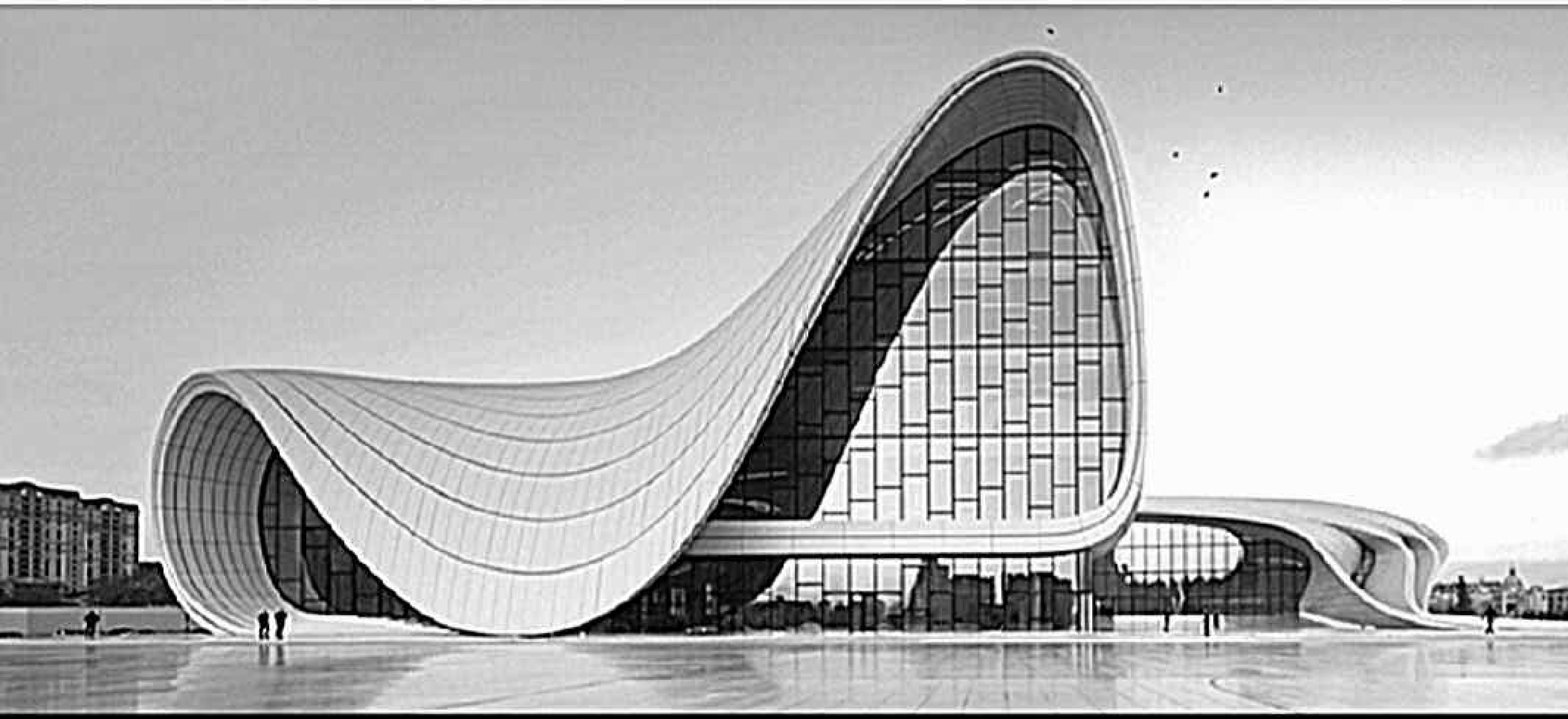
8 Application of Integrals



"Curvilinear designs are the symbol of modern architecture, the notion of freeform and innovation but curves bring an emotional connection as curves are found all through nature. Definite integrals help in finding the areas under a curve or between two curves so that these structures are both theoretically and practically stable for years and years.

Topic Notes

- *Application of Definite Integrals*

APPLICATION OF DEFINITE INTEGRALS 1

TOPIC 1

AREA OF BOUNDED REGIONS

In the previous chapter, we have studied about indefinite integrals and discussed few methods of finding them including integrals of some special functions. We also studied what is called definite integral of a function and we now know a definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)\,dx$, where a is called lower limit of the integral and b is called upper limit of the integral.

If the definite integral has an anti-derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points of the interval, *i.e.*,

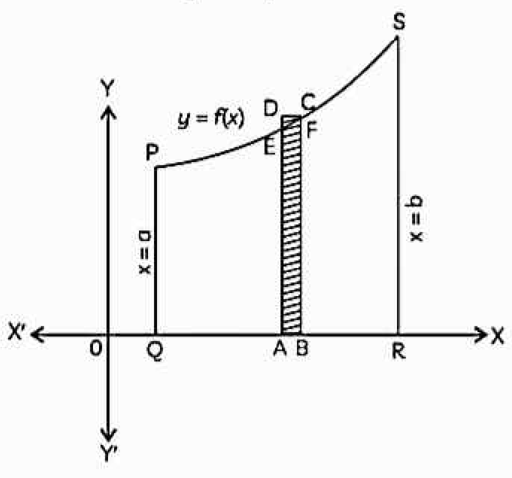
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Here, in this chapter, we shall study a specific application of integrals to find the area under simple curves specially, lines, circles, parabolas and ellipses (standard forms only).

Theorem 1: Let f be a continuous function defined on $[a, b]$. Then, the area bounded by the curve $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

Proof: Let PS be a curve $y = f(x)$ between the lines $x = a$ (PQ) and $x = b$ (RS). Then the required area is the area of the closed region PQRS.



Let $E(x, y)$ and $C(x + \Delta x, y + \Delta y)$ be two points on the curve $y = f(x)$. We draw two lines EA and CB perpendicular to x-axis (see the figure).

Then, $AB = \Delta x$, $BC = y + \Delta y$ and $EA = y$.

Let, area(QBCPQ) = $A + \Delta A$ and area(QAEPQ) = A.

Then, ΔA = area(ABCEA)

And area(ABFEA) = $y\Delta x$; area (ABCDA) = $(y + \Delta y)\Delta x$

Now, area (ABFEA) ≤ area (ABCEA) ≤ area (ABCDA)

$$y\Delta x \le \Delta A \le (y + \Delta y)\Delta x$$

$$y \le \frac{\Delta A}{\Delta x} \le (y + \Delta y)$$

$$\Rightarrow \quad \lim_{\Delta x \to 0} y \le \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \le \lim_{\Delta x \to 0} (y + \Delta y)$$

$$\Rightarrow \quad y \le \frac{dA}{dx} \le y$$

$$\Rightarrow \quad y = \frac{dA}{dx}$$

$$\Rightarrow \quad y\,dx = \frac{dA}{dx}dx$$

$$\Rightarrow \quad \int_a^b y dx = \int_a^b \frac{dA}{dx} dx$$

$$\Rightarrow \quad \int_a^b y dx = [A]_a^b$$

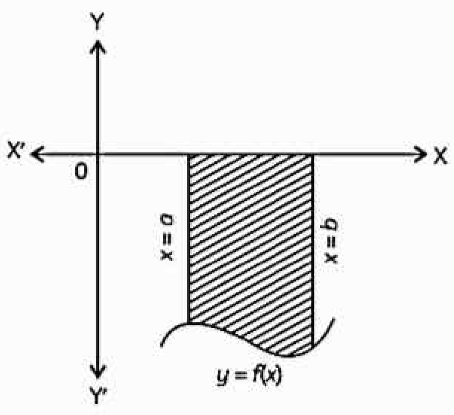
= (area A when $x = b$) – (area A when $x = a$)

= area (PQRSP) – 0

$$\Rightarrow \quad \int_a^b y dx = \text{area (PQRSP)}$$

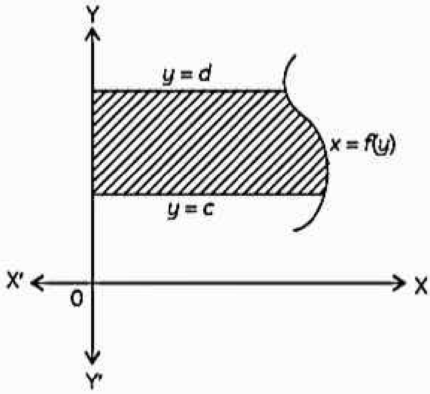
⚠ Caution

➥ *If the curve $y = f(x)$ lies below the x-axis, then the area bounded by $y = f(x)$, $x = a$ and $x = b$ will be negative. In this situation, the area is represented by* $\left|\int_a^b y dx\right|$.



Theorem 2: Let f be a continuous function defined on $[a, b]$. Then, the area bounded by the curve $x = f(y)$, the y-axis and the straight lines $y = c$ and $y = d$ is given by

$$\int_c^d f(y)\,dy \text{ or } \int_c^d x\,dy.$$



Steps to follow while obtaining Area of a Bounded Region

(1) Draw a rough sketch of the curve.

(2) If the curve is symmetrical about one of the axes, then find the area of one part and make it double. If the curve is symmetrical about both the axes, then find the area of one part and multiply it by 4.

(3) If the required area is bounded between abscissa $x = a$ and $x = b$, lies below x-axis, then use the formula:

$$\left|\int_a^b y\,dx\right|$$

(4) If y changes sign in $[a, b]$, then obtain the area of each part separately and then add after taking their moduli.

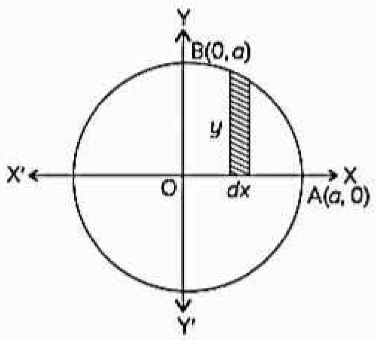
(5) If the required area is bounded between ordinates $y = c$ and $y = d$, and y-axis then use the formula:

$$\left|\int_c^d x\,dy\right|$$

(6) If x changes sign in $[c, d]$, then obtain the area of each part separately and then add after taking their moduli.

Illustration: To find the area enclosed by the circle $x^2 + y^2 = a^2$.

The rough sketch of the circle $x^2 + y^2 = a^2$ is drawn below.



The curve is symmetrical about both the axes. So,

Required area $= 4 \times$ area (quad. OABO)

$$= 4 \times \int_0^a y\,dx$$

(Taking vertical strip)

$$= 4 \times \int_0^a \sqrt{a^2 - x^2}\,dx$$

{As the region OABO lies in the first quadrant, y is taken positive.}

$$= 4\times\left[\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\times\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) \text{or } \pi a^2 \text{ sq. units}$$

Alternate Solution:

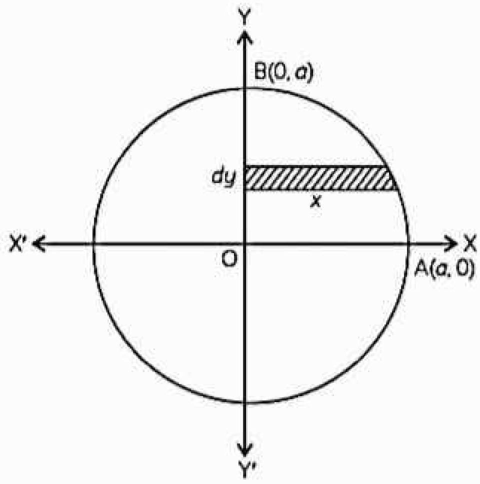
The curve is symmetrical about both the axes. So,

Required area

$= 4 \times$ area (quad. OABO)

$= 4\times\int_0^a x\,dy$ (Taking horizontal strip)

$$= 4\times\int_0^a \sqrt{a^2 - y^2}\,dy$$



{As the region OABO lies in the first quadrant, x is taken positive.}

$$= 4\times\left[\frac{1}{2}y\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

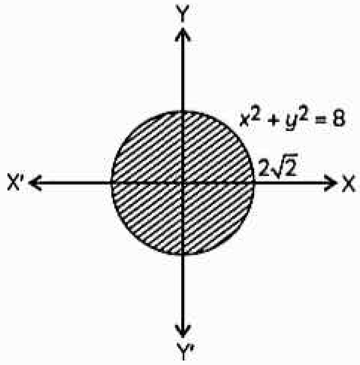
$$= 4\times\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right), \text{ or } \pi a^2 \text{ sq. units}$$

Example 1.1: Find the area of the circle $x^2 + y^2 = 8$ by using definite integration.

Ans. Given equation of circle is:

$$x^2 + y^2 = 8$$

Here centre and radius of a circle are (0, 0) and $2\sqrt{2}$ units.



$\therefore$ Area of circle

= 4 × Area of quadrant of a circle

$= 4\int_0^{2\sqrt{2}} y\,dx$

$= 4\int_0^{2\sqrt{2}} \sqrt{8 - x^2}\,dx$

$= 4\left[\frac{x}{2}\sqrt{8 - x^2} + \frac{8}{2}\sin^{-1}\frac{x}{2\sqrt{2}}\right]_0^{2\sqrt{2}}$

$= 4[\,0 + 4 \times \sin^{-1} 1]$

$= 16 \times \frac{\pi}{2} = 8\pi$ sq. units

Example 1.2: Using integration, find the area of the triangular region whose sides have equations: $y = 2x + 1$, $y = 3x + 1$ and $x = 4$ **[NCERT]**

Ans. Equations of sides of ΔABC are :

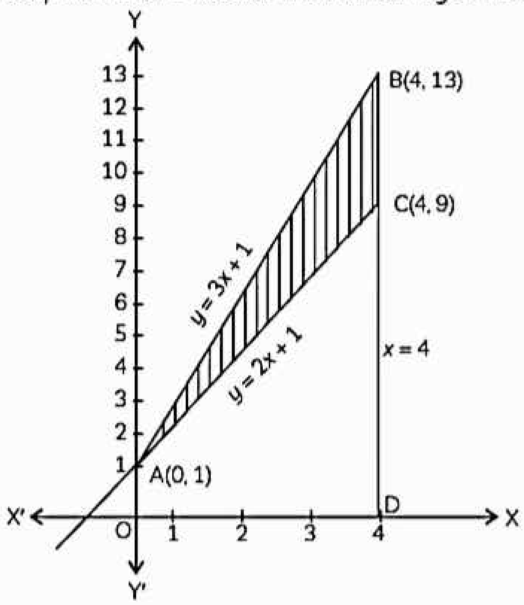
$$y = 2x + 1 \quad \text{...(i)}$$
$$y = 3x + 1 \quad \text{...(ii)}$$
$$x = 4 \quad \text{...(iii)}$$

Solving (i) and (ii) , intersecting point is A(0, 1).

Solving (ii) and (iii) , intersecting point is B(4, 13).

Solving (i) and (iii) , intersecting point is C(4, 9).

Required area = Area of the shaded region ABC



= Area of ODBAO – Area of ODCAO

$= \int_0^4 (3x + 1)\,dx - \int_0^4 (2x + 1)\,dx$

$= \left[\frac{3x^2}{2} + x\right]_0^4 - [x^2 + x]_0^4$

= (24 + 4 – 0) – (16 + 4), = 8 sq. units

Example 1.3: Using integration, find the area of the triangular region whose vertices are A(– 1, 0), B(1, 3) and C(3,2) **[NCERT]**

Ans. Let A(– 1, 0), B(1, 3) and C(3, 2) be the vertices of ΔABC. Then,

Equation of AB:

$$y - 0 = \frac{3 - 0}{1 - (-1)}(x + 1),$$

$$\Rightarrow \quad y = \frac{3}{2}(x + 1)$$

Equation of BC:

$$y - 3 = \frac{2 - 3}{3 - 1}(x - 1),$$

$$\Rightarrow \quad y = \frac{-1}{2}x + \frac{7}{2}$$

Equation of CA:

$$y - 0 = \frac{2 - 0}{3 + 1}(x + 1),$$

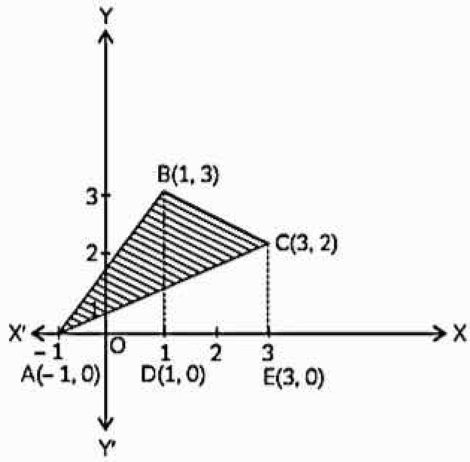
$$\Rightarrow \quad y = \frac{1}{2}x + \frac{1}{2}$$

Required area

= Area of the shaded region ABC

= Area of ADBA + Area of DECBD

– Area of AECA



$$= \frac{3}{2}\int_{-1}^{1}(x + 1)\,dx + \int_1^3\left(\frac{-1}{2}x + \frac{7}{2}\right)dx$$

$$- \int_{-1}^{3}\left(\frac{1}{2}x + \frac{1}{2}\right)dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1} + \frac{1}{2}\left[-\frac{x^2}{2}+7x\right]_{1}^{3} - \frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{3}$$

$$= \frac{3}{2}\left(\frac{1}{2}+1-\frac{1}{2}+1\right)+\frac{1}{2}\left[-\frac{9}{2}+21+\frac{1}{2}-7\right]$$

$$-\frac{1}{2}\left(\frac{9}{2}+3-\frac{1}{2}+1\right)$$

$$= \frac{3}{2}\cdot(2)+\frac{1}{2}\cdot(10)-\frac{1}{2}\cdot(8) = 4 \text{ sq. units}$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$, is:

(a) 2 sq. units (b) $\frac{9}{4}$ sq. units

(c) $\frac{9}{3}$ sq. units (d) $\frac{9}{2}$ sq. units

2. The area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0$, $x = \frac{\pi}{2}$ and the x-axis is:

(a) 2 sq. units (b) 4 sq. units

(c) 3 sq. units (d) 1 sq. unit

[NCERT Exemplar]

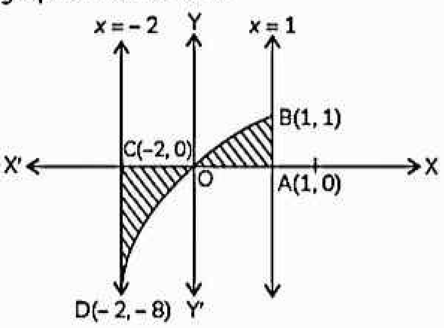
3. Area bounded by the curves $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$, is:

(a) -9 sq. units (b) $\frac{-15}{4}$ sq. units

(c) $\frac{15}{4}$ sq. units (d) $\frac{17}{4}$ sq. units

Ans. *(d) $\frac{17}{4}$ sq. units*

Explanation: The given curve $y = x^3$ represents the graph shown below.



x	1	− 1	− 2
y	1	− 1	− 8

Required area

= Area of the region CDOC

+ Area of the region OABO

$= \left|\int_{-2}^{0} -x^3 dx\right| + \int_0^1 x^3\, dx$

$= \left|\frac{x^4}{4}\right|_{-2}^{0} + \left|\frac{x^4}{4}\right|_{0}^{1}$

$= |0-4| + \left(\frac{1}{4}-0\right)$, i.e., $\frac{17}{4}$ sq. units

4. The area of the region bounded by the curve $x = 2y + 3$ and the lines, $y = 1$ and $y = -1$ is:

(a) 4 sq. units (b) $\frac{3}{2}$ sq. units

(c) 6 sq. units (d) 8 sq. units

[NCERT Exemplar]

5. Area bounded by the curves $y = x|x|$, the x-axis and the ordinates $x = -1$ and $x = 1$, is:

(a) 0 (b) $\frac{1}{3}$ sq. units

(c) $\frac{2}{3}$ sq. units (d) $\frac{4}{3}$ sq. units

6. Area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$ is:

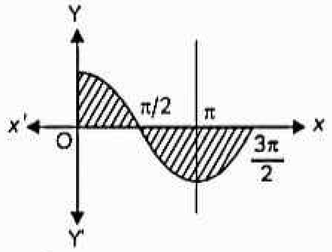
(a) 2 sq. units (b) 4 sq. units

(c) 3 sq. units (d) 1 sq. unit

[NCERT Exemplar]

Ans. *(a) 2 sq. units*

Explanation: The graph of cosine function is positive from 0 to $\frac{\pi}{2}$ and negative from $\frac{\pi}{2}$ to π.



$\therefore$ Area $= \int_0^{\pi} |\cos x|\, dx$

$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x) dx$

$= [\sin x]_0^{\pi/2} + [-\sin x]_{\pi/2}^{\pi}$

$= \left(\sin\frac{\pi}{2} - \sin 0\right) + \left(-\sin\pi + \sin\frac{\pi}{2}\right)$

$= (1 - 0) + (0 + 1)$

$= 1 + 1 = 2$ sq. units

⚠ Caution

Remember that area can never be negative.

7. **Area of the region bounded by the curve $y^2 = 4x$, Y-axis and the line $y = 3$ is:**

(a) $\frac{8}{3}$ sq. units (b) $\frac{9}{2}$ sq. units

(c) $\frac{9}{4}$ sq. units (d) None of these

8. **If we draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval [1, 5], then the area under the curve and between the lines $x = 1$ and $x = 5$ is:**

(a) $\frac{32}{3}$ sq. units (b) $\frac{16}{3}$ sq. units

(c) $\frac{32}{9}$ sq. units (d) $\frac{16}{9}$ sq. units

Ans. *(a) $\frac{32}{3}$ sq. units*

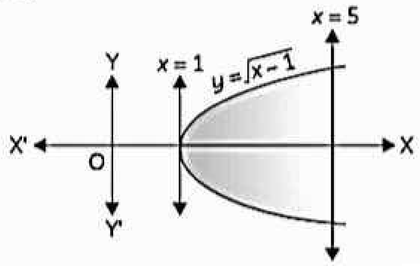
Explanation: Given equation of curve is

$$y = \sqrt{x-1}$$

or

$$y^2 = x - 1$$

It is an equation of parabola opening side wards towards positive x-axis with vertex at (1, 0).



$\therefore$ Area of shaded region $= 2\int_1^5 y\, dx$

$= 2\int_1^5 \sqrt{x-1}\, dx$

$= 2\left[\frac{(x-1)^{3/2}}{3/2}\right]_1^5$

$= \frac{4}{3}[(x-1)^{3/2}]_1^5$

$= \frac{4}{3}\left[(5-1)^{3/2} - 0\right]$

$= \frac{4}{3} \times (4)^{3/2}$

$= \frac{4}{3} \times 8 = \frac{32}{3}$ sq. units

9. **The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is:**

(a) 8π sq. units (b) 20π sq. units

(c) 16π sq. units (d) 256π sq. units

[NCERT Exemplar]

10. **The area of region bounded by the lines $y = x$, $x = 0$, $x = 3$ and x-axis is:**

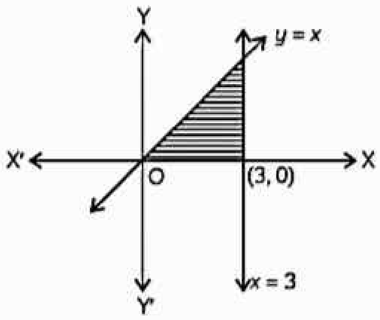
(a) $\frac{1}{5}$ sq. units (b) $\frac{9}{4}$ sq. units

(c) $\frac{9}{2}$ sq. units (d) $\frac{4}{5}$ sq. units

Ans. *(c) $\frac{9}{2}$ sq. units*

Explanation: The area of shaded region is

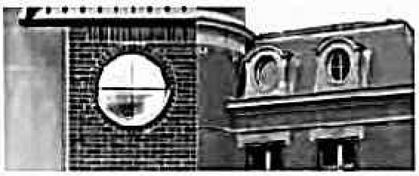
$$\int_0^3 y\, dx = \int_0^3 x\, dx$$



$= \left[\frac{x^2}{2}\right]_0^3$

$= \frac{1}{2}[3^2 - 0] = \frac{9}{2}$ sq. units

11. While going through certain pictures of glass windows on the internet, Shefali and her friend came across rather uncommon round glass windows as shown below. They discussed the application of definite integrals in finding the area of the glass bounded by the circle and the line passing through its centre.



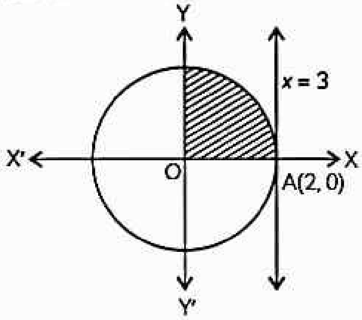
Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$, is

(a) π sq. units (b) $\frac{\pi}{4}$ sq. units

(c) $\frac{\pi}{2}$ sq. units (d) $\frac{\pi}{3}$ sq. units

Ans. *(a) π sq. units*

Explanation:



Required area $= \int_0^2 \sqrt{4 - x^2}$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

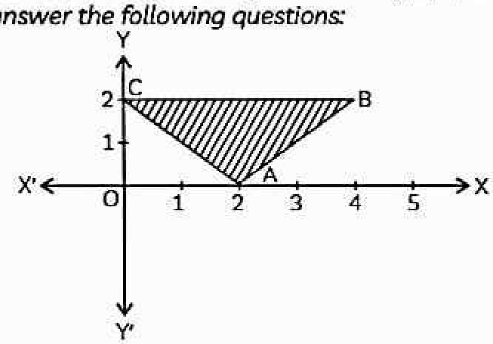
$$= 2 \times \frac{\pi}{2} = \pi \text{ sq. units}$$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

12. *Look at the shaded region on the graph, and answer the following questions:*



(A) The shaded region is bounded by the lines:

(a) $y = x - 2, x = 2$ and x-axis

(b) $x = y - 2, y = 2$ and x-axis

(c) $y = |x - 2|$ and $y = 2$

(d) $x = |y - 2|$ and $x = 2$

(B) Which of the following is not a corner point of the bounded region?

(a) (0, 2) (b) (2, 0)

(c) (4, 2) (d) (2, 4)

(C) The area A of the shaded region is given by:

(a) $A = \int_0^4 2\,dx - \int_0^2 (2-x)\,dx - \int_2^4 (x-2)\,dx$

(b) $A = \int_0^4 2\,dx - \int_0^2 (x-2)\,dx - \int_2^4 (2-x)\,dx$

(c) $A = \int_0^4 2\,dx + \int_0^2 (x-2)\,dx - \int_2^4 (2-x)\,dx$

(d) $A = \int_0^4 2\,dx + \int_0^2 (x-2)\,dx + \int_2^4 (2-x)\,dx$

(D) The area of the shaded region is:

(a) 2 sq. units (b) 4 sq. units

(c) 6 sq. units (d) 8 sq. units

(E) Which is the point of intersection of the lines $y = 2 - x$ and $y = 2$?

(a) (0, 2) (b) (2, 0)

(c) (4, 2) (d) (2, 4)

Ans. (A) *(c) $y = |x - 2|$ and $y = 2$*

Explanation: Equations of the three lines AC, CB and AB of the shaded region are $y = 2 - x$, $y = 2$ and $y = x - 2$.

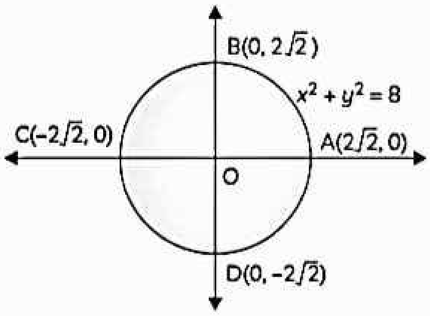
(C) *(a) $A = \int_0^4 2\,dx - \int_0^2 (2-x)\,dx - \int_2^4 (x-2)\,dx$*

Explanation: Area of the shaded region is given by

$$\int_0^4 y_{BC}\,dx - \int_0^2 y_{CA}\,dx - \int_2^4 y_{AB}\,dx$$

i.e., $\int_0^4 2\,dx - \int_0^2 (2-x)\,dx - \int_2^4 (x-2)\,dx$

13. *A farmer has a field in the shape of a circle, in which he wishes to grow four varieties of vegetables as shown below.*



(A) Evaluate $\sqrt{8-x^2}dx$.

(B) Using integration, find the area of the field.

Ans. (A) $\int\sqrt{8-x^2dx} = \sqrt{(2\sqrt{2})^2-x^2}dx$

$$= \frac{x}{2}\sqrt{\left(2\sqrt{2}\right)^2-x^2}+\frac{\left(2\sqrt{2}\right)^2}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)$$

$$= \frac{x}{2}\sqrt{8-x^2}+4\sin^{-1}\frac{x}{2\sqrt{2}}$$

(B) Area of the field $= 4\times\int_0^{2\sqrt{2}} y\,dx$

$$= 4\int_0^{2\sqrt{2}}\sqrt{8-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{8-x^2}+4\sin^{-1}\frac{x}{2\sqrt{2}}\right]_0^{2\sqrt{2}}$$

[Using part (A)]

$$= 4\left[\left(\frac{2\sqrt{2}}{2}\sqrt{8-\left(2\sqrt{2}\right)^2}+4\sin^{-1}\frac{2\sqrt{2}}{2\sqrt{2}}\right)-\left(\frac{0}{2}\sqrt{8-(0)^2}+4\sin^{-1}\frac{0}{2\sqrt{2}}\right)\right]$$

$$= 4[(0+4\sin^{-1}1)-(0+4\sin^{-1}0)]$$

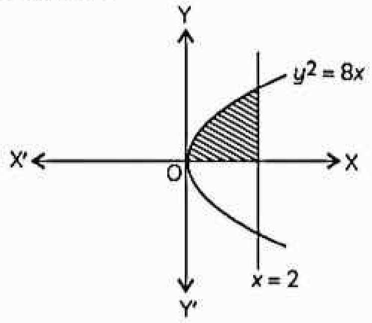
$$= 4\left[4\times\frac{\pi}{2}-0\right]$$

$= 8\pi$ sq. units.

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

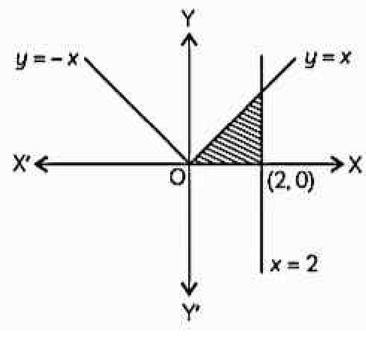
14. Find the area of the shaded region in the following figure.



15. Find the area of the curve $y = |x|$ bounded by the lines $x = 0$, $x = 2$ and x-axis.

Ans. Given curve is $y = |x|$

$$= \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



∴ Area of shaded region

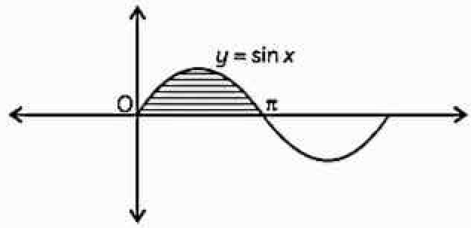
$$= \int_0^2 y\,dx$$

$$= \int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{4}{2} = 2 \text{ sq. units}$$

16. Find the area of shaded region in the figure, shown below.



Ans. Area of shaded region

$$= \int_0^{\pi} y\,dx$$

$$= \int_0^{\pi}(\sin x)\,dx$$

$$= [-\cos x]_0^{\pi}$$

$= -[\cos \pi - \cos 0]$

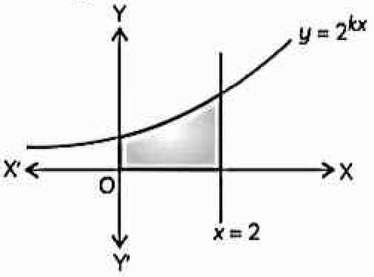
$= -[-1-1] = 2$ sq. units

17. **Find the area bounded by the line $x + y = 3$ and the axes.**

18. **Find the area of the region lying in the first quadrant and bounded by $y = 6x^2$, $x = 0$, $y = 2$ and $y = 4$.**

19. **If the area above the x-axis, bounded by the curve $y = 2^{kx}$ and $x = 0$ and $x = 2$ is $\frac{3}{\log_e 2}$, then find the value of k.**

Ans. Given curve is $y = 2^{kx}$



$\therefore$ Area of bounded region

$$= \int_0^2 y\,dx$$

$$= \int_0^2 2^{kx}\,dx$$

$$= \left[\frac{2^{kx}}{k\log 2}\right]_0^2$$

$$= \frac{1}{k\log 2}[2^{2k} - 2^0]$$

$$\Rightarrow \quad \frac{3}{\log_e 2} = \frac{1}{k\log 2}[2^{2k} - 1]$$

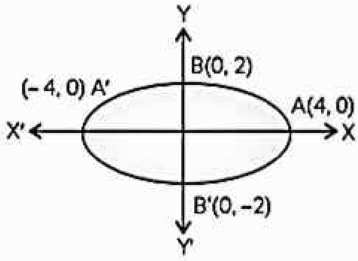
$$\Rightarrow \quad k = 1$$

20. *The Ellipse (sometimes referred to as President's Park South) is a 52-acre park south of the White House fence and north of Constitution Avenue and the National Mall in Washington, D.C. The entire park, which features monuments, is open to the public and is a part of President's Park. The Ellipse is the location for many annual events.*

Find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$

Ans. Equation of ellipse is:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$



Here, $a = 4$, $b = 2$

$\therefore$ The vertices of an ellipse are A(4, 0) and A'(– 4, 0)

Since, this curve is symmetrical about both sides.

$\therefore$ Area of ellipse

$= 4 \times$ Area of ellipse in first quadrant

$$= 4 \times \int_0^4 y\,dx$$

$$= 4 \times \int_0^4 2\sqrt{1 - \frac{x^2}{16}}\,dx$$

$$= 8\int_0^4 \frac{\sqrt{16 - x^2}}{4}\,dx$$

$$= 2\int_0^4 \sqrt{16 - x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{(4)^2}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= 2\left[\frac{4}{2} \times 0 + 8\sin^{-1}\left(\frac{4}{4}\right) - 0 + 8\sin^{-1} 0\right]$$

$$= 2\left[0 + 8 \times \frac{\pi}{2} - 0\right] = 8\pi \text{ sq. units}$$

Hence, area of the ellipse is 8π sq. units.

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

21. **Find the area between the parabola $y^2 = 6x$ and its latus rectum.**

22. **Find the area bounded by $y = \tan x$, $y = 0$ and $x = \frac{\pi}{4}$ in the first quadrant.**

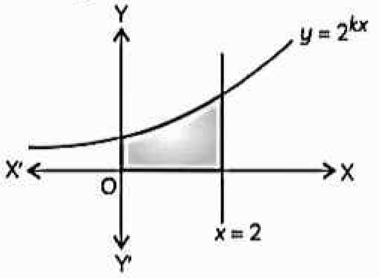
$= -[\cos \pi - \cos 0]$
$= -[-1 - 1] = 2$ sq. units

17. **Find the area bounded by the line $x + y = 3$ and the axes.**

18. **Find the area of the region lying in the first quadrant and bounded by $y = 6x^2$, $x = 0$, $y = 2$ and $y = 4$.**

19. **If the area above the x-axis, bounded by the curve $y = 2^{kx}$ and $x = 0$ and $x = 2$ is $\frac{3}{\log_e 2}$, then find the value of k.**

Ans. Given curve is $y = 2^{kx}$



$\therefore$ Area of bounded region

$$= \int_0^2 y\,dx$$

$$= \int_0^2 2^{kx}\,dx$$

$$= \left[\frac{2^{kx}}{k\log 2}\right]_0^2$$

$$= \frac{1}{k\log 2}[2^{2k} - 2^0]$$

$$\Rightarrow \quad \frac{3}{\log_e 2} = \frac{1}{k\log 2}[2^{2k} - 1]$$

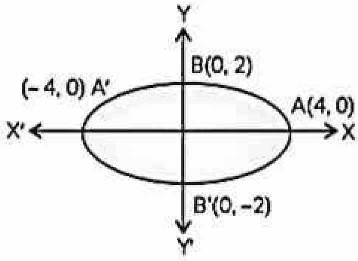
$$\Rightarrow \quad k = 1$$

20. *The Ellipse (sometimes referred to as President's Park South) is a 52-acre park south of the White House fence and north of Constitution Avenue and the National Mall in Washington, D.C. The entire park, which features monuments, is open to the public and is a part of President's Park. The Ellipse is the location for many annual events.*

Find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$

Ans. Equation of ellipse is:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$



Here, $a = 4$, $b = 2$

$\therefore$ The vertices of an ellipse are A(4, 0) and A'(– 4, 0)

Since, this curve is symmetrical about both sides.

$\therefore$ Area of ellipse

$= 4 \times$ Area of ellipse in first quadrant

$$= 4 \times \int_0^4 y\,dx$$

$$= 4 \times \int_0^4 2\sqrt{1 - \frac{x^2}{16}}\,dx$$

$$= 8\int_0^4 \frac{\sqrt{16 - x^2}}{4}\,dx$$

$$= 2\int_0^4 \sqrt{16 - x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{(4)^2}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= 2\left[\frac{4}{2} \times 0 + 8\sin^{-1}\left(\frac{4}{4}\right) - 0 + 8\sin^{-1} 0\right]$$

$$= 2\left[0 + 8 \times \frac{\pi}{2} - 0\right] = 8\pi \text{ sq. units}$$

Hence, area of the ellipse is 8π sq. units.

SHORT ANSWER Type-I Questions (SA-I)

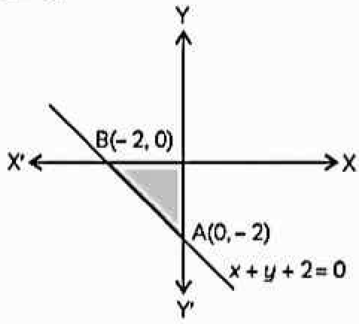
[2 marks]

21. **Find the area between the parabola $y^2 = 6x$ and its latus rectum.**

22. **Find the area bounded by $y = \tan x$, $y = 0$ and $x = \frac{\pi}{4}$ in the first quadrant.**

23. Find the area bounded by the lines $x = 0$, $y = 0$ and $x + y + 2 = 0$.

Ans. Given equation of lines are $x = 0$, $y = 0$ and $x + y + 2 = 0$.



Line $x + y + 2 = 0$ intersect the coordinate axes at points A(0, –2) and B(–2, 0).

∴ Area of shaded region

$$= \left|\int_0^{-2} |y|\,dx\right|$$

$$= \left|\int_0^{-2} (-2 - x)\,dx\right|$$

$$= \left|\left[-2x - \frac{x^2}{2}\right]_0^{-2}\right|$$

$$= \left|\left[+4 - \frac{4}{2} - (0 - 0)\right]\right|$$

$$= |4 - 2| = 2 \text{ sq. units}$$

24. Find the area between the minor axis and latus rectum of positive side of x-axis of an ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

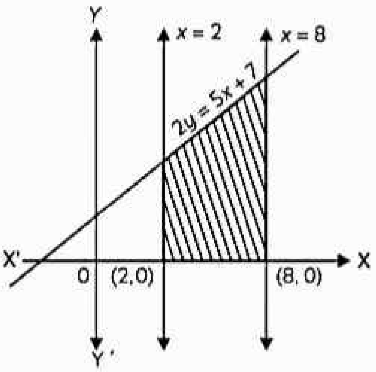
25. Find the area of region bounded by $y = -1$, $y = 2$, $x = y^3$ and $x = 0$.

26. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$. [NCERT Exemplar]

Ans. Given curve is,

$$2y = 5x + 7$$

$$\Rightarrow \quad y = \frac{5x+7}{2}$$



∴ Area of shaded region

$$= \int_2^8 y\,dx$$

$$= \frac{1}{2}\int_2^8 (5x+7)\,dx$$

$$= \frac{1}{2}\left[5\times\frac{x^2}{2} + 7x\right]_2^8$$

$$= \frac{1}{2}[(5 \times 32 + 7 \times 8) - (10 + 14)]$$

$$= \frac{1}{2}(160 + 56 - 24)$$

$$= \frac{192}{2} = 96 \text{ sq. units}$$

27. Find the area bounded by the curve $|x| + |y| = 1$.

Ans. Given curve is

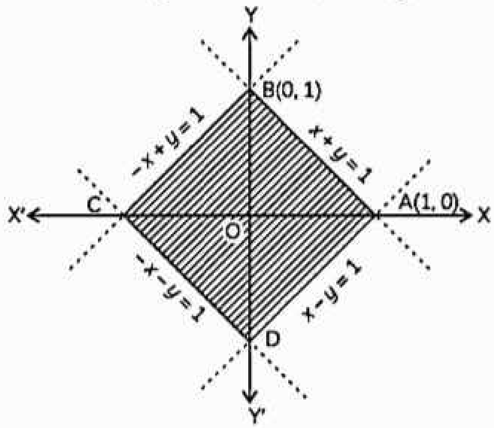
$$|x| + |y| = 1$$

$\Rightarrow$ $x + y = 1$ when, $x \geq 0, y \geq 0$

$x - y = 1$ when, $x \geq 0, y < 0$

$-x + y = 1$ when, $x < 0, y \geq 0$

and $-x - y = 1$ when, $x < 0, y < 0$



Since, the figure is symmetrical.

$\therefore$ Required Area = 4 × Area of shaded region OABO

$$= 4\int_0^1 (1-x)\,dx$$

$$= 4\left[x - \frac{x^2}{2}\right]_0^1$$

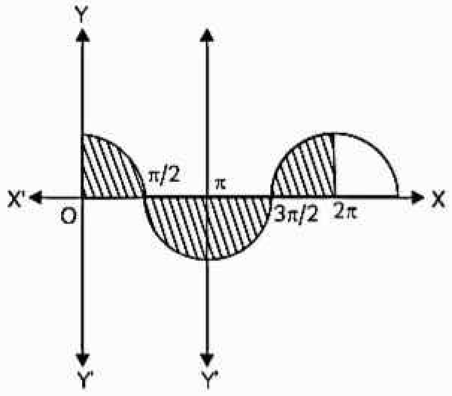
$$= 4\left[1 - \frac{1}{2}\right]$$

= 2 sq. units

28. Find the area bounded by the curve $y = 2\cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.

[NCERT Exemplar]

Ans. The graph of a cosine function is positive from 0 to $\frac{\pi}{2}$, negative for $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ and again positive from $\frac{3\pi}{2}$ to 2π.



$$\therefore \text{ Required area} = \int_0^{2\pi} |2\cos x|\,dx$$

$$= \int_0^{\pi/2} 2\cos x\,dx + \int_{\pi/2}^{3\pi/2} (-2\cos x)\,dx + \int_{3\pi/2}^{2\pi} 2\cos x\,dx$$

$$= 2[\sin x]_0^{\pi/2} + 2[-\sin x]_{\pi/2}^{3\pi/2} + 2[\sin x]_{3\pi/2}^{2\pi}$$

$= 2(1-0) + 2[-(-1)+1] + 2[0-(-1)]$

$= 2 + 4 + 2$

= 8 sq. units

Caution

➡ *Draw the appropriate curve for cos x.*

LONG ANSWER Type Questions (LA)

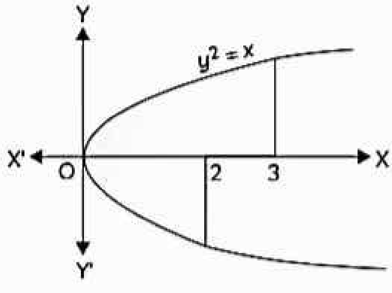
[**4** & **5** marks]

29. **Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.**

[NCERT Exemplar]

30. **Find the area of the region bounded by the triangle whose vertices are (–1, 1), (0, 5) and (3, 2) using integration. [NCERT Exemplar]**

31. **Shown below is a parabola**



Find the area of the shaded region. [Use $\sqrt{2}$ as 1.4 and $\sqrt{3}$ as 1.7]

Ans. Area of shaded region $= \int_0^3 \sqrt{x}\,dx + \left|\int_0^2 \sqrt{x}\,dx\right|$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 + \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2$$

$$= \frac{2}{3}(3^{3/2} - 0) + \frac{2}{3}(2^{3/2} - 0)$$

$$= \frac{2}{3} \times 3\sqrt{3} + \frac{2}{3} \times 2\sqrt{2}$$

$$= \frac{2}{3} \times 3 \times 1.7 + \frac{2}{3} \times 2 \times 1.4$$

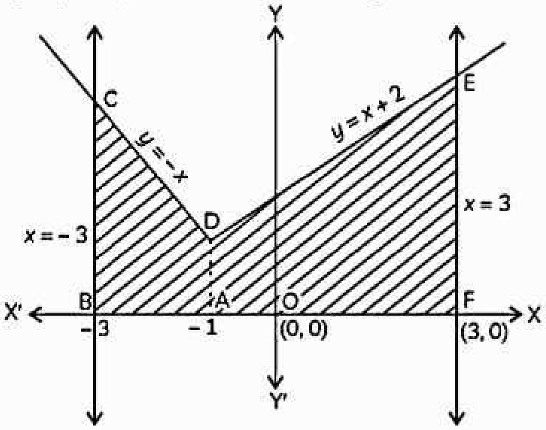
$= 3.4 + 1.867$

= 5.267 sq. units

32. **Using integration, find the area bounded by the curves: $y = |x + 1| + 1$, $x = -3$, $x = 3$ and $y = 0$. [CBSE 2014]**

Ans. Given curves,

$y = |x + 1| + 1, x = -3, x = 3$ and $y = 0$



Required Area = Area of shaded region

= Area of shaded region ABCDA + Area of shaded region ADEFA

$= \int_{-3}^{-1} -x\,dx + \int_{-1}^{3} (x+2)\,dx$

$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$

$= -\left[\frac{1}{2} - \frac{9}{2}\right] + \left[\frac{9}{2} + 6 - \frac{1}{2} + 2\right]$

$= 4 + 12 = 16$ sq. units

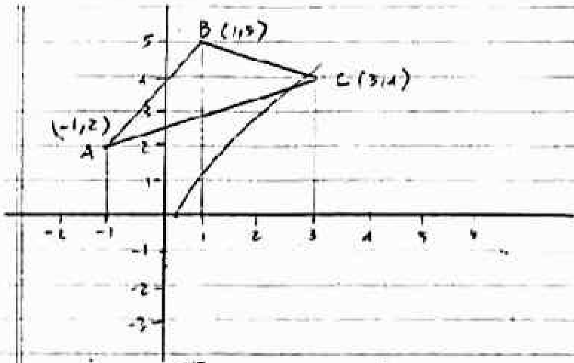
TOPPER'S CORNER

LONG ANSWER Type Questions (LA)

[**4** & **5** marks]

1. Using integration, find the area of the region bounded by the triangle whose vertices are (–1, 2), (1, 5) and (3, 4).

Ans.



$\frac{1}{2} \times 2 \times |\,\cdot\,(+a)|$

$\frac{1}{2} \times 2 \times (2+2)$

$2(6) = 12$

Equation of $\overrightarrow{AB}$:

$(y-5) = \frac{3}{2}(x-1)$

$2y - 10 = 3x - 3$

$3x - 2y + 7 = 0 \qquad y = \frac{3x+7}{2}$

Eq. of $\overrightarrow{BC}$:

$(y-5) = \frac{1}{-2}(x-1)$

$-2y + 10 = x - 1$

$x + 2y - 11 = 0 \qquad y = \frac{11-x}{2}$

Equation of AC :

$(y-4) = \frac{1}{2}(x-3)$

$2y - 8 = x - 3$

$x - 2y + 5 = 0 \qquad y = \frac{x+5}{2}$

(I) Area under AB :

$A(I) = \int_{-1}^{1} y\,dx$

$= \int_{-1}^{1} \frac{1}{2}(3x+7)\,dx$

$= \frac{1}{2}\left(\frac{3x^2}{2} + 7x\right)\Big]_{-1}^{1}$

$= \frac{1}{2}\left[\frac{3}{2} + 7 - \frac{3}{2} + 7\right]$

$= 7$ sq. units

(II) Area under BC :

$A(II) = \int_{1}^{3} y\,dx$

$= \int_{1}^{3} \left(\frac{11-x}{2}\right) dx$

$= \frac{1}{2}\left[11x - \frac{x^2}{2}\right]_{1}^{3}$

$$= \frac{1}{2}\left[35 - \frac{9}{2} - 11 + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[22 - \frac{8}{2}\right]$$

$$= \frac{1}{2}(22 - 4)$$

$$= 9 \text{ sq. units}$$

III Area under AC :

$$A(\text{III}) = \int_{-1}^{3} y\,dx$$

$$= \int_{-1}^{3} \frac{x+5}{2}\,dx$$

$$= \frac{1}{2}\int_{-1}^{3} (x+5)\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} + 5x\right)\Big]_{-1}^{3}$$

$$= \frac{1}{2}\left[\frac{9}{2} + 15 - \frac{1}{2} + 5\right]$$

$$= \frac{1}{2}\left[20 + 4\right] \quad \left(= \frac{24}{2} = 12\right)$$

$$= 12 \text{ sq. units}$$

∴ required area of the triangle $= A(\text{I}) + A(\text{II}) - A(\text{III})$

$$= (7 + 9 - 12) \text{ sq. units}$$

$$= (16 - 12) \text{ sq. units}$$

$$= 4 \text{ sq. units}$$

[CBSE Topper 2014]