Let x be one part of the number and y be the other part.

∴ x + y = 20

∴ y = (20 - x) ...(i)

The product of two numbers is xy.

∴ f(x) = xy = x(20 - x) = 20x - x^2

∴ f'(x) = 20 - 2x and f''(x) = - 2

Consider, f '(x) = 0

∴ 20 - 2x = 0

∴ 20 = 2x

∴ x = 10

For x = 10,

f ''(10) = - 2 < 0

∴ f(x), i.e., product is maximum at x = 10

and 10 + y = 20 ....[from (i)]

i.e., y = 10.

Exercise 4.3 | Q 3 | Page 109

A metal wire of 36cm long is bent to form a rectangle. Find it's dimensions when it's area is maximum.

Solution: Let the length and breadth of a rectangle be l and b.

∴ Perimeter of rectangle = 2 (l + b) = 36cm

∴ l + b = 18 ....(i)

Area of rectangle = l × b = l (18 - l)

Let f(l) = 18l - l^2

∴ f'(l) = 18 - 2l

and f''(l) = - 2

Consider, f '(l) = 0

∴ 18 - 2l = 0

∴ 18 = 2l

∴ l = 9

For l = 9,

f''(9) = - 2 < 0

∴ f(x), i.e. area is maximum when l = 9 cm

and b = 18 - 9 ....[From (i)]

= 9 cm

Exercise 4.3 | Q 4 | Page 109

The total cost of producing x units is ₹ (x^2 + 60x + 50) and the price is ₹ (180 − x) per unit. For what units is the profit maximum?

Solution: Given, no. of units = x,

selling price of each unit = ₹ (180 – x)

∴ selling price of x unit = ₹ (180 - x).x

= ₹ (180x - x^2)

Also, cost price of x units = ₹ $(x^2 + 60x + 50)$

Now, Profit = P = Selling price – Cost price

$= 180x - x^2 - (x^2 + 60x + 50)$

$= 180x - x^2 - x^2 - 60x - 50$

$\therefore P = -2x^2 + 120x - 50$

$$\therefore \frac{dP}{dx} = -4x + 120$$

and $\frac{d^2P}{dx^2} = -4$

Consider, $\frac{dP}{dx} = 0$

$\therefore -4x + 120 = 0$

$\therefore -4x = -120$

$\therefore x = 30$

For x = 30,

$\frac{d^2P}{dx^2} = -4 < 0$

∴ P, i.e. profit is maximum at x = 30.

EXERCISE 4.4 [PAGES 112 - 113]

Exercise 4.4 | Q 1 | Page 112

The demand function of a commodity at price P is given as, D = $40 - \frac{5P}{8}$. Check whether it is increasing or decreasing function.

Solution: Given, the demand function is

$$D = 40 - \frac{5P}{8}$$

$$\therefore \frac{dD}{dP} = 0 - \frac{5}{8}(1) = \frac{-5}{8} < 0$$

∴ The given function is a decreasing function.

Exercise 4.4 | Q 2 | Page 112

Price P for demand D is given as $P = 183 + 120D - 3D^2$ Find D for which the price is increasing

Solution: Price function P is given by

$$P = 183 + 120D - 3D^2$$

Differentiating w.r.t. D

$$\frac{dP}{dD} = 120 - 6D$$

If price is increasing then we have $\frac{dP}{dD} > 0$

∴ $120 - 6D > 0$

∴ $6D < 120$

∴ $D < 20$

∴ The price is increasing for $D < 20$.

Exercise 4.4 | Q 3 | Page 112

The total cost function for production of x articles is given as $C = 100 + 600x - 3x^2$. Find the values of x for which total cost is decreasing.

Solution: Given, the cost function is

$C = 100 + 600x - 3x^2$

$$\therefore \frac{dC}{dx} = 0 + 600 - 6x$$

= 600 - 6x

= 6(100 - x)

Since total cost C is a decreasing function,

$$\frac{dC}{dx} < 0$$

∴ 6(100 - x) < 0

∴ 100 - x < 0

∴ 100 < x

∴ x > 100

∴ The total cost is decreasing for x > 100.

Exercise 4.4 | Q 4.1 | Page 112

The manufacturing company produces x items at the total cost of ₹ 180 + 4x. The demand function for this product is P = (240 – x). Find x for which revenue is increasing.

Solution: Let C be the total cost function and R be the revenue

∴ C = 180 + 4x

Now, Revenue = Price × Demand

∴ R = P × x = (240 - x)x

∴ R = 240x - x^2

$$\therefore \frac{dR}{dx} = 240 - 2x = 2(120 - x)$$

Since revenue R is an increasing function, $\frac{dR}{dx} > 0$

$\therefore 2(120 - x) > 0$

$\therefore 120 - x > 0$

$\therefore 120 > x$

$\therefore x < 120$

$\therefore$ The revenue is increasing for $x < 120$.

Exercise 4.4 | Q 4.2 | Page 112

A manufacturing company produces x items at the total cost of Rs (180+4x). The demand function of this product is P=(240 - x). Find x for which profit is increasing.

Solution: Total cost function C = 180 + 4x

Demand function P = 240 - x

Where x Is the number of items produced.

Total revenue R=P.D = x (240 - x)

$\therefore R = 240x - x^2$

Profit function π = R - C

$= (240x - x^2) - (180 + 4x)$

$= 240x - x^2 - 4x - 180$

$\therefore \pi = -x^2 + 236x - 180$

Differentiating w.r.t. x

$$\frac{d\pi}{dx} = -2x + 236$$

Profit π is increasing if $\frac{d\pi}{dx} > 0$

i.e. if $-2x + 236 > 0$

i.e. if $236 > 2x$

i.e. if $x < \frac{236}{2}$

i.e. if $x < 118$

∴ The profit is increasing for $x < 118$.

Exercise 4.4 | Q 5.1 | Page 112

For manufacturing x units, labour cost is 150 - 54x, processing cost is x^2 and revenue R = $10800x - 4x^3$. Find the value of x for which Total cost is decreasing.

Solution: Total cost C(x) = Processing cost + labour cost

$C(x) = x^2 + 150 - 54x$

$C(x) = x^2 - 54x + 150$

$$\frac{dC}{dx} = 2x - 54$$

Total cost is decreasing

If $\frac{dC}{dx} < 0$

i.e if $2x - 54 < 0$

i.e if $2x < 54$

i.e if $x < 27$

Total cost C is decreasing for $x < 27$.

Exercise 4.4 | Q 5.2 | Page 112

For manufacturing x units, labour cost is $150 – 54x$ and processing cost is x^2. Price of each unit is $p = 10800 – 4x^2$. Find the values of x for which Revenue is increasing.

Solution: Revenue = Price × Demand

$\therefore R = p \times x$

$\therefore R = (10800 - 4x^2)x$

$\therefore R = 10800 - 4x^3$

$$\therefore \frac{dR}{dx} = 10800 - 12x^2 = 12(900 - x^2)$$

Since revenue R is an increasing function,

$$\frac{dR}{dx} > 0$$

$\therefore 12(900 - x^2) > 0$

$\therefore 900 - x^2 > 0$

$\therefore 900 > x^2$

$\therefore x^2 < 900$

$\therefore -30 < x < 30$

$\therefore x > -30$ and $x < 30$

But $x > -30$ is not possible ....[$\because x > 0$]

$\therefore x < 30$

$\therefore$ The revenue R is increasing for $x < 30$.

Exercise 4.4 | Q 6.1 | Page 112

The total cost of manufacturing x articles is C = ($47x + 300x^2 - x^4$). Find x, for which average cost is increasing.

Solution: $C = 47x + 300x^2 - x^4$

$$\text{Average cost } C_A = \frac{C}{x} = 47 + 300x - x^3$$

Differencing w.r.t. x,

$$\frac{dC_A}{dx} = 300 - 3x^2$$

$$\text{Now } C_A \text{ is increasing if } \frac{dC_A}{dx} > 0$$

∴ $300 - 3x^2 > 0$

∴ $300 > 3x^2$

∴ $100 > x^2$

∴ $x^2 < 100$

∴ $-10 < x < 10$

∴ $x > -10$ and $x < 10$

But $x > -10$ is not possible ....[∵ $x > 0$]

∴ $x < 10$

∴ The average cost C_A is increasing for $x < 10$.

Exercise 4.4 | Q 6.2 | Page 112

The total cost of manufacturing x articles $C = 47x + 300x^2 – x^4$. Find x, for which average cost is decreasing.

Solution: $C = 47x + 300x^2 - x^4$

Average cost C_A = $\frac{C}{x} = 47 + 300x - x^3$

Differencing w.r.t. x,

$$\frac{dC_A}{dx} = 300 - 3x^2$$

Now C_A is decreasing if $\frac{dC_A}{dx} < 0$

$\therefore 300 - 3x^2 < 0$

$\therefore 300 < 3x^2$

$\therefore 100 < x^2$

$\therefore x^2 > 100$

$\therefore x > 10$ or $x < -10$

But $x < -10$ is not possible .....[$\because x > 0$]

$\therefore x > 10$

Hence C_A is decreasing for $x > 10$.

Exercise 4.4 | Q 7.1 | Page 112

Find the marginal revenue if the average revenue is 45 and elasticity of demand is 5.

Solution: Given, average revenue (R_A) = 45 and

elasticity of demand (η) = 5

$$R_m = R_A\left(1 - \frac{1}{\eta}\right)$$

$\therefore R_m = 45\left(1 - \frac{1}{5}\right) = 45\left(\frac{4}{5}\right)$

$\therefore R_m = 36$

$\therefore$ Marginal revenue $(R_m) = 36$

Exercise 4.4 | Q 7.2 | Page 112

Find the price, if the marginal revenue is 28 and elasticity of demand is 3.

Solution: Given, marginal revenue $(R_m) = 28$ and

elasticity of demand $(\eta) = 3$

$$R_m = P\left(1 - \frac{1}{\eta}\right)$$

$$\therefore 28 = P\left(1 - \frac{1}{3}\right)$$

$$\therefore 28 = P\left(\frac{2}{3}\right)$$

$$\therefore \frac{28 \times 3}{2} = P$$

$\therefore P = 42$

$\therefore$ price = ₹ 42

Exercise 4.4 | Q 7.3 | Page 112

Find the elasticity of demand, if the marginal revenue is 50 and price is Rs 75.

Solution:

Given, marginal revenue $(R_m) = 50$ and price (P) = ₹ 75 using, $R_m = p\left(1 - \frac{1}{\eta}\right)$

$\therefore 50 = 75\left(1 - \frac{1}{\eta}\right)$

$$\therefore \frac{50}{75} = 1 - \frac{1}{\eta}$$

$$\therefore \frac{2}{3} = 1 - \frac{1}{\eta}$$

$$\therefore \frac{1}{\eta} = \frac{1}{3}$$

$\therefore \eta = 3$

$\therefore$ elasticity of demand = 3

Exercise 4.4 | Q 8 | Page 112

If the demand function is D = (p+6/p-3), find the elasticity of demand at p = 4.

Solution: Given, demand function is

$D = \left(\frac{p+6}{p-3}\right)$

$$\therefore \frac{dD}{dp} = \frac{(p-3)\frac{d}{dp}(p+6) - (p+6)\frac{d}{dp}(p-3)}{(p-3)^2}$$

$$= \frac{(p-3)(1+0) - (p+6)(1-0)}{(p-3)^2}$$

$$\therefore \frac{dD}{dp} = \frac{p-3-p-6}{(p-3)^2}$$

$$= \frac{-9}{(p-3)^2}$$

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$\therefore \eta = \frac{-p}{\left(\frac{p+6}{p-3}\right)} \cdot \frac{-9}{(p-3)^2}$$

$$\therefore \eta = \frac{9p}{(p+4)(p-3)}$$

Substituting p = 4, we get

$$\eta = \frac{9 \times 4}{(4+6)(4-3)} = \frac{36}{10(1)}$$

∴ η = 3.6

∴ elasticity of demand at p = 4 is 3.6

Exercise 4.4 | Q 9 | Page 113

Find the price for the demand function D = $\left(\frac{2p+3}{3p-1}\right)$, when elasticity of demand is $\frac{11}{14}$.

Solution:

Given, elasticity of demand (η) = $\frac{11}{14}$ and demand function is D = $\left(\frac{2p+3}{3p-1}\right)$

$$\therefore \frac{dD}{dp} = \frac{(3p-1)\frac{d}{dp}(2p+3) - (2p+3)\frac{d}{dp}(3p-1)}{(3p-1)^2}$$

$$= \frac{(3p - 1)(2 + 0) - (2p + 3)(3 - 0)}{(3p - 1)^2}$$

$$\therefore \frac{dD}{dp} = \frac{6p - 2 - 6p - 9}{(3p - 1)^2} = \frac{-11}{(3p - 1)^2}$$

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$\therefore \frac{11}{14} = \frac{-p}{\frac{2p+3}{3p-1}} \cdot \frac{-11}{(3p - 1)^2}$$

$$\therefore \frac{11}{14} = \frac{11p}{(2p + 3)(3p - 1)}$$

∴ 11 (2p + 3) (3p - 1) = 11p × 14

∴ $6p^2$ - 2p + 9p - 3 = 14p

∴ $6p^2$ + 7p - 14p - 3 = 0

∴ $6p^2$ - 7p - 3 = 0

∴ (2p - 3)(3p + 1) = 0

∴ 2p - 3 = 0 or 3p + 1 = 0

∴ p = $\frac{3}{2}$ or p = $-\frac{1}{3}$

But, p ≠ $-\frac{1}{3}$

∴ p = $\frac{3}{2}$

∴ The price for elasticity of demand (η) = $\frac{11}{14}$ is $\frac{3}{2}$.

Exercise 4.4 | Q 10.1 | Page 113

If the demand function is D = 50 – 3p – p^2. Find the elasticity of demand at p = 5 comment on the result.

Solution: Given, demand function is D = 50 - 3p - p^2.

$$\therefore \frac{dD}{dp} = 0 - 3 - 2p = -3 - 2p$$

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$\therefore \eta = \frac{-p}{50 - 3p - p^2} \cdot (-3 - 2p)$$

$$\therefore \eta = \frac{3p + 2p^2}{50 - 3p - p^2}$$

When p = 5

$$\eta = \frac{3(5) + 2(5)^2}{50 - 3(5) - (5)^2} = \frac{15 + 50}{50 - 15 - 25} = \frac{65}{10}$$

∴ η = 6.5

∴ elasticity of demand at p = 5 is 6.5

Here, η > 0

∴ The demand is elastic.

Exercise 4.4 | Q 10.2 | Page 113

If the demand function is D = 50 – 3p – p^2. Find the elasticity of demand at p = 2 comment on the result.

Solution: Given, demand function is D = 50 - 3p - p^2.

$$\therefore \frac{dD}{dp} = 0 - 3 - 2p = -3 - 2p$$

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$\therefore \eta = \frac{-p}{50 - 3p - p^2} \cdot (-3 - 2p)$$

$$\therefore \eta = \frac{3p + 2p^2}{50 - 3p - p^2}$$

When p = 2

$$\eta = \frac{3(2) + 2(2)^2}{50 - 3(2) - (2)^2} = \frac{6 + 8}{50 - 6 - 4} = \frac{14}{40}$$

$$\therefore \eta = \frac{7}{20}$$

$\therefore$ elasticity of demand at p = 2 is $\frac{7}{20}$

Here, $\eta < 0$

$\therefore$ The demand is elastic.

Exercise 4.4 | Q 11.1 | Page 113

For the demand function D = 100 – p^2/2. Find the elasticity of demand at p = 10 and comment on the results.

Solution:

Given, demand function is D = 100 - $\frac{p^2}{2}$

$$\therefore \frac{dD}{dp} = 0 - \frac{2p}{2} = -p$$

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$\therefore \eta = \frac{-p}{100 - \frac{p^2}{2}} \cdot (-p)$$

$$= \frac{p^2}{\frac{200 - p^2}{2}}$$

$$\therefore \eta = \frac{2p^2}{200 - p^2}$$

When p = 10,

$$\eta = \frac{2(10)^2}{200 - (10)^2} = \frac{200}{100} = 2$$

∴ elasticity of demand at p = 10 is 2

Here, η > 0

∴ The demand is elastic.

Exercise 4.4 | Q 11.2 | Page 113

For the demand function D = 100 – p^2/2. Find the elasticity of demand at p = 6 and comment on the results.

Solution:

Given, demand function is D = 100 - $\frac{p^2}{2}$

$$\therefore \frac{dD}{dp} = 0 - \frac{2p}{2} = -p$$

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$\therefore \eta = \frac{-p}{100 - \frac{p^2}{2}} \cdot (-p)$$

$$= \frac{p^2}{\frac{200-p^2}{2}}$$

$$\therefore \eta = \frac{2p^2}{200 - p^2}$$

When p = 6,

$$\eta = \frac{2(6)^2}{200 - (6)^2} = \frac{72}{164} = \frac{18}{41}$$

∴ elasticity of demand at p = 6 is $\frac{18}{41}$

Here, η > 0

∴ The demand is inelastic.

Exercise 4.4 | Q 12.1 | Page 113

A manufacturing company produces x items at a total cost of ₹ 40 + 2x. Their price is given as p = 120 – x. Find the value of x for which revenue is increasing.

Solution: Let C be the total cost function.

∴ C = 40 + 2x

Revenue = Price × Demand

$$\therefore R = p \times x = (120 - x) \cdot x$$

∴ R = 120x - x^2

$$\therefore \frac{dR}{dx} = 120 - 2x = 2(60 - x)$$

Since revenue R is an increasing function, $\frac{dR}{dx} > 0$

∴ 2(60 - x) > 0

∴ 60 - x > 0

∴ 60 > x

∴ x < 60

∴ The revenue R is increasing for x < 60.

Exercise 4.4 | Q 12.2 | Page 113

A manufacturing company produces x items at a total cost of ₹ 40 + 2x. Their price is given as p = 120 – x. Find the value of x for which profit is increasing.

Solution: Let C be the total cost function.

∴ C = 40 + 2x

Profit = Revenue - Cost

∴ π = R - C

∴ π = $120x - x^2 - (40 + 2x)$

$= 120x - x^2 - 40 - 2x$

∴ π = $-x^2 + 118x - 40$

$$\therefore \frac{d\pi}{dx} = -2x + 118 = 2(-x + 59)$$

Since profit π is an increasing function, $\frac{d\pi}{dx} > 0$

∴ 2(- x + 59) > 0

∴ - x + 59 > 0

∴ 59 > x

∴ x < 59

∴ The profit π is increasing for x < 59.

Exercise 4.4 | Q 12.3 | Page 113

A manufacturing company produces x items at a total cost of ₹ 40 + 2x. Their price is given as p = 120 – x. Find the value of x for which also find an elasticity of demand for price 80.

Solution: Given, the price is p = 120 - x

∴ x = 120 - p

where, x = demand

$$\therefore \frac{dx}{dp} = 0 - 1 = -1$$

$$\eta = \frac{-p}{x} \cdot \frac{dx}{dp}$$

$$\therefore \eta = \frac{-p}{120 - p} \cdot (-1)$$

$$\therefore \eta = \frac{p}{120 - p}$$

p = 80 ....(Given)

$$\therefore \eta = \frac{80}{120 - 80} = \frac{80}{40} = 2$$

∴ The elasticity of demand for p = 80 is η = 2.

Exercise 4.4 | Q 13 | Page 113

Find MPC, MPS, APC and APS, if the expenditure E_c of a person with income I is given as $E_c = (0.0003)\ I^2 + (0.075)\ I$ When I = 1000.

Solution: Given, $E_c = (0.0003)\ I^2 + (0.075)\ I$

$\therefore$ MPC = $\frac{dE_c}{dI} = (0.0003)(2I) + 0.075$

$\therefore$ MPC = 0.0006 I + 0.075

I = 1000 ....[Given]

$\therefore$ MPC = 0.0006(1000) + 0.075

= 0.6 + 0.075

$\therefore$ MPC = 0.675

Since MPC + MPS = 1,

0.675 + MPS = 1

$\therefore$ MPS = 1 - 0.675

$\therefore$ MPS = 0.325

Now, APC = $\frac{E_c}{I}$

$$= \frac{(0.0003)I^2 + (0.075)I}{I}$$

$$= \frac{I(0.0003I + 0.075)}{I}$$

$\therefore$ APC = 0.0003 I + 0.075

I = 1000 ....[Given]

$\therefore$ APC = 0.0003(1000) + 0.075

= 0.3 + 0.075

$\therefore$ APC = 0.375

Also, APC + APS = 1

$\therefore$ 0.375 + APS = 1

∴ APS = 1 - 0.375

∴ APS = 0.625

∴ For I = 1000,

MPC = 0.675, MPS = 0.325

APC = 0.375, APS = 0.625

MISCELLANEOUS EXERCISE 4 [PAGES 113 - 114]

Miscellaneous Exercise 4 | Q 1.1 | Page 113

Choose the correct alternative.

The equation of tangent to the curve $y = x^2 + 4x + 1$ at (-1, -2) is

1. **2x - y = 0**
2. 2x + y - 5 = 0
3. 2x - y - 1 = 0
4. x + y - 1 = 0

Solution: 2x - y = 0

Miscellaneous Exercise 4 | Q 1.2 | Page 113

Choose the correct alternative.

The equation of tangent to the curve $x^2 + y^2 = 5$ where the tangent is parallel to the line 2x – y + 1 = 0 are

1. **2x – y + 5 = 0; 2x – y – 5 = 0**
2. 2x + y + 5 = 0; 2x + y – 5 = 0
3. x – 2y + 5 = 0; x – 2y – 5 = 0
4. x + 2y + 5 = 0; x + 2y – 5 = 0

Solution: 2x – y + 5 = 0; 2x – y – 5 = 0

Miscellaneous Exercise 4 | Q 1.3 | Page 113

Choose the correct alternative.

If elasticity of demand η = 1, then demand is

1. constant
2. inelastic
3. **unitary elastic**

4. elastic

Solution: unitary elastic

Miscellaneous Exercise 4 | Q 1.4 | Page 113

Choose the correct alternative.

If $0 < \eta < 1$, then demand is

1. constant
2. **inelastic**
3. unitary elastic
4. elastic

Solution: inelastic

Miscellaneous Exercise 4 | Q 1.5 | Page 113

Choose the correct alternative.

The function $f(x) = x^3 - 3x^2 + 3x - 100$, $x \in R$ is

1. **increasing for all $x \in R, x \neq 1$**
2. decreasing
3. neither, increasing nor decreasing
4. decreasing for all $x \in R, x \neq 1$

Solution: increasing for all $x \in R, x \neq 1$

Explanation:

$f(x) = x^3 - 3x^2 + 3x - 100$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 6x + 3$

$= 3(x^2 - 2x + 1)$

$= 3(x - 1)^2$

Note that $(x - 1)^2 > 0$ for all $x \in R, x \neq 1$.

$\therefore 3(x - 1)^2 > 0$ for all $x \in R, x \neq 1$

$\therefore f(x)$ is increasing for all $x \in R, x \neq 1$.

Miscellaneous Exercise 4 | Q 1.6 | Page 113

Choose the correct alternative.

If $f(x) = 3x^3 - 9x^2 - 27x + 15$ then

1. f has maximum value 66
2. f has minimum value 30
3. **f has maxima at x = –1**
4. f has minima at x = –1

Solution: f has maxima at x = –1

Explanation:

$f(x) = 3x^3 - 9x^2 - 27x + 15$

$\therefore f'(x) = 9x^2 - 18x - 27$

$\therefore f''(x) = 18x - 18$

Consider, $f'(x) = 0$

$\therefore 9x^2 - 18x - 27 = 0$

$\therefore x^2 - 2x - 3 = 0$

$\therefore (x - 3)(x + 1) = 0$

$\therefore x = 3$ or $x = -1$

For $x = 3$, $f''(x) = 18(3) - 18 = 36 > 0$

$\therefore$ f(x) has minimum value at x = 3

$\therefore$ Minimum value = f(3) = – 66

For $x = -1$, $f''(x) = 18(-1) - 18 = -36 < 0$

$\therefore$ f(x) has maximum value at x = –1

$\therefore$ Maximum value = f(–1) = 30.

Miscellaneous Exercise 4 | Q 2.1 | Page 114

Fill in the blank:

The slope of tangent at any point (a, b) is called as ________.

Solution: The slope of tangent at any point (a, b) is called a **gradient**.

Miscellaneous Exercise 4 | Q 2.2 | Page 114

Fill in the blank:

If $f(x) = x - 3x^2 + 3x - 100$, $x \in R$ then $f''(x)$ is _______

Solution: If $f(x) = x - 3x^2 + 3x - 100$, $x \in R$ then $f''(x)$ is **6(x - 1)**

Explanation:

$f(x) = x^3 - 3x^2 + 3x - 100$

$\therefore f'(x) = 3x^2 - 6x + 3$

$\therefore f''(x) = 6x - 6$

$= 6(x - 1)$

Miscellaneous Exercise 4 | Q 2.3 | Page 114

Fill in the blank:

If $f(x) = \frac{7}{x} - 3$, $x \in R$ $x \neq 0$ then $f''(x)$ is ______

Solution:

If $f(x) = \frac{7}{x} - 3$, $x \in R$ $x \neq 0$ then $f''(x)$ is $\mathbf{14x^{-3}}$.

Explanation:

$f(x) = \frac{7}{x} - 3$

$\therefore f'(x) = \frac{-7}{x^2}$

$\therefore f''(x) = \frac{14}{x^3}$

$= 14x^{-3}$

Miscellaneous Exercise 4 | Q 2.4 | Page 114

Fill in the blank:

A road of 108 m length is bent to form a rectangle. If the area of the rectangle is maximum, then its dimensions are _______.

Solution: A road of 108 m length is bent to form a rectangle. If area of the rectangle is maximum, then its dimensions are **x = 27, y = 27.**

Explanation:

Let the length and breadth of a rectangle be x and y.

∴ Perimeter of rectangle = 2(x + y) = 108

∴ x + y = 54

∴ y = 54 - x ....(i)

Let A = Area of rectangle = x × y

= x (54 - x) = 54x - x^2

Differentiating w.r.t. we get

$$\frac{dA}{dt} = 54 - 2x$$

Consider, $\frac{dA}{dt} = 0$

∴ 54 - 2x = 0

∴ x = 27

∴ y = 27 ....[from (i)]

x = 27, y = 27

Miscellaneous Exercise 4 | Q 2.5 | Page 114

Fill in the blank:

If f(x) = x log x, then its minimum value is______

Solution:

If f(x) = x log x, then its minimum value is $\frac{-1}{e}$

Miscellaneous Exercise 4 | Q 3.1 | Page 114

State whether the following statement is True or False:

The equation of tangent to the curve $y = 4xe^x$ at $\left(-1, \frac{-4}{e}\right)$ is ye + 4 = 0

1. **True**
2. False

Solution: True.

Explanation:

$y = 4x\ e^x$

$\therefore \frac{dy}{dx} = 4e^x + 4xe^x$

Slope of the tangent at $\left(-1, \frac{-4}{e}\right)$ is

$$\left(\frac{dy}{dx}\right)_{\left(-1, \frac{-4}{e}\right)} = 4e^{-1} + 4(-1)e^{-1}$$

$$= \frac{4}{e} - \frac{4}{e} = 0$$

$\therefore$ Equation of the tangent at $\left(-1, \frac{-4}{e}\right)$ is $\left(y + \frac{4}{e}\right) = 0(x + 1)$

$\therefore$ ye + 4 = 0

Miscellaneous Exercise 4 | Q 3.2 | Page 114

State whether the following statement is True or False:

x + 10y + 21 = 0 is the equation of normal to the curve $y = 3x^2 + 4x - 5$ at (1, 2).

1. True
2. **False**

Solution: False.

Explanation:

At (1, 2) equation of the line x + 10y + 21 = 0 is

(1) + 10(2) + 21 = 1 + 20 + 21 = 42 ≠ 0

i.e., (1, 2) does not lie on line x + 10y + 21 = 0

Miscellaneous Exercise 4 | Q 3.3 | Page 114

State whether the following statement is True or False:

An absolute maximum must occur at a critical point or at an end point.

1. **True**
2. False

Solution: True.

Miscellaneous Exercise 4 | Q 3.4 | Page 114

State whether the following statement is True or False:

The function f(x) = $x \cdot e^{x(1-x)}$ is increasing on $\left(\frac{-1}{2}, 1\right)$.

1. **True**
2. False

Solution: True.

Explanation:

f(x) = $x \cdot e^{x(1-x)}$

∴ f '(x) = $e^{x(1-x)} + x \cdot e^{x(1-x)}[1 - 2x]$

$= e^{x(1-x)}\left[1 + x - 2x^2\right]$

If f(x) is increasing, then f '(x) > 0.

Consider f '(x) > 0

∴ $e^{x(1-x)}\left(1 + x - 2x^2\right) > 0$

∴ $2x^2 - x - 1 < 0$

∴ (2x + 1)(x - 1) < 0

ab < 0 ⇔ a > 0 and b < 0 or a < 0 or b > 0

∴ Either (2x + 1) > 0 and (x – 1) < 0 or

(2x + 1) < 0 and (x – 1) > 0

Case 1: (2x + 1) > 0 and (x – 1) < 0

$\therefore x > -\frac{1}{2}$ and $x < 1$

i.e., $x \in \left(-\frac{1}{2}, 1\right)$

Case 2: (2x + 1) < 0 and (x – 1) > 0

$\therefore x < -\frac{1}{2}$ and $x > 1$

which is not possible.

∴ f(x) is increasing on $\left(-\frac{1}{2}, 1\right)$

Miscellaneous Exercise 4 | Q 4.1 | Page 114

Find the equation of tangent and normal to the following curve.

$xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ where t is parameter.

Solution: Equation of the curve is $xy = c^2$

Differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0$$

$$\therefore \frac{dy}{dx} = \frac{-y}{x}$$

$\therefore$ slope of tangent at $\left(ct, \frac{c}{t}\right)$ is

$$\left(\frac{dy}{dx}\right)_{\left(ct, \frac{c}{t}\right)} = \frac{\frac{-c}{t}}{ct} = \frac{-1}{t^2}$$

Equation of tangent at $\left(ct, \frac{c}{t}\right)$ is

$$\left(y - \frac{c}{t}\right) = \frac{-1}{t^2}(x - ct)$$

$\therefore$ $yt^2 - ct = - x + ct$

$\therefore$ $x + yt^2 - 2ct = 0$

Slope of normal $= \frac{-1}{\frac{-1}{t^2}} = t^2$

Equation of normal at $\left(ct, \frac{c}{t}\right)$ is

$$\left(y - \frac{c}{t}\right) = t^2(x - ct)$$

$\therefore$ $yt - c = xt^3 - ct^4$

$\therefore$ $t^3x - yt - (t^4 - 1)c = 0$

Miscellaneous Exercise 4 | Q 4.1 | Page 114

Find the equation of tangent and normal to the following curve.

$y = x^2 + 4x$ at the point whose ordinate is -3.

Solution: Equation of the curve is $y = x^2 + 4x$ ....(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2x + 4$$

$y = - 3$ .....[Given]

Putting the value of y in (i), we get

$- 3 = x^2 + 4x$

$\therefore x^2 + 4x + 3 = 0$

$\therefore (x + 1)(x + 3) = 0$

$\therefore x = - 1$ or $x = -3$

For $x = -1$, $y = (- 1)^2 + 4(-1) = - 3$

$\therefore$ Point is $(x, y) = (- 1, -3)$

Slope of tangent at $(-1, -3)$ is $\frac{dy}{dx} = 2(-1) + 4 = 2$

Equation of tangent at $(- 1, - 3)$ is

$y + 3 = 2(x + 1)$

$\therefore y + 3 = 2x + 2$

$\therefore 2x - y - 1 = 0$

Slope of normal at $(-1, -3)$ is $\frac{-1}{\frac{dy}{dx}} = \frac{-1}{2}$

Equation of normal at $(-1, -3)$ is

$y + 3 = \frac{-1}{2}(x + 1)$

$\therefore 2y + 6 = - x - 1$

$\therefore x + 2y + 7 = 0$

For $x = - 3$, $y = (- 3)^2 + 4(- 3) = - 3$

$\therefore$ Point is $(x, y) = (- 3, - 3)$

Slope of tangent at $(- 3, - 3) = 2(- 3) + 4 = - 2$

Equation of tangent at $(- 3, -3)$ is

$y + 3 = - 2(x + 3)$

$\therefore y + 3 = - 2x - 6$

$\therefore 2x + y + 9 = 0$

Slope of normal at (– 3, – 3) is $\frac{-1}{\frac{dy}{dx}} = \frac{1}{2}$

Equation of normal at (- 3, - 3) is

y + 3 = $\frac{1}{2}$(x + 3)

∴ 2y + 6 = x + 3

∴ x - 2y - 3 = 0

Miscellaneous Exercise 4 | Q 4.1 | Page 114

Find the equation of tangent and normal to the following curve.

x = $\frac{1}{t}$, $y = t - \frac{1}{t}$, at t = 2

Solution:

x = $\frac{1}{t}$, $y = t - \frac{1}{t}$

$$\therefore \frac{dx}{dt} = -\frac{1}{t^2}, \frac{dy}{dt} = 1 + \frac{1}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \frac{1}{t^2}}{-\frac{1}{t^2}} = -t^2 - 1$$

Slope of tangent at t = 2 is

$\left(\frac{dy}{dx}\right)_{t=2}$ = - $(2)^2$ - 1 = - 5

∴ Point is (x_1, y_1) = $\left(\frac{1}{2}, 2 - \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{3}{2}\right)$

Equation of tangent at $\left(\frac{1}{2}, \frac{3}{2}\right)$

$y - \frac{3}{2} = -5\left(x - \frac{1}{2}\right)$

∴ 2y - 3 = - 5(2x - 1)

∴ 10x + 2y = 8

∴ 5x + y = 4

∴ 5x + y - 4 = 0

Slope of normal at t = 2 is $\frac{-1}{\left(\frac{dy}{dx}\right)_{t=2}}$ = $\frac{-1}{-5} = \frac{1}{5}$

Equation of normal at $\left(\frac{1}{2}, \frac{3}{2}\right)$ is

$y - \frac{3}{2} = \frac{1}{5}\left(x - \frac{1}{2}\right)$

∴ $\frac{2y - 3}{2} = \frac{2x - 1}{10}$

∴ 10y - 15 = 2x - 1

∴ 2x - 10y + 14 = 0

∴ x - 5y + 7 = 0

Miscellaneous Exercise 4 | Q 4.1 | Page 114

Find the equation of tangent and normal to the following curve.

$y = x^3 - x^2 - 1$ at the point whose abscissa is -2.

Solution: Equation of the curve is $y = x^3 - x^2 - 1$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 2x$$

If x = - 2, ...[Given]

Putting the value of x in (i), we get

$y = (-2)^3 - (-2)^2 - 1 = -8 - 4 - 1 = -13$

∴ Point is $P(x_1, y_1) \equiv (-2, -13)$

Slope of tangent at (- 2,- 13) is

$$\left(\frac{dy}{dx}\right)_{(-2,-13)} = 2(-2)^2 - 2(-2) = 12 + 4 = 16$$

Equation of tangent at (- 2, -13) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x=-2)} (x - x_1)$$

$\therefore y - (-13) = 16[x - (-2)]$

$\therefore y + 13 = 16x + 32$

$\therefore 16x - y + 19 = 0$

Slope of normal at (- 2, - 13) is $\dfrac{-1}{\left(\frac{dy}{dx}\right)_{(-2,-13)}} = -\dfrac{1}{16}$

Equation of normal at (-2, - 13) is

$$\therefore y + 13 = \frac{-1}{16}(x + 2)$$

$$\therefore x + 16y + 210 = 0$$

Miscellaneous Exercise 4 | Q 4.2 | Page 114

Find the equation of tangent to the curve

$y = \sqrt{x - 3}$

which is perpendicular to the line 6x + 3y – 4 = 0.

Solution:

Equation of the curve is $y = \sqrt{x - 3}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x - 3}}$$

Slope of the tangent at $P(x_1, y_1)$ is

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1) = \frac{1}{2\sqrt{x_1 - 3}}}$$

Slope of the line 6x + 3y – 4 = 0 is –2.

According to the given condition, tangent to the curve is perpendicular to the line 6x + 3y – 4 = 0.

$$\therefore \text{slope of the tangent} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{1}{2}$$

$$\therefore \frac{1}{2\sqrt{x_1 - 3}} = \frac{1}{2}$$

$\therefore \sqrt{x_1 - 3} = 1$

$\therefore x_1 - 3 = 1$

$\therefore x_1 = 4$

$P(x_1, y_1)$ lies on the curve $y = \sqrt{x - 3}$

$\therefore y_1 = \sqrt{4 - 3}$

$\therefore y_1 = 1$

∴ The point on the given curve is (4, 1).

∴ Equation of the tangent at (4, 1) is

$(y - 1) = \frac{1}{2}(x - 4)$

$\therefore 2y - 2 = x - 4$

$\therefore x - 2y - 2 = 0$

Miscellaneous Exercise 4 | Q 4.3 | Page 114

Show that function $f(x) = \frac{x - 2}{x + 1}$, $x \neq -1$ is increasing.

Solution:

$f(x) = \frac{x - 2}{x + 1}$, $x \neq 0$

For function to be increasing, $f'(x) > 0$

Then $f'(x) = \frac{(x + 1)\frac{d}{dx}(x - 2) - (x - 2)\frac{d}{dx}(x + 1)}{(x + 1)^2}$

$$= \frac{(x+1)-(x-2)}{(x+1)^2} = \frac{x+1-x+2}{(x+1)^2}$$

$$= \frac{3}{(x+1)^2} > 0 \quad ....[\because (x+1) \neq 0, (x+1)^2 > 0]$$

Thus, f(x) is an increasing function for x ≠ - 1.

Miscellaneous Exercise 4 | Q 4.4 | Page 114

Show that function f(x) =3/x + 10, x ≠ 0 is decreasing.

Solution:

$$f(x) = \frac{3}{x} + 10$$

For function to be decreasing, f '(x) < 0

$$\text{Then } f'(x) = \frac{-3}{x^2} < 0 \quad ....[\because x \neq 0, -x^2 < 0]$$

Negative sign indicates that it always decreases as x^2 never becomes negative.

Thus, f(x) is a decreasing function for x ≠ 0.

Miscellaneous Exercise 4 | Q 4.5 | Page 114

If x + y = 3 show that the maximum value of x^2y is 4.

Solution: x + y = 3

∴ y = 3 - x

Let $T = x^2y = x^2(3 - x) = 3x^2 - x^3$

Differentiating w.r.t. x, we get

$$\frac{dT}{dx} = 6x - 3x^2 \quad ....(i)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2T}{dx^2} = 6 - 6x \quad ...(ii)$$

Consider, $\frac{dT}{dx} = 0$

∴ $6x - 3x^2 = 0$

∴ $x = 2$

For x = 2,

$$\left(\frac{d^2T}{dx^2}\right)_{(x=2)} = 6 - 6(2) = 6 - 12 = -6 < 0$$

Thus, T, i.e., x^2y is maximum at x = 2

For x = 2, y = 3 - x = 3 - 2 = 1

∴ Maximum value of $T = x^2y = (2)^2(1) = 4$

Miscellaneous Exercise 4 | Q 4.6 | Page 114

Examine the function for maxima and minima $f(x) = x^3 - 9x^2 + 24x$

Solution: $f(x) = x^3 - 9x^2 + 24x$

∴ $f'(x) = 3x^2 - 18x + 24$

∴ $f''(x) = 6x - 18$

Consider, $f'(x) = 0$

∴ $3x^2 - 18x + 24 = 0$

∴ $3(x^2 - 6x + 8) = 0$

∴ $3(x - 4)(x - 2) = 0$

∴ $(x - 4)(x - 2) = 0$

∴ x = 2 or x = 4

For x = 4,

$f''(4) = 6(4) - 18 = 24 - 18 = 6 > 0$

∴ f(x) is minimum at x = 4

∴ Minima = $f(4) = (4)^3 - 9(4)^2 + 24(4)$

$= 64 - 144 + 96 = 16$

For x = 2,

$f''(2) = 6(2) - 18 = 12 - 18 = -6 < 0$

∴ f(x) is maximum at x = 2

∴ Maxima = $f(2) = (2)^3 - 9(2)^2 + 24(2) = 8 - 36 + 48 = 20$

∴ Maxima = 20 and Minima = 16