Chapter 5

Quadratic Equations

Solutions

SECTION - A

Objective Type Questions (One option is correct)

1. If a and b are rational and α, β be the roots of $x^2 + 2ax + b = 0$, then the equation with rational coefficients one of whose roots is $\alpha+\beta+\sqrt{\alpha^2+\beta^2}$ is

(1) $x^2 + 4ax - 2b = 0$ (2) $x^2 + 4ax + 2b = 0$ (3) $x^2 - 4ax + 2b = 0$ (4) $x^2 - 4ax - 2b = 0$

Sol. Answer (2)

$\alpha+\beta = -2a,\ \alpha\beta = b$

$\alpha+\beta+\sqrt{\alpha^2+\beta^2} = -2a+\sqrt{4a^2-2b}$

The other root of equation will be $\alpha+\beta-\sqrt{\alpha^2+\beta^2}$

i.e., $-2a-\sqrt{4a^2-2b}$

$\therefore$ Sum of roots, $S = -4a$

Product of roots, $P = 4a^2-(4a^2-2b) = 2b$

$\therefore$ required equation is $x^2 - Sx + P = 0$

i.e., $x^2 + 4ax + 2b = 0$

2. Let α, β be the roots of $ax^2 + bx + c = 0$, γ, δ be the roots of $px^2 + qx + r = 0$ and D_1 and D_2 be their respective discriminant. If α, β, γ, δ are in A.P., then the ratio $D_1 : D_2$ is equal to

(1) $\frac{a^2}{b^2}$ (2) $\frac{a^2}{p^2}$ (3) $\frac{b^2}{q^2}$ (4) $\frac{c^2}{r^2}$

Sol. Answer (2)

$\alpha+\beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a},\ D_1 = b^2-4ac$

$\gamma+\delta = -\frac{q}{p},\ \gamma\delta = \frac{r}{p},\ D_2 = q^2-4rp$

Let common difference of A.P. be k.

$k = |\alpha - \beta| = |\gamma - \delta|$

$$\Rightarrow \frac{\sqrt{b^2 - 4ac}}{a} = \frac{\sqrt{q^2 - 4pr}}{p} \quad \Rightarrow \frac{\sqrt{b^2 - 4ac}}{\sqrt{q^2 - 4pr}} = \frac{a}{p} \quad \Rightarrow \frac{\sqrt{D_1}}{\sqrt{D_2}} = \frac{a}{p}$$

$$\Rightarrow \frac{D_1}{D_2} = \frac{a^2}{p^2}$$

3. The value of k if

(1) The roots of $5x^2 + 13x + k = 0$ are reciprocal to each other is 5

(2) The roots of $x^2 + x + k = 0$ are consecutive integer is 1

(3) The roots of $x^2 - 6x + k = 0$ are in the ratio 2 : 1 is 7

(4) The roots of the equation $x^2 + kx - 1 = 0$ are real, equal in magnitude but opposite in sign is 1

Sol. Answer (1)

(1) Let roots are α, β

$\alpha + \beta = -\frac{13}{5}$ ...(i)

$\alpha\beta = \frac{k}{5}$ ...(ii)

For reciprocal roots $\alpha\beta = 1 \Rightarrow k = 5$.

(2) If roots are cosecutive integer then

$|\alpha - \beta| = 1$

$\Rightarrow |\alpha - \beta|^2 = 1$

$(\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow 1 - 4k = 1$

$\Rightarrow k = 0$

(3) Let roots are 2α, α

$\Rightarrow 2\alpha + \alpha = 6$ ...(i)

$2\alpha \cdot \alpha = k$ ...(ii)

By (i), (ii)

$\alpha = 2$, $k = 8$

(4) In this case

$\alpha + \beta = 0$

$\Rightarrow k = 0$

4. If the difference of the roots of the equation $x^2 + ax + b = 0$ is equal to the difference of the roots of the equation $x^2 + bx + a = 0$, then

(1) $a + b = 4$ (2) $a + b = -4$ (3) $a - b = 4$ (4) $a - b = -4$

Sol. Answer (2)

$$x^2 + ax + b = 0 \Rightarrow \alpha + \beta = -a, \ \alpha\beta = b, \ |\alpha - \beta| = \sqrt{a^2 - 4b}$$

$$x^2 + bx + a = 0 \Rightarrow \gamma + \delta = -b, \ \gamma\delta = a, \ |\gamma - \delta| = \sqrt{b^2 - 4a}$$

Now, $|\alpha - \beta| = |\gamma - \delta| \Rightarrow \sqrt{a^2 - 4b} = \sqrt{b^2 - 4a}$

$\Rightarrow a^2 - 4b = b^2 - 4a \Rightarrow a^2 - b^2 = 4(b - a)$

$\Rightarrow (a - b)(a + b + 4) = 0 \Rightarrow a + b = -4 (a \neq b)$

5. If the equations $px^2 + 2qx + r = 0$ and $px^2 + 2rx + q = 0$ $(q \neq r)$ have a common root, then $p + 4q + 4r$ equals

(1) 0 (2) 1 (3) 2 (4) –2

Sol. Answer (1)

Let α be common root, $p\alpha^2 + 2q\alpha + r = 0$ …(1)

and $p\alpha^2 + 2r\alpha + q = 0$ …(2)

Now (1) – (2) $\Rightarrow 2\alpha(q - r) + r - q = 0 \Rightarrow \alpha = \frac{1}{2}$

Common root is $\alpha = \frac{1}{2}$, substituting in (1)

$p\left(\frac{1}{2}\right)^2 + 2q\left(\frac{1}{2}\right) + r = 0 \Rightarrow 4r + 4q + p = 0$

6. Consider that $f(x) = ax^2 + bx + c$, $D = b^2 - 4ac$, then which of the following is not true?

(1) If $a > 0$, then minimum value of $f(x)$ is $\frac{-D}{4a}$

(2) If $a < 0$, then maximum value of $f(x)$ is $\frac{-D}{4a}$

(3) If $a > 0$, $D < 0$, then $f(x) > 0$ for all $x \in R$

(4) If $a > 0$, $D > 0$, then $f(x) > 0$ for all $x \in R$

Sol. Answer (4)

$f(x) = ax^2 + bx + c$

$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$

$f(x) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{b^2 - 4ac}{4a^2}\right)$

$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$

$f(x) = a\left(x + \frac{b}{2a}\right) - \frac{D}{4a}$

Clearly if $a > 0$ the minimum value of $f(x) = \frac{-D}{4a}$

Similarly of $a < 0$ the maximum value $= \frac{-D}{4a}$

If $ax^2 + bx + c > 0$ then $a > 0$, $D < 0$ for all $x \in R$

Hence option (4) is not true.

7. If one root of the equation $(l - m)x^2 + lx + 1 = 0$ is double the other and if l is real, then the greatest value of m is $(l \neq m)$

(1) $\frac{9}{8}$ (2) $\frac{7}{8}$ (3) $\frac{8}{9}$ (4) $\frac{5}{9}$

Sol. Answer (1)

$$(l-m)x^2 + lx + 1 = 0$$

$$\alpha + 2\alpha = \frac{l}{m-l} \Rightarrow \alpha = \frac{l}{3(m-l)} \quad \ldots(1)$$

$$\alpha \cdot (2\alpha) = \frac{1}{l-m} \Rightarrow \alpha^2 = \frac{1}{2(l-m)} \quad \ldots(2)$$

From (1) and (2),

$$\frac{1}{2(l-m)} = \frac{l^2}{9(l-m)^2}$$

$$\Rightarrow 2l^2 - 9l + 9m = 0$$

For real l, $81 - 8 \times 9m \geq 0 \Rightarrow m \leq \frac{81}{72}$

$$\Rightarrow m \leq \frac{9}{8}$$

$\therefore$ Greatest value of m is $\frac{9}{8}$

8. If p, q, r are real numbers satisfying the condition $p + q + r = 0$, then the roots of the quadratic equation $3px^2 + 5qx + 7r = 0$ are

(1) Positive (2) Negative (3) Real and distinct (4) Imaginary

Sol. Answer (3)

$$3px^2 + 5qx + 7r = 0$$

$$\Delta = (5q)^2 - 4(3p)(7r)$$

$$= 25q^2 - 84pr$$

$$= 25(p+r)^2 - 84pr$$

$$= 25p^2 - 34pr + 25r^2$$

$$= \left(5p - \frac{17}{5}r\right)^2 + \frac{336}{25}r^2 > 0$$

$\therefore$ roots are real and distinct.

9. If x is real, then the expression $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$

(1) Lies between 4 and 7
(2) Lies between 5 and 9
(3) Has no value between 4 and 7
(4) Has no value between 5 and 9

Sol. Answer (4)

Let $y = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$

$\Rightarrow\ x^2(y-1) + x(2y-34) + 71 - 7y = 0$

For real x, discriminant should be ≥ 0

$\Rightarrow\ (2y-34)^2 - 4(y-1)(71-7y) \geq 0\ \Rightarrow\ 4(y-17)^2 - 4(y-1)(71-7y) \geq 0$

$\Rightarrow\ (y-17)^2 - (-7y^2 + 78y - 71) \geq 0\ \Rightarrow\ 8y^2 - 112y + 360 \geq 0$

$\Rightarrow\ y^2 - 14y + 45 \geq 0\ \Rightarrow\ (y-9)(y-5) \geq 0$

$\Rightarrow\ y \geq 9$ or $y \leq 5$

10. Let $a, b, c \in R$ and $a \neq 0$ be such that $(a + c)^2 < b^2$, then the quadratic equation $ax^2 + bx + c = 0$ has

(1) Imaginary roots (2) Real roots

(3) Exactly one real root lying in the interval (–1, 1) (4) Exactly two roots in (–1, 1)

Sol. Answer (3)

Here we observe that $(a + c)^2 < b^2$

$\Rightarrow\ (a - b + c)\,(a + b + c) < 0$

$\Rightarrow$ Exactly one real root of the given equation lies in (–1, 1).

$ax^2 + bx + c = 0$

$D = b^2 - 4ac > (a+c)^2 - 4ac = (a-c)^2$

$\Rightarrow\ \Delta \geq 0$

$\therefore$ Roots are real.

11. If $a_1, a_2, a_3, a_4, \ldots\ldots, a_{n-1}, a_n$ are distinct non-zero real numbers such that $(a_1^2 + a_2^2 + a_3^2 + \ldots\ldots + a_{n-1}^2)x^2 + 2(a_1a_2 + a_2a_3 + a_3a_4 + \ldots\ldots + a_{n-1}a_n)x + (a_2^2 + a_3^2 + a_4^2 + \ldots\ldots + a_n^2) \leq 0$ then $a_1, a_2, a_3, \ldots\ldots, a_{n-1}, a_n$ are in

(1) A.P. (2) G.P. (3) H.P. (4) A.G.P.

Sol. Answer (2)

We have, given expression

$(a_1^2 + a_2^2 + a_3^2 + \ldots.. + a_{n-1}^2)x^2 + 2(a_1a_2 + a_2a_3 + a_3a_4 + \ldots.. + a_{n-1}a_n)x + (a_2^2 + a_3^2 + a_4^2 + \ldots\ldots + a_n^2) \leq 0$

$\Rightarrow\ (a_1x + a_2)^2 + (a_2x + a_3)^2 + (a_3x + a_4)^2 + \ldots\ldots. + (a_{n-1}x + a_n)^2 \leq 0$

$\Rightarrow\ (a_1x + a_2)^2 + (a_2x + a_3)^2 + (a_3x + a_4)^2 + \ldots\ldots. + (a_{n-1}x + a_n)^2 = 0,$

as sum of square cann't be negative.

$\Rightarrow\ a_1x + a_2 = 0 = a_2x + a_3 = a_3x + a_4 = \ldots\ldots. = a_{n-1}x + a_n$

$\Rightarrow\ -x = \dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \dfrac{a_4}{a_3} = \ldots\ldots. = \dfrac{a_n}{a_{n-1}}$

$\Rightarrow\ a_1, a_2, a_3, \ldots\ldots., a_{n-1}, a_n$ are in G.P.

12. The roots of $ax^2 + bx + c = 0$, whose $a \neq 0$, $b, c \in R$, are non-real complex and $a + c < b$. Then

(1) $4a + c > 2b$ (2) $4a + c < 2b$ (3) $4a + c = 2b$ (4) None of these

Sol. Answer (2)

$f(x) = ax^2 + bx + c$, given $f(-1) = a - b + c < 0$

$\therefore\ f(x) < 0,\ \forall\ x \in R$ as roots are non-real complex

$\therefore\ f(-2) < 0 \Rightarrow 4a - 2b + c < 0 \Rightarrow 4a + c < 2b$

13. The number of irrational roots of the equation
$(x - 1)(x - 2)(3x - 2)(3x + 1) = 21$ is
(1) 0 (2) 2 (3) 3 (4) 4

Sol. Answer (2)

$(x-1)(3x-2)(3x+1)(x-2) = 21$

$(3x^2 - 5x + 2)(3x^2 - 5x - 2) = 21$

Put, $3x^2 - 5x = t$

$(t+2)(t-2) = 21 \Rightarrow t^2 = 25 \Rightarrow t = 5,\ t = -5$

Now, $3x^2 - 5x = 5$ and $3x^2 - 5x = -5$

$\Rightarrow 3x^2 - 5x - 5 = 0$ and $3x^2 - 5x + 5 = 0$

$3x^2 - 5x - 5 = 0$ has two irrational roots.

whereas roots of $3x^2 - 5x + 5 = 0$ are imaginary.

14. If α, β are the roots of the equation $ax^2 - bx + c = 0$, then equation $(a + cy)^2 = b^2y$ in y has the roots

(1) $\frac{1}{\alpha}, \frac{1}{\beta}$ (2) α^2, β^2 (3) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (4) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

Sol. Answer (4)

Since, α, β are the roots of the equation $ax^2 - bx + c = 0$

So, $\alpha + \beta = \frac{b}{a},\ \alpha\beta = \frac{c}{a}$

Now, we have to observe root of the equation

$(a + cy)^2 = b^2y$

$\Rightarrow a^2 + 2acy + c^2y^2 = b^2y$

$\Rightarrow c^2y^2 + (2ac - b^2)y + a^2 = 0$

$$\Rightarrow y^2 + \left(\frac{2ac - b^2}{c^2}\right)y + \frac{a^2}{c^2} = 0$$

$$\Rightarrow y^2 - \left(\frac{b^2}{c^2} - \frac{2a}{c}\right)y + \frac{a^2}{c^2} = 0$$

$$\Rightarrow y^2 - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)y + \frac{1}{\alpha^2\beta^2} = 0$$

Hence the equation $(a + cy)^2 = b^2y$ has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

15. If a, b, c are in G.P., then the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(1) A.P. (2) G.P. (3) H.P. (4) $ab = cd$

Sol. Answer (1)

Since a, b, c are in G.P.,

So, $b^2 = ac$

$\Rightarrow$ $4b^2 - 4ac = 0$

$D = 0$ for the equation $ax^2 + 2bx + c = 0$

Hence, it will have equal roots, and root will be

$$x = -\frac{b}{a}$$

Now, $ax^2 + 2bx + c$ and $dx^2 + 2ex + f = 0$ have a common root,

So, $x = -\frac{b}{a}$ will satisfy the equation

$dx^2 + 2ex + f = 0$

$$\Rightarrow d.\frac{b^2}{a^2} - 2e.\frac{b}{a} + f = 0$$

$$\Rightarrow \frac{db^2 - 2aeb + a^2 f}{a^2} = 0$$

$\Rightarrow db^2 - 2aeb + a^2f = 0$

$\Rightarrow dac - 2aeb + a^2f = 0$

$\Rightarrow dc + af = 2eb$

$$\frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$$

So, $\frac{d}{a}, \frac{c}{b}, \frac{f}{c}$ are in A.P.

16. If $a < b < c < d$ and $k > 0$, then the quadratic equation $(x - a)(x - c) + k(x - b)(x - \mathrm{d}) = 0$ has

(1) All roots real and distinct (2) All roots real but not necessarily distinct

(3) All roots real and negative (4) May be imaginary

Sol. Answer (1)

$f(x) = (x - a)(x - c) + k(x - b)(x - d)$

$f(a) = k(a - b)(a - d)$ which is positive

$f(b) = (b - a)(b - c)$ which is negative

$f(c) = k(c - b)(c - d)$ which is negative

$f(d) = (d - a)(d - c)$ which is positive

So, $f(x) = 0$ has a root in the interval (a, b) and another in (c, d). So the roots are real and distinct.

17. If the roots of the quadratic equation $x^2 - ax + 2b = 0$ are prime numbers, then the value of $(a - b)$ is

(1) 0 (2) 2 (3) –2 (4) 4

Sol. Answer (2)

$x^2 - ax + 2b = 0$ (Here *a*, *b* are integers)

Let α, β be roots

Now, sum of roots = *a*

product of roots = 2*b* (an even number)

$\therefore$ '2' is one root

Now, $4 - 2a + 2b = 0 \Rightarrow a - b = 2$

18. The interval to which *x* belongs if

(1) $3^{72}\left(\frac{1}{3}\right)^x\left(\frac{1}{3}\right)^{\sqrt{x}} \ge 1$ is [1, 64]

(2) $x^2 - |x + 2| + x > 0$ is $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(3) $x^2 - x + 1 > 0$ is $(0, \infty)$ only

(4) $x^2 - 4x + 3 < 0$ is (1, 4)

Sol. Answer (2)

(1) The given expression can be written as

$3^{72-x-\sqrt{x}} \ge 1$

$\Rightarrow \quad 72 - x - \sqrt{x} \ge 0$

$\Rightarrow \quad x + \sqrt{x} - 72 \le 0$

$\Rightarrow \quad (\sqrt{x} + 9)(\sqrt{x} - 8) \le 0$

$\Rightarrow \quad x \le 64$

$\Rightarrow \quad 0 \le x \le 6$

(2) If $x \ge -2$

$\Rightarrow \quad x^2 - (x + 2) + x > 0$

$x^2 - 2 > 0$

$\Rightarrow \quad x \in \left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$

But $x \ge -2$

Hence $x \in \left[-2, -\sqrt{2}\right] \cup \left[\sqrt{2}, \infty\right)$

(3) $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0, \forall x \in R$

(4) $x^2 - 4x + 3 < 0$

$\Rightarrow \quad (x - 1)(x - 3) < 0$

$1 < x < 3.$

19. Which of the following is false?

(1) $(\log_5 x)^2 + \log_5 x - 2 < 0$ then $x \in \left(\frac{1}{25}, 5\right)$

(2) The greatest negative integer satisfying $x^2 - 4x - 77 < 0$ and $x^2 > 4$ is –3

(3) $x^{12} - x^9 + x^4 - x + 1 > 0$ then $x \in (-\infty, \infty)$

(4) $\frac{x}{x} = 1, x \in R$

Sol. Answer (4)

(1) $(\log_5 x + 2)(\log_5 x - 1) < 0$

$\Rightarrow \quad -2 < \log_5 x < 1$

$= \dfrac{1}{25} < x < 5$

(2) $x^2 - 4x - 77 < 0$

$\Rightarrow \quad (x-1)(x+7) < 0$

$\Rightarrow \quad -7 < x < 11 \qquad \ldots(i)$

$x^2 - 4 > 0$

$(x - 2)(x + 2) > 0$

$\Rightarrow \quad x \in (-\infty, -2) \cup (2, \infty)$

Hence the greatest negative integer = –3

(3) $y = x^{12} - x^9 + x^4 - x + 1 > 0$

(i) If $x < 0 \Rightarrow y = 0$

(ii) If $x > 1 \Rightarrow y > 1$

(iii) If $x \in (0, 1)$

$\Rightarrow y = (1 - x) + (x^4 - x^9) + x^{11} > 0$

Hence $x \in (-\infty, \infty)$

(4) $\dfrac{x}{x} = 1 \Rightarrow x \neq 0$

SECTION - B

Objective Type Questions (More than one options are correct)

1. If sin α, cos α are the roots of the equation $ax^2 + bx + c = 0$ ($c \neq 0$), then

(1) $a^2 - b^2 + 2ac = 0$ (2) $(a + c)^2 = b^2 + c^2$ (3) $\dfrac{b}{a} \in [-\sqrt{2}, \sqrt{2}]$ (4) $\dfrac{c}{a} \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

Sol. Answer (1, 2, 3, 4)

$\Rightarrow \quad \sin\alpha + \cos\alpha = -\dfrac{b}{a}$

$\Rightarrow \quad \sin\alpha.\cos\alpha = \dfrac{c}{a}$

$\Rightarrow \quad \sin^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \quad \left(-\dfrac{b}{a}\right)^2 - \dfrac{2c}{a} = 1 \Rightarrow b^2 - 2ca = a^2$

$\Rightarrow \quad a^2 - b^2 + 2ac = 0$

Also $(a + c)^2 = b^2 + c^2$.

2. Let a, b, c be real numbers in G.P. such that a and c are positive, then the roots of the equation $ax^2 + bx + c = 0$

(1) Are real and are in the ratio $b : ac$

(2) Are real

(3) Are imaginary and are in ratio $1 : \omega$ where ω is a non-real complex cubic root of constant

(4) Are imaginary and are in the ratio $\omega^2 : 1$ with usual notation

Sol. Answer (3, 4)

$ax^2 + bx + c = 0$

Let a, b, c is a, ar, ar^2

Now, $ax^2 + arx + ar^2 = 0$

$\Rightarrow \quad x^2 + rx + r^2 = 0 \;\Rightarrow\; x = r\left(\dfrac{-1 \pm \sqrt{3}i}{2}\right) \Rightarrow x = \omega\, r$ or $\omega^2 r$

$\therefore$ Roots are imaginary and are in the ratio $1 : \omega$ or $\omega^2 : 1$.

3. Let $\cos\alpha$ be a root of the equation $25x^2 + 5x - 12 = 0$, $-1 < x < 0$, then the value of $\sin 2\alpha$ can be

(1) $\dfrac{20}{25}$ (2) $-\dfrac{12}{25}$ (3) $\dfrac{24}{25}$ (4) $-\dfrac{24}{25}$

Sol. Answer (3, 4)

$25x^2 + 5x - 12 = 0 \;\Rightarrow\; (5x - 3)(5x + 4) = 0$

$\Rightarrow x = \dfrac{3}{5},\; x = -\dfrac{4}{5} \quad \left(x \neq \dfrac{3}{5} \text{ as } -1 < x < 0\right)$

$\therefore \quad \cos\alpha = -\dfrac{4}{5}$

$\Rightarrow \sin\alpha = \dfrac{3}{5}$ or $-\dfrac{3}{5}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \times \dfrac{3}{5}\left(-\dfrac{4}{5}\right) = -\dfrac{24}{25}$

or $2\left(-\dfrac{3}{5}\right)\left(-\dfrac{4}{5}\right) = \dfrac{24}{25}$

4. If the quadratic equations $x^2 + pqx + r = 0$ and $x^2 + prx + q = 0$ have a common root then the equation containing their other roots is/are

(1) $x^2 - p(q + r)x + p^2qr = 0$ (2) $x^2p(q + r) + (q + r)x - pqr = 0$

(3) $p(q + r)x^2 - (q + r)x + pqr = 0$ (4) $x^2 + p(q + r)x - p^2qr = 0$

Sol. Answer (1, 2)

Let α be common root

$\alpha^2 + pq\alpha + r = 0, \qquad \alpha + \beta = -pq,\; \alpha\beta = r$

$\alpha^2 + pr\alpha + q = 0, \qquad \alpha + \gamma = -pr,\; \alpha\gamma = q$

$$\overline{\alpha p(q-r)+r-q=0} \quad \Rightarrow \quad \alpha=\frac{1}{p}$$

Common root is $\alpha = \dfrac{1}{p}$

Other roots are, $\beta = rp$ and $\gamma = qp$

$\therefore$ Equation containing other roots is

$x^2 - p(r+q)x + p^2rq = 0$

$\because$ $\dfrac{1}{p}$ is common root $\Rightarrow \left(\dfrac{1}{p}\right)^2 + pq\left(\dfrac{1}{p}\right) + r = 0$

$\Rightarrow \dfrac{1}{p^2} = -(q+r)$

Now, $x^2 - p(q+r)x + p^2qr = 0$

$\Rightarrow -p\left[-\dfrac{p}{p^2}x^2 + (q+r)x - pqr\right] = 0$

$\Rightarrow p(q+r)x^2 + (q+r)x - pqr = 0$

5. The quadratic equation $x^2 - (m-3)x + m = 0$ has

(1) Real and distinct roots if and only if $m \in (-\infty, 1) \cup (9, \infty)$

(2) Both roots positive if and only if $m \in (9, \infty)$

(3) Both roots negative if and only if $m \in (0, 1)$

(4) Both roots negative if $m \in (0, 3)$

Sol. Answer (1, 2, 4)

$x^2 - (m-3)x + m = 0$

For real distinct roots, $(m-3)^2 - 4m > 0$

$\Rightarrow \quad m^2 - 10m + 9 > 0$

$\Rightarrow \quad (m-9)(m-1) > 0 \quad \Rightarrow \quad m \in (-\infty, 1) \cup (9, \infty)$ …(i)

For positive roots,

Sum > 0, product > 0

$\Rightarrow m - 3 > 0$, $m > 0$ …(ii)

From (i) and (ii), $m \in (9, \infty)$

For negative roots

sum < 0, product > 0

$\Rightarrow m - 3 < 0$, $m > 0$ …(iii)

From (i) and (iii), $m \in (0, 1)$

6. If both roots of the equation $x^2 - 2ax + a^2 - 1 = 0$ lies strictly between –3 and 4, then $[a]$ can be, where [ ] represents the greatest integer function

(1) 1 (2) –1 (3) 2 (4) 0

Sol. Answer (1, 2, 3, 4)

$x^2 - 2ax + a^2 - 1 = 0$

$(x - a)^2 = 1 \Rightarrow x = a + 1,\ a - 1.$

Now, $-3 \le [a+1] < 4$ and $-3 \le [a-1] \le 4$

$\Rightarrow -3 \le [a] + 1 < 4$ and $-3 \le [a] - 1 \le 4$

$\Rightarrow -4 \le [a] < 3$ and $-2 \le [a] \le 5$

$\Rightarrow -2 \le [a] < 3 \Rightarrow [a] = -2, -1, 0, 1, 2$

7. Let α, β be the roots of $x^2 - 4x + A = 0$ and γ, δ be the roots of $x^2 - 36x + B = 0$. If $\alpha, \beta, \gamma, \delta$ forms an increasing G.P. Then

(1) $B = 81A$ (2) $A = 3$ (3) $B = 243$ (4) $A + B = 251$

Sol. Answer (1, 2, 3)

$\alpha + \beta = 4,\ \alpha\beta = A$

$\gamma + \delta = 36,\ \gamma\delta = B$

Let $\alpha, \beta, \gamma, \delta$ be a, ar, ar^2, ar^3

$a + ar = 4$

$ar^2 + ar^3 = 36 \Rightarrow \frac{1+r}{r^2(1+r)} = \frac{1}{9}$

$\Rightarrow r^2 = 9 \Rightarrow r = \pm 3,\ a = 1.$

$A = \alpha\beta \Rightarrow a(ar) = A \Rightarrow A = 3$

$B = \gamma\delta \Rightarrow B = (ar^2)(ar^3) \Rightarrow B = 243 \Rightarrow B = 81A.$

8. For the equation $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$, which one of the following is true?

(1) Has at least one real solution (2) Has exactly three real solutions

(3) Has exactly one irrational solutions (4) Has non-real complex roots

Sol. Answer (1, 2, 3)

$x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$

Taking log with base 2 on both side.

$$\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]\log_2 x = \log_2 \sqrt{2} = \frac{1}{2}$$

Put $\log_2 x = t$, $\left(\frac{3}{4}t^2 + t - \frac{5}{4}\right)t = \frac{1}{2}$

$\Rightarrow 3t^3 + 4t^2 - 5t - 2 = 0 \Rightarrow (t-1)(3t^2 + 7t + 2) = 0$

$\Rightarrow \ (t-1)(3t+1)(t+2) = 0$

$\Rightarrow \ t = 1, -\frac{1}{3}, -2 \Rightarrow \ \log_2 x = 1, -\frac{1}{3}, -2 \Rightarrow \ x = 2, \ 2^{-1/3}, 2^{-2}$

9. If $f(x) = ax^2 + bx + c$, $g(x) = -ax^2 + bx + c$, where $ac \neq 0$ then $f(x).g(x) = 0$ has

(1) At least three real roots
(2) No real roots
(3) At least two real roots
(4) At most two imaginary roots

Sol. Answer (3, 4)

Let D_1 and D_2 be the respective discriminates.

then $D_1 = b^2 - 4ac$ and $D_2 = b^2 + 4ac$

Adding we get $D_1 + D_2 = 2b^2$

$\because \ D_1 + D_2$ is positive

$\Rightarrow$ at least one of D_1 or D_2 is positive.

$\Rightarrow$ at least 2 real roots.

10. The value of a for which the equation $x^2 + ax + a^2 + 6a < 0$ is satisfied for all $x \in (1, 2)$

(1) $\left(\frac{-7-\sqrt{45}}{2}, -2\right)$
(2) $\left(\frac{-7-\sqrt{45}}{2}, -3\right)$
(3) $\left(\frac{-7-\sqrt{45}}{2}, -4+2\sqrt{3}\right)$
(4) $\left(\frac{-7+\sqrt{45}}{2}, 4+3\sqrt{3}\right)$

Sol. Answer (1, 2, 3)

Let α, β be the roots of the corresponding equation

$x^2 + ax + a^2 + 6a = 0$ ...(i)

As the coefficient of $x^2 = 1 > 0$ $x^2 + ax + a^2 + 6x < 0$ will be satisfied for all values of $x \in (\alpha, \beta)$ if α, β are real and unequal (let $\alpha < \beta$).

Hence the inequality will hold for all real $x \in (1, 2)$ if the interval (1, 2) is a subject of the interval (α, β). Thus for (1) we should have $D > 0$ and $\alpha < 1$, $\beta > 1$ as well as $\alpha < 2$, $\beta > 2$.

Now, $D > 0 \Rightarrow a^2 - 4(a^2 + 6a) > 0$

$\Rightarrow a^2 + 8a < 0$

$\Rightarrow a \in (-8, 0)$ ...(ii)

$\alpha < 1, \beta > 1 \ \Rightarrow \ \alpha - 1 < 0, \beta - 1 > 0$

$(\alpha - 1)(\beta - 1) < 0$

$\Rightarrow a^2 + 7a + 1 < 0$

$a \in \left(\frac{-7-\sqrt{45}}{2}, \frac{-7+\sqrt{45}}{2}\right)$ ...(iii)

$\alpha < 2, \beta > 2 \Rightarrow (\alpha - 2)(\beta - 2) < 0$

$\Rightarrow a^2 + 8a + 4 < 0$

$\Rightarrow a \in (-4 - 2\sqrt{3}, -4 + 2\sqrt{3})$ ...(iv)

Common values of a satisfying (ii), (iii) and (iv) are

$a \in \left(\dfrac{-7 - \sqrt{45}}{2}, -4 + 2\sqrt{3}\right)$ ...(v)

Hence answer is (1), (2), (3) those are subset of (v)

11. If the roots of the equation $\dfrac{1}{x+p} + \dfrac{1}{x+q} = \dfrac{1}{r}$ are equal in magnitude but opposite in sign and its product is α

(1) $p + q = r$ (2) $p + q = 2r$ (3) $\alpha^2 = \dfrac{p^2 + q^2}{2}$ (4) $\alpha = -\left(\dfrac{p^2 + q^2}{2}\right)$

Sol. Answer (2, 4)

The given equation is

$$\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$$

$$\Rightarrow \frac{x+q+x+p}{(x+p)(x+q)} = \frac{1}{r}$$

$\Rightarrow (2x + p + q)r = x^2 + (p + q)x + pq$

$\Rightarrow x^2 + (p + q - 2r)x + (pq - qr - rp) = 0$

According to the question the given equation has roots equal in magnitude but opposite in sign, hence

Coefficient of $x = 0$

$\Rightarrow p + q - 2r = 0$

$\Rightarrow r = \dfrac{p+q}{2}$

Product of roots

$= -[(p + q)r - pq]$

$= -\left(\dfrac{(p+q)^2}{2} - pq\right)$

$= -\dfrac{1}{2}(p^2 + q^2)$

12. The integral values of a for which the equation $(a + 2)x^2 + 2(a + 1)x + a = 0$ will have both roots integers

(1) 0 (2) – 1 (3) – 2 (4) – 3

Sol. Answer (1, 2, 4)

$(a + 2)x^2 + 2(a + 1)x + a = 0$

Let α, β be roots

$\alpha + \beta = \dfrac{-2(a+1)}{a+2}$ (integer)

$\alpha\beta = \frac{a}{a+2}$ (integer)

$\frac{a}{a+2}$ will integer if

For $a = 0$, $a = -1$, $a = -3$

Also for $a = 0$, $a = -1$, $a = -3$

$\alpha + \beta = \frac{-2(a+1)}{a+2}$ is integer

13. If $(x - 1)^2$ is a factor of $ax^3 + bx^2 + c$, then roots of the equation $cx^3 + bx + a = 0$ may be

(1) 1 (2) – 1 (3) – 2 (4) 0

Sol. Answer (1, 3)

Since 1 is the repeated roots of $ax^3 + bx^2 + c = 0$

So, $1 + 1 + \alpha = -\frac{b}{a}$

$1.1 + \alpha + \alpha = 0 \Rightarrow \alpha = -\frac{1}{2}$

$1.1.\alpha = -\frac{c}{a} = -\frac{1}{2} \Rightarrow \frac{c}{a} = \frac{1}{2}$

$\frac{b}{a} = -\frac{3}{2}$

$\therefore \frac{b}{c} = -3$

Now, by the equation,

$cx^3 + bx + a = 0$

$\Rightarrow x^3 + \frac{b}{c}x + \frac{a}{c} = 0$

$\Rightarrow x^3 - 3x + 2 = 0$

$x^3 - x^2 + x^2 - x - 2x + 2 = 0$

$\Rightarrow x^2(x - 1) + x(x - 1) - 2(x - 1) = 0$

$\Rightarrow (x - 1)(x^2 + x - 2) = 0$

$\Rightarrow (x - 1)(x^2 + 2x - x - 2) = 0$

$\Rightarrow (x - 1)(x - 1)(x + 2) = 0$

$\Rightarrow x = +1, -2$

Hence answer is (1), (3)

14. If $b^2 \geq 4ac$ for the equation $ax^4 + bx^2 + c = 0$ then all roots of the equation will be real if

(1) $b > 0$, $a < 0$, $c > 0$ (2) $b > 0$, $a > 0$, $c > 0$

(3) $b < 0$, $a > 0$, $c > 0$ (4) $b > 0$, $a < 0$, $c < 0$

Sol. Answer (3, 4)

Let $x^2 = y$

So the equation $ay^2 + by + c = 0$ should have both roots non-negative in order to all roots of the equation $ax^4 + bx^2 + c = 0$ are real for this

$$\alpha + \beta = -\frac{b}{a} > 0 \Rightarrow \frac{b}{a} < 0 \qquad \text{...(i)}$$

$$\alpha\beta = \frac{c}{a} > 0 \qquad \text{...(ii)}$$

From (i) and (ii)

$b > 0,\ a < 0,\ c < 0$

or $b < 0,\ a > 0,\ c > 0$

15. The difference between the roots of the equation $x^2 + kx + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of k is

(1) $(-3, 0)$ (2) $(0, 3)$ (3) $(-3, 3)$ (4) $(3, \infty)$

Sol. Answer (1, 2, 3)

$|\alpha - \beta| \propto \sqrt{5}$

$\Rightarrow (\alpha - \beta)^2 < 5$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta < 5$

$\Rightarrow k^2 - 4 < 5$

$\Rightarrow k^2 < 9 \Rightarrow k \in (-3, 3)$

Hence answers is (1, 2, 3)

16. The set of real values of a for which $a^2 + 2a$, $2a + 3$ and $a^2 + 3a + 8$ are the sides of a triangle may be

(1) $\left(6, \frac{13}{2}\right)$ (2) $(5, 7)$ (3) $(5, \infty)$ (4) $(0, 5)$

Sol. Answer (1, 2, 3)

We know that in a triangle sum of two sides of a triangle is greater than third side.

So, $a^2 + 2a + 2a + 3 > a^2 + 3a + 8 \Rightarrow 4a > 3a + 5 \Rightarrow a > 5$

$a^2 + 2a + a^2 + 3a + 8 > 2a + 3 \Rightarrow 2a^2 + 3a + 5 > 0 \Rightarrow a \in R$

$2a + 3 + a^2 + 3a + 8 > a^2 + 2a \Rightarrow 3a > -11 \Rightarrow a > -\frac{11}{3}$

Combining these three,

$a \in (5, \infty)$

Hence answer is (1, 2, 3)

SECTION - C

Linked Comprehension Type Questions

Comprehension-I

Let us consider an equation $f(x) = x^3 - 3x + k = 0$. Then the values of k for which the equation has

1. Exactly one root which is positive, then k belongs to

(1) $(-\infty, -2)$ (2) $(2, \infty)$ (3) $(0, 2)$ (4) $(-2, 0)$

Sol. Answer (1)

$f(x) = x^3 - 3x + k$

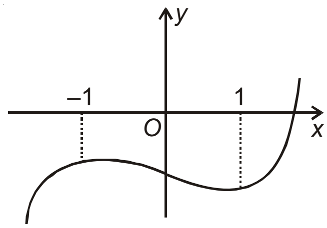
$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

For exactly one positive root

$f(-1) < 0$ and $f(1) < 0$

$\Rightarrow -1 + 3 + k < 0$ and $1 - 3 + k < 0$

$\Rightarrow k < -2$ and $k < 2 \Rightarrow k \in (-\infty, -2)$



2. Exactly one root which is negative, then k belongs to

(1) (2, ∞) (2) (0, 2) (3) (–2, 0) (4) (–∞, –2)

Sol. Answer (1)

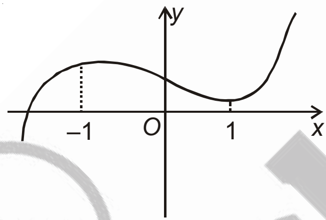
For exactly one negative root,

$f(-1) > 0$, $f(1) > 0$

$\Rightarrow -1 + 3 + k > 0, 1 - 3 + k > 0$

$\Rightarrow k > -2, k > 2$

$\Rightarrow k \in (2, \infty)$



3. One negative and two positive root if k belongs to

(1) (2, ∞) (2) (0, 2) (3) (–2, 0) (4) (2, 3)

Sol. Answer (2)

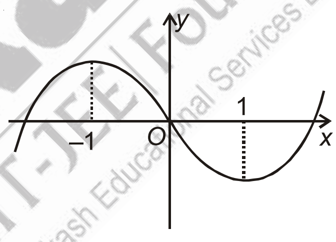
For one negative and two positive root

$f(-1) > 0$, $f(0) > 0$, $f(1) < 0$

$\Rightarrow -1 + 3 + k > 0, k > 0, 1 - 3 + k < 0$

$\Rightarrow k > -2, k > 0, k < 2$

$\Rightarrow 0 < k < 2$ *i.e.*, $k \in (0, 2)$



Comprehension-II

The values of 'k' for which the equation $|x|^2 (|x|^2 - 2k + 1) = 1 - k^2$, has

1. No real root, when k belongs to

(1) (–1, 1) (2) $\left(-1, \frac{5}{4}\right)$ (3) $(-\infty, -1) \cup \left(\frac{5}{4}, \infty\right)$ (4) R

Sol. Answer (3)

$|x|^2 (|x|^2 - 2k + 1) = 1 - k^2$

$\Rightarrow x^2(x^2 - 2k + 1) = 1 - k^2 \Rightarrow x^4 - (2k-1)x^2 + k^2 - 1 = 0$

All roots are imaginary, if $D = b^2 - 4ac < 0$

$\Rightarrow (2k-1)^2 - 4(k^2 - 1) < 0$

$\Rightarrow k > \frac{5}{4}$ ...(1)

Also roots are imaginary if $D > 0$, but x^2 is negative, i.e. roots of $(x^2)^2 - (2k-1)(x^2) + k^2 - 1 = 0$ are both negative.

$\therefore$ Sum < 0, and product > 0

$\Rightarrow\ 2k - 1 < 0$ and $k^2 - 1 > 0 \Rightarrow k \in (-\infty, -1)$

$\therefore$ All roots are imaginary if $k \in (-\infty, -1) \cup \left(\frac{5}{4}, \infty\right)$

2. Exactly two real roots, when k belongs to

(1) $(-\infty, -1)$ (2) $(-1, 1)$ (3) $\left(1, \frac{5}{4}\right)$ (4) R

Sol. Answer (2)

For exactly two real roots of

$t^2 - (2k-1)t + k^2 - 1 = 0$

D > 0 and one value of $t = x^2$ is positive and one is negative.

$\Rightarrow\ (2k-1)^2 - 4(k^2 - 1) > 0$

$\Rightarrow\ k < \frac{5}{4}$ …(i)

Product = $k^2 - 1 < 0 \ \Rightarrow\ -1 < k < 1$ …(ii)

From (1) and (2) $k \in (-1, 1)$.

3. Repeated roots, when k belongs to

(1) {1, –1} (2) {0, 1} (3) {0, –1} (4) {2, 3}

Sol. Answer (1)

For repeated roots Δ = 0, $k = \frac{5}{4}$

or $P = 0, \ \Rightarrow\ k^2 - 1 = 0, k = \pm 1$

When product = 0, x = 0 is repeated root.

Comprehension-III

Let $f(x) = px^2 + qx + r$, $p \neq 0$, $p, q, r \in Z$. Suppose that $f(1) = 0$, $50 < f(7) < 60$ and $70 < f(8) < 80$.

1. The least value of $px^2 + qx + r$ is

(1) $\frac{3}{4}$ (2) $\frac{9}{2}$ (3) $-\frac{9}{8}$ (4) $\frac{2}{5}$

Sol. Answer (3)

Given, $f(1) = p + q + r = 0$, $50 < 49p + 7q + r < 60$ and $70 < 64p + 8q + r < 80$

$\Rightarrow\ 8p + q = 9$ and $9p + q = 11 \quad \Rightarrow\ p = 2$, $q = -7$ and $r = 5$

$\Rightarrow\ f(x) = 2x^2 - 7x + 5$

Least value = $\frac{-D}{4a} = \frac{-(49-40)}{4.2} = \frac{-9}{8}$

2. Number of integral values of x for which $f(x) < 0$ is

(1) 0 (2) 1 (3) 2 (4) 3

Sol. Answer (2)

Given, $f(1) = p + q + r = 0$, $50 < 49p + 7q + r < 60$ and $70 < 64p + 8q + r < 80$

$\Rightarrow$ $8p + q = 9$ and $9p + q = 11 \Rightarrow$ $p = 2$, $q = -7$ and $r = 5$

$\Rightarrow$ $f(x) = 2x^2 - 7x + 5$

$f(x) < 0 \Rightarrow 2x^2 - 7x + 5 < 0 \Rightarrow x \in \left(1, \frac{5}{2}\right)$

$\Rightarrow$ $x = 2$ only integral solution.

SECTION - D

Matrix-Match Type Questions

1. For the quadratic equation $x^2 - (k - 3)x + k = 0$, match the condition in column I with the corresponding values of k in column II.

Column-I	Column-II
(A) Both the roots are positive	(p) ϕ
(B) Both the roots are negative	(q) $(-\infty, 1) \cup (9, \infty)$
(C) Both the roots are real	(r) $[10, \infty)$
(D) One root is less than –1 and other is greater than 1	(s) $(0, 1]$
	(t) $[125, 1250]$

Sol. Answer A(r, t), B(s), C(q, r, s, t), D(p)

$x^2 - (k - 3)x + k = 0$

For roots to be real

$(k - 3)^2 - 4k \geq 0$

$\Rightarrow$ $k^2 - 6k + 9 - 4k \geq 0 \Rightarrow k^2 - 10k + 9 \geq 0 \Rightarrow (k - 1)(k - 9) \geq 0$

$\Rightarrow$ $k \in (-\infty, 1] \cup [9, \infty]$ …(i)

(A) For both roots to be positive,

$f(0) > 0$ and $\frac{k-3}{2} > 0$

$k > 0$ …(ii)

and $k > 3$ …(iii)

From (i), (ii) and (iii)

$k \in [9, \infty]$

(B) For both roots to be negative

$D \geq 0$

$k > 0$, $(k - 3) / 2 < 0$, $\Rightarrow k < 3$, $\Rightarrow k \in (0, 1]$

(C) For both roots to be real $k \in (-\infty, 1] \cup [9, \infty)$

(D) $f(-1) < 0$, $f(1) < 0$

$1 + (k - 3) + k < 0$ also $1 - (k - 3) + k < 0$

$\Rightarrow 2k - 2 < 0 \Rightarrow k < 1$, $4 < 0$

No such value is possible

2. Match the following

Column-I	Column-II
(A) If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then a belongs to	(p) a does not exist
(B) If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and greater than 3, then a belongs to	(q) $\left(\frac{3}{2}, 3\right)$
(C) If exactly one root of the above equation lies in the interval (1, 3), then a belongs to	(r) $(-1, 2) \cup (2, 3)$
(D) If the roots of the above equation are such that one root is greater than 3 and other roots is smaller than 1, then a belongs to	(s) $(-\infty, 2)$

Sol. Answer A(s), B(p), C(r), D(p)

(A) For real roots, $D \geq 0$

$\Rightarrow (-2a)^2 - 4(a^2 + a - 3) \geq 0$

$\Rightarrow 4[a^2 - a^2 - a + 3) \geq 0 \quad \Rightarrow a \leq 3$ …(i)

$f(3) > 0 \Rightarrow 9 - 6a + a^2 + a - 3 > 0$

$\Rightarrow a^2 - 5a + 6 > 0$

$\Rightarrow a \in (-\infty, 2) \cup (3, \infty)$ …(ii)

$-\frac{b}{2a} < 3 \Rightarrow \frac{2a}{2} < 3 \Rightarrow a < 3$ …(iii)

from (i), (ii) and (iii) $a \in (-\infty, 2)$

(B) $D \geq 0 \Rightarrow a \leq 3$ …(i)

$f(3) > 0 \Rightarrow a \in (-\infty, 2) \cup (3, \infty)$ …(ii)

For greater than 3,

$-\frac{b}{2a} > 3 \Rightarrow a > 3$ …(iii)

From (i), (ii) and (iii), $a \in \phi$.

(C) $D > 0 \Rightarrow a < 3$ …(i)

$f(1)\, f(3) < 0$

$(1 - 2a + a^2 + a - 3)(9 - 6a + a^2 + a - 3) < 0$

$\Rightarrow (a^2 - a - 2)(a^2 - 5a + 6) < 0 \Rightarrow (a-2)^2(a+1)(a-3) < 0$

$\Rightarrow a \in (-1, 3) - \{2\}$ …(ii)

From (i) and (ii)

$a \in (-1, 2) \cup (2, 3)$

(D) $D \geq 0 \Rightarrow a \leq 3$ …(i)

$f(1) < 0 \Rightarrow (1-2a+a^2+a-3)<0$

$\Rightarrow (a^2-a-2)<0$

$\Rightarrow -1 < a < 2$ …(ii)

$f(3) < 0 \Rightarrow (9-6a+a^2+a-3)<0$

$\Rightarrow a^2-5a+6<0$

$\Rightarrow 2 < a < 3$ …(iii)

From (i), (ii) and (iii)

$a \in \phi$.

3. For given equation $x^2 - ax + b = 0$, match conditions in column I with possible values in column II

Column-I	Column-II
(A) If roots differ by unity then a^2 is equal to	(p) $b(ab + 2)$
(B) If roots differ by unity then $1 + a^2$ is equal to	(q) $1 + 4b$
(C) If one of the root be twice the other then $2a^2$ is equal to	(r) $2(1 + 2b)$
(D) If the sum of roots of the equation equal to the sum of squares of their reciprocal then a^2 is equal to	(s) $9b$

Sol. Answer A(q), B(r), C(s), D(p)

(A) $\rightarrow$ (q), (B) $\rightarrow$ (r)

$x^2-ax+b=0$

$\alpha+\beta=a$, $\alpha\beta=b$

$|\alpha-\beta|=1=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\sqrt{a^2-4b}$

$\Rightarrow a^2-4b=1$

$\Rightarrow a^2=1+4b$

$\Rightarrow a^2+1=2+4b=2(1+2b)$

(C) $\alpha+2\alpha=a \Rightarrow \alpha=\dfrac{a}{3}$

$2\alpha.\alpha=b \Rightarrow \alpha^2=\dfrac{b}{2}$

$\left(\dfrac{a}{3}\right)^2=\dfrac{b}{2} \Rightarrow a^2=\dfrac{9b}{2} \Rightarrow 2a^2=9b$.

(D) $\alpha+\beta=\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$

$(\alpha+\beta)(\alpha^2\beta^2)=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\Rightarrow a(b)^2=a^2-2b$

$\Rightarrow a^2=ab^2+2b=b(ab+2)$

4. If α, β, γ be the roots of the equation $x(1 + x^2) + x^2(6 + x) + 2 = 0$, then match the entries of column-I with those of column-II.

Column I	Column II
(A) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ is less than or equal to	(p) 8
(B) $\alpha^2 + \beta^2 + \gamma^2$ equals	(q) $-\frac{1}{2}$
(C) $(\alpha^{-1} + \beta^{-1} + \gamma^{-1}) - (\alpha + \beta + \gamma)$ is less than or equal to	(r) –1
(D) $[\alpha^{-1} + \beta^{-1} + \gamma^{-1}]$ equals where $[\cdot]$ denotes the greatest integer less than or equal to	(s) 3
	(t) $\frac{5}{2}$

Sol. Answer A(p, q, s, t), B(p), C(p, s, t), D(r)

α, β, γ be the roots of the equation

$x(1 + x^2) + x^2(6 + x) + 2 = 0$

$\Rightarrow x^3 + x + 6x^2 + x^3 + 2 = 0$

$\Rightarrow 2x^3 + 6x^2 + x + 2 = 0$

So, $\alpha + \beta + \gamma = -3$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{1}{2}$

$\alpha\beta\gamma = -1$

Now,

(A) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}=-\frac{1}{2}$

(B) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 9-2\cdot\frac{1}{2}=8$

(C) $(\alpha^{-1} + \beta^{-1} + \gamma^{-1}) - (\alpha + \beta + \gamma) = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}-(\alpha+\beta+\gamma) = -\frac{1}{2}+3=\frac{5}{2}$

(D) $[\alpha^{-1} + \beta^{-1} + \gamma^{-1}] = \left[-\frac{1}{2}\right]=-1$

SECTION - E

Assertion-Reason Type Questions

1. STATEMENT-1 : A root of the equation $(2^{10} - 3)x^2 - 2^{11}x + (2^{10} + 3) = 0$ is 1.

and

STATEMENT-2 : The sum of the coefficients of a quadratic equation is zero, then 1 is a root of the equation.

Sol. Answer (1)

$ax^2 + bx + c = 0$

$x = 1$, $a + b + c = 0$

If sum of coefficient is 0 then 1 is the root of the equation.

$(2^{10} - 3) - 2^{11} + 2^{10} + 3 = 0$

$\therefore$ Both are true and Statement-2 is correct explanation of Statement-1

2. STATEMENT-1 : The equation whose roots are reciprocal of the roots of the equation $10x^2 - x - 5 = 0$ is $5x^2 + x - 10 = 0$.

and

STATEMENT-2 : To obtain a quadratic equation whose roots are reciprocal of the roots of the given equation $ax^2 + bx + c = 0$ change the coefficients a, b, c to c, b, a. ($c \neq 0$)

Sol. Answer (1)

For reciprocal roots, replacing x by $\frac{1}{x}$ in $ax^2 + bx + c = 0$ $\frac{a}{x^2} + \frac{b}{x} + c = 0 \Rightarrow cx^2 + bx + a = 0$

Statement-2 is correct and is correct explanation of Statement-1

$$10x^2 - x - 5 = 0 \Rightarrow \frac{10}{x^2} - \frac{1}{x} - 5 = 0 \Rightarrow 5x^2 + x - 10 = 0$$

3. STATEMENT-1 : The equation $x^2 - 2009x + 2008 = 0$ has rational roots.

and

STATEMENT-2 : The quadratic equation $ax^2 + bx + c = 0$ has rational roots iff $b^2 - 4ac$ is a perfect square.

Sol. Answer (3)

The equation in first statement is $x^2 - 2009x + 2008 = 0$ can be written as $(x - 2008)(x - 1) = 0$

$\Rightarrow$ $x = 1, 2008$ are roots of the equation where are rationals also.

$\Rightarrow$ Statement 1 is True.

Statement 2 is not always true. When $D = b^2 - 4ac$ = a perfect square than roots of the equation $ax^2 + bx + c = 0$ are rational only when a, b, c are rationals, otherwise roots are irrationals.

To this end, let us consider an equation $4x^2 - 4\sqrt{3}x - 1 = 0$

whose discriminant = 48 + 16 = 64 = 8^2 = a perfect square but roots are

$$x = \frac{4\sqrt{3} \pm \sqrt{48+16}}{2 \times 4} = \frac{\sqrt{3} \pm 2}{2}$$ which are not rationals.

Thus statement 2 is false.

Hence option (3) is correct.

4. STATEMENT-1 : The quadratic equation $ax^2 + bx + c = 0$ has real roots if $(a + c)^2 > b^2$, $\forall\, a, b, c \in R$.

and

STATEMENT-2 : The quadratic equation $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac \geq 0$.

Sol. Answer (4)

We observe that

$(a + c)^2 > b^2$

$\Rightarrow (a + c)^2 - b^2 > 0$

$\Rightarrow (a - b + c)(a + b + c) > 0$

$\Rightarrow f(-1)f(1) > 0$, where $f(x) = ax^2 + bx + c$

$\Rightarrow f(x) = ax^2 + bx + c = 0$ has either no root in $(-1, 1)$ or if real roots exist, then both roots lie in $(-1, 1)$

$\Rightarrow$ Statement 1 is not necessarily true

Hence statement 1 is false.

5. STATEMENT-1 : There is just one quadratic equation with real coefficients, one of whose roots is $\frac{1}{3+\sqrt{7}}$.

and

STATEMENT-2 : In a quadratic equation with rational coefficients the irrational roots occur in pair.

Sol. Answer (4)

Statement-1 is wrong.

$\frac{1}{3+\sqrt{7}}$ can be root of infinite equations with real coefficients, *e.g.,* $(x-1)\left(x-\frac{1}{3}+\sqrt{7}\right)=0$

$(x-2)\left(x-\frac{1}{3}+\sqrt{7}\right)=0...$

6. STATEMENT-1 : The roots of $x^2+2\sqrt{2008}x+501=0$ are irrational.

and

STATEMENT-2 : If the discriminant of the equation $ax^2 + bx + c = 0$, $a \neq 0$ (a, b, c, $\in R$) is a perfect square, then the roots are rational.

Sol. Answer (3)

$x^2+2\sqrt{2008}x+501=0$

$x=\frac{-2\sqrt{2008}\pm\sqrt{4\times 2008-4(501)}}{2} = -\sqrt{2008}\pm\sqrt{1507}$

Roots are rational.

STATEMENT-2 is wrong as roots are rational only when coefficients are rational and $b^2 = 4ac$ is perfect square.

7. STATEMENT-1 : If a, b, c not all equal and $a \neq 0$, $a^3 + b^3 + c^3 = 3abc$, then the equation $ax^2 + bx + c = 0$ has two real roots of opposite sign.

and

STATEMENT-2 : If roots of a quadratic equation $ax^2 + bx + c = 0$ are real and of opposite sign then $ac < 0$.

Sol. Answer (4)

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$,

$a + b + c = 0 \therefore a, b, c$ are not equal. The sign of other root depends on sign of ac.

Hence (4) is answer.

8. STATEMENT-1 : Let a quadratic equation has a root $3 - 9i$ then the sum of roots is 6.

and

STATEMENT-2 : If one root of $ax^2 + bx + c = 0$, $a \neq 0$ $a, b, c \in R$ is $\alpha + i\beta$, $\alpha, \beta \in R$ then the other root must be $\alpha - i\beta$.

Sol. Answer (4)

For, $ax^2+bx+c=0$, roots are conjugate of each other if a, b, c are real, which is not mentioned in statement-1.

SECTION - F

Integer Answer Type Questions

1. The number of value(s) of k, for which both the roots of the equation $x^2 - 6kx + 9(k^2 - k + 1) = 0$ are real, distinct and have values atmost 3 is ________.

Sol. Answer (0)

$x^2 - 6kx + 9(k^2 - k + 1) = 0$

For real and distinct roots

$D > 0$

$\Rightarrow k^2 - (k^2 - k + 1) > 0$

$\Rightarrow k - 1 > 0$ ...(i)

$f(3) \geq 0$

$9 - 18k + 9(k^2 - k + 1) \geq 0$

$\Rightarrow 1 - 2k + k^2 - k + 1 \geq 0$

$\Rightarrow k^2 - 3k + 2 \geq 0$

$k \in (-\infty, 1] \cup [2, \infty)$ ...(ii)

Also $\dfrac{\text{sum of roots}}{2} < 3$

$\Rightarrow 3k < 3$

$k < 1$ ...(iii)

From (i), (ii), (iii)

We observe that there does not exist any real value of k.

2. The possible greatest integral value of a for which the expression $\dfrac{ax^2+3x+4}{x^2+2x+2}$ is less than 5 for all real x is ________.

Sol. Answer (2)

We have

$$y = \frac{ax^2 + 3x + 4}{x^2 + 2x + 2}$$

As $\dfrac{ax^2 + 3x + 4}{x^2 + 2x + 2} < 5$

$$\Rightarrow \frac{ax^2 + 3x + 4 - 5x^2 - 10x - 10}{x^2 + 2x + 2} < 0$$

$\Rightarrow (a - 5)x^2 - 7x - 6 < 0$ (as $x^2 + 2x + 2 > 0, \forall x \in R$)

It is satisfied for all x if

$a - 5 < 0, 49 + 24(a - 5) < 0$

$a < \dfrac{71}{24}$

$\Rightarrow a < 3$

The possible greatest integral value of a is 2.

3. Let $f(x) = ax^2 + bx + c$, where a, b, c are real numbers. If the numbers $2a$, $a + b$ and c are all integers, then the number of integral values between 1 and 5 that $f(x)$ can take is ________.

Sol. Answer (5)

Firstly, let $f(x) = ax^2 + bx + c$; $a, b, c \in R$ be an integer whenever x is an integer

$\Rightarrow f(0), f(1), f(-1)$ are integers

$\Rightarrow$ c, $a + b + c$, $a - b + c$ are integers

$\Rightarrow c$, $a + b + c - c$, $a - b + c - c$ are integers

$\Rightarrow c$, $a + b$, $a - b$ are integers

$\Rightarrow c$, $a + b$, $a + b + a - b$ are integers

$\Rightarrow c$, $a + b$, $2a$ are integers

Secondly let $2a$, $a + b$ and c be integers. Let x be an integer.

Then $f(x) = ax^2 + bx + c = 2a\left(\frac{x(x-1)}{2}\right) + (a+b)x + c$

Since x is an integer $\Rightarrow x(x - 1)$ is an even integer.

$\Rightarrow 2a\left(\frac{x(x-1)}{2}\right) + (a+b)x + c$ is an integer as $2a$, $a + b$, c are integers.

$\Rightarrow f(x)$ is an integer for all integer x.

4. If $x_1, x_2, x_3, x_4, x_5, x_6$ are non-zero distinct real roots of the equation $x^6 + ax^4 + bx + \lambda = 0$ for some $a, b, \lambda \in R$, then the value of $\prod_{i=1}^{6}(1 - x_i) - \prod_{i=1}^{6}(1 + x_i) = kb$, then k is equal to

Sol. Answer (2)

$(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6) = x^6 + ax^4 + bx + \lambda$

$\Rightarrow (1 - x_1)(1 - x_2) \ldots. (1 - x_6) = 1 + a + b + \lambda$ and $(1 + x_1)(1 + x_2) \ldots. (1 + x_6) = 1 + a - b + \lambda$

$\Rightarrow \underset{i=1}{\overset{6}{\pi}}(1 - x_i) - \underset{i=1}{\overset{6}{\pi}}(1 + x_i) = 2b$

5. If α, β are the roots of the equation $ax^2 - 15x + b = 0$ and $T_n = \alpha^n + \beta^n$, then $\frac{aT_{2019} + bT_{2017}}{5T_{2018}}$ is equal to

Sol. Answer (3)

$$\frac{a(\alpha^{2019} + \beta^{2019}) + b(\alpha^{2017} + \beta^{2017})}{5(\alpha^{2018} + \beta^{2018})} = 3$$

6. If n_1, n_2, n_3, n_4 are distinct integers such that $\prod_{i=1}^{4}(x - n_i) = 4$ has an integral root 15. Then sum of digits of $\prod_{i=1}^{4} n_i$ is equal to

Sol. Answer (4)

$(\alpha - n_1)(\alpha - n_2)(\alpha - n_3)(\alpha - n_4) = 4$

Let $n_1 < n_2 < n_3 < n_4$

Then $\alpha - n_1 > \alpha - n_2 > \alpha - n_3 > \alpha - n_4$

It follows that

$\alpha - n_4 = -2,\ \alpha - n_3 = -1,\ \alpha - n_2 = 1,\ \alpha - n_4 = 2$

7. If α, β are roots of the equation $x^2 + x + 2 = 0$ and γ, δ are the roots of the equation $x^2 - x + 7 = 0$, then the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ and having unity as leading coefficient will have constant term k, then value of $(50 + k)$ is

Sol. Answer (3)

$(\alpha\gamma + \beta\delta)\ (\alpha\delta + \beta\gamma)$

$= \gamma\delta\ (\alpha^2 + \beta^2) + \beta\alpha\ (\gamma^2 + \delta^2)$

$= 7\ (-3) + 2\ (-13)$

$= -47 \Rightarrow 50 + k = 3$

8. The number of solutions for the equation $\sum_{i=1}^{100} |x - i| = 2450$ will be

Sol. Answer (0)

As we can see that:

$$|x - k| + |x - (101 - k)| \geq |101 - 2k|$$

Adding all:-

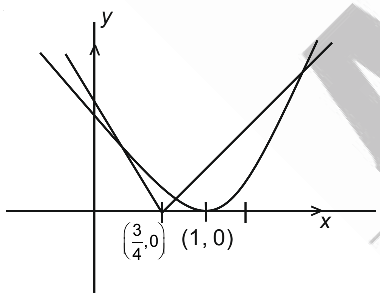
$$\sum_{i=1}^{i=100} |x - i| \geq 2500$$

9. The sum of the values of λ if the equation $(x - 1)^2 = |x - \lambda|$ has exactly three solutions, is

Sol. Answer (3)

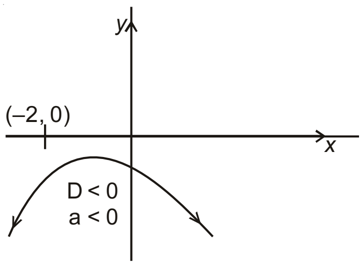
As we can see graphically such these cases at

$\lambda = \frac{3}{4}, \frac{5}{4}, 1$ exits



10. If $ax^2 + bx + c = 0$ does not have any real root and $4a + c < 2b$ then the number of real solutions of the equation $x^4 + 3x^2 - a = 0$ is

Sol. Answer (0)



As $ax^2 + bx + c < 0$ for all $x \in R$

hence $D < 0$ and $a < 0$

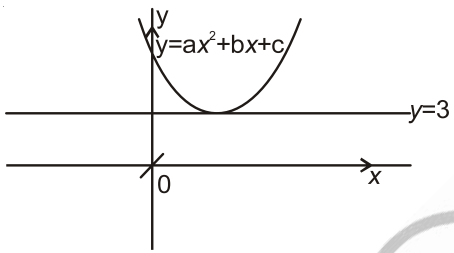
$\Rightarrow x^4 + 3x^2 \geq 0 \ \forall \ x \in R$

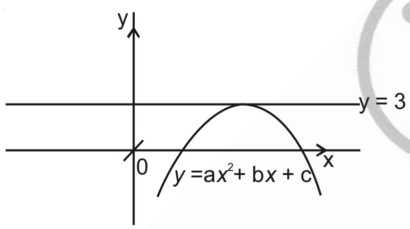
$\Rightarrow x^4 + 3x^2 \neq a$ for any $x \in R$

11. The value of $\frac{-D}{a}$ for $D = b^2 - 4ac$ of $ax^2 + bx + c = 0$ and $a, b, c \in R - \{0\}$, if $y = ax^2 + bx + c$ and $|y - 3| \leq 0$ have exactly one intersection point is

Sol. Answer (12)

Hence $\frac{-D}{4a} = 3 \Rightarrow \frac{D}{a} = -12$





12. The number of solutions of equation $|\sqrt{x} - 2| + \sqrt{x}(\sqrt{x} - 4) + 2 = 0$, is

Sol. Answer (2)

We have,

$$|\sqrt{x} - 2| + \sqrt{x}(\sqrt{x} - 4) + 2 = 0$$

$$\Rightarrow |\sqrt{x} - 2| + (\sqrt{x})^2 - 4\sqrt{x} + 2 = 0$$

$$\Rightarrow |\sqrt{x} - 2| + |\sqrt{x} - 2|^2 - 2 = 0$$

$$\Rightarrow |\sqrt{x} - 2| = -2, 1$$

$\therefore x = 1, 9$

Exactly two solutions are possible.

13. If $a + b + c = -46$ and the roots α_1, α_2 and α_3 of $x^3 + ax^2 + bx + c = 0$ are integers and greater than 2 then $(\alpha_1 - \alpha_2 + \alpha_3)$ is equal to

Sol. Answer (6)

$\because$ Roots are $\alpha_1, \alpha_2, \alpha_3$

then $\alpha_1 + \alpha_2 + \alpha_3 = -a$, $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = b$

and $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = -c$

Now, $a + b + c + 1 = -45 \Rightarrow -a - b - c - 1 = 45$

i.e., $(\alpha_1+\alpha_2+\alpha_3)-(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)+\alpha_1\alpha_2\alpha_3-1=45$

$\Rightarrow\ (\alpha_1-1)(\alpha_2-1)(\alpha_3-1)=3\times3\times5$

$a_1 = 4,\ a_2 = 4,\ a_3 = 6$

$\therefore\ \alpha_1-\alpha_2+\alpha_3=6$

14. Consider the following equation in real *x*, *y* : $(x-2y-1)^2+(4x-3y-4)^2+(x-2y-1)(4x-3y-4)=0$. Then the number of solutions (*x*, *y*) is/are

Sol. Answer (1)

The given expression is of form

$$\lambda^2+\mu^2+\lambda\mu=0$$

$\therefore$ It has no real solution.

But $\lambda = 0$ and $\mu = 0$ are two lines which intersect each other at a fixed point (1, 0) *i.e.*, solution.

15. If *p* and *q* are distinct zeroes of polynomial $x^3 - 2x + r$ and $p^2(2p^2 + 4pq + 3q^2) = 3$, then $q^2(3p^2 + 4pq + 2q^2)$ is equal to

Sol. Answer (5)

$(p^3-2p+r)-(q^3-2q+r)=0 \Rightarrow p^2+pq+q^2=2$

Let $\lambda=p^2(2p^2+4pq+3q^2)$, $\mu=(q^2(3p^2+4pq+2q^2))$

$\therefore\ \lambda+\mu-8=2(pq+2)(p^2+pq+q^2-2)=0$

$\Rightarrow\ 3+\mu-8=0$

$\therefore\ \mu=5$

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