Sequences and Series

ARITHMETIC PROGRESSION — Section - 1

1.1 Definition

When the sequence of (or numbers) a_1, a_2, a_3, a_4,..... increases or decreases by a fixed quantity, then the sequence is in *arithmetic progression* (A.P.). The fixed quantity is called as *common difference*. For an A.P., we define its first term as a and the common difference as d; the general expression for an A.P. is :

$$a, a + d, a + 2d, a + 3d, a + 4d \ldots\ldots\ldots\ldots$$

If T_r represents the general term of an $A.P.$ (or *the* r^{th} term), then

$$T_r = a + (r - 1)\,d \qquad \text{where } r \in \{1, 2, 3\ldots\ldots\ldots, n\}$$

It is also denoted by a_r. If the total number of terms be n, then n^{th} term is also known as the last term of $A.P.$ and is denoted by l, *i.e.*,

$$l = T_n = a + (n - 1)\,d$$

In an $A.P.$, the difference of any two consecutive terms is d and is given by:

$$d = T_r - T_{r-1}$$

1.2 Sum of *n* terms of an A.P

Consider n terms of an $A.P.$ with first term as a and the common difference as d. Let S_n denotes the sum of the first n terms, then

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots\ldots + (a + (n - 1)d)$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)\,d\right]$$

or $$S_n = \frac{n}{2}(a + l) \quad \text{as} \quad l = a + (n - 1)\,d$$ [where a be the first term and l be the last term]

1.3 Arithmetic mean (A)

When three quantities are in $A.P$, then the middle one is called as *arithmetic mean* of other two. If a and b are two numbers and A be the arithmetic mean of a and b, then a, A, b are in $A.P.$

$$\Rightarrow \quad A - a = b - A \quad \Rightarrow \quad A = \frac{a + b}{2}$$

1.4 Inserting *n* arithmetic means between two numbers

A_1, A_2, A_3, ..., A_n are called n arithmetic means between two numbers a and b, if the sequence a, A_1, A_2, A_3, A_4,.........., A_n, b in an A.P.

For this $A.P.$, first term is a, number of terms is $(n + 2)$, & the last term is $= b = T_{n+2}$.

Let d be the common difference of this A.P., then

$\Rightarrow \quad T_{n+2} = b = a + \{(n + 2) - 1\}\, d$

$\Rightarrow \quad b = a + (n + 1)\, d$

$\Rightarrow \quad d = \dfrac{b-a}{n+1} \quad \Rightarrow \quad A_1 = a + d = a + \dfrac{b-a}{n+1}, \quad A_2 = a + 2d = a + 2\left(\dfrac{b-a}{n+1}\right)$ and so on.

In general k^{th} arithmetic mean is $\equiv A_k = a + kd = a + k\left(\dfrac{b-a}{n+1}\right)$

1.5 Important Points

(i) If a, b, c are in A.P., then : ak, bk, ck are also in A.P. $(k \neq 0)$

$a/k, b/k, c/k$ are also in A.P. $(k \neq 0)$

$a \pm k, b \pm k, c \pm k$ are also in A.P.

(ii) Three terms in an A.P. are taken as : $a - d, a, a + d$

(iii) Four terms in an A.P. are taken as : $a - 3d, a - d, a + d, a + 3d$

(iv) The sum of any two terms (of an A.P.) equidistant from beginning and end is equal to the sum of the first and the last term. Hence, $(a + md) + (\ell - md) = a + \ell$

Illustrating the Concept :

➤ *The third term of an* A.P. *is* 18 *and* 7^{th} *term is* 30. *Find the* 17^{th} *term.*

Let the first term of A.P. is a and the common difference be d.

$T_r = a + (r - 1)\, d$

$\Rightarrow \quad T_3 = a + 2d = 18 \quad$ . . . **(i)**

$T_7 = a + 6d = 30 \quad$ . . . **(ii)**

Solve **(i)** and **(ii)** to get : $a = 12$ and $d = 3$

$T_{17} = a + 16d = 12 + 16 \times 3 = 60$

$\Rightarrow \quad T_{17} = 60$

➤ *Find the sum of n terms of a series whose 7^{th} term is 30 and 13^{th} term is 54. Hence or otherwise find the sum of r terms and 50 terms of the series. Assume the series is A.P.*

Let the first term of A.P. is a and the common difference be d.

$T_r = a + (r-1)d$

$T_7 = a + 6d = 30$ . . . (i)

$T_{13} = a + 12d = 54$ . . . (ii)

On solving (i) and (ii) we get, $a = 6$ and $d = 4$

Now using :

$$S_n = \frac{n}{2}\left[2a + (n-1)\,d\right]$$

$$S_n = \frac{n}{2}\left[2\times 6 + (n-1)\times 4\right] = 2n\,(n+2)$$

Now find the sum of r terms, S_r.

In $S_n = 2n\,(n+2)$, we replace n by r to get S_r.

$\Rightarrow \quad S_r = 2r(r+2)$

To find the sum of 50 terms, we can use

$S_n = 2n(n+2)$.

$\Rightarrow \quad S_{50} = 2 \times 50\,(50+2) = 5200$

Illustration - 1 *The sum of n 'terms of an A.P. whose first term is x, the last term is y and the common difference is* 1, *is :*

(A) $\frac{1}{2}(x+y)(1-x+y)$ (B) $(x+y)(1-x+y)$

(C) $\frac{1}{2}(x-y)(1-x+y)$ (D) $(x-y)(1-x+y)$

SOLUTION : (A)

$$S_n = \frac{n}{2}[a+\ell]$$

[where a is the first term and l is the last term]

$\Rightarrow \quad S_n = \frac{n}{2}[x+y]$ . . . (i)

Now in the required answer, note that term n is not there, so eliminate n.

Using : $l = a + (n-1)\,d$

$\Rightarrow \quad y = x + (n-1)\times 1$

$\Rightarrow \quad n = 1 - x + y$

Substitute the value of n in (i), we get :

$$S_n = \frac{1}{2}(x+y)(1-x+y)$$

Illustration - 2 *If x, y, z are in A.P., then* $(x+2y-z)(2y+z-x)(z+x-y)$ *is :*

(A) xyz (B) $4\,xyz$ (C) $2\,xyz$ (D) $8xyz$

SOLUTION : (B)

Since x, y and z are in A.P.

$\Rightarrow \quad 2y = x + z$

Substituting for $2y$ in the given expression, to get :

$(x + x + z - z)\,(x + z + z - x)\,(2y - y)$

$= (2x)\,(2z)\,(y) = 4xyz$

Illustration - 3 *If a, b, c are the x^{th}, y^{th} and z^{th} terms of an A.P., then :*

(i) $a(y-z)+b(z-x)+c(x-y)$ is :

(A) 0 **(B)** abc **(C)** $a+b+c$ **(D)** $a(b+c)$

(ii) $x(b-c)+y(c-a)+z(a-b)$ is :

(A) abc **(B)** 0 **(C)** $a+b+c$ **(D)** $a(b+c)$

SOLUTION : (i) - (A) (ii) - (B)

Let A be the first term and D be the common difference,

$\Rightarrow$ $T_x = A+(x-1)D = a$ ...(i)

$T_y = A+(y-1)D = b$ ...(ii)

$T_z = A+(z-1)D = c$ ...(iii)

Operating [(ii) – (iii)], [(iii) – (i)] and [(i) – (ii)] we get :

$$b-c=(y-z)D,\quad c-a=(z-x)D,$$
$$a-b=(x-y)D$$

$$\Rightarrow \quad y-z=\frac{b-c}{D},\qquad z-x=\frac{c-a}{D},$$
$$x-y=\frac{a-b}{D}$$

(i) Now substituting the values of $(y-z)$, $(z-x)$ and $(x-y)$ in part (i) of the question to get :

$$\frac{a(b-c)}{D}+\frac{b(a-c)}{D}+\frac{c(a-b)}{D}=0$$

(ii) Now substitute the values of $(b-c)$, $(c-a)$ and $(a-b)$ in part (ii) of the question to get :

$$x(y-z)D+y(z-x)D+z(x-y)D$$
$$=\{xy-xz+yz-yx+zx-zy\}D=0$$

Illustration - 4 *The sum of n terms of two series in A.P. are in the ratio $5n+4 : 9n+6$. Find the ratio of their 13^{th} terms.*

(A) $\frac{129}{231}$ **(B)** $\frac{12}{13}$ **(C)** $\frac{131}{229}$ **(D)** $\frac{13}{12}$

SOLUTION : (A)

Let a_1, a_2 be the first terms of two A.P.'s and d_1, d_2 are their respective common differences.

Now $\dfrac{S_n}{S_n'}=\dfrac{5n+4}{9n+6}$

$$\Rightarrow \quad \frac{\frac{n}{2}\left[2a_1+(n-1)d_1\right]}{\frac{n}{2}\left[2a_2+(n-1)d_2\right]}=\frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{a_1+\frac{(n-1)}{2}d_1}{a_2+\frac{(n-1)}{2}d_2}=\frac{5n+4}{9n+6} \quad \text{...(i)}$$

Now the ratio of 13^{th} terms $=\dfrac{a_1+12d_1}{a_2+12d_2}$

$\Rightarrow$ Substitute $\dfrac{(n-1)}{2}=12$

$\Rightarrow$ $n=25$ in (i)

$$\Rightarrow \quad \frac{a_1+12d_1}{a_2+12d_2}=\frac{5(25)+4}{9(25)+6}=\frac{129}{231}$$

Illustration - 5 *If the sum of n terms of a series is $S_n = n(5n-3)$, then find the n^{th} term.*

(A) $10n - 8$ **(B)** $10n + 8$ **(C)** $10n + 4$ **(D)** $10n - 4$

SOLUTION : (A)

Let $S_n = T_1 + T_2 + T_3 + T_4 + \dots + T_{n-1} + T_n$

$S_n = \{\text{sum of } (n-1) \text{ terms}\} + T_n$

$\Rightarrow \quad T_n = S_n - S_{n-1}$

[Learn it as standard result]

Now in the given problem :

$S_n = n(5n-3)$

and $S_{n-1} = (n-1)[5(n-1)-3]$

$\Rightarrow \quad T_n = S_n - S_{n-1} = 10n - 8$

Illustration - 6 *If a, b, c are in A.P., then b + c, c + a, a + b are also in :*

(A) A.P. **(B)** G.P. **(C)** H.P. **(D)** *None of these*

SOLUTION : (A)

If a, b, c are in A.P., then

$a-(a+b+c), b-(a+b+c), c-(a+b+c)$ are also in A.P. (subtracting $a + b + c$ from each term)

$\Rightarrow$ $-(b+c), -(c+a), -(a+b)$ are also in A.P.

$\Rightarrow$ $(b+c), (c+a), (a+b)$ are also in A.P.

Note : If a, b, c are in A.P., then $a \pm k,\ b \pm k,\ c \pm k$ are also in A.P. and $-a, -b, -c$ are also in A.P.

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Illustration - 7 *If p times the p^{th} term of an A.P. be equal to q times the q^{th} term, then (p + q)th term is:*

(A) $p + q$ **(B)** 0 **(C)** $p - q$ **(D)** $2p + 3q$

SOLUTION : (B)

Let a be the first term and d be the common difference of A.P.

$\Rightarrow \quad T_p = a + (p-1)d$

and $T_q = a + (q-1)d$

Now to find $T_{(p+q)}$, we have

$T_{(p+q)} = a + (p+q-1)d \quad \dots$ **(i)**

Since : $p(T_p) = q(T_q)$ (Given)

$\Rightarrow \quad p[a+(p-1)d] = q[a+(q-1)d]$

$\Rightarrow \quad pa + p(p-1)d = qa + q(q-1)d$

$\Rightarrow \quad (p-q)a + (p^2-q^2)d - (p-q)d = 0$

$\Rightarrow \quad (p-q)[a+(p+q-1)d] = 0$

$\Rightarrow \quad a + (p+q-1)d = 0$ (since $p \neq q$)

$\Rightarrow \quad T_{(p+q)} = 0$ [from (i)]

Illustration - 8 *The sum of three consecutive terms of an A.P. is 15 and the sum of their squares is 83, find the terms.*

(A) 3, 6, 9 **(B)** 2, 4, 6 **(C)** 3, 5, 7 **(D)** 2, 7, 10

SOLUTION : (C)

Three consecutive terms of an A.P are taken as :

$(a-d), a, (a+d)$.

$\Rightarrow \quad (a-d)+a+(a+d)=15 \quad \ldots$(i)

$(a-d)^2+a^2+(a+d)^2=83 \quad \ldots$(ii)

From eqn (i), we get : $a=5$.

Substituting for $a=5$ in eqn (ii), we get :

$d^2=4. \quad \Rightarrow \quad d=\pm 2$

Hence the terms are :

3, 5, 7 or 7, 5, 3.

Illustration - 9 *How many terms of the series : 24 + 20 + 16. . . . . . totals 72 ?*

(A) 9 **(B)** 3 **(C)** 6 **(D)** 8

SOLUTION : (A)

$$S_n=\frac{n}{2}[2a+(n-1)\,d]$$

Now $a=24$, $d=-4$ and $S_n=72$.

$$\Rightarrow \quad 72=\frac{n}{2}\left[2\times 24+(n-1)\times(-4)\right]$$

On solving, we get, $n=4$ or 9.

Note : Since the series is decreasing with $d=-4$, it contains some negative numbers, so the sum of first 4 terms is same as the sum of 9 terms.

Illustration - 10 *If the sum of first p, q, r terms of an A.P. is a, b, c respectively, then :*

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q) \text{ is equal to:}$$

(A) –1 **(B)** 0 **(C)** 1 **(D)** *None of these*

SOLUTION : (B)

Let A be the first term and D be the common difference of the A.P.

$$\Rightarrow \quad a=\frac{p}{2}\left[2A+(p-1)\,D\right]$$

We can write

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)$$

$$=\sum\frac{a}{p}(q-r)$$

$$\text{L.H.S.}=\sum\frac{a}{p}(q-r)$$

$$=\sum\frac{1}{2}(q-r)\left[2A+(p-1)\,D\right]$$

$$=\frac{1}{2}\sum 2A(q-r)+\frac{1}{2}\sum(q-r)\,D\,(p-1)$$

$$=A\sum(q-r)+\frac{D}{2}\sum\left[p\,(q-r)\right]-\frac{D}{2}\sum(q-r)$$

$$=0+0-0=0$$

GEOMETRICAL PROGRESSION — Section - 2

If the sequence of numbers $a_1, a_2, a_3, \ldots\ldots$ decrease or increase by a constant factor, they are said to be in *geometrical progression* (G.P.). The constant factor is called as *common ratio*. For a G.P., we define the first term as a and the common ratio as r ; the general expression for a G.P. is $a, ar, ar^2, ar^3, \ldots\ldots\ldots$

In G.P., the common ratio (r) is the ratio of any two consecutive terms. If T_k represents the general term (or k^{th} term) of a G.P., then $r = \dfrac{T_k}{T_{k-1}}$ and $T_k = a\,r^{k-1}$ where $k \in \{1, 2, \ldots\ldots, n\}$

2.2 Sum of *n* terms of a G.P.

Consider n terms of a $G.P.$ with a as the first term and r as the common ratio. Let S_n denote the sum of n terms. Then $S_n = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-1}$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}, (r \neq 1)$$

2.3 Geometric Mean (G)

When three quantities are in G.P., the middle one is called as *geometric mean* between the other two. If a and b are two numbers, G is the geometric mean between the two, then a, G, b follows a G.P.

$$\Rightarrow \quad \frac{b}{G} = \frac{G}{a} \quad \Rightarrow \quad G^2 = ab \quad \Rightarrow \quad G = \sqrt{ab}$$

2.4 Inserting *n* geometric means between two numbers :

$G_1, G_2, G_3, \ldots\ldots, G_n$ are called n geometric means between two numbers a and b,

If the sequence $a, G_1, G_2, G_3, \ldots\ldots\ldots, G_n, b$ is a $G.P.$

For this G.P. : first term $= a$, total number of terms $= n + 2$,

Last term $= b = T_{n+2} = (n + 2)^{th}$ term, let r be the common ratio of this G.P.

$$\Rightarrow \quad T_{n+2} = b = ar^{n+1} \quad \Rightarrow \quad \frac{b}{a} = r^{n+1} \quad \Rightarrow \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Hence $\quad G_1 = ar = a\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$ and $G_2 = ar^2 = a\left(\dfrac{b}{a}\right)^{\frac{2}{n+1}}$ . . . .

In general k^{th} geometric mean $G_k = ar^k = a\left(\dfrac{b}{a}\right)^{\frac{k}{n+1}}$

2.5 Infinite Geometric series :

If for *G.P.*, the common ratio r lies between – 1 and 1 i.e., $-1 < r < 1$, it is called as *decreasing geometric series*. The sum of the infinite terms of such a sequence (called as infinite geometric series) is denoted by S_∞ and is given as :

$$S_\infty = \frac{a}{1-r}$$ [where a is the first term and r is the common ratio]

Note : $S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$. If $n \to \infty$ then $r^n \to 0$ for $-1 < r < 1$ $\Rightarrow$ $S_\infty = \frac{a}{1-r}$

2.6 Important Points

(i) If a, b, c are in *G.P.*, then : ak, bk, ck are also in *G.P.* $(k \neq 0)$

$a/k.\ b/k,\ c/k$ are also on *G.P.* $(k \neq 0)$

(ii) Three terms in a *G.P.* are taken as : $ar, a, a/r$

(iii) Four terms in a *G.P.* are taken as : $ar^3, ar, a/r, a/r^3$

(iv) If a, b, c, d are in *G.P.*, then : $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ $\Rightarrow$ $b^2 = ac$, $c^2 = bd$, $ad = bc$

Also $b = ar,\ c = ar^2,\ d = ar^3$ (where r is the common ratio)

Illustrating the Concept :

➤ *If 5^{th} term of a G.P is 1/3 and the 9^{th} term is 16/243, then find the 4^{th} term. Also find the sum of first 10 terms of the G.P.*

Let a be the first term and r be the common ratio.

$$T_5 = ar^4 = \frac{1}{3} \quad \ldots \text{(i)}$$

$$T_9 = ar^8 = \frac{16}{243} \quad \ldots \text{(ii)}$$

Divide (i) and (ii) to get $r = \pm\frac{2}{3}$

Substitute for r in eqn (i) to get $a = \frac{27}{16}$

$$\Rightarrow \quad T_4 = ar^3 = \frac{27}{16}\left(\pm\frac{2}{3}\right)^3 = \pm\frac{1}{2}$$

We have, $S_n = \frac{a\left(1-r^n\right)}{1-r}$

$$S_{10} = \frac{\frac{27}{16}\left[1-\left(\frac{2}{3}\right)^{10}\right]}{1-\frac{2}{3}},\ \frac{\frac{27}{16}\left[1-\left(\frac{-2}{3}\right)^{10}\right]}{1-\left(\frac{-2}{3}\right)}$$

$$= \frac{81}{16}\left(\frac{3^{10}-2^{10}}{3^{10}}\right),\ \frac{81}{80}\left(\frac{3^{10}-2^{10}}{3^{10}}\right)$$

- *Find the sum of the series :*
  $3, \sqrt{3}, 1, \ldots\ldots\ldots$ *to infinity.*

  First term, $a = 3$ and the common ratio,

  $$r = \frac{1}{\sqrt{3}} \quad (|r| < 1)$$

  $$S_\infty = \frac{a}{1-r} \Rightarrow S_\infty = \frac{3}{1-\frac{1}{\sqrt{3}}} = \frac{3\sqrt{3}}{\sqrt{3}-1} = \frac{3(3+\sqrt{3})}{2}$$

Illustration - 11 *How many terms of the series : $\sqrt{3}, 3, 3\sqrt{3}, \ldots\ldots\ldots$ amounts to $39+13\sqrt{3}$?*

(A) 8 **(B)** 9 **(C)** 6 **(D)** 10

SOLUTION : (C)

Here, the first term a is $\sqrt{3}$ and the common ratio r is $\sqrt{3}$.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow 39+13\sqrt{3} = \frac{\sqrt{3}\left[1-(\sqrt{3})^n\right]}{1-\sqrt{3}}$$

$$\Rightarrow \frac{(39+13\sqrt{3})(1-\sqrt{3})}{\sqrt{3}} = 1-(\sqrt{3})^n$$

$$\Rightarrow (\sqrt{3})^n - 1 = 26 \Rightarrow n = 6$$

Illustration - 12 *If a, b, c, d are in G.P.,then find the following :*

(i) $(b-c)^2 + (c-a)^2 + (d-b)^2$

(A) $(a-d)^2$ **(B)** $(a-b)^2$ **(C)** $(c-d)^2$ **(D)** *None of these*

(ii) $(a^2-b^2)(c^2-d^2)$

(A) $(a^2-b^2)^2$ **(B)** $(b^2-c^2)^2$ **(C)** $(b^2-d^2)^2$ **(D)** $(a-d)^2$

SOLUTION : (i) - (A) (ii) - (B) :

(a) a, b, c, d are in *G.P.*,

$\Rightarrow b^2 = ac, \quad c^2 = bd, \quad bc = ad \quad \ldots$ **(i)**

Now expanding the given terms, we get:

$(b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ac) + (d^2 + b^2 - 2bd)$

$= 2(b^2 - ac) + 2(c^2 - bd) + a^2 + d^2 - 2bc$

$= 2(0) + 2(0) + a^2 + d^2 - 2ad$ [Using (i)]

$= (a-d)^2$

(b) Now expanding the given terms :

$(a^2-b^2)(c^2-d^2) = a^2c^2 - b^2c^2 - a^2d^2 + b^2d^2$

$= b^4 - b^2c^2 - b^2c^2 + c^4$ [Using (i)]

$= (b^2-c^2)^2$

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Illustration - 13 *If a, b, c are in A.P. and x, y, z are in G.P., then find the value of* $x^{b-c}\, y^{c-a}\, z^{a-b}$.

(A) 0 **(B)** 1 **(C)** 4 **(D)** 2

SOLUTION : (B)

a, b, c are in A.P. $\Rightarrow$ $a - b = b - c$

x, y, z are in G.P. $\Rightarrow$ $y^2 = xz$

Now, the given expression becomes

$x^{b-c} z^{b-c} y^{c-a}$ $\quad (\because\ b - c = a - b)$

$= (xz)^{b-c} y^{c-a} = y^{2(b-c)} y^{c-a}$ $\quad (\because\ xz = y^2)$

$= y^{\{2(b-c)+(c-a)\}}$

$= y^{2b-a-c} = y^0 = 1$ $\quad (\because\ 2b = a + c)$

Illustration - 14 *The sum of infinite terms of a G.P is 15 and the sum of their squares is 45. Find the series.*

(A) $1, \frac{2}{3}, \frac{4}{9}, \ldots.$ **(B)** 5,15, 45 **(C)** $15, \frac{15}{2}, \frac{45}{4}, \ldots.$ **(D)** $5, \frac{10}{3}, \frac{20}{9}, \ldots.$

SOLUTION : (D)

Let the first term of infinite series be a and the common ratio be r. Now for the series with squares of each term, the first term will be a^2 and the common ratio will be r^2.

We have, $S_\infty = \frac{a}{1-r}$ (where a is the term and r is common ratio ($|r| < 1$))

$\Rightarrow \quad \frac{a}{1-r} = 15 \quad \ldots$ **(i)**

and $\quad \frac{a^2}{1-r^2} = 45 \quad \ldots$ **(ii)**

On dividing equation (ii) by equation (i), we get :

$\frac{a}{1+r} = 3 \quad \ldots$ **(iii)**

From eqn. (i) and eqn (iii), we get :

$\frac{1+r}{1-r} = 5 \qquad r = \frac{2}{3}$ and $a = 5$

Hence the series is : $5, \frac{10}{3}, \frac{20}{9}, \ldots\ldots.$

Illustration - 15 *Find the value of* $x^{1/2} . x^{1/4} . x^{1/8}$ .... *upto infinity.*

(A) x^3 **(B)** x^2 **(C)** x **(D)** x^{-1}

SOLUTION : (C)

The given expression can be written as : $x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\ldots\infty}$

Now, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\infty$ is an infinite geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

$\Rightarrow \quad x^{\left(\frac{1/2}{1-1/2}\right)} = x^1 = x$ $\quad$ [Using $S_\infty = \frac{a}{1-r}$]

Illustration - 16 *If the continued product of three numbers in G.P. is 216 and the sum of the products taken in pairs is 156, then find the numbers.*

(A) 2, 6, 18 **(B)** 18, 6, 4 **(C)** 3, 8, 9 **(D)** None of these

SOLUTION : (A)

Let $\frac{a}{r}$, a, ar be the three numbers.

$\Rightarrow \frac{a}{r} \cdot a \cdot ar = 216 \quad \Rightarrow a = 6$

Also $\frac{a}{r}(a) + a(ar) + \frac{a}{r}(ar) = 156$

$\Rightarrow a^2\left(\frac{1}{r} + r + 1\right) = 156$

$\Rightarrow 6^2 (r^2 + r + 1) = 156\, r$

$\Rightarrow 3r^2 + 3r + 3 = 13r \quad \Rightarrow \quad r = 3, \frac{1}{3}$

Hence the numbers are 2, 6, 18 or 18, 6, 2.

Illustration - 17 *If a, b, c are in G.P., then*

$$\frac{a^2 + ab + b^2}{bc + ca + ab} \text{ is :}$$

(A) $\frac{b+c}{a+b}$ **(B)** $\frac{a+c}{a+b}$ **(C)** $\frac{a+b}{b+c}$ **(D)** $\frac{a-c}{a-b}$

SOLUTION : (C)

As a, b, c are in $G.P.$, let us consider $b = ar$

and $c = ar^2$ where r is common ratio

$$\frac{a^2+ab+b^2}{bc+ca+ab} = \frac{a^2+a^2r+a^2r^2}{a^2r^3+a^2r^2+a^2r}$$

$$= \frac{a^2\left(1+r+r^2\right)}{a^2r\left(r^2+r+1\right)} = \frac{1}{r}$$

$$= \frac{a+ar}{ar+ar^2} = \frac{a+b}{b+c}$$

Illustration - 18 *If $a^x = b^y = c^z$ and x, y, z are in G.P., then :*

(A) $log_b\, a = log_c\, b$ **(B)** $log_a\, a = log_c\, b$ **(C)** $log_b\, a = log_b\, c$ **(D)** *None of these*

SOLUTION : (A)

Given $a^x = b^y = c^z$. Operating log, we get:

$x \log a = y \log b = z \log c$

$\Rightarrow \frac{x}{y} = \frac{\log b}{\log a}$ and $\frac{y}{z} = \frac{\log c}{\log b}$

As x, y, z are in G.P.,

$\Rightarrow \frac{x}{y} = \frac{y}{z} \quad \Rightarrow \quad \frac{\log b}{\log a} = \frac{\log c}{\log b}$

$\Rightarrow \frac{\log a}{\log b} = \frac{\log b}{\log c} \quad \Rightarrow \quad \log_b a = \log_c b$

Illustration - 19 *If S_n is the sum of first n terms of a G.P. : $\{a_n\}$ and S_n' is the sum of another G.P. : $\{1/a_n\}$, then show that : $S_n = kS_n'$ where k is :*

(A) $\dfrac{1}{a_1 a_n}$ **(B)** $a_1\, a_n$ **(C)** $\dfrac{a_1}{a_n}$ **(D)** $\dfrac{a_n}{a_1}$

SOLUTION : (B)

$$S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$$

$$S_{n'} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{a_n}$$

For the first $G.P.\ (a_n),\ a_n = a_1\, r^{n-1}$ . . . (i)

$$S_n = \frac{a_1\,(1 - r^n)}{1 - r},$$

[where r is the common ratio]

For the second G.P. $\left(\dfrac{1}{a_n}\right)$, common ratio $= \dfrac{1}{r}$

$$S'_n = \frac{1}{a_1}\frac{\left(1 - \dfrac{1}{r^n}\right)}{\left(1 - \dfrac{1}{r}\right)} = \frac{\left(r^n - 1\right)}{a_1\left(r - 1\right) r^{n-1}} = \frac{r^n - 1}{a_n\left(r - 1\right)}$$

$$\Rightarrow \quad S'_n = \frac{1}{a_1 a_n} \times \frac{a_1\left(r^n - 1\right)}{r - 1} \quad \text{[using (i)]}$$

$$\Rightarrow \quad S'_n = \frac{1}{a_1 a_n} S_n \qquad \Rightarrow \quad S_n = S'_n\, a_1 a_n.$$

HARMONICAL PROGRESSION — Section - 3

A sequence $a_1, a_2, a_3, a_4, \ldots\ldots$ is said to be in *harmonic progression* (H.P.) if

$\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \dfrac{1}{a_4}$ forms an arithmetic progression

i.e., $$\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \frac{1}{a_4} - \frac{1}{a_3} = \ldots\ldots\ldots$$

Three quantities a, b, c are said to be in H.P., when $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

$$\Rightarrow \quad \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} \quad \Rightarrow \quad \frac{a-b}{ab} = \frac{b-c}{bc} \quad \Rightarrow \quad \frac{a}{c} = \frac{a-b}{b-c}$$

3.2 Harmonic mean(H)

When three quantities are in H.P., the middle one is called as the *harmonic mean* between the other two. If a and b are two numbers, H is harmonic mean between the two, then a, H, b must be in H.P.

$$\Rightarrow \quad \frac{1}{a}, \frac{1}{H}, \frac{1}{b} \text{ must be in } A.P. \quad \Rightarrow \quad \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\Rightarrow \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b} \qquad \Rightarrow \quad H = \frac{2ab}{a+b}$$

3.3 Inserting *n* harmonic means between two numbers

$H_1, H_2, H_3, \ldots\ldots., H_n$ are called n harmonic means between two numbers a and b, if the sequence :

$a, H_1, H_2, H_3, \ldots\ldots., H_n, b$ is an H.P.

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots\ldots\ldots\ldots\ldots \frac{1}{H_n}, \frac{1}{b} \quad \text{forms an A.P.}$$

For this A.P., first term = $1/a$, number of terms = $n + 2$ and the last term = $1/b = T_{n+2}$.

$$\Rightarrow \quad T_{n+2} = \frac{1}{b} = \frac{1}{a} + \left[(n+2) - 1\right] d \qquad \Rightarrow \qquad d = \frac{\frac{1}{b} - \frac{1}{a}}{n} + \frac{1}{1} = \frac{a-b}{ab(n+1)}$$

$$\Rightarrow \quad \frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)} \qquad \text{and} \qquad \frac{1}{H_2} = \frac{1}{a} + 2d = \frac{1}{a} + 2\frac{a-b}{ab(n+1)} \;\cdots\cdots$$

If H_k be the k^{th} harmonic mean, then :

$$\frac{1}{H_k} = \frac{1}{a} + kd = \frac{1}{a} + k\frac{a-b}{ab(n+1)} \qquad \Rightarrow \qquad H_k = \frac{ab(n+1)}{b(n+1) + k(a-b)}$$

3.4 Relation among A, G, H

If A, G, H are arithmetic, geometric and harmonic means respectively between two positive numbers a and b, then

$$A = \frac{a+b}{2}, \qquad G = \sqrt{ab} \quad \text{and} \quad H = \frac{2ab}{a+b} \quad \Rightarrow \quad AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab = G^2$$

$\Rightarrow \quad G^2 = AH$ i.e. G is the geometric mean between A and H.

Hence, G lies between A and H.

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a + b - 2\sqrt{ab}}{2}$$

$$\Rightarrow \quad A - G = \left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{2}}\right)^2$$ now R.H.S. is positive if a and b are positive

$$\Rightarrow \quad A - G > 0 \qquad \Rightarrow \qquad A > G$$

As G lies between A and H, this means $\quad A > G > H$

Illustration - 20 *The 10th term of the series :* $8, \frac{8}{3}, \frac{8}{5}$..................

(A) 8/17 **(B)** 8/15 **(C)** 8/19 **(D)** 8/21

SOLUTION : (C)

The given series is an H.P.

$\Rightarrow \quad \frac{1}{8}, \frac{3}{8}, \frac{5}{8}$ is an A.P. with first term $a = \frac{1}{8}$

and common difference $d = \frac{1}{4}$.

Now *n*th term of the H.P.

=1/(*n*th term of the A.P)

$$\Rightarrow \quad T_n \text{(H.P.)} = \frac{1}{T_n \text{ (A.P.)}}$$

$$= \frac{1}{\frac{1}{8} + \frac{n-1}{4}} = \frac{8}{2n-1}$$

$$\Rightarrow \quad T_{10} = \frac{8}{2 \times 10 - 1} = \frac{8}{19}$$

Illustration - 21 *The mth term of an H.P. is n and nth term is m .The ratio of* $(m+n)^{th}$ *and* $(mn)^{th}$ *terms of H.P. is :*

(A) $\frac{mn}{m+n}$ **(B)** $\frac{m+n}{mn}$ **(C)** 1 **(D)** 2

SOLUTION : (A)

Assume *a* and *d* be the first term and common difference of corresponding A.P. respectively.

Using $T_r \text{ (H.P.)} = \frac{1}{T_r \text{ (A.P.)}}$

$$T_m = n = \frac{1}{a + (m-1)\, d}$$

$$\Rightarrow \quad a + (m-1)\, d = \frac{1}{n} \qquad \text{. . . (i)}$$

$$T_n = m = \frac{1}{a + (n-1)\, d}$$

$$\Rightarrow \quad a + (n-1)\, d = \frac{1}{m} \qquad \text{. . . (ii)}$$

Subtracting (ii) from (i), we get :

$$(n-m)\, d = \frac{n-m}{mn} \quad \Rightarrow \quad d = \frac{1}{mn}$$

Substituting for *d* in eqn (i), we get :

$$a = \frac{1}{mn}$$

$$T_{m+n} = \frac{1}{a + (m+n-1)\, d}$$

$$= \frac{1}{\frac{1}{mn} + (m+n-1)\frac{1}{mn}} = \frac{mn}{m+n}$$

Similarly, $T_{mn} = \frac{mn}{mn} = 1$

Hence, $\frac{T_{m+n}}{T_{mn}} = \frac{mn}{m+n}$

Illustration - 22 *If the sum of three numbers in H.P. is 26 and the sum of their reciprocals is 3/8. then the number are :*

(A) 6, 8, 10 **(B)** 6, 9, 12 **(C)** 6, 8, 12 **(D)** 8, 10, 12

SOLUTION : (C)

Three numbers in H.P. are taken as :

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

$$\Rightarrow \quad \frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 26 \quad \text{...(i)}$$

Also $(a-d) + a + (a+d) = 3/8$ ...(ii)

From (i) and (ii)

$$a = \frac{1}{8} \text{ and } d = \pm\frac{1}{24}$$

$\Rightarrow$ The numbers are 12, 8, 6 or 6, 8, 12.

Illustration - 23 *In an H.P., the p^{th} term is qr and q^{th} term is rp. Show that the r^{th} term is pq.*

(A) $\frac{p}{q}$ **(B)** pq **(C)** $\frac{1}{pq}$ **(D)** $\frac{q}{p}$

SOLUTION: (B)

Let A, D be the first term and the common difference of the *A.P.* formed by the reciprocals of given *H.P.*

p^{th} term of A.P. is $\frac{1}{qr}$ and q^{th} term of A.P. is $\frac{1}{rp}$

$$\Rightarrow \quad \frac{1}{qr} = A + (p-1)D$$

and $\frac{1}{rp} = A + (q-1)D$

We will solve these two equation to get *A* and *D*.

Subtracting, we get,

$$\frac{p-q}{pqr} = (p-q)D \quad \Rightarrow \quad D = \frac{1}{pqr}$$

Hence, $\frac{1}{qr} = A + \frac{p-1}{pqr} \quad \Rightarrow \quad A = \frac{1}{pqr}$

Now the r^{th} term of A.P. $= T_r = A + (r-1)D$

$$\Rightarrow \quad T_r = \frac{1}{pqr} + \frac{r-1}{pqr} = \frac{1}{pq}$$

Hence r^{th} term of the given H.P. is *pq*.

Illustration - 24 *If a, b, c are respectively p^{th}, q^{th}, and r^{th} terms of H.P., then :*

$bc(q-r) + ca(r-p) + ab(p-q)$ *is equal to :*

(A) abc (B) 0 (C) pqr (D) None of these

SOLUTION : (B)

Let A and D be the first term and common difference of the A.P. formed by the reciprocals of the given H.P.

$$\Rightarrow \frac{1}{a} = A + (p-1)D \quad \ldots \text{(i)}$$

$$\frac{1}{b} = A + (q-1)D \quad \ldots \text{(ii)}$$

$$\frac{1}{c} = A + (r-1)D \quad \ldots \text{(iii)}$$

Subtracting (iii) from (ii) we get:

$$\frac{c-b}{bc} = (q-r)D$$

$$\Rightarrow bc(q-r) = -\frac{(b-c)}{D}$$

Now, $\sum bc(q-r)$

$$= -\sum \frac{b-c}{D} = -\frac{1}{D}\sum(b-c)$$

$$= -\frac{1}{D}[b-c+c-a+a-b] = 0$$

★

Illustration - 25 *If* $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$, *then which of the following is (are) true?*

(A) $a + c = 2b$ (B) $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$ (C) $\frac{1}{a} - \frac{1}{c} = \frac{2}{b}$ (D) $a - c = 2b$

SOLUTION : (B)

$$\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$$

$$\Rightarrow \left(\frac{1}{a} + \frac{1}{c-b}\right)\left(\frac{1}{c} + \frac{1}{a-b}\right) = 0$$

$$\Rightarrow \frac{a+c-b}{a(c-b)} + \frac{c+a-b}{c(a-b)} = 0$$

$$\Rightarrow (a+c-b)\left\{\frac{1}{a(c-b)} + \frac{1}{c(a-b)}\right\} = 0$$

$$\Rightarrow a(c-b) = -c(a-b) \quad or \quad a+c-b = 0$$

$$\Rightarrow 2ac = ab + bc \quad or \quad a+c = b$$

$$\Rightarrow \frac{2}{b} = \frac{1}{c} + \frac{1}{a} \quad or \quad a+c = 2b$$

$\Rightarrow$ a, b, c are in $H.P.$ or $a + c = b$.

Illustration - 26 *If $a_1, a_2, a_3, ..., a_n$ are in H.P., then $a_1 a_2 + a_2 a_3 + ... + a_{n-1}a_n = k a_1 a_n$ where k is:*

(A) $n-1$ **(B)** n **(C)** 1 **(D)** $1/n$

SOLUTION : (A)

Let D be the common difference of the A.P. corresponding to the given H.P.

$$\Rightarrow \quad \frac{1}{a_n} = \frac{1}{a_1} + (n-1)D \qquad ...(i)$$

Now $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$.......... are in A.P.

$$\Rightarrow \quad \frac{1}{a_2} - \frac{1}{a_1} = D \quad \Rightarrow \quad a_1 a_2 = \frac{a_1 - a_2}{D}$$

and $a_2 a_3 = \frac{a_2 - a_3}{D}$ and so on.

$$\Rightarrow \quad a_{n-1}a_n = \frac{a_{n-1} - a_n}{D}$$

Adding all such expressions we get :

$$\Rightarrow \quad a_1a_2 + a_2a_3 + a_3a_4 + \text{........} a_{n-1}a_n = \frac{a_1 - a_n}{D}$$

$$\Rightarrow \quad a_1a_2 + a_2a_3 + \text{.........} + a_{n-1}a_n = \frac{a_1 - a_n}{D}$$

$$\Rightarrow \quad a_1a_2 + a_2a_3 + \text{..............} + a_{n-1}a_n = \frac{a_1 a_n}{D}\left[(n-1)D\right] \quad \text{[Using (i)]}$$

$$\Rightarrow \quad a_1a_2 + a_2a_3 + \text{.........} + a_{n-1}a_n = (n-1)a_1 a_n$$

Hence, $k = n - 1$

Illustration - 27 *If A and G are arithmetic mean (AM) and geometric mean (GM) between two numbers a and b, then roots of the equation :* $x^2 - 2Ax + G^2 = 0$ *are :*

(A) $a, 2b$ **(B)** $2a, b$ **(C)** a, b **(D)** $2a, 2b$

SOLUTION : (C)

Since A is the A.M. between a and b and G is G.M. between a and b.

$$\Rightarrow \quad A = \frac{a+b}{2} \quad \text{and} \quad G = \sqrt{ab}$$

Equation: $x^2 - 2Ax + G^2 = 0$

$\Rightarrow \quad x^2 - (a+b)x + ab = 0$

$\Rightarrow \quad (x - a)(x - b) = 0$

Hence, roots are a and b.

Illustration - 28 *If a, b, c are in an A.P., x is the GM of a, b and y is GM of b, c, then the AM of x^2 and y^2 is :*

(A) a^2 **(B)** b^2 **(C)** c^2 **(D)** *None of these*

SOLUTION : (B)

a, b, c are in A.P.

$\Rightarrow \quad 2b = a + c \qquad \textbf{. . . (i)}$

x is G.M. of a, b

$\Rightarrow \quad x = \sqrt{ab} \qquad \textbf{. . . (ii)}$

y is G.M. of b, c

$\Rightarrow \quad y = \sqrt{bc} \qquad \textbf{. . . (iii)}$

Squaring (ii) and (iii) and adding, we get :

$\Rightarrow \quad x^2 + y^2 = ab + bc = b(a + c)$

From (i) $\quad a + c = 2b$

$$\Rightarrow \quad x^2 + y^2 = 2b^2 \quad \Rightarrow \quad b^2 = \frac{x^2 + y^2}{2}$$

Hence b^2 is arithmetic mean (AM) of x^2 and y^2.

SUMMATION OF SERIES Section - 4

4.1 Sum of first *n* natural numbers

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \text{...........} + n$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

4.2 Sum of squares of first *n* natural numbers

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^3 + 3^3 + \text{............} + n^2$$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

4.3 Sum of cubes of first *n* natural numbers:

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \text{.............} + n^3$$

$$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4.4 Arithmetico-Geometric Series :

A series in which each term is the product of corresponding terms in an arithmetic and geometric series.

The general expression for such a series :

$$a, (a+d)\,r, (a+2d)\,r^2, (a+3d)\,r^3, \text{............}$$

Note **:** a : first term of A.P. d : common difference of A.P. r : common ratio of $G.P.$

k^{th} term of such a series : $T_k = [a + (k-1)\,d]\, r^{k-1}$

Sum of n terms of arithmetico - geometric series :

Let S = sum of series.

$$S = a + (a+d)\,r + (a+2d)\,r^2 + \text{........} + [a + (n-1)\,d]\, r^{n-1} \quad \text{. . . (i)}$$

Multiply each term by r (on both the sides) and write as follows.

$$rS = 0 + ar + (a+d)\,r^2 + (a+2d)\,r^3 + \text{....} + [a + (n-1)\,d]\, r^n \quad \text{. . . (ii)}$$

Operating (i) - (ii) we get :

$\Rightarrow \quad S(1-r) = a + dr + dr^2 + dr^3 + \ldots\ldots + dr^{n-1} - [a + (n-1)d]\, r^n$

$\Rightarrow \quad S(1-r) = a + d(r + r^2 + r^3 + \ldots\ldots + r^{n-1}) - [a + (n-1)d]\, r^n$

$$\Rightarrow \quad S(1-r) = a + \frac{dr\left(1-r^{n-1}\right)}{1-r} - [a + (n-1)d]\, r^n$$

$$\Rightarrow \quad S = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]\, r^n}{1-r}$$

Sum of an infinite Arithmetico - Geometric series

If $|r| < 1$ i.e., $-1 < r < 1$ then, $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$

Illustration - 29 *The sum of the series :* 1.2 + 2.3 + 3.4 + ... *upto n terms is :*

(A) $\dfrac{n(n+2)(n+3)}{6}$ **(B)** $\dfrac{n(n+1)(n+2)}{3}$ **(C)** $\dfrac{n(n+1)(n+2)}{6}$ **(D)** $\dfrac{n(n+2)(n+3)}{3}$

SOLUTION : (B)

In such questions, find out n^{th} term and try to use the formulae for $\sum r,\ \sum r^2,\ \sum r^3$.

In the given problem, the n^{th} term is

$n(n+1) = n^2 + n$

$T_1 = 1^2 + 1,\ T_2 = 2^2 + 2$

$T_3 = 3^2 + 3$ and so on......

Let S_n denotes the sum of n terms.

$S_n = T_1 + T_2 + T_3 + \ldots\ldots\ldots + T_n$

$\Rightarrow \quad S_n = (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + \ldots\ldots\ldots\ldots + (n^2 + n)$

$\Rightarrow \quad S_n = (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2) + (1 + 2 + 3 + \ldots\ldots\ldots\ldots + n)$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$S_n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$S_n = \frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 1\right] = \frac{n(n+1)(n+2)}{3}$$

Illustration - 30 *The sum of the series* : $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots$ *n terms is* :

(A) $\frac{n}{3}(4n^2-1)$ (B) $\frac{n^2}{3}(4n-1)$ (C) $\frac{n^2}{3}(4n+1)$ (D) $\frac{n}{3}(4n^2+1)$

SOLUTION : (A)

First try to determine the n^{th} term of the given expression.

Note that each term of the series is equal to the square of the corresponding term of series,

1, 3, 5, 7, ...

So the n^{th} term of the given series should be the square of the n^{th} term of the series

1, 3, 5, 7 .....

The n^{th} term of the series 1, 3, 5, 7 ....is $(2n-1)$.

$\Rightarrow \quad T_n = (2n-1)^2 = 4n^2 - 4n + 1$

Now $\quad T_1 = 4(1^2) - 4(1) + 1$

$T_2 = 4(2^2) - 4(2) + 1$

$\vdots \quad \vdots \quad \vdots$

$T_n = 4(n^2) - 4(n) + 1$

On adding, we get :

$S_n = T_1 + T_2 + T_3 + \ldots\ldots\ldots\ldots + T_n$

$S_n = 4(1^2 + 2^2 + 3^3 + \ldots\ldots\ldots\ldots + n^2)$

$- 4(1 + 2 + 3 + \ldots\ldots\ldots\ldots + n)$

$+(1 + 1 + 1 + \ldots\ldots\ldots\ldots \; n \text{ times})$

$$S_n = 4\sum_{r=1}^{n} n^2 - 4\sum_{r=1}^{n} r + n$$

$$S_n \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$S_n = \frac{n}{3}\left(4n^2-1\right)$$

Illustration - 31 *The sum of the series* : $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots\ldots\ldots$ *to n terms is* :

(A) $\frac{n(n+1)(n+2)}{12}$ (B) $\frac{n(n+1)(n+2)^2}{12}$ (C) $\frac{n(n+1)^2(n+2)}{12}$ (D) $\frac{n^2(n+1)(n+2)}{12}$

SOLUTION : (C)

First determine the r^{th} term.

$$\Rightarrow \quad T_r = (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + r^2) \quad \Rightarrow \quad T_r = \sum_{r=1}^{r} r^2 = \frac{n(n+1)(2n+1)}{6} \quad \Rightarrow \quad T_r = \frac{1}{3}r^3 + \frac{1}{2}r^2 + \frac{1}{6}r$$

Now
$$S_n = \sum_{r=1}^{n} T_r = \frac{1}{3}\sum_{r=1}^{n} r^3 + \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{6}\sum_{r=1}^{n} r = \frac{1}{3}\frac{n^2(n+1)^2}{4} + \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\frac{n(n+1)}{2}$$

Simplify to get :
$$S_n = \frac{n(n+1)^2(n+2)}{12}$$

Illustration - 32 *The Sum the nth term of the series :* 3.7 + 5.10 + 7.13 + ... *to n terms is :*

(A) $\frac{n}{2}\left(4n^2+17n+21\right)$ (B) $\frac{n^2}{4}\left(4n+17n+19\right)$ (C) $\frac{n}{2}\left(4n^2+17n+19\right)$ (D) $\frac{n}{4}\left(4n^2+17n+21\right)$

SOLUTION : (A)

As usual, find the r^{th} term. Note that 3, 5, 7, ... i.e., the first number of each term is in A.P. and 7, 10, 13, . . . i.e. second number of each term is also in *A.P.*

$\Rightarrow \quad T_r = (2r + 1)(3r + 4)$

[product of *r* terms of two A.P.(s)]

$\Rightarrow \quad T_r = 6r^2 + 11r + 4$

Taking summation of both sides,

$$S_n = \sum_{r=1}^{n} T_r = 6\sum_{r=1}^{n} r^2 + 11\sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 1$$

$$\Rightarrow \quad S_n = \frac{6n(n+1)(2n+1)}{6} + \frac{11n(n+1)}{2} + 4n$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left(4n^2 + 17n + 21\right)$$

Illustration - 33 *Sum the series :* $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots\ldots\ldots$

(i) *to n terms*

(A) $\frac{1}{1-x} + \frac{2x\left(1-x^n\right)}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$ (B) $\frac{1}{1-x} + \frac{2x\left(1-x^{n-1}\right)}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$

(C) $\frac{1}{1-x} + \frac{2x\left(1-x^n\right)}{(1-x)^2} + \frac{(2n-1)x^n}{1-x}$ (D) $\frac{1}{1-x} + \frac{2x\left(1-x^{n-1}\right)}{(1-x)^2} + \frac{(2n-1)x^n}{1-x}$

(ii) *to infinity* *if* (| *x* | < 1)

(A) $\frac{1+x}{1-x}$ (B) $\frac{1-x}{1+x}$ (C) $\frac{1+x}{(1-x)^2}$ (D) $\frac{1-x}{(1-x)^2}$

SOLUTION : (i) - (B) (ii) - (C)

Note that the given series is an Arithmetico-Geometric series.

1, 3, 5, ... are in A.P.

$\Rightarrow \quad T_r = 2r - 1$

$1, x, x^2$, ... are in G.P.

$\Rightarrow \quad T_r = x^{r-1}$

(a) This means that r^{th} term of A-G series $= (2r - 1)\,x^{r-1}$

$S_n = 1 + 3x + 5x^2 + \ldots\ldots\ldots + (2n-3)\,x^{n-2} + (2n-1)\,x^{n-1}$ . . . (i)

$xS_n = x + 3x^2 + 5x^3 + \ldots\ldots\ldots + (2n-3)\,x^{n-1} + (2n-1)\,x^n$ . . . (ii)

$\Rightarrow \quad (1-x)\,S_n = 1 + 2x + 2x^2 + \ldots\ldots\ldots + 2x^{n-1} - (2n-1)\,x^n$

$$\Rightarrow \quad (1-x)S_n = 1 + \frac{2x\left(1-x^{n-1}\right)}{1-x} - (2n-1)\,x^n$$

$$\Rightarrow \quad S_n = \frac{1}{1-x} + \frac{2x\left(1-x^{n-1}\right)}{\left(1-x^2\right)} - \frac{(2n-1)\,x^n}{1-x}$$

(b) $S_\infty = 1 + 3x + 5x^2 + \ldots\ldots\ldots\ldots\ldots\text{to } \infty$

$xS_\infty = x + 3x^2 + 5x^3 + \ldots\ldots\ldots\ldots\ldots\text{to } \infty$

$\Rightarrow \quad (1-x)\,S_\infty = 1 + 2x + 2x^2 + \ldots\ldots\ldots\text{to } \infty$

$\Rightarrow \quad (1-x)\,S_\infty = 1 + 2x\,(1 + x + x^2 + \ldots\ldots \text{ to } \infty)$

$$\Rightarrow \quad (1-x)\,S_\infty = 1 + 2x\left(\frac{1}{1-x}\right) = \frac{1+x}{1-x}$$

$$\Rightarrow \quad S_\infty = \frac{1+x}{(1-x)^2}$$

Illustration - 34 *The sum of the series :* $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots\ldots$ *to* ∞ *terms is :*

(A) $\frac{35}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{7}{16}$ **(D)** $\frac{35}{16}$

SOLUTION : (D)

The given series is arithmetico - geometric series.

Let $S = 1 + \frac{4}{5} + \frac{7}{5^2} + \ldots\ldots + \frac{3n-2}{5^{n-1}}$ . . . (i)

$\frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \ldots\ldots + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$ . . . (ii) [$\because$ Multiply both side $r = \frac{1}{5}$]

Subtracting (i) from (ii)

$$\Rightarrow \quad \frac{4}{5}S_n = 1 + \left(\frac{3}{5} + \frac{3}{5^2} + \ldots\ldots + \frac{3}{5^{n-1}}\right) - \left(\frac{3n-2}{5^n}\right)$$

$$\Rightarrow \quad S_n = \frac{5}{4} + \frac{5}{4} \times \frac{3}{5}\left(\frac{1 - \frac{1}{5^{n-1}}}{1 - \frac{1}{5}}\right) - \left(\frac{3n-2}{5^n} \times \frac{5}{4}\right)$$

$$\Rightarrow \quad S_n = \frac{5}{4} + \frac{3}{4}\left(\frac{5^{n-1}-1}{4}\right)\frac{1}{5^{n-2}} - \frac{3n-2}{4.5^{n-1}}$$

$$\Rightarrow \quad S_n = \frac{5}{4} + \frac{15}{16} - \frac{3}{16.5^{n-2}} - \frac{3n-2}{20.5^{n-2}}$$

$$\Rightarrow \quad S_n = \frac{35}{16} - \left(\frac{12n+7}{80\left(5^{n-2}\right)}\right)$$

$$\text{Now } S_\infty = 1 + \frac{4}{5} + \frac{7}{5^2} + \ldots\ldots\ldots \infty \;;$$

$$\frac{1}{5} S_\infty = \frac{1}{5} + \frac{4}{5^2} + \ldots\ldots\ldots\ldots\ldots \infty$$

$$\Rightarrow \quad \frac{4}{5} S_\infty = 1 + \frac{3}{5} + \frac{3}{5^2} + \ldots\ldots\ldots\ldots\ldots \infty$$

$$\Rightarrow \quad \frac{4}{5} S_\infty = 1 + \frac{3/5}{1 - 1/5} \qquad \Rightarrow \quad S_\infty = \frac{5}{4}\left(1 + \frac{3}{4}\right) = \frac{35}{16}$$

Illustration - 35

The sum of the series : $\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \ldots\ldots\ldots\ldots$ *to n terms is :*

(A) $\frac{n}{12}\left(2n^2 + 9n + 13\right)$ **(B)** $\frac{n}{24}\left(2n^2 + 9n + 13\right)$ **(C)** $\frac{n}{12}\left(2n^2 + 6n + 13\right)$ **(D)** $\frac{n}{24}\left(n^2 + 6n + 13\right)$

SOLUTION : (B)

$$T_r = \frac{1^3 + 2^3 + \ldots\ldots\ldots\ldots r^3}{1 + 3 + 5 + \ldots\ldots\text{to } r \text{ terms}}$$

$$T_r = \frac{\sum r^3}{\frac{r}{2}\left[2 + (r-1)\,2\right]} = \frac{(r+1)^2}{4}$$

$$\Rightarrow \quad T_r = \frac{r^2 + 2r + 1}{4}$$

$$S_n = \sum_{r=1}^{r=n} T_r = \frac{1}{4}\left[\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + n(n+1) + n\right]$$

$$= \frac{n}{24}\left[2n^2 + 3n + 1 + 6n + 6 + 6\right]$$

$$\Rightarrow \quad S_n = \frac{n}{24}\left[2n^2 + 9n + 13\right]$$

Illustration - 36 *Find the sum of the products of every pair of the first n natural numbers .*

(A) $\dfrac{n(n+1)^2(3n-2)}{24}$ (B) $\dfrac{n(n+1)(3n-2)(n-1)}{24}$

(C) $\dfrac{n(n+1)(3n+2)(n-1)}{24}$ (D) $\dfrac{n(n+1)^2(3n+2)}{24}$

SOLUTION : (C)

The required sum is given as follows.

$S = 1.2 + 1.3 + 1.4 + \ldots\ldots + 2.3 + 2.4 + \ldots\ldots$

$+ 3.4 + 3.5 + \ldots\ldots + \ldots\ldots + (n-1)\,n$

Using : $S = \dfrac{\left(\sum n\right)^2 - \sum n^2}{2}$, we get :

$$S = \frac{1}{2}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{n(n+1)}{24}\left[3n(n+1) - 2(2n+1)\right]$$

$$= \frac{n(n+1)}{24}\left[3n^2 - n - 2\right]$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(3n+2)(n-1)}{24}$$

Illustration - 37 *Sum the series to n terms : 4 + 44 + 444 + 4444 + ... is :*

(A) $\dfrac{4}{81}\left[10(10^n - 1) - 9n\right]$ (B) $\dfrac{4}{81}\left[10(10^n + 1) - 9n\right]$

(C) $\dfrac{4}{81}\left[10(10^n - 1) + 9n\right]$ (D) $\dfrac{4}{81}\left[10(10^n + 1) + 9n\right]$

SOLUTION : (A)

Let $S_n = 4 + 44 + 444 + 4444$ + ... to n terms.

$\Rightarrow \quad S_n = 4\,(1 + 11 + 111 + 1111$ + ... n terms)

$\Rightarrow \quad S_n = 4/9\ \{(10 - 1) + (100 - 1)$ + (1000 – 1) + ... n terms}

$\Rightarrow \quad S_n = 4/9\ \{\ (10 + 10^2 + 10^3 +$ .... n terms) – (1 + 1 + 1 + ... n terms)}

$$S_n = \frac{4}{9}\left(\frac{10\left(10^n - 1\right)}{10 - 1} - n\right)$$

$$\Rightarrow \quad S_n = \frac{4}{81}\left[10\left(10^n - 1\right) - 9n\right]$$

Illustration - 38 *If a, b, c are distinct such that $ab + bc + ca \neq 0$ and in A.P., then $a^2(b + c), b^2(c + a), c^2(a + b)$ are in :*

(A) A.P. **(B)** G.P. **(C)** H.P. **(D)** *None of these*

SOLUTION : (A)

As a, b, c are in A.P., then $a(ab + bc + ca)$, $b(ab + bc + ca)$, $c(ab + bc + ca)$ are in A.P.

$\Rightarrow$ $a^2(b + c) + abc$, $b^2(a + c) + abc$, $c^2(a + b) + abc$ are in A.P.

$\Rightarrow$ $a^2(b + c) + abc - abc$, $b^2(a + c) + abc - abc$, $c^2(a + b) + abc - abc$ are in A.P.

Hence, $a^2(b + c)$, $b^2(a + c)$, $c^2(a + b)$ are in A.P.

Alternate Approach :

As a, b, c are in A.P., we get : $a - b = b - c$ **. . . (i)**

Consider $a^2(b + c) - b^2(c + a) = (a^2b - b^2a) + (a^2c - b^2c) = (a - b)(ab + ac + bc)$ **. . . (ii)**

Also $b^2(c + a) - c^2(a + b) = (b^2c - c^2b) + (b^2a - c^2a) = (b - c)(bc + ba + ca)$ **. . . (iii)**

Form (i), (ii), (iii) we get :

$$a^2(b + c) - b^2(c + a) = b^2(c + a) - c^2(a + b)$$

$\Rightarrow$ $a^2(b + c), b^2(c + a), c^2(a + b)$ are in A.P.

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Illustration - 39 *If $\sqrt[x]{a} = \sqrt[y]{b} = \sqrt[z]{c}$ and if a, b, c are in G.P., then x, y, z are in :*

(A) A.P. **(B)** G.P. **(C)** H.P. **(D)** *None of these*

SOLUTION : (A)

Let $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$

$$\Rightarrow \frac{\log a}{x} = \frac{\log b}{y} = \frac{\log c}{z} = k$$

$\Rightarrow$ $\log a = kx$, $\log b = ky$, $\log c = kz$ $[\because b^2 = ac]$

$\Rightarrow$ $2\log b = \log a + \log c$ $[\because b^2 = ac]$

We have $2ky = kx + kz$

$\Rightarrow$ $2y = x + z$ $\Rightarrow$ x, y, z are in A.P.

Illustration - 40 *At what values of parameter a are there values of x such that the numbers :*

$(5^{1+x} + 5^{1-x}), (a/2), (25^x + 25^{-x})$ *form an A.P. ?*

(A) $a \geq 18$ **(B)** $a \geq 12$ **(C)** $a \geq 15$ **(D)** $a \leq 3$

SOLUTION : (B)

For the given numbers to be in A.P.,

$$2\left(\frac{a}{2}\right) = 5^{1+x} + 5^{1-x} + 25^x + 25^{-x}$$

Let $5^x = k$

$$\Rightarrow \quad a = 5k + \frac{5}{k} + k^2 + \frac{1}{k^2}$$

$$\Rightarrow \quad a = 5\left(k + \frac{1}{k}\right) + \left(k^2 + \frac{1}{k^2}\right)$$

As the sum of a positive number and its reciprocal is always greater than or equal to 2,

$$k + \frac{1}{k} \geq 2 \quad \text{and} \quad k^2 + \frac{1}{k^2} \geq 2$$

Hence $a \geq 5(2) + 2 \quad \Rightarrow \quad a \geq 12$

Illustration - 41 *The series of natural numbers is divided into groups :* (1) ; (2, 3, 4) ; (5, 6, 7, 8, 9) *and so on. The sum of the numbers in the* n^{th} *group is :*

(A) $n^3 + (n+1)^3$ **(B)** $(n-1)^3$ **(C)** n^3 **(D)** $n^3 + (n-1)^3$

SOLUTION : (C)

Note that the last term of each group is the square of a natural number.

Hence first term in the n^{th} group is $= (n-1)^2 + 1 = n^2 - 2n + 2$

There is 1 term in **I**st group, 3 terms in **II**nd, 5 terms in **III**rd, 7 terms in **IV**th,..............

No. of terms in the n^{th} group $= n^{th}$ term of (1, 3, 5, 7.....) $= 2n - 1$

Common difference in the n^{th} group = 1

$$\text{Sum} = \frac{2n-1}{2}\left[2\left(n^2 - 2n + 2\right) + (2n-2)1\right]$$

$$= \frac{2n-1}{2}\left[2n^2 - 2n + 2\right]$$

$$= (2n-1)\left(n^2 - n + 1\right)$$

$$= 2n^3 - 3n^2 + 3n - 1 = n^3 + (n-1)^3$$

Illustration - 42 *If a, b, c are in G.P., and the equations* $ax^2 + 2bx + c = 0$ *and* $dx^2 + 2ex + f = 0$ *have a common root then d/a , e/b , f/c are in :*

(A) A.P. **(B)** G.P. **(C)** H.P. **(D)** None of these

SOLUTION : (A)

a, b, c are in $G.P.$ $\Rightarrow$ $b^2 = ac$

Hence the first equation has equal roots because its discriminant $= 4b^2 - 4ac = 0$.

The roots are $x = \dfrac{-2b}{2a} = -\dfrac{b}{a}$.

As the two equations have a common roots, $-b/a$ is also a root of the second equation .

$$\Rightarrow \quad d\left(-\frac{b}{a}\right)^2 + 2e\left(-\frac{b}{a}\right) + f = 0$$

$$\Rightarrow \quad db^2 - 2abe + a^2f = 0$$

Dividing by ab^2

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{a^2 f}{ab^2} = 0$$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{a^2 f}{a(ac)} = 0$$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0$$

$$\Rightarrow \quad \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

Illustration - 43 *If the first and the last term of an A.P., a G.P. and a H.P. are equal and their* n^{th} *terms are a, b and c respectively, then, find the correct choice(s) : [Given : Total number of terms = 2n - 1]*

(A) $a = b = c$ **(B)** $a \geq b \geq c$ **(C)** $a + c = b$ **(D)** $ac - b^2 = 0$

SOLUTION : (B D)

Let the first term $= A$

The last term $= [(2n-1)^{th}$ term$] = L$

No. of terms $= 2n - 1$ i.e. odd number

Middle term $= \dfrac{(2n-1)+1}{2} = n^{th}$ term

$\Rightarrow$ T_n is the middle term for all the three progressions. In an $A.P.$ the middle term is the arithmetic mean of first and the last terms.

$$\Rightarrow \quad a = \frac{A+L}{2}$$

In a G.P. the middle term is the geometric mean of first and last terms.

$$\Rightarrow \quad b = \sqrt{AL}$$

In an H.P. the middle term is the harmonic mean of first and last terms.

$$\Rightarrow \quad c = \frac{2\,AL}{A+L}$$

Hence a, b, c are AM, GM & HM between the numbers A and L.

As $(GM)^2 = (AM)\,(HM)$

We have $b^2 = ac$

Hence, (B) and (D) are the correct choices.

★

Illustration - 44 *Find the sum of n terms of series :* $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots\ldots\ldots$

(A) $\frac{1}{x+y}\left[\frac{x^2(1-x^n)}{1-x}\right] - \frac{1}{x+y}\left[\frac{y^2(1-y^n)}{1-y}\right]$

(B) $\frac{1}{x-y}\left[\frac{x(1-x^n)}{1-x}\right] - \frac{1}{x-y}\left[\frac{y(1-y^n)}{1-y}\right]$

(C) $\frac{1}{x+y}\left[\frac{x(1-x^n)}{1-x}\right] - \frac{1}{x+y}\left[\frac{y(1-y^n)}{1-y}\right]$

(D) $\frac{1}{x-y}\left[\frac{x^2(1-x^n)}{1-x}\right] - \frac{1}{x-y}\left[\frac{y^2(1-y^n)}{1-y}\right]$

SOLUTION : (D)

Let $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots\ldots$

$$S_n = \frac{x^2-y^2}{x-y} + \frac{x^3-y^3}{x-y} + \frac{x^4-y^4}{x-y} + \ldots\ldots\ldots$$

$$= \frac{1}{x-y}\left(x^2 + x^3 + x^4 + \ldots\ldots n \text{ terms}\right) - \frac{1}{x-y}\left(y^2 + y^3 + y^4 + \ldots\ldots n \text{ terms}\right)$$

$$= \frac{1}{x-y}\left(\frac{x^2\left(1-x^n\right)}{1-x}\right) - \frac{1}{x-y}\left(\frac{y^2\left(1-y^n\right)}{1-y}\right)$$

Note : The following results can be very useful.

(i) $\frac{x^n - y^n}{x-y} = x^{n-1}y^0 + x^{n-2}y^1 + x^{n-3}y^2 + \ldots\ldots + xy^{n-2} + x^0y^{n-1}$ (n is any natural number)

(ii) $\frac{x^n + y^n}{x+y} = x^{n-1}y^0 - x^{n-2}y^1 + x^{n-3}y^2 - x^{n-4}y^3 + \ldots\ldots + x^0y^{n-1}$ (n is odd natural number)

Illustration - 45

Find the sum of first n terms of the series : $3 + 7 + 13 + 21 + 31 + ...$

(A) $\frac{n}{2}\left(n^2+3n+5\right)$ (B) $\frac{n}{4}\left(n^2+3n+5\right)$ (C) $\frac{n}{3}\left(n^2+3n+5\right)$ (D) $\frac{n}{6}\left(n^2+3n+5\right)$

SOLUTION : (C)

The given series is neither an *A.P.* nor a *G.P.*, but the differences of the successive terms are in *A.P.*

Series : 3 7 13 21 31.......

Differences : 4 6 8 10................

In such cases, we find the n^{th} term as follows:

Let S be the sum of the first n terms.

$S = 3 + 7 + 13 + 21 + 31 + ...+ T_n$

$S = 3 + 7 + 13 + 21 + 31 + ...+ T_{n-1} + T_n$

On subtracting, we get :

$0 = 3 + \{4 + 6 + 8 + 10 + ...\} - T_n$

$\Rightarrow T_n = 3 + \{4 + 6 + 8 + 10 + ...(n-1)\text{ terms}\}$

$\Rightarrow T_n = 3 + \frac{n-1}{2}\left[2(4) + (n-2)2\right]$

$\Rightarrow T_n = n^2 + n + 1$

$\Rightarrow S = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$

$\Rightarrow S = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$

$= \frac{n}{3}\left(n^2 + 3n + 5\right)$

Illustration - 46

Find the series : $1 + 4 + 10 + 22 + 46 + ...$ *to n terms .*

(A) $3.\,2^n + 2n - 3$ (B) $3.\,2^n + 2n + 3$ (C) $3.\,2^n - 2n + 3$ (D) $3.\,2^n - 2n - 3$

SOLUTION : (D)

The differences of successive terms are in G.P. :

Series : 1 4 10 22 46..........

Differences : 3 6 12 24..................

Let S = sum of first n terms.

$\Rightarrow S = 1 + 4 + 10 + 22 + 46 + ... + T_n$

$\Rightarrow S = 1 + 4 + 10 + 22 + 46 + ... + T_{n-1} + T_n$

On subtracting, we get :

$0 = 1 + \{3 + 6 + 12 + 24 + ...\} - T_n$

$T_n = 1 + \{3 + 6 + 12 + 24 + ... + (n-1)\text{ terms}\}$

$\Rightarrow T_n = 1 + \frac{3\left(2^{n-1} - 1\right)}{2-1} = 3.2^{n-1} - 2$

$\Rightarrow S = \sum_{k=1}^{n} T_k = \frac{3}{2}\sum 2^k - \sum 2$

$S = \frac{2}{3}\left(2 + 4 + 8 + ... + 2^n\right) - 2n$

$= \frac{3}{2}\,\frac{2\left(2^n - 1\right)}{2-1} - 2n = 3.2^n - 2^n - 3$

★

Illustration - 47 *Find the sum of the series :*

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots\ldots \text{ to } n \text{ terms.}$$

(A) $\frac{1}{3n-1}$ **(B)** $\frac{n}{3n+1}$ **(C)** $\frac{n}{3n-1}$ **(D)** $\frac{1}{3n+1}$

SOLUTION : (B)

$$\text{Let } S=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots\ldots+\frac{1}{(3n-2)(3n+1)}$$

$$\Rightarrow \quad 3S=\frac{3}{1.4}+\frac{3}{4.7}+\frac{3}{7.10}+\ldots\ldots\ldots\ldots+\frac{3}{(3n-2)(3n+1)}$$

$$\Rightarrow \quad 3S=\frac{4-1}{1.4}+\frac{7-4}{4.7}+\frac{10-7}{7.10}+\ldots\ldots\ldots\ldots+\frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)}$$

$$\Rightarrow \quad 3S=\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\ldots\ldots\ldots+\left(\frac{1}{(3n-2)}-\frac{1}{3n+1}\right)$$

$$\Rightarrow \quad S=\frac{1}{1}-\frac{1}{3n+1} \qquad \Rightarrow \quad S=\frac{n}{3n+1}$$

Note : The above method works in the case when the n^{th} term of a series can be expressed as the difference of the two quantities of the type : $T_n=f(n)-f(n-1)$ or $T_n=f(n)-f(n+1)$

In the above example, $T_n=\frac{1}{(3n-2)(3n+1)}=\frac{1}{3}\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)$

It is the form $f(n)-f(n+1)$.

★ **Illustration - 48** *Find the sum of first n terms of the series :*

$1(1)! + 2(2)! + 3(3)! + 4(4)! + ...$

(A) $n!-1$ **(B)** $(n+1)!+1$ **(C)** $(n+1)!-1$ **(D)** $n!+1$

SOLUTION : (C)

The n^{th} term, $T_n = n(n)!$

T_n can be written as

$$T_n = (n + 1 - 1)(n)!$$

$\Rightarrow \quad T_n = (n + 1)! - (n)! \quad$ **. . . (i)**

This is in the form $f(n) - f(n-1)$

$$S = \sum_{n=1}^{n} T_n = T_1 + T_2 + T_3 + T_4 + ...+ T_n$$

Using (i),

$$S = (2! - 1!) + (3! - 2!) + (4! - 3!) + ... + \{(n + 1)! - n!\}$$

$\Rightarrow \quad S = -1! + (n + 1)!$

$\Rightarrow \quad S = (n + 1)! - 1.$

THINGS TO REMEMBER

1. If T_r represents the general term of an $A.P.$ (or rth term), then

$$T_r = a + (n-1)\,d \qquad \text{where } r \in \{1,2,3..........,n\}$$

It is also denoted by a_r. If the total number of terms be n, then n^{th} term is also known as the last term of $A.P.$ and is denoted by l, *i.e.*, $\quad l = T_n = a + (n-1)\,d$

In an $A.P.$, the difference of any two consecutive terms is d and is given by :

$$d = T_r - T_{r-1}$$

2. Consider n terms of an $A.P.$ with first term as a and the common difference as d. Let S_n denotes the sum of the first n terms, then $\quad S_n = \dfrac{n}{2}\left[2a + (n-l)\,d\right]$

or $\quad S_n = \dfrac{n}{2}(a + l) \quad$ as $\quad l = a + (n - l)\,d$

3. Arithmetic mean (A) :

When three quantities are in $A.P$, then the middle one is called as *arithmetic mean* of other two. If a and b are two numbers and A be the arithmetic mean of a and b, then a, A, b are in $A.P.$

$$\Rightarrow \quad A - a = b - A \qquad \Rightarrow \quad A = \frac{a+b}{2}$$

4. Important Points

(i) If a, b, c are in A.P., then : $ak, \; bk, \; ck$ are also in A.P. $(k \neq 0)$

$a/k, \; b/k, \; c/k$ are also in A.P. $(k \neq 0)$

$a \pm k, \; b \pm k, \; c \pm k$ are also in A.P.

(ii) Three terms in an A.P. are taken as : $a-d, \; a, \; a+d$

(iii) Four terms in an A.P. are taken as : $a-3d, a-d, a+d, a+3d$

(iv) The sum of any two terms (of an A.P.) equidistant from beginning and end is equal to the sum of the first and the last term.

$$(a + md) + (\ell - md) = a + \ell$$

5. In G.P., the common ratio (r) is the ratio of any two consecutive terms. If T_k represents the general term (or k^{th} term) of a G.P., then $\quad r = \dfrac{T_k}{T_{k-1}} \quad$ and $\quad T_k = a\,r^{k-1} \qquad$ where $k \in \{1,2,..........n\}$

6. Sum of *n* terms of a G.P.

Consider n terms of a G.P. with a as the first term and r as the common ratio. Let S_n denote the sum of n terms.

Then $S_n = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots + ar^{n-1}$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

7. Infinite Geometric series :

If for G.P., the common ratio r lies between – 1 and 1 i.e., $-1 < r < 1$, it is called as *decreasing geometric series*. The sum of the infinite terms of such a series is denoted by S_∞ and is given as

$$S_\infty = \frac{a}{1-r}$$ [where a is the first term and r is the common ratio]

Note : $$S_\infty = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

If $n \to \infty$ then $r^n \to 0$ for $-1 < r < +1$ $\Rightarrow$ $S_\infty = \dfrac{a}{1-r}$

8. Important Points

(i) If a, b, c are in G.P., then : $ak,\ bk,\ ck$ are also in G.P. $(k \neq 0)$

$a/k,\ b/k,\ c/k$ are also on G.P. $(k \neq 0)$

(ii) Three terms in a G.P. are taken as : $ar,\ a,\ a/r$

(iii) Four terms in a G.P. are taken as : $ar^3,\ ar,\ a/r,\ a/r^3$

(iv) If a, b, c, d are in G.P., then : $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$ $\Rightarrow$ $b^2 = ac$, $c^2 = bd$, $ad = bc$

Also, $b = ar,\ c = ar^2$, $d = ar^3$ where r is the common ratio

9.

A series a_1, a_2, a_3, a_4, ... is said to be a *harmonical progression* (H.P.) if

$\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \dfrac{1}{a_4}$ forms an arithmetical progression

i.e., $$\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \frac{1}{a_4} - \frac{1}{a_3} = \ldots\ldots\ldots\ldots$$

Three quantities a, b, c are said to be in H.P., when

$\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P. $\Rightarrow$ $\dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b}$ $\Rightarrow$ $\dfrac{a-b}{ab} = \dfrac{b-c}{bc}$ $\Rightarrow$ $\dfrac{a}{c} = \dfrac{a-b}{b-c}$

10. Harmonic mean(H)

When three quantities are in H.P., the middle one is called as the *harmonic mean* between the other two.

If a and b are two numbers, H is the harmonic mean between the two, then a, H, b must be in H.P.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ must be in $A.P.$ $\quad \Rightarrow \quad \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$

$\Rightarrow \quad \frac{2}{H} = \frac{1}{a} + \frac{1}{b} \quad \Rightarrow \quad H = \frac{2ab}{a+b}$

11. Relation among A, G, H

If A, G, H are arithmetic, geometric and harmonic means respectively between two positive numbers a and b, then $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$ $\Rightarrow$ $G^2 = AH$ i.e. G is the geometric mean between A and H.

Also G lies between A and H, such that $A > G > H$

12. Sum of first *n* natural numbers :

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + ... + n \quad \Rightarrow \quad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

13. Sum of squares of first *n* natural numbers

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^3 + 3^3 + ... + n^2 \quad \Rightarrow \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

14. Sum of cubes of first *n* natural numbers

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 \quad \Rightarrow \quad \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

15. Arithmetico-Geometric Series :

A series in which each term is the product of corresponding terms in an arithmetic and geometric series.

The general expression for such a series : $a, (a + d)\, r, (a + 2d)\, r^2, (a + 3d)\, r^3, ...$

Note : a : first term of A.P. $\quad d$: common difference of A.P. $\quad r$: common ratio of $G.P.$

k^{th} term of such a series : $T_k = [a + (k - 1)\, d]\, r^{k-1}$

Sum of n terms of arithmetico - geometric series :

$$S = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{\left[a+(n-1)d\right]r^n}{1-r}$$

Sum of an infinite Arithmetico - Geometric series

When $|r| < 1$ i.e., $-1 < r < 1$ $\Rightarrow$ $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$