STRAIGHT LINE

1. RELATION BETWEEN CARTESIAN CO-ORDINATE & POLAR CO-ORDINATE SYSTEM

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

2. DISTANCE FORMULA AND ITS APPLICATIONS :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note :

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if $AB = BC = CD = DA$ & $AC = BD$; $AC \perp BD$

(b) Rhombus if $AB = BC = CD = DA$ and $AC \neq BD$; $AC \perp BD$

(c) Parallelogram if $AB = DC$, $BC = AD$; $AC \neq BD$; $AC \not\perp BD$

(d) Rectangle if $AB = CD$, $BC = DA$, $AC = BD$; $AC \not\perp BD$

3. SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio m : n is given by :

(a) For internal division : $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

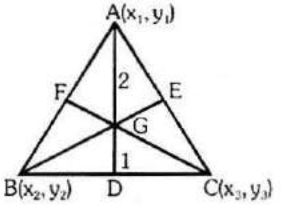
(b) For external division : $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$

(c) Line $ax + by + c = 0$ divides line joining points $P(x_1 y_1)$ & $Q(x_2, y_2)$ in ratio $= -\frac{(ax_1 + by_1 + c)}{(ax_2 + by_2 + c)}$

4. CO-ORDINATES OF SOME PARTICULAR POINTS :

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of any triangle ABC, then

(a) Centroid :



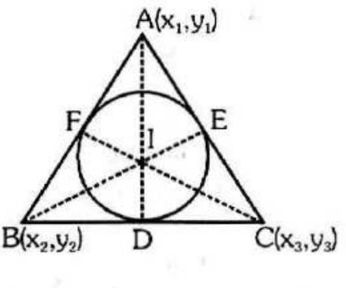
(i) The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices).

(ii) Centroid divides the median in the ratio of 2 : 1.

(iii) Co-ordinates of centroid $G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

(iv) If P is any internal point of triangle such that area of ΔAPB, ΔAPC and ΔBPC are same then P must be centroid.

(b) Incenter :



The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of a circle touching all the sides of a triangle.

Co-ordinates of incenter $I\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Where a, b, c are the sides of triangle ABC.

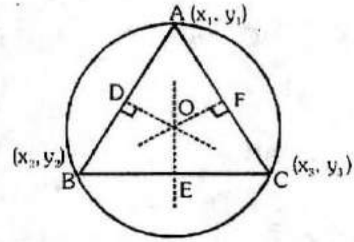
Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a, (c+a):b, (a+b):c$

(c) Circumcenter :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2 = OB^2 = OC^2$.



Also it is a centre of a circle touching all the vertices of a triangle.

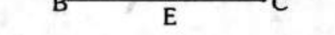
Note :

(i) If a triangle is right angle, then its circumcenter is mid point of hypotenuse.

(ii) Find perpendicular bisector of any two sides and solve them to find circumcentre.

(d) Orthocenter :

It is the point of intersection of perpendicular drawn from vertices on opposite sides of a triangle and can be obtained by solving the equation of any two altitudes.



Note :

If a triangle is right angled triangle, then orthocenter is the point where right angle is formed.

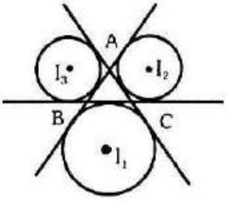
Remarks :

(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincides.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining. orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

(e) Ex-centers :



The centre of the circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively, and

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right),$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

5. AREA OF TRIANGLE :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\,|[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]|$$

To remember the above formula, take the help of the following method :

$$=\frac{1}{2}\left[\begin{matrix} x_1 \\ y_1 \end{matrix}\times\begin{matrix} x_2 \\ y_2 \end{matrix}\times\begin{matrix} x_3 \\ y_3 \end{matrix}\times\begin{matrix} x_1 \\ y_1 \end{matrix}\right] = \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) Area of Equilateral triangle

If altitude of any equilateral triangle is P, then its area $= \frac{P^2}{\sqrt{3}}$.

If 'a' be the side of equilateral triangle, then its area $= \left(\frac{a^2\sqrt{3}}{4}\right)$

(iii) Area of quadrilateral whose consecutive vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) & (x_4, y_4) is $\frac{1}{2}\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_4 & y_2 - y_4 \end{vmatrix}$

6. CONDITION OF COLLINEARITY FOR THREE POINTS :

Three points (x_1,y_1), (x_2,y_2) and (x_3,y_3) are collinear if any one of the given point lies on the line passing through the remaining two points. Thus the required condition is -

$$\frac{y_3 - y_1}{y_2 - y_1} = \frac{x_3 - x_1}{x_2 - x_1} \text{ or } \frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3} \text{ or } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

7. EQUATION OF STRAIGHT LINE :

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here remember that every one degree equation in variable x and y always represents a straight line i.e. $ax + by + c = 0$; a & $b \neq 0$ simultaneously.

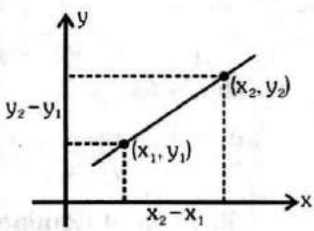
(a) Equation of a line parallel to x-axis at a distance a is $y = a$ or $y = -a$

(b) Equation of x-axis is $y = 0$

(c) Equation of line parallel to y-axis at a distance b is $x = b$ or $x = -b$

(d) Equation of y-axis is $x = 0$

8. SLOPE OF LINE :



If a given line makes an angle θ $(0° \leq \theta < 180°, \theta \neq 90°)$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter **m** i.e. $m = \tan\theta$. Obviously the slope of the x-axis and line parallel to it is zero and y-axis and line parallel to it does not exist.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line AB = $\frac{y_2 - y_1}{x_2 - x_1}$

9. STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE :

(a) Slope Intercept form : Let m be the slope of a line and c its intercept on y-axis. then the equation of this straight line is written as : $\mathbf{y = mx + c}$

(b) Point Slope form : If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as : $\mathbf{y - y_1 = m(x - x_1)}$

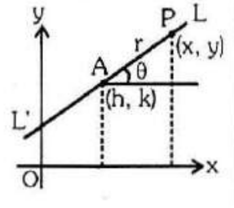
(c) Two point form : Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as :

$$\mathbf{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) Intercept form : If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\mathbf{\frac{x}{a} + \frac{y}{b} = 1}$

(e) Normal form : If p is the length of perpendicular on a line from the origin and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $\mathbf{x\cos\alpha + y\sin\alpha = p}$ (p is always positive), where $0 \le \alpha < 2\pi$.

(f) Parametric form : To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'. Let AP = r then $x - h = r\cos\theta$, $y - k = r\sin\theta$ & $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ is the equation of the straight line LAL'.



Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

(g) General form : We know that a first degree equation in x and y, $\mathbf{ax + by + c = 0}$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and intercept by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2 + b^2}$.

10. ANGLE BETWEEN TWO LINES :

(a) If θ be the angle between two lines : $y = m_1x + c$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

(i) Parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iv) Intersecting $\Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

11. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

Length of perpendicular from a point (x_1,y_1) on the line $ax+by+c=0$ is $= \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \frac{|c|}{\sqrt{a^2 + b^2}}$

12. DISTANCE BETWEEN TWO PARALLEL LINES :

(a) The distance between two parallel lines $ax + by + c_1=0$ and $ax+by+c_2=0$ is $= \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram = $\frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

13. EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE :

(a) Equation of line parallel to line $ax + by + c = 0$

$ax + by + \lambda = 0$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$bx - ay + k = 0$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

14. STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line $y=mx+c$ is written as :

$$\mathbf{y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)}$$

15. POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE :

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

16. CONCURRENCY OF LINES :

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

are concurrent, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Note :

If lines are concurrent then $\Delta = 0$ but if $\Delta = 0$ then lines may or may not be concurrent {lines may be parallel}.

17. REFLECTION OF A POINT :

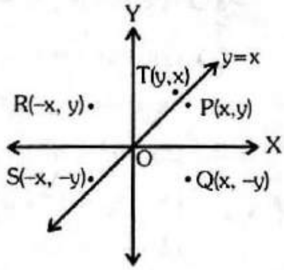
Let P(x,y) be any point, then its image with respect to

(a) x-axis is Q(x, −y)

(b) y-axis is R(−x, y)

(c) origin is S(−x, −y)

(d) line y=x is T(y, x)



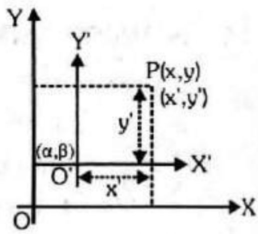
18. TRANSFORMATION OF AXES

(a) Shifting of origin without rotation of axes :

If coordinates of any point P(x, y) with respect to new origin (α, β) will be (x', y')

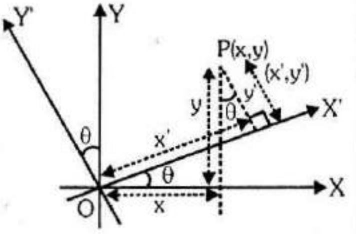
then $x = x' + \alpha, \quad y = y' + \beta$

or $\quad x' = x - \alpha, \quad y' = y - \beta$



Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(b) Rotation of axes without shifting the origin :



Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY', where $\angle X'OX = \angle YOY' = \theta$

then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

Old / New	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

19. EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \qquad \text{.........(1)}$$

(a) Equation of bisector of angle containing origin :

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (1)

(b) Equation of bisector of acute/obtuse angles :

See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by −1 to make the constant terms positive

Determine the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use − sign in eq. (1)
−	use − sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

20. FAMILY OF LINES :

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

21. GENERAL EQUATION AND HOMOGENEOUS EQUATION OF SECOND DEGREE :

(a) A general equation of second degree $\mathbf{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$ represent a pair of straight lines if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(b) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$

Obviously these lines are

(i) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(ii) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(c) Homogeneous equation of 2^{nd} degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \equiv y = m_1x \;\&\; y = m_2x$$

and $m_1 + m_2 = -\frac{2h}{b}$; $m_1m_2 = \frac{a}{b}$

These straight lines passes through the origin and for finding the angle between these lines same formula as given for general equation is used.

The condition that these lines are :

(i) At right angles to each other is $a + b = 0$. i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$.

(ii) Coincident is $h^2 = ab$.

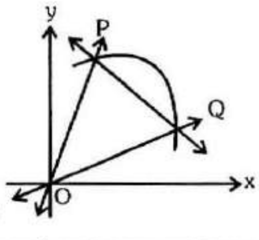
(iii) Equally inclined to the axis of x is $h = 0$. i.e. coeff. of $xy = 0$.

(d) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2^{nd} degree is given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, $a \neq b$, $h \neq 0$.

(e) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(f) If lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel then distance between them is $= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$

22. EQUATIONS OF LINES JOINING THE POINTS OF INTERSECTION OF A LINE AND A CURVE TO THE ORIGIN :



Let the equation of curve be :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ......(i)

and straight line be

$\ell x + my + n = 0$ ......(ii)

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus required equation is given by -

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

23. STANDARD RESULTS :

(a) Area of rhombus formed by lines $a|x|+b|y|+c=0$

or $\pm ax \pm by + c = 0$ is $\frac{2c^2}{|ab|}$.

(b) Area of triangle formed by line $ax+by+c=0$ and axes is $\frac{c^2}{2|ab|}$.

(c) Co-ordinate of foot of perpendicular (h, k) from (x_1, y_1) to the line

$ax+by+c=0$ is given by $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-(ax_1+by_1+c)}{a^2+b^2}$

(d) Image of point (x_1, y_1) w.r. to the line $ax+by+c=0$ is given by

$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$