CHAPTER – 9

DIFFERENTIAL EQUATIONS

Miscellaneous Exercise

Question 1 A: For each of the differential equations given below, indicate its order and degree (if defined).

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$$

Answer:

It is given that equation is $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y - \log x = 0$$

We can see that the highest order derivative present in the differential is $\frac{d^2y}{dx^2}$.

Thus, its order is two. It is polynomial equation in $\frac{d^2y}{dx^2}$. The highest power raised to $\frac{d^2y}{dx^2}$ is 1.

Therefore, its degree is one.

Question 1 B: For each of the differential equations given below, indicate its order and degree (if defined).

$$\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$$

Answer:

It is given that equation is $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$

$$\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y - \sin x = 0$$

We can see that the highest order derivative present in the differential is $\frac{dy}{dx}$.

Thus, its order is one. It is polynomial equation in $\frac{dy}{dx}$The highest power raised to $\frac{dy}{dx}$ is 3.

Therefore, its degree is three.

Question 1 C: For each of the differential equations given below, indicate its order and degree (if defined).

$$\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

Answer:

It is given that equation is $\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y - \log x = 0$$

We can see that the highest order derivative present in the differential is $\frac{d^4y}{dx^4}$.

Thus, its order is four. The given differential equation is not a polynomial equation.

Therefore, its degree is not defined.

Question 2 A: For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

$xy = ae^x + be^{-x} + x^2$; $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

Answer:

It is given that xy = $ae^x + b\,e^{-x} + x^2$

Now, differentiating both sides w.r.t. x, we get,

$\frac{dy}{dx} = a\frac{d}{dx}(e^x) + b\frac{d}{dx}(e^{-x}) + \frac{d}{dx}(x^2)$

$\Rightarrow \frac{dy}{dx} = ae^x + be^{-x} + 2x$

Now, Again differentiating both sides w.r.t. x, we get,

$\frac{d}{dx}(y') = \frac{d}{dx}(ae^x + be^{-x} + 2x)$

$\Rightarrow \frac{d^2y}{dx^2} = ae^x + be^{-x} + 2$

Now, Substituting the values of $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equations, we get,

LHS = $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2$

= x $(ae^x + be^{-x} + 2) + 2(ae^x - be^{-x} + 2) - x(ae^x + be^{-x} + x^2) + x^2 - 2$

= $(axe^x + bxe^{-x} + 2x) + 2(ae^x - be^{-x} + 2) - x(ae^x + be^{-x} + x^2) + x^2 - 2$

= $2ae^x - 2be^{-x} + x^2 + 6x - 2$

$\neq 0$

$\Rightarrow$ LHS $\neq$ RHS.

Therefore, the given function is not the solution of the corresponding differential equation.

Question 2 B: For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

$y = e^x(a\cos x + b\sin x); \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Answer:

It is given that $= e^x(a\cos x + b\sin x) = ae^x\cos x + be^x\sin x$

Now, differentiating both sides w.r.t. x, we get,

$\frac{dy}{dx} = a\frac{d}{dx}(e^x\cos x) + b\frac{d}{dx}(e^x\sin x)$

$\Rightarrow \frac{dy}{dx} = a(e^x\cos x - e^x\sin x) + b(e^x\sin x - e^x\cos x)$

$\Rightarrow \frac{dy}{dx} = (a+b)e^x\cos x + (b-a)e^x\sin x$

Now, again differentiating both sides w.r.t. x, we get,

$\frac{d^2y}{dx^2} = (a+b).\frac{d}{dx}(e^x\cos x) + (b-a)\frac{d}{dx}(e^x\sin x)$

$= (a+b).[e^x\cos x - e^x\sin x] + (b-a)[e^x\sin x - e^x\cos x]$

$= e^x[a\cos x - a\sin x + b\cos x - b\sin x + b\sin x + b\cos x$

$-a\sin x - a\cos x]$

$= [2e^x(b\cos x - a\sin x)]$

Now, Substituting the values of $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equations, we get,

LHS $= \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y$

$=2e^x(b\cos x - a\sin x) - 2e^x[(a+b)\cos x + (b-a)\sin x] + 2e^x(a\cos x + b\sin x)$

$= e^x$ [(2bcosx – 2asinx) - (2acosx + 2bcosx) - (2bsinx – 2asinx) + (2acosx + 2bsinx)]

$= e^x$ [(2b – 2a – 2b + 2a) cos x] + e^x [(-2a – 2b + 2a + 2bsinx]

= 0 = RHS.

Therefore, the given function is the solution of the corresponding differential equation.

Question 2 C: For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

$y = x \sin 3x ; \frac{d^2y}{dx^2} + 9y - 6 \cos 3x = 0$

Answer:

It is given that $= y = x \sin 3x$

Now, differentiating both sides w.r.t. x, we get,

$\frac{dy}{dx} = \frac{d}{dx}(x \sin 3x) = \sin 3x + x. \cos 3x . 3$

$\Rightarrow \frac{dy}{dx} = \sin 3x + 3x \cos 3x$

Now, again differentiating both sides w.r.t. x, we get,

$\frac{d^2y}{dx^2} = \frac{d}{dx}(x \sin 3x) + 3\frac{d}{dx}(x \cos 3x)$

$\Rightarrow \frac{d^2y}{dx^2} = 3x \cos 3x + 3[\cos 3x + x(-\sin 3x). 3]$

$\Rightarrow \frac{d^2y}{dx^2} = 6 \cos 3x - 9x \sin 3x$

Now, substituting the value of $\frac{d^2y}{dx^2}$ in the LHS of the given differential equation, we get,

$\frac{d^2y}{dx^2} + 9y - 6\cos 3x$

= (6.cos3x – 9xsin3x) + 9xsin3x – 6cos3x

= 0 = RHS

Therefore, the given function is the solution of the corresponding differential equation.

Question 2 D: For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

$x^2 = 2y^2 \log y\, ; (x^2 + y^2)\frac{dy}{dx} - xy = 0$

Answer:

It is given that $x^2 = 2y^2 \log y$

Now, differentiating both sides w.r.t. x, we get,

$2x = 2.\frac{d}{dx}(y^2 \log y)$

$\Rightarrow x = \left[2y. \log y. \frac{dy}{dx} + y^2. \frac{1}{y}. \frac{dy}{dx}\right]$

$\Rightarrow x = \frac{dy}{dx}(2y \log y + y)$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y(1+2\log y)}$

Now, substituting the value of $\frac{dy}{dx}$ in the LHS of the given differential equation, we get,

$(x^2 + y^2)\frac{dy}{dx} - xy = (2y^2 log + y^2). \frac{x}{y(1+2\log y)} - xy$

$= y^2(1 + 2\log y). \frac{x}{y(1+2\log y)} - xy$

$= xy - xy$

$= 0$

Therefore, the given function is the solution of the corresponding differential equation.

Question 3: Form the differential equation representing the family of curves given by $(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.

Answer:

It is given that $(x - a)^2 + 2y^2 = a^2$

$\Rightarrow x^2 + a^2 - 2ax + 2y^2 = a^2$

$\Rightarrow 2y^2 = 2ax - x^2$ ………. (1)

Now, differentiating both sides w.r.t. x, we get,

$$2y\frac{dy}{dx} = \frac{2a-2x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a-x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2ax-2x^2}{4xy} \quad \text{………… (2)}$$

So, equation (1), we get,

$2ax = 2y^2 + x^2$

On substituting this value in equation (3), we get,

$$\frac{dy}{dx} = \frac{2y^2+x^2-2x^2}{4xy}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2-x^2}{4xy}$$

Therefore, the differential equation of the family of curves is given as $\frac{dy}{dx} = \frac{2y^2 - x^2}{4xy}$.

Question 4: Prove that $x^2 - y^2 = c\,(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, where c is a parameter.

Answer:

It is given that $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \qquad \ldots\ldots\ldots \text{(1)}$$

Now, let us take y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now, substituting the values of y and $\frac{dv}{dx}$ in equation (1), we get,

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x(vx)^2}{(vx)^3 - 3x^2(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2 - v\left(v^3 - 3v\right)}{v^3 - 3v}$$

$$\Rightarrow \frac{v^3 - 3v}{1 - 3v^4}dv = \frac{dx}{x}$$

On integrating both sides we get,

$$\int \frac{v^3 - 3v}{1 - 3v^4}dv = \log x + \log C' \qquad \ldots\ldots\ldots\ldots \text{(2)}$$

Now, $\int \frac{v^3-3v}{1-3v^4} dv = \int \frac{v^3}{1-v^4} dv - 3\int \frac{vdv}{1-v^4}$

$\Rightarrow \int \frac{v^3-3v}{1-3v^4} dv = I_1 - 3I_2$ where $I_1 \int \frac{v^3}{1-v^4} dv$ and $I_2 = \int \frac{vdv}{1-v^4}$ ....(3)

Let $1 - v^4 = t$

$\Rightarrow \frac{d}{dv}(1 - v^4) = \frac{dt}{dv}$

$\Rightarrow -4v^3 = \frac{dt}{dv}$

$\Rightarrow v^3 dv = -\frac{dt}{4}$

Now, $I_1 = \int -\frac{dt}{4} = -\frac{1}{4}\log t = -\frac{1}{4}\log(1 - v^4)$

and $I_2 = \int \frac{vdv}{1-v^4} = \int \frac{vdv}{1-(v^2)^2}$

Let $v^2 = p$

$\frac{d}{dv}(v^2) = \frac{dp}{dv}$

$\Rightarrow 2v = \frac{dp}{dv}$

$\Rightarrow vdv = \frac{dp}{2}$

$\therefore\ I_2 = \frac{1}{2}\int \frac{dt}{1-p^2} = \frac{1}{2\times 2}\log\left|\frac{1+p}{1-p}\right| = \frac{1}{4}\left|\frac{1+v^2}{1-v^2}\right|$

Now, substituting the values of I_1 and I_2 in equation (3), we get,

$\int \left(\frac{v^3-3y}{1-v^4}\right) dv - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$

Thus, equation (2), becomes,

$-\frac{1}{4}\log(1 - v^4) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log x + \log C'$

$\Rightarrow -\frac{1}{4}\log\left[(1-v^4)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log C'x$

$\Rightarrow \frac{(1+v^2)^4}{(1-v^2)^2} = (C'x)^{-4}$

$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2} = \frac{1}{C'^4x^4}$

$\Rightarrow (x^2 - y^2)^2 = C'^4(x^2 + y^2)^4$

$\Rightarrow (x^2 - y^2) = C'^2(x^2 + y^2)$

$\Rightarrow (x^2 - y^2) = C(x^2 + y^2)$, where $C = C'^2$

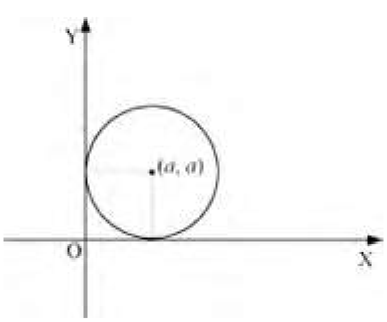
Therefore, the result is proved.

Question 5: Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

Answer:

We know that the equation of a circle in the first quadrant with centre (a, a) and radius a which touches the coordinate axes is:

$(x - a)^2 + (y - a)^2 = a^2$ -----------(1)



Now differentiating above equation w.r.t. x, we get,

$2(x - a) + 2(y - a)\frac{dy}{dx} = 0$

$\Rightarrow$ (x – a) + (y – a) y’ = 0

$\Rightarrow$ x – a +yy’ – ay’ = 0

$\Rightarrow$ x + yy’ –a(1+y’) = 0

$\Rightarrow$ a = $\frac{x+yy\prime}{1+y\prime}$

Now, substituting the value of a in equation (1), we get,

$$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow\left[\frac{(x-y)y'}{1+y'}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$\Rightarrow$ $(x - y)^2 . y'^2 + (x – y)^2 = (x + yy')^2$

$\Rightarrow$ $(x – y)^2[1 + (y')^2] = (x + yy')^2$

Therefore, the required differential equation of the family of circles is

$(x – y)^2[1 + (y')^2] = (x + yy')^2$

Question 6: Find the general solution of the differential equation

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$

Answer:

It is given that $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

$$\Rightarrow\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$$

$$\Rightarrow\frac{dy}{\sqrt{1-y^2}}=-\frac{dx}{\sqrt{1-x^2}}$$

On integrating, we get,

$\sin^{-1} y = \sin^{-1} x + C$

$\Rightarrow \sin^{-1} x = \sin^{-1} y + C$

Question 7: Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ is given by (x + y + 1) = A (1 – x – y – 2xy), where A is parameter.

Answer:

It is given that $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{y^2+y+1}{x^2+x+1}\right)$

$\Rightarrow \frac{dy}{y^2+y+1} = \frac{-dx}{x^2+x+1}$

$\Rightarrow \frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0$

On integrating both sides, we get,

$\int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = C$

$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = C$

$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = C$

$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right] + \tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] = \frac{\sqrt{3}}{2}C$

$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{2y+1}{\sqrt{3}}\cdot\frac{2x+1}{\sqrt{3}}}\right] = \frac{\sqrt{3}}{2}C$

$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{3-4xy-2x-2y-1}\right] = \frac{\sqrt{3}}{2}\text{C}$

$\Rightarrow \tan^{-1}\left[\frac{2\sqrt{3}(x+y+1)}{2(1-x-y-2xy)}\right] = \frac{\sqrt{3}}{2}\text{C}$

$\Rightarrow \frac{\sqrt{3}(x+y+1)}{(1-x-y-2xy)} = \tan\left(\frac{\sqrt{3}}{2}\text{C}\right)$

Let $\tan\left(\frac{\sqrt{3}}{2}\text{C}\right) = \text{B}$

Then,

$x + y + 1 = \frac{2\text{B}}{\sqrt{3}}(1 - x - y - 2xy)$

Now, let A = $\frac{2\text{B}}{\sqrt{3}}$, then, we have,

$x + y + 1 = A(1 - x - y - 2xy)$

Question 8: Find the equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$ whose differential equation is sin x cos y dx + cos x sin y dy = 0.

Answer:

It is given that sin x cos y dx + cos x sin y dy = 0

$\Rightarrow \frac{\sin x \cos y\, dx + \cos x \sin y dy}{\cos x \cos y} = 0$

⇒ tan x dx + tan y dy = 0

So, on integrating both sides, we get,

log (sec x) + log (sec y) = log C

⇒ log (sec x. sec y) = log C

⇒ sec x. sec y = C

The curve passes through point $\left(0, \frac{\pi}{4}\right)$

Thus, $1 \times \sqrt{2} = C$

$\Rightarrow C = \sqrt{2}$

On substituting $C = \sqrt{2}$ in equation (1), we get,

sec x. sec y = $\sqrt{2}$

$\Rightarrow \sec x \,.\, \frac{1}{\cos y} = \sqrt{2}$

$\Rightarrow \cos y = \frac{\sec x}{\sqrt{2}}$

Therefore, the required equation of the curve is $\cos y = \frac{\sec x}{\sqrt{2}}$

Question 9: Find the particular solution of the differential equation $(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0$, given that y = 1 when x = 0.

Answer:

It is given that $(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0$

$\Rightarrow \frac{dy}{1+y^2} + \frac{e^x dx}{1+e^{2x}} = 0$

On integrating both sides, we get,

$\tan^{-1} y + \int \frac{e^x dx}{1+e^{2x}} = C$ .... (1)

Let $e^x = t$

$\Rightarrow e^{2x} = t^2$

$\Rightarrow \frac{d}{dx}(e^x) = \frac{dt}{dx}$

$\Rightarrow e^x = \frac{dt}{dx}$

$\Rightarrow e^x dx = dt$

Substituting the value in equation (1), we get,

$\tan^{-1} y + \int \frac{dt}{1+t^2} = C$

$\Rightarrow \tan^{-1} y + \tan^{-1} t = C$

$\Rightarrow \tan^{-1} y + \tan^{-1} (e^x) = C$ ……….. (2)

Now, y =1 at x = 0

Therefore, equation (2) becomes:

$\tan^{-1} 1 + \tan^{-1} 1 = C$

$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = C$

$\Rightarrow C = \frac{\pi}{4}$

Substituting $C = \frac{\pi}{4}$ in (2), we get,

$\tan^{-1} y + \tan^{-1} (e^x) = \frac{\pi}{4}$

Question 10: Solve the differential equation $ye^{\frac{x}{y}} = \left(xe^{\frac{x}{y}} + y^2\right) dy (y \neq 0)$

Answer:

It is given that $ye^{\frac{x}{y}} = \left(xe^{\frac{x}{y}} + y^2\right) dy$

$\Rightarrow ye^{\frac{x}{y}} \frac{dx}{dy} = xe^{\frac{x}{y}} + y^2$

$\Rightarrow e^{\frac{x}{y}} \left[y.\frac{dx}{dy} - x\right] = y^2$

$\Rightarrow e^{\frac{x}{y}} \frac{\left[y.\frac{dx}{dy} - x\right]}{y^2} = 1$ ……… (1)

Let $e^{\frac{x}{y}} = z$

Differentiating it w.r.t. y, we get,

$$\frac{d}{dy}\left(e^{\frac{x}{y}}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}.\frac{d}{dy}\left(\frac{x}{y}\right) = \frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}\left[\frac{y.\frac{dx}{dy}-x}{y^2}\right] = \frac{dz}{dy} \qquad \text{......... (2)}$$

From equation (1) and equation (2), we get,

$$\frac{dz}{dy} = 1$$

⇒ dz = dy

On integrating both sides, we get,

z = y + C

$$\Rightarrow e^{\frac{x}{y}} = y + C$$

Question 11: Find a particular solution of the differential equation (x – y) (dx + dy) = dx – dy, given that y = –1, when x = 0. (Hint: put x – y = t).

Answer:

It is given that (x – y) (dx +dy) = dx – dy

⇒ (x –y + 1) dy = (1- x + y) dx

$$\Rightarrow \frac{dy}{dx} = \frac{1-x+y}{x-y+1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-(x-y)}{x-y+1} \quad \text{-----------------(1)}$$

Let x – y = t

$$\Rightarrow \frac{d(x-y)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 1 - \frac{dt}{dx} = \frac{dy}{dx}$$

Now, let us substitute the value of x-y and $\frac{dy}{dx}$ in equation (1), we get,

$$1 - \frac{dt}{dx} = \frac{1-t}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = 1 - \left(\frac{1-t}{1+t}\right)$$

$$\Rightarrow \frac{dt}{dx} = \frac{(1+t)-(1-t)}{1+t}$$

$$\Rightarrow \frac{dt}{dx} = \frac{2t}{1+t}$$

$$\Rightarrow \left(\frac{1+t}{t}\right) dt = 2dx$$

$$\Rightarrow \left(1 + \frac{1}{t}\right) dt = 2dx \quad \text{-------(2)}$$

On integrating both side, we get,

t + log |t| = 2x + C

⇒ (x – y) + log |x – y| = 2x + C

⇒ log |x – y| = x + y + C --------(3)

Now, y = -1 at x = 0

Then, equation (3), we get,

log 1 = 0 – 1 + C

$\Rightarrow$ C = 1

Substituting C = 1in equation (3), we get,

log |x – y| = x + y + 1

Therefore, a particular solution of the given differential equation is log |x – y| = x + y + 1.

Question 12: Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1 (x \neq 0)$.

Answer:

It is given that $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This is equation in the form of $\frac{dy}{dx} + py = Q$ (where, p = $\frac{1}{\sqrt{x}}$ and Q = $\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$)

Now, I.F. = $e^{\int p dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$

Thus, the solution of the given differential equation is given by the relation:

y(I.F.) = $\int (Q \times \text{I. F.}) dx + C$

$$\Rightarrow ye^{2\sqrt{x}} = \int \left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} \times e^{2\sqrt{x}}\right) dx + C$$

$$\Rightarrow ye^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}} dx + C$$

$$\Rightarrow ye^{2\sqrt{x}} = 2\sqrt{x} + C$$

Question 13: Find a particular solution of the differential equation $\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$(x $\neq$ 0), given that y = 0 when $x = \frac{\pi}{2}$

Answer:

It is given that $\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$

This is equation in the form of $\frac{dy}{dx} + py = Q$ (where, p = cot x and Q = 4xcosecx)

Now, I.F. = $e^{\int pdx} = e^{\int \cot x\, dx} = e^{\log|\sin x|} = \sin x$

Thus, the solution of the given differential equation is given by the relation:

y(I.F.) = $\int (Q \times \text{I. F.})dx + C$

$\Rightarrow y \sin x = \int 2x \operatorname{cosec} x\, dx + C$

= $4 \int xdx + C$

= $4.\frac{x^2}{2} + C$

$\Rightarrow y \sin x = 2x^2 + C$ -----------(1)

Now y = 0 at x = $\frac{\pi}{2}$

Therefore, equation (1), we get,

$0 = 2 \times \frac{\pi^2}{4} + C$

$\Rightarrow C = \frac{\pi^2}{4}$

Now, substituting C = $\frac{\pi^2}{4}$in equation (1), we get, y sin x = $2x^2 - \frac{\pi^2}{4}$

Therefore, the required particular solution of the given differential equation is

y sin x = $2x^2 - \frac{\pi^2}{4}$

Question 14: Find a particular solution of the differential equation $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$, given that y = 0 when x = 0.

Answer:

It is given that $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$

$\Rightarrow \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$

$\Rightarrow \frac{e^y dy}{2-e^y} = \frac{dx}{x+1}$

On integrating both sides, we get,

$\int \frac{\mathrm{e^y dy}}{2-\mathrm{e^y}} = \log|\mathrm{x}+1| + \log \mathrm{C}$ ----------------(1)

Let $2 - \mathrm{e^y} = t$

$\therefore \frac{d}{dt}(2 - \mathrm{e^y}) = \frac{dt}{dy}$

$\Rightarrow -\mathrm{e^y} = \frac{dt}{dy}$

⇒ e^ydt = -dt

Substituting value in equation (1), we get,

$\int \frac{-\mathrm{dt}}{\mathrm{t}} = \log|\mathrm{x}+1| + \log \mathrm{C}$

⇒ -log |t| = log |C(x+1) |

$\Rightarrow$ -log|2 – e^y| = log |C (x + 1) |

$\Rightarrow \frac{1}{2-e^y}$ = C (x+1)

$\Rightarrow 2 - e^y = \frac{1}{c(x+1)}$ ------------------(2)

Now, at x = 0 and y = 0, equation (2) becomes,

$\Rightarrow 2 - 1 = \frac{1}{C}$

$\Rightarrow$ C = 1

Now, substituting the value of C I equation (2), we get,

$\Rightarrow 2 - e^y = \frac{1}{(x+1)}$

$\Rightarrow e^y = 2 - \frac{1}{(x+1)}$

$\Rightarrow e^y = \frac{2x+2-1}{(x+1)}$

$\Rightarrow e^y = \frac{2x+1}{(x+1)}$

$\Rightarrow y = \log\left|\frac{2x+1}{x+1}\right|.(x \neq -1)$

Therefore, the required particular solution of the given differential equation is

$y = \log\left|\frac{2x+1}{x+1}\right|.(x \neq -1)$

Question 15: The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20, 000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Answer:

Let the population at any instant (t) be y.

Now it is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$\therefore \frac{dy}{dt} \alpha y$

$\Rightarrow \frac{dy}{dt} = ky (k \text{ is a constant})$

$\Rightarrow \frac{dy}{y} = kdt$

Now, integrating both sides, we get,

log y = kt + C ------(1)

According to given conditions,

In the year 1999, t = 0 and y = 20000

⇒ log 20000 = C ----(2)

Also, in the year 2004, t = 5 and y = 25000

⇒ log 25000 = k.5 + C

⇒ log 25000 = 5k + log 20000

$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$

$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right)$ -------(3)

Also, in the year 2009, t = 10

Now, substituting the values of t, k and c in equation (1), we get

$\log\ y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$

$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$

$\Rightarrow y = 20000 \times \frac{5}{4} \times \frac{5}{4}$

$\Rightarrow$ y = 31250

Therefore, the population of the village in 2009 will be 31250.

Question 16: The general solution of the differential equation $\frac{ydx - xdx}{y} = 0$ is

A. xy = C
B. $x = Cy^2$
C. y = Cx
D. $y = Cx^2$

Answer:

It is given that $\frac{ydx - xdx}{y} = 0$

$\Rightarrow \frac{ydx - xdx}{y} = 0$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$

Integrating both sides, we get,

log |x| - log |y| = log k

$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$

$\Rightarrow \frac{x}{y} = k$

$\Rightarrow y = \frac{1}{k}x$

$\Rightarrow$ y = Cx where $C = \frac{1}{k}$.

Question 17: The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is

A. $xe^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dx + C$

B. $ye^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dx + C$

C. $xe^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dy + C$

D. $xe^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dx + C$

Answer:

The integrating factor of the given differential equation

$\frac{dx}{dy} + P_1 x = Q_1$ is $e^{\int P_1 dy}$.

Thus, the general solution of the differential equation is given by,

x(I.F.) = $\int (Q \times \text{I.F.})\, dy + C$

$\Rightarrow x.e^{\int P_1 dy} = \int \left(Q_1 e^{\int P_1 dy}\right) dy + C$

Question 18: The general solution of the differential equation e^x dy + (y e^x + 2x) dx = 0 is

A. x ey + x^2 = C

B. x ey + y^2 = C

C. y ex + x^2 = C

D. y ey + x^2 = C

Answer:

It is given that $e^x dy + (ye^x + 2x)\, dx = 0$

$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$

$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$

This is equation in the form of $\frac{dy}{dx} + py = Q$ (where, p = 1 and Q = $-2xe^{-x}$)

Now, I.F. = $e^{\int p dx} = e^{\int dx} = e^{x}$

Thus, the solution of the given differential equation is given by the relation:

y(I.F.) = $\int (Q \times \text{I. F.}) dx + C$

$\Rightarrow ye^{x} = \int (-2xe^{-x} \times e^{x})\, dx + C$

$\Rightarrow ye^{2\sqrt{x}} = \int 2x\, dx + C$

$\Rightarrow ye^{x} = -x^{2} + C$

$\Rightarrow ye^{x} + x^{2} = C$