Chapter 14

Complex Numbers-II

Solutions (Set-1)

Short Answer Type Questions :

1. Find the square root of the complex number 7 + 24i.

Sol. Let $\sqrt{7+24i} = x + iy$, then

$7 + 24i = x^2 - y^2 + 2xyi$

$\Rightarrow \quad x^2 - y^2 = 7$ and $2xy = 24$

$\Rightarrow \quad x^2 + y^2 = 25$

$\Rightarrow \quad x^2 = 16$ and $y^2 = 9$

$\Rightarrow \quad \sqrt{7+24i} = \pm(4+3i) \qquad (\because \ \text{Im}(z) > 0)$

2. Find the square root of the complex number 12 – 5i.

Sol. Let $\sqrt{12-5i} = x + iy$

$\Rightarrow \quad 12 - 5i = x^2 - y^2 + 2ixy$

$\Rightarrow \quad x^2 - y^2 = 12$ and $2xy = -5$

$\Rightarrow \quad x^2 + y^2 = 13$

$\Rightarrow \quad x^2 = \frac{25}{2}$ and $y^2 = \frac{1}{2}$

$\Rightarrow \quad \sqrt{12-5i} = \pm\left(\frac{5-1}{\sqrt{2}}\right), \qquad (\because \ \text{Im}(z) < 0)$

3. Find the value of 1 + ω^{2019} + ω^{2022}, where ω is an imaginary cube root of unity.

Sol. $1+\omega^{2019}+\omega^{2022} = 1+(\omega^3)^{673}+(\omega^3)^{674}$; $(\because \ \omega^3 = 1)$

$= 1+1+1 = 3$

4. Find the value of $\left(-1-\sqrt{3}i\right)^8+\left(-1+\sqrt{3}i\right)^8$.

Sol. Let $-1+\sqrt{3}i=2\omega$ and $-1-\sqrt{3}i=2\omega^2$

$\therefore \quad \left(-1-\sqrt{3}i\right)^8+\left(-1+\sqrt{3}i\right)^8$

$=\left(2\omega^2\right)^8+\left(2\omega\right)^8$

$=2^8\left(\omega^{16}+\omega^8\right)=2^8\left(\omega^{15}\cdot\omega+\omega^6\cdot\omega^2\right)$

$=2^8\left(\omega+\omega^2\right)$

$=-2^8$

5. Let z be a complex number which satisfy $|z-1-i|=\sqrt{2}$. Then find the locus of z.

Sol. Put $z = x + iy$ in $|z-1-i|=\sqrt{2}$

$\Rightarrow \quad |(x-1)+i(y-1)|=\sqrt{2}$

$\Rightarrow \quad (x-1)^2+(y-1)^2=2$

$\Rightarrow \quad x^2+y^2-2x-2y=0$

which is the equation of circle.

6. Let z be a non-zero complex number, if $\text{Im}\left(\frac{1}{z}\right)$ is zero, then find the locus of z.

Sol. $\text{Im}\left(\frac{1}{z}\right)=0$

$\Rightarrow \quad \text{Im}\left(\frac{1}{x+iy}\right)=0 \qquad \Rightarrow \quad \text{Im}\left(\frac{x-iy}{x^2+y^2}\right)=0$

$\Rightarrow \quad \frac{-y}{x^2+y^2}=0 \qquad \Rightarrow \quad y=0$

$\Rightarrow$ x-axis (except origin)

Long Answer Type Questions :

7. Solve the equation $x^7 + x^4 + x^3 + 1 = 0$.

Sol. $x^7+x^4+x^3+1=0 \;\Rightarrow\; (x^4+1)(x^3+1)=0$

$\Rightarrow$ $x^4 = -1$, $x^3 = -1$

$x = -1, -\omega, -\omega^2$ (where ω is non-real cube root of unity)

$\Rightarrow \quad x^4 = \cos(2r+1)\pi + i\sin(2r+1)\pi$

$x = \cos(2r+1)\frac{\pi}{4} + i\sin(2r+1)\frac{\pi}{4}$; ($r$ = 0, 1, 2, 3)

$x = \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}$

$\therefore$ Values of x are –1, $-\omega$, $-\omega^2$, $\frac{1+i}{\sqrt{2}}$, $\frac{-1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$ and $\frac{-1-i}{\sqrt{2}}$

8. Solve the equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

Sol. $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

$\Rightarrow \quad -x^7 + x^3 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

$\Rightarrow \quad -x^7 + x(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) + 1 = 0$

$\Rightarrow \quad x^7 = 1$

Now, $x^7 = \cos 2r\pi + i\sin 2r\pi$

$x = \cos\frac{2r\pi}{7} + i\sin\frac{2r\pi}{7}$

$r = 0$, $x = 1$

$r = 1$, $\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$, $r = 2$, $\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$

$r = 3$, $\cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$, $r = 4$, $\cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}$

$r = 5$, $\cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$, $r = 6$, $\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$

But $x = 1$ is not satisfied original equation.

9. If the real part of $\frac{2z+1}{iz+1}$ is zero, then show that the locus of the point representing z in the argand plane is a straight line.

Sol. Let $z = x + iy$

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+2iy}{(1-y)+ix} = \frac{(2x+1-y)+i(2y-2y^2-2x^2-x)}{(1-y)^2+x^2}$$

$\text{Re}\left(\frac{2z+1}{iz+1}\right) = 0 \quad \Rightarrow \quad 2x + 1 - y = 0$

$2x - y = 1$

10. Let z be a complex number, then what is the locus of z if amplitude of $(z - 3 - 4i)$ is $\frac{\pi}{4}$.

Sol. $\text{amp}(z-3-4i)=\frac{\pi}{4}$

$$\tan^{-1}\left(\frac{y-4}{x-3}\right)=\frac{\pi}{4}$$

$$\Rightarrow \frac{y-4}{x-3}=\tan\frac{\pi}{4}$$

$\Rightarrow y - 4 = x - 3$

$\Rightarrow y = x + 1$

The locus of z is a straight line.

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Chapter 14

Complex Numbers-II

Solutions (Set-2)

[De Moivre's Theorem, Square Root, Cube Root of Unity, *n*th roots of Unity and its Properties]

1. The square root of –8*i* is

(1) ± 2(1 – *i*) (2) 2(1 + *i*) (3) ± (1 – *i*) (4) ± (1 + *i*)

Sol. Answer (1)

$\sqrt{0-8i} = x + iy$

$-8i = x^2 - y^2 + 2ixy$

$x^2 - y^2 = 0 \qquad xy = -4$

$x = \pm y \qquad x(\pm x) = -4 \qquad x^2 = 4$

$x = \pm 2$

$y = \pm 2$

2. If $a = \cos\theta + i\sin\theta$, $b = \cos\phi + i\sin\phi$, $c = \cos\Psi + i\sin\Psi$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 2$, then

$\sin(\theta - \phi) + \sin(\phi - \psi) + \sin(\psi - \theta)$ equals

(1) 3 (2) $-\frac{3}{2}$ (3) 0 (4) $\frac{3}{2}$

Sol. Answer (3)

From question

$\cos(\theta-\varphi) + i\sin(\theta-\varphi) + \cos(\varphi-\psi) + i\sin(\varphi-\psi) + \cos(\psi-\theta) + i\sin(\psi-\theta) = 2 + 0i$

$\Rightarrow \sin(\theta-\varphi) + \sin(\varphi-\psi) + \sin(\psi-\theta) = 0$

3. The value of $(i+\sqrt{3})^{100} + (i-\sqrt{3})^{100} + 2^{100} =$

(1) 1 (2) –1 (3) 0 (4) 2

Sol. Answer (3)

$(i+\sqrt{3})^{100}+(i-\sqrt{3})^{100}+2^{100}$

$=(-i)^{100}\left(\frac{-1-i\sqrt{3}}{2}\right)^{100}\times 2^{100}+(-i)^{100}\left(\frac{-1+i\sqrt{3}}{2}\right)^{100}.2^{100}+2^{100}$

$= 2^{100}((\omega^2)^{100} + (\omega)^{100} + 1)$

$= 2^{100}(\omega^2 + \omega + 1)$ {where ω is complex cube root of unity}

$= 0$

4. Let $a = i^i$ and consider the following statements

S_1: $a = e^{-\frac{\pi}{2}}$

S_2: The value of $\sin(\ln a) = -1$

S_3: $\text{Im}(a) + \arg(a) = 0$

Now identify the correct combination of the true statements.

(1) S_1, S_2 only (2) S_1, S_3 only (3) S_1, S_2, S_3 (4) S_1 only

Sol. Answer (3)

$a=(i)^i$

$\log_e a = i\log_e i$

$\log_e a = i\log_e\left(e^{-\frac{\pi}{2}}\right)$

$\log_e a = -\frac{\pi}{2}$

$a = e^{-\frac{\pi}{2}}$

Therefore $\sin(\ln a) = \sin\left(-\frac{\pi}{2}\right) = -1$

$\text{Im}(a)+\arg(a) = 0+0 = 0$

Therefore S_1, S_2, S_3 are correct.

5. If $z^2 + z + 1 = 0$ then the value of $\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\ldots+\left(z^{21}+\frac{1}{z^{21}}\right)^2$ is equal to

(1) 21 (2) 42 (3) 0 (4) 11

Sol. Answer (2)

if $z^2 + z + 1 = 0$

$\Rightarrow (z-\omega)(z-\omega^2) = 0$

$\Rightarrow z = \omega, \omega^2$

if $z = \omega$, then $\frac{1}{z} = \omega^2$

To find the value of $\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+......+\left(z^{21}+\frac{1}{z^{21}}\right)^2$

Now, $z+\frac{1}{z}=\omega+\frac{1}{\omega}=-1,\ z^2+\frac{1}{z^2}=\omega^2+\frac{1}{\omega^2}=-1,\ z^3+\frac{1}{z^3}=2$

$z^4+\frac{1}{z^4}=\omega^4+\frac{1}{\omega^4}=\omega+\frac{1}{\omega}=-1,\ \ z^5+\frac{1}{z^5}=\omega^2+\frac{1}{\omega^2}=-1$ and

$z^6+\frac{1}{z^6}=2$ ...... and so on

Therefore,

$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+......+\left(z^{21}+\frac{1}{z^{21}}\right)^2$

$= \{(-1)^2 + (-1)^2 + (2)^2\} + \{(-1)^2 + (-1)^2 + (2)^2\} \times$ ...... 7 times

$= (1 + 1 + 4) + (1 + 1 + 4) \times$ .... 7 times

$= 6 + 6 \times$ ..... 7 times

$= 6 \times 7 = 42$

6. If 1, α_1, α_2,............. α_{3n} be the roots of equation $x^{3n+1} - 1 = 0$, and ω be an imaginary cube root of unity, then $\frac{(\omega^2-\alpha_1)(\omega^2-\alpha_2).....(\omega^2-\alpha_{3n})}{(\omega-\alpha_1)(\omega-\alpha_2).....(\omega-\alpha_{3n})}$

(1) ω (2) $-\omega$ (3) 1 (4) ω^2

Sol. Answer (3)

We have

$x^{3n+1} - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2)\ ....\ (x - \alpha_{3n})$

Thus, $\frac{(\omega^2-\alpha_1)(\omega^2-\alpha_2)........(\omega^2-\alpha_{3n})}{(\omega-\alpha_1)(\omega-\alpha_2)........(\omega-\alpha_{3n})}$

$=\frac{1}{(\omega+1)}.\frac{(\omega^2-1)(\omega^2-\alpha_1)(\omega^2-\alpha_2)......(\omega^2-\alpha_{3n})}{(\omega-1)(\omega-\alpha_1)(\omega-\alpha_2)......(\omega-\alpha_{3n})}$

$=\frac{1}{\omega+1}.\frac{(\omega^2)^{3n+1}-1}{\omega^{3n+1}-1}=\frac{1}{\omega+1}.\frac{(\omega^2)^{3n}.\omega^2-1}{\omega^{3n}.\omega-1}$

$\frac{1}{\omega+1}.\frac{\omega^2-1}{\omega-1}$ $[\because\ \omega^{3n}=\omega^{6n}=1]$

$=\frac{\omega^2-1}{\omega^2-1}=1$

7. The value of

$(x + \omega + \omega^2)(x + \omega^2 + \omega^4)(x + \omega^4 + \omega^8)$ .... till $2n$ factors

(1) $(x-1)^{2n}$ (2) $(x-1)^{2n+1}$ (3) $(x-1)^{2n-1}$ (4) $(x-1)^{2n+2}$

Sol. Answer (1)

The given expression is $(x - 1)(x - 1)$ ..... $(x - 1)$ ..... till $2n$ factors.

$= (x - 1)^{2n}$

8. If α, β and γ be the roots of the equation $x^3 - 3x^2 + 3x + 7 = 0$ and ω be non-real cube root of unity, then the modulus of the expression $\left(\dfrac{\alpha-1}{\beta-1}+\dfrac{\beta-1}{\gamma-1}+\dfrac{\gamma-1}{\alpha-1}\right)$ is equal to

(1) 1 (2) 2 (3) 3 (4) 4

Sol. Answer (3)

$x^3 - 3x^2 + 3x + 7 = 0$

$(x - 1)^3 + 8 = 0$

$\Rightarrow (x - 1) = -2, -2\omega, -2\omega^2$

$x = -1, 1 - 2\omega, 1 - 2\omega^2$

Now, $\left|\dfrac{\alpha-1}{\beta-1}+\dfrac{\beta-1}{\gamma-1}+\dfrac{\gamma-1}{\alpha-1}\right|$

$= \left|\dfrac{-2}{-2\omega}+\dfrac{-2\omega}{-2\omega^2}+\dfrac{-2\omega^2}{-2}\right|$

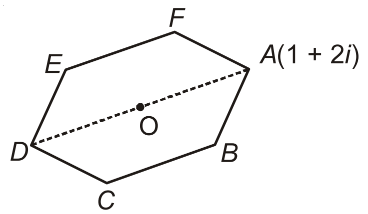
$= \left|\dfrac{1}{\omega}+\dfrac{1}{\omega}+\omega^2\right| = |3\omega^2| = 3$

[Rotation of a Complex Number and Locus Problems]

9. If centre of a regular hexagon is at origin and one of the vertex on Argand diagram is $1 + 2i$, then its perimeter is

(1) $6\sqrt{5}$ (2) $4\sqrt{5}$ (3) $6\sqrt{2}$ (4) $2\sqrt{5}$

Sol. Answer (1)



In regular hexagon $OA = AB = BC = CD = ED = EF = FA$

Length of perimeter $= 6 \times |OA|$

$= 6 \times \sqrt{1+4}$

$= 6\sqrt{5}$

10. If $\log_{\sqrt{3}}\left(\dfrac{|z|^2-|z|+1}{2+|z|}\right) < 2$, then the locus of z is

(1) $|z| = 5$ (2) $|z| < 5$ (3) $|z| > 5$ (4) $|z| = 0$

Sol. Answer (2)

$$\log_{\sqrt{3}}\left(\frac{|z|^2-|z|+1}{2+|z|}\right)<2$$

$$\Rightarrow \frac{z^2-|z|+1}{2+|z|}<(\sqrt{3})^2$$

$\Rightarrow$ $|z|^2 - |z| + 1 < 6 + 3|z|$

$\Rightarrow |z|^2 - 4|z| - 5 < 0$

$\Rightarrow (|z| + 1)(|z| - 5) < 0$

but $|z| + 1 > 0$

$\Rightarrow |z| - 5 < 0$

$\Rightarrow |z| < 5$

11. If $\text{Re}\left(\dfrac{z+4}{2z-1}\right)=\dfrac{1}{2}$, then z is represented by a point lying on

(1) A circle (2) An ellipse (3) A straight line (4) No real locus

Sol. Answer (3)

$$\text{Re}\left(\frac{z+4}{2z-1}\right)=\frac{1}{2}$$

$$\Rightarrow \left(\frac{z+4}{2z-1}\right)+\overline{\left(\frac{z+4}{2z-1}\right)}=1$$

$$\Rightarrow \frac{z+4}{2z-1}+\frac{\bar{z}+4}{2\bar{z}-1}=1$$

$$\Rightarrow \frac{2z\bar{z}-z+8\bar{z}-4+2z\bar{z}+8z-\bar{z}-4}{(2z-1)(2\bar{z}-1)}=1$$

$$\Rightarrow 4z\bar{z}+7z+7\bar{z}-8=4z\bar{z}-2z-2\bar{z}+1$$

$$\Rightarrow 9z+9\bar{z}-9=0$$

$$\Rightarrow \boxed{z+\bar{z}=1}$$

Hence point z lies on a straight line.

12. If z is a complex number satisfying $|2008z - 1| = 2008|z - 2|$, then locus z is

(1) y - axis (2) x - axis (3) Circle (4) A line parallel to y-axis

Sol. Answer (4)

$z = x + iy$

$|2008z - 1| = 2008|z - 2|$

$$\Rightarrow \left|z-\frac{1}{2008}\right|=|z-2|$$

Put $z = x + iy$

$$\left(x-\frac{1}{2008}\right)^2+(y)^2=(x-2)^2+y^2$$

$$\Rightarrow \quad \left(\frac{1}{2008}\right)^2-2\times x\times\frac{1}{2008}=4-4x$$

$$\Rightarrow \quad 4x-\frac{x}{1004}=4-\left(\frac{1}{2008}\right)^2$$

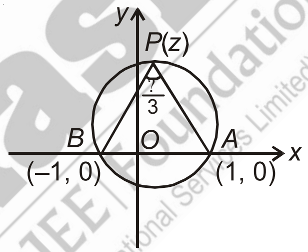
$$x=\frac{4-\left(\frac{1}{2008}\right)^2}{4-\frac{1}{1004}}$$

a line parallel to y-axis.

13. The locus of the point z satisfying the condition $\arg\frac{z-1}{z+1}=\frac{\pi}{3}$ is

(1) A straight line (2) Circle (3) A parabola (4) Ellipse

Sol. Answer (2)



$$\arg\left(\frac{z-1}{z+1}\right)=\angle APB=\frac{\pi}{3}$$

$\Rightarrow$ z lies on a circle

Alternatively

put $z = x + iy$

$$\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{(x-1)+iy}{(x+1)+iy}\right]=\frac{\pi}{3} \quad \Rightarrow \arg\left[\frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}\right]=\frac{\pi}{3}$$

$$\Rightarrow \arg\left[\frac{(x^2+y^2-1)+i(2y)}{(x+1)^2+y^2}\right]=\frac{\pi}{3} \Rightarrow \tan^{-1}\left[\frac{2y}{x^2+y^2-1}\right]=\frac{\pi}{3}$$

$$\Rightarrow \frac{2y}{x^2+y^2-1}=\sqrt{3} \quad \Rightarrow \sqrt{3}(x^2+y^2)-\sqrt{3}-2y=0$$

$$\Rightarrow x^2+y^2-\frac{2}{\sqrt{3}}y-1=0$$

$\Rightarrow$ Which represents a circle having centre at $\left(0,\frac{1}{\sqrt{3}}\right)$ and radius $\sqrt{\frac{1}{3}+1}=\frac{2}{\sqrt{3}}$

14. If $|z - 2 - 3i| + |z + 2 - 6i| = 4$, $i = \sqrt{-1}$, then locus of z is

(1) An ellipse (2) A point

(3) Segment joining the points $(2+3i)$ and $(-2+6i)$ (4) Empty

Sol. Answer (4)

$|z - (2 + 3i)| + |z - (-2 + 6i)| = 4$

Let $z_1 = 2 + 3i$, $z_2 = -2 + 6i$

$|z_1 - z_2| = |4 - 3i| = 5 > k$

$\Rightarrow |z - z_1| + |z - z_2| = 2a$, where $k < |z_1 - z_2|$

$\Rightarrow$ This does not represent any curve

$\Rightarrow$ Locus of z is an empty set.

Alternatively : If we put $z = x + iy$, then we got an equation in x and y which does not have any solution.

[Miscellaneous Questions]

15. If z is a complex number satisfying the equation $z^3 - (2 + 4i)z^2 - 3(1 - 3i)z + 14 - 2i = 0$, where $i = \sqrt{-1}$, then which of the following is not a value of $|z|$?

(1) 2 (2) $\sqrt{5}$ (3) $\sqrt{10}$ (4) 3

Sol. Answer (4)

$z_1 z_2 z_3 = 2i - 14$

$|z_1|\,|z_2|\,|z_3| = 10$

$\Rightarrow |z| \neq 3$

16. The complex number z_1, z_2 satisfies the equation $z + 1 + 8i = |z|(1 + i)$, where $i = \sqrt{-1}$, then the equation whose roots are $|z_1|$ and $|z_2|$ is

(1) $|z|^2 - 18|z| + 65 = 0$

(2) $|z|^2 - 7|z| + 12 = 0$

(3) $|z|^2 - 1 = 0$

(4) $|z|^2 - 17|z| + 60 = 0$

Sol. Answer (1)

$x + iy + 1 + 8i = \sqrt{x^2 + y^2}(1 + i)$

$\Rightarrow \quad x + 1 = \sqrt{x^2 + y^2}$

$x^2 + 1 + 2x = x^2 + y^2$

$y^2 = 2x + 1$ ...(1)

$\Rightarrow \quad y + 8 = \sqrt{x^2 + y^2}$

$y^2 + 64 + 16y = x^2 + y^2$

$y^2 = 16(y + 4)$ ...(2)

From (1) and (2) $x = 4$, $y = -3$ or $x = 12$, $y = 5$

$\Rightarrow z_1 = 4 - 3i$, $z_2 = 12 + 5i$

$\Rightarrow |z_1| = 5$, $|z_2| = 13$

17. The complex number z and ω are such that $|z| = |\omega| = 1$. Then the locus of $\frac{z+\omega}{1+z\omega}$ is

(1) A Circle (2) A Parabola (3) The x-axis (4) The y-axis

Sol. Answer (3)

$z = e^{i\theta_1}, \omega^{i\theta_2}$

$$\frac{z+\omega}{1+z\omega} = \frac{e^{i\theta_1}+e^{i\theta_2}}{1+e^{i(\theta_1+\theta_2)}} = e^{i\theta_1}\left(\frac{1+e^{i(\theta_2-\theta_1)}}{1+e^{(\theta_2+\theta_1)}}\right)$$

$$= \frac{e^{i\theta_1}\cdot 2\cos\left(\frac{\theta_2-\theta_1}{2}\right)e^{i\left(\frac{\theta_2-\theta_1}{2}\right)}}{2\cos\left(\frac{\theta_2+\theta_1}{2}\right)e^{i\left(\frac{\theta_2+\theta_1}{2}\right)}}$$

$$= \frac{\cos\left(\frac{\theta_2-\theta_1}{2}\right)}{\cos\left(\frac{\theta_2+\theta_1}{2}\right)}e^{i\left(\theta_1+\frac{\theta_2}{2}-\frac{\theta_1}{2}-\frac{\theta_2}{2}-\frac{\theta_1}{2}\right)}$$

$$= \frac{\cos\left(\frac{\theta_2-\theta_1}{2}\right)}{\cos\left(\frac{\theta_2+\theta_1}{2}\right)}$$

18. If $f(n,\theta) = \sum_{r=0}^{n}\frac{\cos(r\theta)}{2^r}$, then the value of $\lim_{n\to\infty} f\left(n,\frac{\pi}{3}\right)$ is equal to

(1) 2 (2) $\frac{3}{2}$ (3) 1 (4) Zero

Sol. Answer (3)

$$\because \sum_{r=0}^{\infty}\left(\frac{e^{i\theta}}{2}\right)^r = \frac{1}{1-\frac{e^{i\theta}}{2}} = \frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)}$$

$$= \frac{4-2\cos\theta+2i\sin\theta}{5-4\cos\theta}$$

$$\Rightarrow \sum_{r=0}^{\infty}\left(\frac{\cos r\theta}{2^r}\right) = \frac{4-2\cos\theta}{5-4\cos\theta}$$

$$\Rightarrow \lim_{n\to\infty} f\left(n,\frac{\pi}{3}\right) = \left(\sum_{r=0}^{\infty}\left(\frac{\cos r\theta}{2^r}\right)\right)_{\theta=\frac{\pi}{3}}$$

$$= \left(\frac{4-2\cos\theta}{5-4\cos\theta}\right)_{\theta=\frac{\pi}{3}} = \frac{3}{3} = 1$$

19. The number of common solutions of the equations $z^n = 1 + i$ and $z^m = 2 - i$, $n, m \in N$, $z \in C$ is

(1) Zero (2) One

(3) Depends on n and m both (4) n, if $m > n$

Sol. Answer (1)

$\because \quad z^n = 1 + i$

$\Rightarrow \quad |z|^n = \sqrt{2}$

$\because \quad z^m = 2 - i$

$\Rightarrow \quad |z|^m = \sqrt{5}$

$\Rightarrow$ No common solutions

20. Let $\omega = \cos 3° + i\sin 3°$ then $\sum_{r=1}^{10} \text{Re}\left(\omega^{2r-1}\right)$ equals

(1) $\frac{1}{\sin 3°}$ (2) $\frac{2}{\cos 3°}$ (3) $\frac{\sqrt{3}}{4\sin 3°}$ (4) $\frac{\sqrt{3}}{2\cos 3°}$

Sol. Answer (3)

$\omega^{2r-1} = (\cos 3° + i\sin 3°)2^{r-1}$

$= \cos(6r - 3)° + i\sin(6r - 3)°$

$\text{Re}(\omega^{2r-1}) = \cos(6r - 3)°$

$\sum_{r=1}^{10} \text{Re}(\omega^{2r-1}) = \cos 3° + \cos 9° + \cos 15° + ... + \cos 57°$

$= \frac{\sin 30°}{\sin 3°} \cos 30°$

$= \frac{\sqrt{3}}{4\sin 3°}$

21. If 1, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{3n}$ are the roots of equation $z^{3n+1} = 1 (n \in N)$ and ω is an imaginary cube root of unity. Then the value of $|(\alpha_1 - \omega)(\alpha_2 - \omega)(\alpha_3 - \omega)\ldots(\alpha_{3n} - \omega)|$ is

(1) 1 (2) n (3) $3n + 1$ (4) $3n$

Sol. Answer (1)

$\because \quad z^{3n+1} - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_{3n})$

Put $z = \omega$

$\frac{\omega^{3n+1} - 1}{\omega - 1} = (\omega - \alpha_1)(\omega - \alpha_2)\ldots(\omega - \alpha_{3n})$

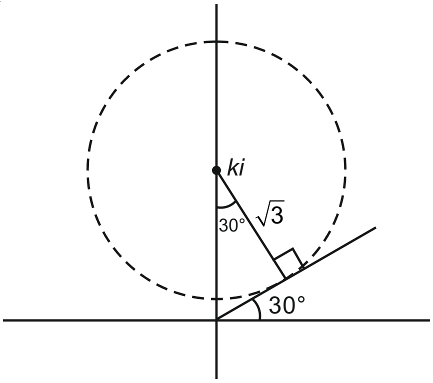
$\Rightarrow \quad (\omega - \alpha_1)(\omega - \alpha_2)\ldots(\omega - \alpha_{3n}) = 1$

$\Rightarrow \quad |(\alpha_1 - \omega)(\alpha_2 - \omega)\ldots(\alpha_{3n} - \omega)| = 1$

22. If there exist exactly one complex number z satisfying the conditions $|z - ki| \le \sqrt{3}$ and arg $(z) \le \frac{\pi}{6}$ (where $k \in R^+$ and $i = \sqrt{-1}$), then the value of k is

(1) 1 (2) 2 (3) $\sqrt{3}$ (4) $\sqrt{2}$

Sol. Answer (2)



$\because$ $|z - ki| \le \sqrt{3}$ represents the interior of a circle with radius $\sqrt{3}$.

$\arg(z) = \frac{\pi}{6}$ represents a line in i^{st} quadrandar shown. There will be exactly one solution if

$\cos 30° = \frac{3}{k} \Rightarrow k = 2$

23. If $|z - 1| = 1$ and $z \ne 0, 2$ then the possible value of $\frac{z-2}{z}$ is

(1) 1 (2) $\sqrt{3}$ (3) 2 (4) $3i$

Sol. Answer (4)

$\because$ z lies on a circle whoes ends of diameter are 0 and 2. So $(z - 2)$ and $(z - 0)$ represents two perpendicular lines then $\frac{z-2}{z}$ is purely imaginary.

24. The number of values of z satisfying the equations $1 + z + z^2 + z^3 + \ldots + z^{17} = 0$ and $1 + z + z^2 + \ldots + z^{11} = 0$ simultaneously is

(1) Zero (2) 1 (3) 5 (4) 6

Sol. Answer (3)

$1 + z + z^2 + \ldots + z^{17}$

$\Rightarrow z^{18} = 1 \cap z \ne 1$

So $z = e^{\frac{n\pi}{9}}$ where $n = 1, 2, 3, \ldots, 17$

Similarily $1 + z + z^2 + \ldots + z^{11} = 0$

$\Rightarrow z^{12} = 1 \cap z \ne 1$

So $z = e^{\frac{n\pi}{6}}$ where $m = 1, 2, 3, \ldots, 11$

Therefore the common roots are

$e^{\frac{\pi}{3}}, e^{\frac{2\pi}{3}}, e^{\pi}, e^{\frac{4\pi}{3}}$ and $e^{\frac{5\pi}{3}}$

25. If $\left[\frac{1+\cos\theta+i\sin\theta}{\sin\theta+i(1+\cos\theta)}\right]^4 = \cos n\theta + i\sin n\theta$ then value of n is

(1) 2 (2) 8 (3) 4 (4) 6

Sol. Answer (3)

$$\left[\frac{(1+\cos\theta)+i\sin\theta}{i(1+\cos\theta)-i\sin\theta}\right]^4$$

$$=\frac{1}{e^4}\left(\frac{2\cos^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)=\left(\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}\right)^4$$

$= (e^{i\theta})^4 = e^{i4\theta}$

$n = 4$

26. The value of $\sum_{n=1}^{7}\left(x^n+\frac{1}{x^n}\right)^2$ is (where x is a root of $x^2 - x + 1 = 0$)

(1) 17 (2) Zero (3) 13 (4) 3

Sol. Answer (3)

$x^2 - x + 1 = 0$

$x = -\omega, -\omega^2$

Now $\left(x^n+\frac{1}{x^n}\right)^2 = \left(\omega^n+\omega^{2n}\right)^2$

$\because$ $\omega^n + \omega^{2n} = -1$ when $n \neq 3m$, $m \in N$

$\omega^n + \omega^m = 2$ when $n = 3m$, $m \in N$

$$\sum_{n=1}^{7}\left(x^n+\frac{1}{x^n}\right)^2 = (-1)^2 + (-1)^2 + (2)^2 + (-1)^2 + (-1)^2 + (-2)^2 + (-1)^2 = 1 + 1 + 4 + 1 + 1 + 4 + 1$$

$= 13$

27. If the locus of a point z lying on $|z - 3| + |z - i| = k$ represent on ellipse, then k can be

(1) 2 (2) $\sqrt{5}$ (3) 3 (4) $\sqrt{15}$

Sol. Answer (4)

$|z - 3| + |z - i| = k$ represent on ellipse if $k > |z_1 - z_2|$

$\Rightarrow$ $k > |3 - i|$

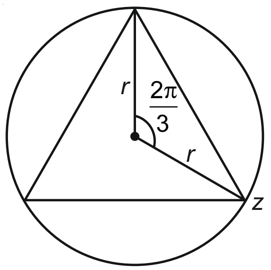
$\Rightarrow$ $k > \sqrt{10}$

28. If one vertex of the triangle having maximum area that can be inscribed in the circle $|z| = 2$ is $1+i\sqrt{3}$, then the other vertex of the triangle can be

(1) $(\sqrt{3}, 1)$ (2) (3, –1) (3) (–2, 0) (4) (0, 2)

Sol. Answer (3)

It is cleared that triangle is an equilateral triangle so



$$\frac{z-0}{(\sqrt{3}i+1)} = e^{\pm i\left(\frac{2\pi}{3}\right)}$$

'+'

$$z = (\sqrt{3}i+1)\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$$

$$z = \frac{-\sqrt{3}}{2}-\frac{1}{2}-\frac{3}{2}+\frac{i\sqrt{3}}{2}$$

$z = -2$

'–'

$$z = (\sqrt{3}i+1)\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$$

$$z = \frac{-\sqrt{3}}{2}i-\frac{1}{2}+\frac{3}{2}-\frac{i\sqrt{3}}{2}$$

$$z = 1-\sqrt{3}i$$

29. If $z_1, z_2, \ldots, z_{100}$ are root of equation $1 + z + z^2 + \ldots + z^{100} = 0$, then the value of $\sum_{r=1}^{100}\frac{1}{z_r-1}$ is equal to

(1) 100 (2) –50 (3) –25 (4) 150

Sol. Answer (2)

$$\frac{1}{z_1-1}+\frac{1}{z_2-1}+\frac{1}{z_3-1}+\ldots+\frac{1}{z_{100}-1} = k$$

Given $1 + z + z^2 + \ldots + z^{100} = (z - z_1)(z - z_2) \ldots (z - z_{100})$

Apply logasition on both side

$\log(1 + z + z^2 + \ldots + z^{100}) = \log(z - z_1) + \log(z - z_2) + \log(z - z_3) + \ldots + \log(z - z_{100})$

Now diff

$$\frac{1}{1+z+z^2+\ldots+z^{100}}(1 + 2z + 3z^2 + \ldots + 100z^{99})$$

$$= \frac{1}{z-z_1}+\frac{1}{z-z_2}+\ldots+\frac{1}{z-z_{100}}$$

$$\frac{1+2z+3z^2+\ldots+100z^{99}}{1+z+z^2+\ldots+z^{100}}$$

$$= -\left[\frac{1}{z_1-z}+\frac{1}{z_2-z}+\ldots+\frac{1}{z_{100}-z}\right]$$

Put $z = 1$

$$\frac{1+2+3+\ldots+100}{101} = -k$$

$$k = -\left(\frac{100\times101}{2\times101}\right) = -50$$

30. If p, q, r are the roots of $x^3 - 3x^2 + 3x + 7 = 0$, then value of $(p - 1)(q - 1)(r - 1)$ equals

(1) 5 (2) –1 (3) 8 (4) –8

Sol. Answer (4)

$x^3 - 3x^2 + 3x + 7 = 0$

$(x - 1)^3 + 8 = 0$

$(x - 1)^3 = (-2)^3$

$x - 1 = -2, -200, -200^2$

$(p - 1)(q - 1)(r - 1) = (-2)(-200)(-200^2) = -8$

31. Value of $\left(\frac{3}{2}+\frac{i\sqrt{3}}{2}\right)^{30}$ equals

(1) 3^{15} (2) -3^{15} (3) $\frac{1}{2}-\frac{\sqrt{3}i}{2}$ (4) $\frac{1}{2}+\frac{\sqrt{3}i}{2}$

Sol. Answer (2)

$$\frac{3}{2}+\frac{\sqrt{3}i}{2} = -i\sqrt{3}\omega$$

$$\Rightarrow \left(-i\sqrt{3}\omega\right) = -3^{15}$$

32. Let $z_1 = 5 + i$ and $z_2 = 2 - 4i$ (where $i = \sqrt{-1}$). Let z be a compex number such that $\arg\left(\frac{z-z_1}{z_2-z}\right) = \frac{\pi}{2}$, then z satisfies

(1) $\left|z-\left(\frac{7-3i}{2}\right)\right| = \sqrt{34}$ (2) $\left|z-\left(\frac{7-3i}{2}\right)\right| = \frac{\sqrt{34}}{2}$

(3) $\left|z-\left(\frac{7+3i}{2}\right)\right| = \sqrt{34}$ (4) $\left|z-\left(\frac{7+3i}{2}\right)\right| = \frac{\sqrt{34}}{2}$

Sol. Answer (2)

z_1, z_2, z_3 lie on a circle

z_1, z_2 are end points of diameter

Centre $(z_0) = \frac{z_1+z_2}{2} = \frac{7-3i}{2}$

and radius $= |z_1 - z_0|$

$$= \left|\frac{3}{2}+\frac{5}{2}i\right| = \frac{\sqrt{34}}{2}$$

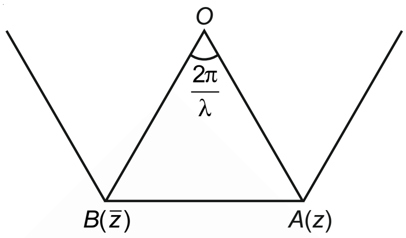
Equation of circle is = $|z - z_0| = r$

$\Rightarrow \left|z - \left(\frac{7-3i}{2}\right)\right| = \frac{\sqrt{34}}{2}$

33. If z and $\bar{z}$ represent adjacent vertices of a regular polygon of λ sides whose centre is at origin and if $\frac{\text{Im}(z)}{\text{Re}(z)} = \sqrt{2}-1$, then λ equals

(1) 4 (2) 6 (3) 8 (4) 12

Sol. Answer (3)



$\frac{z-0}{\bar{z}-0} = e^{\frac{2\pi i}{\lambda}}$

$\Rightarrow \frac{z}{\bar{z}} = e^{\frac{2\pi i}{\lambda}}$

Let $z = re^{i\theta}$

$\Rightarrow e^{2i\theta} = e^{\frac{2\pi i}{\lambda}}$

$\Rightarrow \theta = \frac{\pi}{\lambda}$

$\frac{\text{Im}(z)}{\text{Re}(z)} = \frac{\tan\pi}{\lambda} = \sqrt{2}-1$

$\Rightarrow \lambda = 8$

34. If 1, ω, ω^2 … ω^{n-1} are n, n^{th} roots of unity, the value of $(7-\omega)(7-\omega^2)\ldots(7-\omega^{n-1})$ equals

(1) 7^n (2) $7^n - 1$ (3) $\frac{7^n-1}{6}$ (4) $\frac{7^n}{6}$

Sol. Answer (3)

$x^n - 1 = 0$

$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1})$

$\Rightarrow \frac{7^n-1}{6} = (7-\omega)(7-\omega^2)\ldots(7-\omega^{n-1})$

35. Let u and v are two complex such that $|u| \neq |v|$ and $|u| = 1$ and $z = \frac{au+b}{u-v}$, where $z = x + iy$ and a, b are complex numbers then the locus of z is

(1) A line segment (2) The perpendicular bisector of $-\frac{b}{v}$ and a

(3) A circle (4) Can not be a circle unless $|v| = 1$

Sol. Answer (3)

$\because z = \frac{au+b}{u-v}$

$\Rightarrow u = \frac{b+vz}{z-a}$

$\because \quad |u| = 1$

$\Rightarrow \quad \left|\dfrac{b+vz}{z-a}\right| = 1$

$\Rightarrow \quad \left|\dfrac{z+\frac{b}{v}}{z-a}\right| = \dfrac{1}{|v|}$

$\because \quad |v| \neq 1$

$\therefore \quad z$ is always on circle

36. If 'α' denotes 2020th root of unity then $1 + 2\alpha + 3\alpha^2 + \ldots$ upto 2020 terms is equal to

(1) $\dfrac{2020}{1-\alpha}$ (2) $\dfrac{2021}{1-\alpha}$ (3) $\dfrac{2020}{\alpha-1}$ (4) $\dfrac{2021}{\alpha-1}$

Sol. Answer (3)

$\because$ By sum of A.G.P series we get sum 'S' upto 'n' terms as

$$S = \frac{1}{1-\alpha} + \frac{\alpha-\alpha^n}{(1-\alpha)^2} - \frac{n\alpha^n}{1-\alpha}$$

$\because \quad n = 2020$ and $\alpha^{2020} = 1$

$$\Rightarrow \quad S = \frac{1}{1-\alpha} + \frac{\alpha-1}{(1-\alpha)^2} - \frac{2020}{1-\alpha} = \frac{2020}{\alpha-1}$$

37. A complex number 'z' is rotated in anticlockwise direction by an angle 'α' we get z_1 and if same complex number z is rotated by an angle a in clockwise direction we get z_2 then

(1) z_1, z, z_2 are in G.P. (2) z_1, z, z_2 are in H.P.

(3) $z(\cos\alpha - 1) \geq 0$ (4) $z^2(\cos2\alpha - 1) \leq 0$

Sol. Answer (1)

$z_1 = ze^{i\alpha}$; $z_2 = ze^{-i\alpha}$

$\Rightarrow \quad z_1, z_2 = z^2$

$\Rightarrow \quad z_1, z, z_2$ are in G.P.

38. If $1, z_1, z_2, \ldots.., z_{n-1}$ be the nth roots of unity and 'ω' be a non-real complex cube root of unity, then sum of all possible values of $\prod_{r=1}^{n-1}(\omega - z_r)$ will be equal to

(1) Zero (2) $2 + \omega^2$ (3) $\omega^2 + 1$ (4) $1 - \omega^2$

Sol. Answer (4)

$$\because \quad \frac{x^n-1}{x-1} = (x-z_1)(x-z_2)\ldots(x-z_{n-1})$$

Put $x = \omega$

$$\Rightarrow \quad \prod_{r=1}^{n-1}(\omega - z_r) = \frac{\omega^n - 1}{\omega - 1} = \begin{cases} 0 & \text{if } n = 3k \quad k \in z \\ 1 & \text{if } n = 3k+1 \quad k \in z \\ 1+\omega & \text{if } n = 3k+2 \quad k \in z \end{cases}$$

$\because$ Sum of all possible values $= 2 + \omega = 1 - \omega^2$.

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