Algebraic Expressions and Identities

Chapter 9

9.1 What are Expressions?

In earlier classes, we have already become familiar with what algebraic expressions (or simply expressions) are. Examples of expressions are:

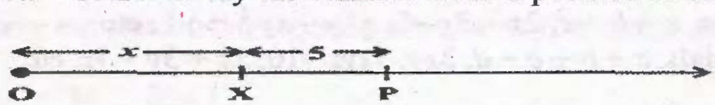
$$x + 3,\ 2y - 5,\ 3x^2,\ 4xy + 7 \text{ etc.}$$

You can form many more expressions. As you know expressions are formed from variables and constants. The expression $2y - 5$ is formed from the variable y and constants 2 and 5. The expression $4xy + 7$ is formed from variables x and y and constants 4 and 7.

We know that, the value of y in the expression, $2y - 5$, may be anything. It can be $2, 5, -3, 0, \frac{5}{2}, -\frac{7}{3}$ etc.; actually countless different values. The value of an expression changes with the value chosen for the variables it contains. Thus as y takes on different values, the value of $2y - 5$ goes on changing. When $y = 2$, $2y - 5 = 2(2) - 5 = -1$; when $y = 0$, $2y - 5 = 2 \times 0 - 5 = -5$, etc. Find the value of the expression $2y - 5$ for the other given values of y.

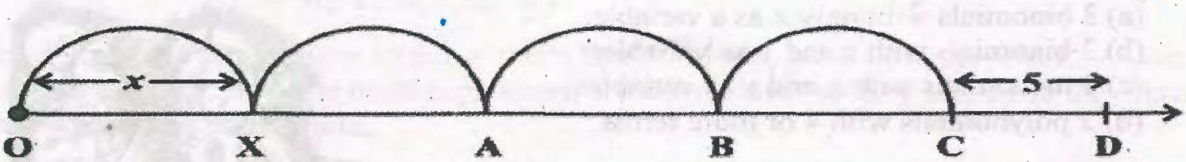
Number line and an expression:

Consider the expression $x + 5$. Let us say the variable x has a position X on the number line;

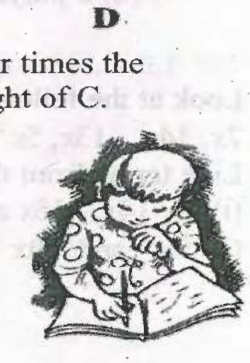


X may be anywhere on the number line, but it is definite that the value of $x + 5$ is given by a point P, 5 units to the right of X. Similarly, the value of $x - 4$ will be 4 units to the left of X and so on.

What about the position of $4x$ and $4x + 5$?



The position of $4x$ will be point C; the distance of C from the origin will be four times the distance of X from the origin. The position D of $4x + 5$ will be 5 units to the right of C.



TRY THESE

1. Give five examples of expressions containing one variable and five examples of expressions containing two variables.
2. Show on the number line x, $x-4$, $2x+1$, $3x-2$.

9.2 Terms, Factors and Coefficients

Take the expression $4x+5$. This expression is made up of two terms, $4x$ and 5. **Terms are added to form expressions. Terms** themselves **can be** formed as **the product of factors.** The term $4x$ is the product of its factors 4 and x. The term 5 is made up of just one factor i.e., 5.

The expression $7xy-5x$ has two terms $7xy$ and $-5x$ The term $7xy$ is a product of factors 7, x and y. The numerical factor of a term is called its **numerical coefficient or simply coefficient.** The coefficient in the term $7xy$ is 7 and the coefficient in the term $-5x$ is -5.

> **TRY THESE**
> Identify the coefficient of each term in the expression $x^2y^2-10x^2y+5xy^2-20$.

9.3 Monomials, Binomials and Polynomials

Expression that contains only one term is called a **monomial.** Expression that contains two terms is called a **binomial.** An expression containing three terms is a **trinomial** and so on. In general, an expression containing, one or more terms with non-zero coefficient (with variables having non negative exponents) is called a **polynomial.** A polynomial may contain any number of terms, one or more than one.

Examples of monomials: $4x^2$, $3xy$, $-7z$, $5xy^2$, $10y$, -9, $82mnp$, etc.

Examples of binomials: $a+b$, $4l+5m$, $a+4$, $5-3xy$, z^2-4y^2, etc.

Examples of trinomials: $a+b+c$, $2x+3y-5$, $x^2y-xy^2+y^2$, etc.

Examples of polynomials: $a+b+c+d$, $3xy$, $7xyz-10$, $2x+3y+7z$, etc.

TRY THESE

1. Classify the following polynomials as monomials, binomials, trinomials.
   $-z+5$, $x+y+z$, $y+z+100$, $ab-ac$, 17
2. Construct
   (a) 3 binomials with only x as a variable;
   (b) 3 binomials with x and y as variables;
   (c) 3 monomials with x and y as variables;
   (d) 2 polynomials with 4 or more terms.

9.4 Like and Unlike terms

Look at the following expressions:

$7x$, $14x$, $-13x$, $5x^2$, $7y$. $7xy$, $-9y^2$, $-9x^2$, $-5yx$

Like terms from these are:

(i) $7x$, $14x$, $-13x$ are like terms.

(ii) $5x^2$ and $-9x^2$ are like terms.

(iii) $7xy$ and $-5yx$ are like terms.
Why are $7x$ and $7y$ not like?
Why are $7x$ and $7xy$ not like?
Why are $7x$ and $5x^2$ not like?

TRY THESE

Write two terms which are like

(i) $7y$ (ii) $4mn^2$ (iii) $2l$

9.5 Addition and Subtraction of Algebraic Expressions

In the earlier classes, we have also learnt how to add and subtract algebraic expressions.
For example, to add $7x^2 - 4x + 5$ and $9x - 10$, we do

$$\begin{array}{r} 7x^2 - 4x + 5 \\ + \quad 9x - 10 \\ \hline 7x^2 + 5x - 5 \end{array}$$

Observe how we do the addition. We write each expression to be added in a separate row. While doing so we write like terms one below the other, and add them, as shown.
Thus $5 + (-10) = 5 - 10 = -5$. Similarly, $-4x + 9x = (-4 + 9)x = 5x$. Let us take some more examples.

Example 1: Add: $7xy + 5yz - 3zx$, $4yz + 9zx - 4y$, $-3xz + 5x - 2xy$.

Solution: Writing the three expressions in separate rows, with like terms one below the other, we have

$$\begin{array}{llllll} & 7xy & +\,5yz & -\,3zx & & \\ + & & 4yz & +\,9zx & & -\,4y \\ + & 2xy & & -\,3zx & +\,5x & \\ \hline & 5xy & +\,9yz & +\,3zx & +\,5x & -\,4y \end{array}$$

(Note xz is same as zx)

Thus, the sum of the expressions is $5xy + 9yz + 3zx + 5x - 4y$. Note how the terms, $-4y$ in the second expression and $5x$ in the third expression, are carried over as they are, since they have no like terms in the other expressions.

Example 2: Subtract $5x^2 - 4y + 6y - 3$ from $7x^2 - 4xy + 8y^2 + 5x - 3y$.

Solution:

$$\begin{array}{lllllll} & 7x^2 & -\,4xy & +\,8y^2 & +\,5x & -\,3y & \\ & 5x^2 & & -\,4y^2 & & +\,6y & -\,3 \\ (-) & & & (+) & & (-) & (+) \\ \hline & 2x^2 & -\,4xy & +\,12y^2 & +\,5x & -\,9y & +\,3 \end{array}$$

> Note that subtraction of a number is the same as addition of its additive inverse. Thus subtracting –3 is the same as adding +3. Similarly, subtracting 6y is the same as adding – 6y; subtracting $-4y^2$ is the same as adding $4y^2$ and so on. The signs in the third row written below each term in the second row help us in knowing which operation has to be performed.

Exercise 9.1

1. Identify the terms, their coefficients for each of the following expressions.
   (i) $5xyz^2 - 3zy$ (ii) $1 + x + x^2$ (iii) $4x^2y^2 - 4x^2y^2z^2 + z^2$
   (iv) $3 - pq + qr - rp$ (v) $\frac{x}{2} + \frac{y}{2} - xy$ (vi) $0.3a - 0.6ab + 0.5b$

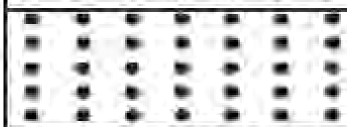
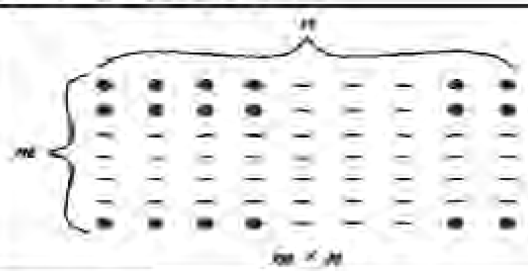
2. Classify the following polynomials as monomials, binomials, trinomials. Which polynomials do not fit in any of these three categories?
   $x + y$, 1000, $x + x^2 + x^3 + x^4$, $7 + y + 5x$, $2y - 3y^2$, $2y - 3y^2 + 4y^3$, $5x - 4y + 3xy$, $4z - 15z^2$, $ab + bc + cd + da$, pqr, $p^2q + pq^2$, $2p + 2q$

3. Add the following.
   (i) $ab - bc$, $bc - ca$, $ca - ab$ (ii) $a - b + ab$, $b - c + bc$, $c - a + ac$
   (iii) $2p^2q^2 - 3pq + 4$, $5 + 7pq - 3p^2q^2$ (iv) $l^2 + m^2$, $m^2 + n^2$, $n^2 + l^2$, $2lm + 2mn + 2nl$

4. (a) Subtract $4a - 7ab + 3b + 12$ from $12a - 9ab + 5b - 3$
   (b) Subtract $3xy + 5yz - 7zx$ from $5xy - 2yz - 2zx + 10xyz$
   (c) Subtract $4p^2q - 3pq + 5pq^2 - 8p + 7q - 10$ from $18 - 3p - 11q + 5pq - 2pq^2 + 5p^2q$

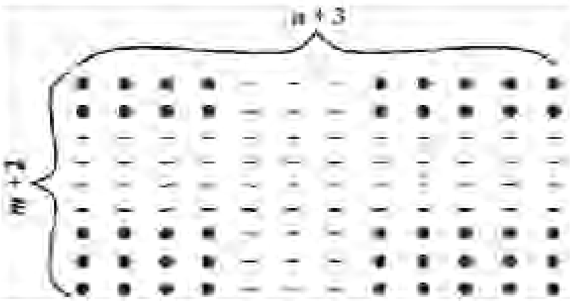
9.6 Multiplication of Algebraic Expressions : Introduction

(i) Look at the following patterns of dots.

Patterns of dots	Total number of dots
	4×9
	5×7
	$m \times n$

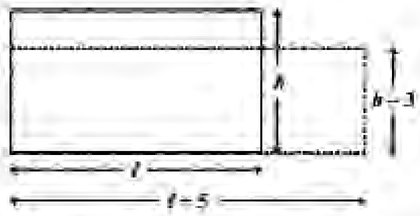
> To find the number of dots we have to multiply the expression for the number of rows by the expression for the number of columns.

Here the number of rows is increased by 2, i.e., $m+2$ and number of columns increased by 3, i.e., $n+3$.



$(m+2) \times (n+3)$

(ii) Can you now think of similar other situations in which two algebraic expressions have to be multiplied?

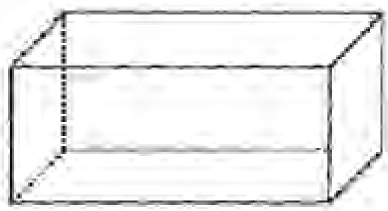


Asma gets up. She says, "We can think of area of a rectangle." The area of a rectangle is $l \times b$, where l is the length, and b is breadth. If the length of the rectangle is increased by 5 units, i.e., $(l+5)$ and breadth is decreased by 3 units, i.e., $(b-3)$ units, the area of the new rectangle will be $(l+5) \times (b-3)$.

To find the area of a rectangle, we have to multiply algebraic expressions like $(l+5) \times (b-3)$.

(iii) Can you think about volume? (The volume of a rectangular box is given by the product of its length, breadth and height).

(iv) Santa points out that when we buy things, we have to carry out multiplication. For example, if
price of bananas per dozen = Rs p
and for the school picnic bananas needed = z dozens,
then we have to pay = Rs $p \times z$
Suppose, the price per dozen was less by Rs 2 and the bananas needed were less by 4 dozens.
Then, price of bananas per dozen = Rs $(p-2)$
and bananas needed = $(z-4)$ dozens,
Therefore, we would have to pay = Rs $(p-2) \times (z-4)$



TRY THESE

Can you think of two more such situation, where we may need to multiply algebraic expressions?

[Hint: Think of speed and time;
Think of interest to be paid, the principal and the rate of simple interest; etc]

In all the above examples, we had to carry out multiplication of two or more quantities. If the quantities are given by algebraic expressions, we need to find their product. This means that we should know how to obtain this product. Let us do this systematically. To begin with we shall look at the multiplication of two monomials.

9.7 Multiplying a Monomial by a Monomial

9.7.1 Multiplying two monomials

We begin with

$4 \times x = x + x + x + x = 4x$ as seen earlier.

Similarly, $4 \times (3x) = 3x + 3x + 3x + 3x = 12x$

Now, observe the following products.

(i) $x \times 3y = x \times 3 \times y = 3 \times x \times y = 3xy$

(ii) $5x \times 3y = 5 \times x \times 3 \times y = 5 \times 3 \times x \times y = 15xy$

(iii) $5x \times (-3y) = 5 \times x \times (-3) \times y$

$= 5 \times (-3) \times x \times y = -15xy$

Notice that all the three products of monomials, $3xy$, $15y$, $-15xy$, are also monomials.

Some more useful examples follow.

(iv) $5x \times 4x^2 = (5 \times 4) \times (x \times x^2)$

$= 20 \times x^3 = 20x^3$

(v) $5x \times (-4xyz) = (5 \times -4) \times (x \times xyz)$

$= -20 \times (x \times x \times yz) = -20x^2yz$

Note that $5 \times 4 = 20$
i.e.; coefficient of product = coefficient of first monomial x coefficient of second monomial;
and $x \times x^2 = x^3$
i.e., algebraic factor of product = algebraic factor of first monomial x algebraic factor of second monomial.

Observe how we collect the powers of different variables in the algebraic parts of the two monomials. While doing so, we use the rules of exponents and powers.

9.7.2 Multiplying three or more monomials

Observe the following examples.

(i) $2x \times 5y \times 7z = (2x \times 5y) \times 7z = 10xy \times 7z = 70xyz$

(ii) $4xy \times 5x^2y^2 \times 6x^3y^3 = (4xy \times 5x^2y^2) \times 6xy = 20x^3y^3 \times 6x^3y^3 = 120x^3y^3 \times x^3y^3$

$= 120(x^3 \times x^3) \times (y^3 \times y^3) = 120x^6 \times y^6 = 120x^6y^6$

It is clear that we first multiply the first two monomials and then multiply the resulting monomial by the third monomial. This method can be extended to the product of any number of monomials.

We can find the product in the other way also
$4xy \times 5x^2y^2 \times 6x^3y^3$
$= (4 \times 5 \times 6) \times (x \times x^2 \times x^3) \times (y \times y^2 \times y^3)$
$= 120x^6y^6$

TRY THESE

Find $4x \times 5y \times 7z$

First find $4x \times 5y$ and multiply it by $7z$;
or first find $5y \times 7z$ and multiply it by $4x$.
Is the result the same? What do you observe?
Does the order in which you carry out the multiplication matter?

Example 3: Complete the table for a rectangle with given length and breadth.

Solution:

length	breadth	area
$3x$	$5y$	$3x \times 5y = 15xy$
$9y$	$4y^2$	....
$4ab$	$5bc$	....
$2l^2 m$	$3lm^2$	....

Example 4: Find the volume of each rectangle box with given length, breadth and height.

length	breadth	area
$2ax$	$3by$	$5cz$
$m^2 n$	$n^2 p$	$p^2 m$
$2q$	$4q^2$	$8q^3$

Solution: Volume = length × breadth × height

Hence, for (i) volume $= (2ax) \times (3by) \times (5cz)$

$= 2 \times 3 \times 5 \times (ax) \times (by) \times (cz) = 30abcxyz$

for (ii) volume $= m^2 \times n \times n^2 p \times p^2 m$

$= (m^2 \times m) \times (n \times n^2) \times (p \times p^2) = m^3 n^3 p^3$

for (iii) volume $= 2q \times 4q^2 \times 8q^3$

$= 2 \times 4 \times 8 \times q \times q^2 \times q^3 = 64q^6$

Exercise 9.2

1. Find the product of the following pairs of monomials.
   (i) $4, 7p$ (ii) $-4p, 7p$ (iii) $-4p, 7pq$ (v) $4p^3, -3p$
   (v) $4p, 0$
2. Find the areas of rectangles with the following pairs of monomials as their lengths and breadths respectively.
   (p, q); $(10m, 5n)$; $(20x^2, 5y^2)$; $(4x, 3x^2)$; $(3mn, 4np)$

3. Complete the table of products.

First monomial → / Seond monomial ↓	$2x$	$-5y$	$3x^2$	$-4xy$	$7x^2y$	-9^2xy^2
$2x$	$4x^2$	…	…	…	…	…
$-5y$	…	…	$-15x^2y$	…	…	…
$3x^2$						
$-4xy$						
$7x^2y$						
$-9x^2y^2$						

4. Obtain the volume of rectangular boxes with the following length, breadth and height respectively.

(i) $5a, 3a^2, 7a^4$ (ii) $2p, 4q, 8r$ (iii) $xy, 2x^2y, 2xy^2$ (iv) $a, 2b, 3c$

5. Obtain the product of

(i) xy, yz, zx (ii) $a, -a^2, a^3$ (iii) $2, 4y, 8y^2, 16y^3$ (iv) $a, 2b, 3c, 6abc$
(v) $m, -mn, mnp$

9.8 Multiplying a Monomial by a Polynomial

9.8.1 Multiplying a monomial by a binomial

Let us multiply the monomial 3x by the binomial 5y +2, i.e., find 3x x (5y +2) =?
Recall that 3x and (5y +2) represent numbers. Therefore, using the distributive law,
$3x \times (5y + 2) = (3x \times 5y) + (3x \times 2) = 15xy + 6x$

> We commonly use distributive law in our calculations. For example:
>
> $7 \times 106 = 7 \times (100 + 6)$
> $= 7 \times 100 + 7 \times 6$ (Here, we used distributive law)
> $= 700 + 42 = 742$
>
> $7 \times 38 = 7 \times (40 - 2)$
> $= 7 \times 40 - 7 \times 2$ (Here, we used distributive law)
> $= 280 - 14 = 266$

Similarly, $(-3x) \times (-5y + 2)$ $(-3x) \times (-.5y) + (-3x) \times (2) = 15xy - 6x$
and $5xy \times (y^2 + 3) = (5xy \times y^2) + (5xy \times 3) = 5xy^3 + 15y.$
What about a binomial x monomial? For example, (5y +2) x 3x =?
We may use commutative law as: $7\times 3 = 3 \times 7$; or in general $a \times b = b \times a$
Similarly, $(5y + 2) \times 3x = 3x \times (5y + 2) = 15xy + 6x$ as before.

TRY THESE

Find the product (i) $2x\,(3x + 5xy)$ (ii) $a^2\,(2ab - 5c)$

9.8.2 Multiplying a monomial by a trinomial

Consider $3p \times (4p^2 + 5p + 7)$. As in the earlier case, we use distributive law;

$$3p \times (4p^2 + 5p + 7) = (3p \times 4p^2) + (3p \times 5p) + (3p \times 7)$$
$$= 12p^3 + 15p^2 + 21p$$

By using the distributive law, we are able to carry out the multiplication term by term.

TRY THESE

Find the product: $(4p^2 + 5p + 7) \times 3p$

Example 5: Simplify the expressions and evaluate them as directed:

(i) $x\,(x - 3) + 2$ for $x = 1$, (ii) $3y\,(2y - 7) - 3\,(y - 4) - 63$ for $y = -2$

Solution:

(i) $x\,(x - 3) + 2 = x^2 - 3x + 2$

For $x = 1, x^2 - 3x + 2 = (1)^2 - 3(1) + 2$
$= 1 - 3 + 2 = 3 - 3 = 0$

(ii) $3y\,(2y - 7) - 3(y - 4) - 63 = 6y^2 - 21y - 3y + 12 - 63$
$= 6y^2 - 21y - 51$

For $y = -2$, $6y^2 - 24y - 51 = 6(-2)^2 - 24(-2) - 51$
$= 6 \times 4 + 24 \times 2 - 51$
$= 24 + 48 - 51 = 72 - 51 = 21$

Example 6: Add

(i) $5m\,(3 - m)$ and $6m^2 - 13m$ (ii) $4y\,(3y^2 + 5y - 7)$ and $2\,(y^3 - 4y^2 + 5)$

Solution:

(i) First expression $= 5m\,(3 - m) = (5m \times 3) - (5m \times m) = 15m - 5m^2$ Now adding the second expression to it, $15m - 5m^2 + 6m^2 - 13m = m^2 + 2m$

(ii) The first expression $= 4y\,(3y^2 + 5y - 7) = (4y \times 3y^2) + (4y \times 5y) + (4y \times (-7))$
$= 12y^3 + 20y^2 - 28y$

The second expression $= 2\,(y^3 - 4y^2 + 5) = + 2 \times (-4y^2) + 2 \times 5$
$= 2y^3 - 8y^2 + 10$

Adding the two expressions,

$$\begin{array}{rl} & 12y^3 + 20y^2 - 28y \\ + & 2y^3 - 8y^2 + 10 \\ \hline & 14y^3 + 12y^2 - 28y + 10 \end{array}$$

Example 7: Subtract $3pq\,(p-q)$ from $2pq\,(p+q)$.

Solution: We have $3pq\,(p-q) = 3p^2q - 3pq^2$ and

$2pq\,(p+q) = 2p^2q + 2pq^2$

Subtracting

$$\begin{array}{r} 2p^2q + 2pq^2 \\ 3p^2q - 3pq^2 \\ - \quad\;\; + \\ \hline -p^2q + 5pq^2 \end{array}$$

Exercise 9.3

1. Carry out the multiplication of the expressions in each of the following pairs.

(i) $4p,\ q+r$ (ii) $ab,\ a-b$ (iii) $a+b,\ 7a^2b^2$ (iv) $a^2-9, 4a$

(v) $pq+qr+rp,\ 0$

2. Complete the table.

	First expression	Second expression	Product
(i)	a	$b+c+d$	...
(ii)	$x+y-5$	$5xy$	...
(iii)	p	$6p^2-7p+5$	...
(iv)	$4p^2q^2$	p^2-q^2	...
(v)	$a+b+c$	abc	...

3. Find the product.

(i) $(a^2)\times(2a^{22})\times(4a^{26})$ (ii) $\left(\frac{2}{3}xy\right)\times\left(\frac{-9}{10}x^2y^2\right)$

(iii) $\left(-\frac{10}{3}pq^3\right)\times\left(\frac{6}{5}p^3q\right)$ (iv) $x\times x^2\times x^3\times x^4$

4. (a) Simplify $3x(4x-5)+3$and find its values for (i) $x=3$ (ii) $x=\frac{1}{2}$.

(b) Simplify $a(a^2+a+1)+5$ and find its value for (i) $a=0$, (ii) $a=1$ (iii) $a=-1$.

5. (a) Add: $p(p-q)$, $q(q-r)$and $r(r-p)$

(b) Add: $2x(z-x-y)$ and $2y(z-y-x)$

(c) Subtract: $31(1-4m+5n)$ from $41(10n-3m+21)$

(d) Subtract: $3a(a+b+c)-2b(a-b+c)$ from $4c(-a+b+c)$

9.9 Multiplying a Polynomial by a Polynomial

9.9.1 Multiplying a binomial by a binomial.

Let us multiply one binomial $(2a + 3b)$ by another binomial, say $(3a + 4b)$. We do this step-by-step, as we did in earlier cases, following the distributive law of multiplication,

$$(3a + 4b) \times (2a + 3b) = 3a \times (2a + 3b) + 4b \times (2a + 3b)$$
$$= (3a \times 2a) + (3a \times 3b) + (4b \times 2a) + (4b \times 3b)$$
$$= 6a^2 + 9ab + 8ba + 12b^2$$
$$= 6a^2 + 17ab + 12b^2 \qquad \text{(Since } ba = ab\text{)}$$

Observe, every term in one binomial multiplies every term in the other binomial.

When we carry out term by term multiplication, we expect $2 \times 2 = 4$ terms to be present. But two of these are like terms, which are combined, and hence we get 3 terms. **In multiplication of polynomials with polynomials, we should always look for like terms, if any, and combine them.**

Example 8: Multiply
(i) $(x - 4)$ and $(2x + 3)$ (ii) $(x - y)$ and $(3x + 5y)$

Solution:

(i) $(x - 4) \times (2x + 3) = x \times (2x + 3) - 4 \times (2x + 3)$
$= (x \times 2x) + (x \times 3) - (4 \times 2x) - (4 \times 3) = 2x^2 + 3x - 8x - 12$
$= 2x^2 - 5x - 12$ (Adding like terms)

(ii) $(x - y) \times (3x + 5y) = x \times (3x + 5y) - y \times (3x + 5y)$
$= (x \times 3x) + (x \times 5y) - (y \times 3x) - (y \times 5y)$
$= 3x^2 + 5xy - 3yx - 5y^2 = 3x^2 + 2xy - 5y^2$ (Adding like terms)

Example 9: Multiply
(i) $(a + 7)$ and $(b - 5)$ (ii) $(a^2 + 2b^2)$ and $(5a - 3b)$

Solution:

(i) $(a + 7) \times (b - 5) = a \times (b - 5) + 7 \times (b - 5) = ab - 5a + 7b - 35$

Note that there are no like terms involved in this multiplication.

(ii) $(a^2 + 2b^2) \times (5a - 3b) = a^2 (5a - 3b) + 2b^2 \times (5a - 3b)$
$= 5a^3 - 3a^2 b + 10ab^2 - 6b^3$

9.9.2 Multiplying a binomial by a trinomial

In this multiplication, we shall have to multiply each of the three terms in the trinomial by each of the two terms in the binomial. We shall get in all $3 \times 2 = 6$ terms, which may reduce to 5 or less, if the term by term multiplication results in like terms. Consider

$(a + 7) \times (a^2 + 3a + 5) = a \times (a^2 + 3a + 5) + 7 \times (a^2 + 3a + 5)$
binomial trinomial [using the distributive law]

$= a^3 + 3a^2 + 5a + 7a^2 + 21a + 35$
$= a^3 + (3a^2 + 7a^2) + (5a + 21a) + 35$
$= a^3 + 10a^2 + 26a + 35$ (Why are there only 4 terms in the final result?)

Example 10: Simplify $(a + b)(2a - 3b + c) - (2a - 3b)c$.

Solution: We have

$(a + b)(2a - 3b + c) = a(2a - 3b + c) + b(2a - 3b + c)$
$= 2a^2 - 3ab + ac + 2ab - 3b^2 + bc$
$= 2a^2 - ab - 3b^2 + bc + ac$ (Note, – 3ab and 2ab are like terms)

and $(2a - 3b)c = 2ac - 3bc$

Therefore,

$(a + b)(2a - 3b + c) - (2a - 3b)c = 2a^2 - ab - 3b^2 + bc + ac - (2ac - 3bc)$
$= 2a^2 - ab - 3b^2 + bc + ac - 2ac + 3bc$
$= 2a^2 - ab - 3b^2 + (bc + 3bc) + (ac - 2ac)$
$= 2a^2 - 3b^2 - ab + 4bc - ac$

Exercise 9.4

1. Multiply the binomials
   (i) $2x + 5)$ and $(4x - 3)$
   (ii) $(y - 8)$ and $(3y - 4)$
   (iii) $(2.5l - 0.5m)$ and $2.5l + 0.5m)$
   (iv) $(a + 3b)$ and $(x + 5)$
   (v) $(2pq + 3q^2)$ and $(3pq - 2q^2)$
   (vi) $\left(\frac{3}{4}a^2 + 3b^2\right)$ and $4\left(a^2 - \frac{2}{3}b^2\right)$
2. Find the product.
   (i) $(5 - 2x)(3 + x)$
   (ii) $(x + 7y)(7x - y)$
   (iii) $(a^2 + b)(a + b^2)$
   (iv) $(p^2 - q^2)(2p + q)$
3. Simplify
   (i) $(x^2 - 5)(x + 5) + 25$
   (ii) $(a^2 + 5)(b^3 + 3) + 5$
   (iii) $(t + s^2)(t^2 - s)$
   (iv) $(a + b)(c - d) + (a - b)(c + d) + 2(ac + bd)$
   (v) $(x + y)(2x + y) + (x + 2y)(x - y)$
   (vi) $(x + y)(x^2 - xy + y^2)$
   (vii) $(1.5x - 4y)(1.5x + 4y + 3) - 4.5x + 12y$
   (viii) $(a + b + c)(a + b - c)$

9.10 What is an Identity?

Consider the equality $(a + 1)(a + 2) = a^2 + 3a + 2$

We shall evaluate both sides of this equality for some value of a, say $a = 10$.

For $a = 10$, LHS $= (a + 1)(a + 2) = (10 + 1)(10 + 2) = 11 \times 12 = 132$
RHS $= a^2 + 3a + 2 = 10^2 + 3 \times 10 + 2 = 100 + 30 + 2 = 132$

Thus, the values of the two sides of the equality are equal for $a = 10$.

Let us now take $a = -5$

LHS = $(a + 1)(a + 2) = (-5 + 1)(-5 + 2) = (-4) \times (-3) = 12$

RHS = $a^2 + 3a + 2 = (-5)^2 + 3(-5) + 2$

$= 25 - 15 + 2 = 10 + 2 = 12$

Thus, for $a = -5$, also LHS = RHS.

We shall find that for any value of a, LHS = RHS **Such an equality, true for every value of the variable in it, is called an identity**. Thus,

$(a + 1)(a + 2) = a^2 + 3a + 2$ is an identity.

An equation is true for only certain values of the variable in it. It is not true for all values of the variable. For example, consider the equation

$$a^2 + 3a + 2 = 132$$

It is true for $a = 10$, as seen above, but it is not true for $a = -5$ or for $a = 0$ etc.

Try it: Show that $a^2 + 3a + 2 = 132$ is not true for $a = -5$ and for $a = 0$.

9.11 Standard Identities

We shall now study three identities which are very useful in our work. These identities are obtained by multiplying a binomial by another binomial.

Let us first consider the product $(a + b)(a + b)$ or $(a + b)^2$.

$$(a + b)^2 = (a + b)(a + b)$$
$$= a(a + b) + b(a + b)$$
$$= a^2 + ab + ba + b^2$$
$$= a^2 + 2ab + b^2 \text{ (since } ab = ba)$$

Thus $\boxed{(a + b)^2 = a^2 + 2ab + b^2}$ **(I)**

Clearly, this is an identity, since the expression on the RHS is obtained from the LHS by actual multiplication. One may verify that for any value of a and any value of b, the values of the two sides are equal.

? Next we consider $(a - b)^2 = (a - b)(a - b) = a(a - b) - b(a - b)$

We have $= a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$

or $\boxed{(a - b)^2 = a^2 - 2ab + b^2}$ **(II)**

? Finally, consider $(a + b)(a - b)$. We have $(a + b)(a - b) = a(a - b) + b(a - b)$

$= a^2 - ab + ba - b^2 = a^2 - b^2$ (since $ab = ba$)

or $\boxed{(a + b)(a - b) = a^2 - b^2}$ **(III)**

The identities **(I)**, **(II)** and **(III)** are known as **standard identities**.

TRY THESE

1. Put $-b$ in place of b in Identity **(I)**. Do you get Identity **(II)**?

? We shall now work out one more useful identity.

$$(x+a)(x+b) = x(x+b) + a(x+b)$$
$$= x^2 + bx + ax + ab$$

or $$\boxed{(x+a)(x+b) = x^2 + (a+b)x + ab} \quad \textbf{(IV)}$$

TRY THESE

1. Verify Identity **(IV)**, for $a = 2$, $b = 3$, $x = 5$.
2. Consider, the special case of Identity **(IV)** with $a = b$, what do you get? Is it related to Identity **(I)**?
3. Consider, the special case of Identity **(IV)** with $a = -c$ and $b = -c$. What do you get? Is it related to Identity **(II)**?
4. Consider the special case of Identity **(IV)** with $b = -a$. What do you get? Is it related to Identity **(III)**?

We can see that Identity **(IV)** is the general form of the other three identities also.

9.11 Applying Identities

We shall now see how, for many problems on multiplication of binomial expressions and also of numbers, use of the identities gives a simple alternative method of solving them.

Example 11: Using the Identity **(I)**, find (i) $(2x + 3y)^2$ (ii) 103^2

Solution:

(i) $(2x+3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2$ [Using the Identity (I)]

$= 4x^2 + 12xy + 9y^2$

We may work out $(2x + 3y)^2$ directly.

$(2x+3y)^2 = (2x+3y)(2x+3y)$

$= (2x)(2x) + (2x)(3y) + (3y)(2x) + (3y)(3y)$

$= 4x^2 + 6xy + 6yx + 9y^2$ (as $xy = yx$)

$= 4x^2 + 12xy + 9y^2$

Using Identity **(I)** gave us an alternative method of squaring $(2x + 3y)$. Do you notice that the Identity method required fewer steps than the above direct method? You will realise the simplicity of this method even more if you try to square more complicated binomial expressions than $(2x + 3y)$.

(ii) $(103)^2 = (100+3)^2$

$= 100^2 + 2 \times 100 \times 3 + 3^2$ (Using Identity I)

$= 10000 + 600 + 9 = 10609$

We may also directly multiply 103 by 103 and get the answer. Do you see that Identity **(I)** has given us a less tedious method than the direct method of squaring 103? Try squaring 1013. You will find in this case, the method of using identities even more attractive than the direct multiplication method.

Example 12: Using Identity **(II)**, find (i) $(4p-3q)^2$ (ii) $(4.9)^2$

Solution:

(i) $(4p-3q)^2 = (4p)^2 - 2(4p)(3q) + (3q)^2$ [Using the Identity **(II)**]

$= 16p^2 - 24pq + 9q^2$

Do you agree that for squaring $(4p-3q)^2$ the method of identities is quicker than the direct method?

(ii) $(4.9)^2 = (5.0-0.1)^2 = (5.0)^2 - 2(5.0)(0.1) + (0.1)^2$

$= 25.00 - 1.00 + 0.01 = 24.01$

Is it not that, squaring 4.9 using Identity (II) is much less tedious than squaring it by direct multiplication?

Example 13: Using Identity **(III)**, find

(i) $\left(\frac{3}{2}m+\frac{2}{3}n\right)\left(\frac{3}{2}m-\frac{2}{3}n\right)$ (ii) 983^2-17^2 (*iii*) 194×206

Solution:

(i) $\left(\frac{3}{2}m+\frac{2}{3}n\right)\left(\frac{3}{2}m-\frac{2}{3}n\right)=\left(\frac{3}{2}m\right)^2-\left(\frac{2}{3}\right)^2$

$=\frac{9}{4}m^2-\frac{4}{9}n^2$

(ii) $983^2-17^2=(983+17)(983-17)$

$[\text{Here } a=983,\ b=17,\ a^2-b^2=(a+b)(a-b)]$

Therefore, $983^2-17^2=1000\times966=966000$

Try doing this directly. You will realise how easy our method of using Identity **(III)** is.

(iii) $194\times206=(200-6)\times(200+6)=200^2-6^2$

$=40000-36=39964$

Example 14: Use the Identity $(x+a)(x+b)=x^2+(a+b)x+ab$ to find the following:

(i) 501×502 (ii) 95×103

Solution:

(i) $501\times502=(500+1)\times(500+2)=500 +(1+2)\times500+1\times2$

$=250000+1500+2=251502$

(ii) $95\times103=(100-5)\times(100+3) = 100^2+(-5+3)\times100+(-5)\times3$

$=10000-200-15=9785$

Exercise 9.5

1. Use a suitable identity to get each of the following products.

(i) $(x+3)(x+3)$ (ii) $(2y+5)(2y+5)$ (iii) $(2a-7)(2a-7)$

(iv) $(3a-\frac{1}{2})(3a-\frac{1}{2})$ (v) $(1.1m-0.4)(1.1m+0.4)$

(vi) $(a^2+b^2)(-a^2+b^2)$ (vii) $(6x-7)(6x+7)$ (viii) $(-a+c)(-a+c)$

(ix) $\left(\frac{x}{2}+\frac{3y}{4}\right)\left(\frac{x}{2}+\frac{3y}{4}\right)$ (x) $(7a-9b)(7a-9b)$

2. Use the identity $(x+a)(x+b)=x^2+(a+b)x+ab$ to find the following products.
(i) $(x+3)(x+7)$ (ii) $(4x+5)(4x+1)$
(iii) $(4x-5)(4x-1)$ (iv) $(4x+5)(4x-1)$
(v) $(2x+5y)(2x+3y)$ (vi) $(2a^2+9)(2a^2+5)$
(vii) $(xyz-4)(xyz-2)$

3. Find the following squares by using the identities.
(i) $(b-7)^2$ (ii) $(xy+3z)^2$ (iii) $(6x^2-5y)^2$
(iv) $\left(\frac{2}{3}m+\frac{3}{2}n\right)^2$ (v) $(0.4p-0.5q)^2$ (vi) $(2xy+5y)^2$

4. Simplify.
(i) $(a^2-b^2)^2$ (ii) $(2x+5)^2-(2x-5)^2$
(iii) $(7m-8n)^2+(7m+8n)^2$ (iv) $(4m+5n)^2+(5m+4n)^2$
(v) $(2.5p-1.5q)^2-(1.5p-2.5q)^2$ (vi) $(ab+bc)^2-2ab^2c$
(vii) $(m^2-n^2m)^2+2m^3n^2$

5. Show that.
(i) $(3x+7)^2-84x=(3x-7)^2$ (ii) $(9p-5q)^2+180pq=(9p+5q)^2$
(iii) $\left(\frac{4}{3}m-\frac{3}{4}n\right)^2+2mn=\frac{16}{9}m^2+\frac{9}{16}n^2$
(iv) $(4pq+3q)^2-(4pq-3q)^2=48pq^2$
(v) $(a-b)(a+b)+(b-c)(b+c)+(c-a)(c+a)=0$

6. Using identities, evaluate.
(i) 71^2 (ii) 99^2 (iii) 102^2 (iv) 998^2
(v) 5.2^2 (vi) 297×303 (vii) 78×82 (viii) 8.9^2
(ix) 1.05×9.5

7. Using $a^2-b^2=(a+b)(a-b)$, find
(i) 51^2-49^2 (ii) $(1.02)^2-(0.98)^2$ (iii) 153^2-147^2
(iv) $12.1^2-7.9^2$

8. Using $(x+a)(x+b)=x^2+(a+b)x+ab$, find
(i) 103×104 (ii) 5.1×5.2 (iii) 103×98 (iv) 9.7×9.8

MISCELLANEOUS EXAMPLE 9:

Example 1: Multiply $(2x - 3y)$ and $4x^2 + 6xy + 9y^2$)

Example 2: Simplify $\left(\frac{4}{3}x + \frac{3}{5}y\right)\left(\frac{16}{9}x^2 - 4xy + \frac{9}{25}y^2\right) - \left(\frac{64x^3}{27} + \frac{27y^3}{125}\right)$

Example3: If $\frac{2}{3}x + \frac{3}{2}y = 5$ and $xy = 6$. Find the value of $\frac{4x^2}{9} + \frac{9y^2}{4}$.

Example 4: (i) Show that $\left(\frac{4}{5}x + \frac{5}{4}y\right)^2 - \left(\frac{4}{5x} - \frac{5}{4}y\right)^2 = 4xy$

(ii) $\left(3pq - \frac{1}{3}q\right)^2 + 2p^2q = 9p^2q^2 + \frac{1}{9}q^2$

Example 5: Factorize using identities:

(i) $x^2 + 6x + 5$ (ii) $\frac{4x^2}{9} - \frac{y^2}{16}$ (iii) $x^2 + 2kx - 3k^2$

Example 6: Using identities evaluate.

(i) 56×44 (ii) 107×103 (iii) 10.5×9.5 (iv) $(9.9)^2$

Example 7: What algebraic expression should be added to $3x^2y + 4xy + 5y^2$ to get $4x^2y + 4y^2$?

Example 8: What should be subtracted from $5x - 3y + 8$ to get $2x + y + 5$.

Example 9: Add $3x^2 + 2xy$ and $3xy + 2xy^2$ and then from the sum subtract $x^2y + 5xy$.

Example10: If $x = 1, y = 2$. Find the value of $4x^2y^2 - 12xy + 9$.

Example 11: What is the difference between an equation and identity? Give an example.

Example 12: Write True or False for the following statements:

(i) The exponent of a variable in a polynomial is always a whole number.

(ii) Polynomial can have one or more variables.

(iii) Algebraic expressions with co-efficient and non-negative integers as exponents of variables are called polynomials.

(iv) A two term polynomial is called a binomial.

(v) There can be irrational Co-efficient in a polynomial.

What Have We Discussed

1. Expressions are formed from **variables** and **constants**.

2. Terms are added to form **expressions**. Terms themselves are formed as product of **factors**.

3. Expressions that contain exactly one, two and three terms are called **monomials**, **binomials** and **trinomials** respectively. In general, any expression containing one or

more terms with non-zero coefficients (and with variables having non- negative exponents) is called a **polynomial**.

4. **Like** terms are formed from the same variables and the powers of these variables are the same, too. Coefficients of like terms need not be the same.
5. While adding (or subtracting) polynomials, first look for like terms and add (or subtract) them; then handle the unlike terms.
6. There are number of situations in which we need to multiply algebraic expressions: for example, in finding area of a rectangle, the sides of which are given as expressions.
7. A monomial multiplied by a monomial always gives a monomial.
8. While multiplying a polynomial by a monomial, we multiply every term in the polynomial by the monomial.
9. In carrying out the multiplication of a polynomial by a binomial (or trinomial), we multiply term by term, i.e., every term of the polynomial is multiplied by every term in the binomial (or trinomial).

   Note that in such multiplication, we may get terms in the product which are like and have to be combined.
10. An **identity** is an equality, which is true for all values of the variables in the equality.

    On the other hand, an equation is true only for certain values of its variables. An equation is not an identity.
11. The following are the standard identities:

    $(a+b)^2 = a^2 + 2ab + b^2$ (I)

    $(a-b)^2 = a^2 - 2ab + b^2$ (II)

    $(a+b)(a-b) = a^2 - b^2$ (III)
12. Another useful identity is $(x+a)(x+b) = x^2 + (a+b)x + ab$ (IV)
13. The above four identities are useful in carrying out squares and products of algebraic expressions.

    They also allow easy alternative methods to calculate products of numbers and so on.