16

Logarithms

INTRODUCTION

Questions based on this chapter are not so frequent in management entrance exams. In exams, most problems that have used the concept of logs have been of an applied nature. However, the aspirants should know the basic concepts of logarithms to ensure there are no surprises in the paper.

While studying this chapter the student should pay particular attention to the basic rules of logarithms as well as develop an understanding of the range of the values of logs.

THEORY

Let a be a positive real number, $a \neq 1$ and $a^x = m$. Then x is called the logarithm of m to the base a and is written as $\log_a m$, and conversely, if $\log_a m = x$, then $a^x = m$.

Note: Logarithm to a negative base is not defined.

Also, logarithm of a negative number is not defined. Hence, in the above logarithmic equation, $\log_a m = x$, and we can say that $m > 0$ and $a > 0$.

Thus $a^x = m \Rightarrow x = \log_a m$ and $\log_a m = x \Rightarrow a^x = m$

In short, $a^x = m \Rightarrow x = \log_a m$.

$x = \log_a m$ is called the logarithmic form and $a^x = m$ is called the exponential form of the equation connecting a, x and m.

Two Properties of Logarithms

1. **$\log_a 1 = 0$ for all $a > 0, a \neq 1$**
   That is, log 1 to any base is zero
   Let $\log_a 1 = x$. Then by definition, $a^x = 1$
   But $a^0 = 1 \therefore a^x = a^0 \Leftrightarrow x = 0$.
   Hence $\log_a 1 = 0$ for all $a > 0, a \neq 1$
2. **$\log_a a = 1$ for all $a > 0, a \neq 1$**
   That is, log of a number to the same base is 1
   Let $\log_a a = x$. Then by definition, $a^x = a$.
   But $a^1 = a \therefore a^x = a^1 \Rightarrow x = 1$.
   Hence $\log_a a = 1$ for all $a > 0, a \neq 1$.

Laws of Logarithms

First Law: $\log_a (mn) = \log_a m + \log_a n$
That is, log of product = sum of logs

Second Law: $\log_a (m/n) = \log_a m - \log_a n$
That is, log of quotient = difference of logs

Note: The first theorem converts a problem of multiplication into a problem of addition and the second theorem converts a problem of division into a problem of subtraction, which are far easier to perform than multiplication or division. That is why logarithms are so useful in all numerical calculations.

Third Law: $\log_a m^n = n \log_a m$

Generalisation

1. $\log (mnp) = \log m + \log n + \log p$
2. $\log (a_1 a_2 a_3 \ldots a_k) = \log a_1 + \log a_2 + \ldots + \log a_k$

Note: *Common logarithms*: We shall assume that the base $a = 10$ whenever it is not indicated. Therefore, we shall denote $\log_{10} m$ by $\log m$ only. The logarithm calculated to base 10 are called common logarithms.

The Characteristic and Mantissa of a Logarithm

The logarithm of a number consists of two parts: the *integral* part and the *decimal* part. The integral part is known as the *characteristic* and the decimal part is called the *mantissa*.

For example,

In log 3257 = 3.5128, the integral part is 3 and the decimal part is .5128; therefore, characteristic = 3 and mantissa = .5128.

It should be remembered that the mantissa is always written as positive.

Rule: To make the mantissa positive (in case the value of the logarithm of a number is negative), subtract 1 from the integral part and add 1 to the decimal part.

Thus, $-3.4328 = -(3 + .4328) = -3 - 0.4328$
$= (-3 - 1) + (1 - 0.4328)$
$= -4 + .5672.$

so the mantissa is = .5672.

Note: The characteristic may be positive or negative. When the characteristic is negative, it is represented by putting a bar on the number.

Thus instead of –4, we write $\bar{4}$.

Hence we may write –4 + .5672 as $\bar{4}.5672$.

Base Change Rule

Till now all rules and theorems you have studied in Logarithms have been related to operations on logs with the same basis. However, there are a lot of situations in Logarithm problems where you have to operate on logs having different basis. The base change rule is used in such situations.

This rule states that

(i) $\log_a (b) = \log_c (b)/\log_c (a)$

It is one of the most important rules for solving logarithms.

(ii) $\log_b (a) = \log_c (a)/\log_b (c)$

A corollary of this rule is

(iii) $\log_a (b) = 1/\log_b (a)$

(iv) log c to the base a^b is equal to $\frac{\log a^c}{b}$.

Results on Logarithmic Inequalities

(a) If $a > 1$ and $\log_a x_1 > \log_a x_2$ then $x_1 > x_2$

(b) If $a < 1$ and $\log_a x_1 > \log_a x_2$ then $x_1 < x_2$

Applied conclusions for logarithms

1. The characteristic of common logarithms of any positive number less than 1 is negative.
2. The characteristic of common logarithm of any number greater than 1 is positive.
3. If the logarithm to any base a gives the characteristic n, then we can say that the number of integers possible is given by $a^{n+1} - a^n$.

   Example: $\log_{10} x = 2.bcde$..., then the number of integral values that x can take is given by: $10^{2+1} - 10^2 = 900$. This can be physically verified as follows.

   Log to the base 10 gives a characteristic of 2 for all 3 digit numbers with the lowest being 100 and the highest being 999. Hence, there are 900 integral values possible for x.
4. If $-n$ is the characteristic of $\log_{10} y$, then the number of zeros between the decimal and the first significant number after the decimal is $n - 1$.

   Thus if the log of a number has a characteristic of –3 then the first two decimal places after the decimal point will be zeros.

   Thus, the value will be –3.00*ab*…

Space for Notes

WORKED-OUT PROBLEMS

Problem 16.1 Find the value of x in $3^{|3x-4|} = 9^{2x-2}$

(a) 8/7 (b) 7/8
(c) 7/4 (d) 16/7

Solution Take the log of both sides, then we get,

$$|3x-4| \log 3 = (2x-2)\log 9$$
$$= (2x-2)\log 3^2$$
$$= (4x-4)\log 3$$

Dividing both sides by log 3, we get

$$|3x-4| = (4x-4) \quad (1)$$

Now, $|3x-4| = 3x-4$ if $x > 4/3$

so if $x > 4/3$

$3x-4 = 4x-4$

or $3x = 4x$

or $3 = 4$

But this is not possible.

Let's take the case of $x < 4/3$

Then $|3x-4| = 4-3x$

Therefore, $4-3x = 4x-4$ from (1)

or $7x = 8$

or $x = 8/7$

Problem 16.2 Solve for x.

$$\log_{10}x - \log_{10}\sqrt{x} = 2\log_x 10$$

Solution Now, $\log_{10}\sqrt{x} = \frac{1}{2} \times \log_{10}x$

Therefore, the equation becomes

$$\log_{10}x - \frac{1}{2}\log_{10}x = 2\log_x 10$$

or $\frac{1}{2}\log_{10}x = 2\log_x 10$ (2)

Using base change rule ($\log_b a = 1/\log_a b$)

Therefore, equation (2) becomes

$$\frac{1}{2}\log_{10}x = 2/\log_{10}x$$

$\Rightarrow$ $(\log_{10}x)^2 = 4$

or $\log_{10}x = 2$

Therefore, $x = 100$

Problem 16.3 If $7^{x+1} - 7^{x-1} = 48$, find x.

Solution Take 7^{x-1} as the common term. The equation then reduces to

$$7^{x-1}(7^2-1) = 48$$

or $7^{x-1} = 1$

or $x-1 = 0$ or $x = 1$

Problem 16.4 Calculate: $\log_2(2/3) + \log_4(9/4)$

$= \log_2(2/3) + (\log_2(9/4)/\log_2 4$

$= \log_2(2/3) + 1/2\log_2(9/4)$

$= \log_2(2/3) + 1/2(2\log_2 3/2)$

$= \log_2 2/3 + \log_2 3/2 = \log_2 1 = 0$

Problem 16.5 Find the value of the expression

$1/\log_3 2 + 2/\log_9 4 - 3/\log_{27}8$

Passing to base 2

we get

$$\log_2 3 + 2\log_{2^2}9 - 3\log_{2^3}27$$
$$= \log_2 3 + \frac{4\log_2 3}{2} - \frac{9\log_2 3}{3}$$
$$= 3\log_2 3 - 3\log_2 3$$
$$= 0$$

Problem 16.6 Solve the inequality.

(a) $\log_2(x+3) < 2$

$\Rightarrow$ $2^2 > x+3$

$\Rightarrow$ $4 > x+3$

$1 > x$

or $x < 1$

But log of negative number is not possible.

Therefore, $x+3 \geq 0$

That is, $x \geq -3$

Therefore, $-3 \leq x < 1$

(b) $\log_2(x^2-5x+5) > 0$

$= x^2-5x+5 > 1$

$\rightarrow x^2-5x+4 > 0$

$\rightarrow (x-4)(x-1) > 0$

Therefore, the value of x will lie outside 1 and 4.

That is, $x > 4$ or $x < 1$.

Space for Rough Work

LEVEL OF DIFFICULTY (I)

1. log 32700 = ?
   (a) log 3.27 + 4 (b) log 3.27 + 2
   (c) 2 log 327 (d) $100 \times \log 327$
2. log . 0867 = ?
   (a) log 8.67 + 2 (b) log 8.67 – 2
   (c) $\frac{\log 867}{1000}$ (d) –2 log 8.67
3. If $\log_{10}2 = .301$ find $\log_{10}125$.
   (a) 2.097 (b) 2.301
   (c) 2.10 (d) 2.087
4. $\log_{32} 8$ = ?
   (a) 2/5 (b) 5/3
   (c) 3/5 (d) 4/5

Find the value of x in equations 5–6.

5. $\log_{0.5}x = 25$
   (a) 2^{-25} (b) 2^{25}
   (c) 2^{-24} (d) 2^{24}
6. $\log_3 x = \frac{1}{2}$
   (a) 3 (b) $\sqrt{3}$
   (c) $\frac{3}{2}$ (d) $\frac{2}{3}$
7. $\log_{15}3375 \times \log_4 1024$ = ?
   (a) 16 (b) 18
   (c) 12 (d) 15
8. $\log_a 4 + \log_a 16 + \log_a 64 + \log_a 256 = 10$. Then a = ?
   (a) 4 (b) 2
   (c) 8 (d) 5
9. $\log_{625} \sqrt{5}$ = ?
   (a) 4 (b) 8
   (c) 1/8 (d) 1/4
10. If log x + log (x + 3) = 1 then the value(s) of x will be, the solution of the equation
    (a) $x + x + 3 = 1$ (b) $x + x + 3 = 10$
    (c) $x(x + 3) = 10$ (d) $x(x + 3) = 1$
11. If $\log_{10}a = b$, find the value of 10^{3b} in terms of a.
    (a) a^3 (b) $3a$
    (c) $a \times 1000$ (d) $a \times 100$
12. 3 log 5 + 2 log 4 – log 2 = ?
    (a) 4 (b) 3
    (c) 200 (d) 1000

Solve equations 13–25 for the value of x.

13. log (3x – 2) = 1
    (a) 3 (b) 2
    (c) 4 (d) 6
14. log (2x – 3) = 2
    (a) 103 (b) 51.5
    (c) 25.75 (d) 26
15. log (12 – x) = –1
    (a) 11.6 (b) 12.1
    (c) 11 (d) 11.9
16. $\log (x^2 - 6x + 6) = 0$
    (a) 5 (b) 1
    (c) Both (a) and (b) (d) 3 and 2
17. $\log 2^x = 3$
    (a) 9.87 (b) 3 log 2
    (c) 3/log 2 (d) 9.31
18. $3^x = 7$
    (a) $1/\log_7 3$ (b) $\log_7 3$
    (c) $1/\log_3 7$ (d) $\log_3 7$
19. $5^x = 10$
    (a) log 5 (b) log 10/log 2
    (c) log 2 (d) 1/log 5
20. Find x, if $0.01^x = 2$
    (a) log 2/2 (b) 2/log 2
    (c) –2/log 2 (d) – log 2/2
21. Find x if log x = log 7.2 – log 2.4
    (a) 1 (b) 2
    (c) 3 (d) 4
22. Find x if log x = log 1.5 + log 12
    (a) 12 (b) 8
    (c) 18 (d) 15
23. Find x if log x = 2 log 5 + 3 log 2
    (a) 50 (b) 100
    (c) 150 (d) 200
24. log (x – 13) + 3 log 2 = log (3x + 1)
    (a) 20 (b) 21
    (c) 22 (d) 24
25. log (2x – 2) – log (11.66 – x) = 1 + log 3
    (a) 452/32 (b) 350/32
    (c) 11 (d) 11.33

Space for Rough Work

LEVEL OF DIFFICULTY (II)

1. Express log $\frac{\sqrt[3]{a^2}}{b^5\sqrt{c}}$ or $\frac{a^{2/3}}{b^5\sqrt{c}}$ in terms of log a, log b and log c.

   (a) $\frac{3}{2}$ log a + 5 log b – 2 log c

   (b) $\frac{2}{3}$ log a – 5 log b – $\frac{1}{2}$ log c

   (c) $\frac{2}{3}$ log a – 5 log b + $\frac{1}{2}$ log c

   (d) $\frac{3}{2}$ log a + 5 log b – $\frac{1}{2}$ log c

2. If log 3 = .4771, find log $(.81)^2 \times \log\left(\frac{27}{10}\right)^{\frac{2}{3}} \div \log 9$.

   (a) 2.689 (b) – 0.0552
   (c) 2.2402 (d) 2.702

3. If log 2 = .301, log 3 = .477, find the number of digits in $(108)^{10}$.

   (a) 21 (b) 27
   (c) 20 (d) 18

4. If log 2 = .301, find the number of digits in $(125)^{25}$.

   (a) 53 (b) 50
   (c) 25 (d) 63

5. Which of the following options represents the value of log $\sqrt{128}$ to the base .625?

   (a) $\frac{2+\log_8 2}{\log_8 5 - 1}$ (b) $\frac{\log_8 128}{2\log_8 0.625}$

   (c) $\frac{2+\log_8 2}{2(\log_8 5 - 1)}$ (d) Both (b) and (c)

6-8. Solve for x:

$$\log\frac{75}{35} + 2\log\frac{7}{5} - \log\frac{105}{x} - \log\frac{13}{25} = 0.$$

   (a) 90 (b) 65
   (c) 13 (d) 45

7. $2\log\frac{4}{3} - \log\frac{x}{10} + \log\frac{63}{160} = 0$

   (a) 7 (b) 14
   (c) 9 (d) 3

8. $\log\frac{12}{13} - \log\frac{7}{25} + \log\frac{91}{3} = x$

   (a) 0 (b) 1
   (c) 2 (d) 3

Questions 9 to 11: Which one of the following is true

9. (a) $\log_{17}275 = \log_{19}375$ (b) $\log_{17}275 < \log_{19}375$
   (c) $\log_{17}275 > \log_{19}375$ (d) Cannot be determined

10. (a) $\log_{11}1650 > \log_{13}1950$
    (b) $\log_{11}1650 < \log_{13}1950$
    (c) $\log_{11}1650 = \log_{13}1950$
    (d) None of these

11. (a) $\frac{\log_2 4096}{3} = \log_8 4096$

    (b) $\frac{\log_2 4096}{3} < \log_8 4096$

    (c) $\frac{\log_2 4096}{3} > \log_8 4096$

    (d) Cannot be determined

12. $\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} = \log x$, x = ?

    (a) 2 (b) 3
    (c) 0 (d) None of these

If log 2 = 0.301 and log 3 = .4771 then find the number of digits in the following.

13. 60^{12}

    (a) 25 (b) 22
    (c) 23 (d) 24

14. 72^9

    (a) 17 (b) 20
    (c) 18 (d) 15

15. 27^{25}

    (a) 38 (b) 37
    (c) 36 (d) 35

Questions 16 to 18: Find the value of the logarithmic expression in the questions below.

16. $\frac{\log\sqrt{27} + \log 8 - \log\sqrt{1000}}{\log 1.2}$

    where, $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.4771213$

    (a) 1.77 (b) 1.37
    (c) 2.33 (d) 1.49

17. $\frac{1}{\log_{xy}(xyz)} + \frac{1}{\log_{yz}(xyz)} + \frac{1}{\log_{zx}(xyz)} =$

    (a) 1 (b) 2
    (c) 3 (d) 4

18. $\log a^n/b^n + \log b^n/c^n + \log c^n/a^n$

    (a) 1 (b) n
    (c) 0 (d) 2

19. $\log_{10}x - \log_{10}\sqrt{x} = 2\log_x 10$ then x = ?
(a) 50 (b) 100
(c) 150 (d) 200

20. $\left(\frac{21}{10}\right)^x = 2$. Then x = ?
(a) $\frac{\log 2}{\log 3 + \log 7 - 1}$ (b) $\frac{\log 2}{\log 3 + \log 7 + 1}$
(c) $\frac{\log 3}{\log 2 + \log 7 - 1}$ (d) $\frac{\log 2}{\log 3 - \log 7 + 1}$

21. $\log(x^3 + 5) = 3\log(x + 2)$ then x = ?
(a) $\frac{-2+\sqrt{2}}{2}$ (b) $\frac{-2-\sqrt{2}}{2}$
(c) Both (a) and (b) (d) None of these

22. $(a^4 - 2a^2b^2 + b^4)^{x-1} = (a - b)^{-2}(a + b)^{-2}$ then x = ?
(a) 1 (b) 0
(c) None of these (d) 2

23. If $\log_{10} 242 = a$, $\log_{10} 80 = b$ and $\log_{10} 45 = c$, express $\log_{10} 36$ in terms of a, b and c.
(a) $\frac{(c-1)(3c+b-4)}{2}$ (b) $\frac{(c-1)(3c+b-4)}{3}$
(c) $\frac{(c-1)(3c-b-4)}{2}$ (d) None of these

24. For the above problem, express $\log_{10} 66$ in terms of a, b and c.
(a) $\frac{(c-1)(3c+b-4)}{8}$ (b) $\frac{3(a+c)+(2b+-5)}{6}$
(c) $\frac{3(a+c)+(2b-5)}{6}$ (d) $\frac{3(c-1)(3c+b-4)}{6}$

25. $\log_2(9 - 2^x) = 10^{\log(3-x)}$. Solve for x.
(a) 0 (b) 3
(c) Both (a) and (b) (d) 0 and 6

26. If $\frac{\log_x}{b-c} = \frac{\log_y}{c-a} = \frac{\log_z}{a-b}$. Mark all the correct options. **IIFT 2006**
(a) $xyz = 1$ (b) $x^a y^b z^c = 1$
(c) $x^{b+c} y^{c+a} z^{a+b} = 1$ (d) All the options are correct.

27. What will be the value of x if it is given that:
$$\log_x\left[\frac{1}{5}+\frac{1}{12}+\frac{1}{21}+\frac{1}{32}+\frac{1}{45}+\ldots+\infty \text{ terms}\right]^2 = 2$$

28. $(\log_4 x^2)(x\log_{27}8)(\log_x 243)$ is equal to:
(a) $2x$ (b) $5x$
(c) $3x$ (d) 1

29. For how many real values of x will the equation $\log_3 \log_6(x^3 - 18x^2 + 108x) = \log_2 \log_4 16$ be satisfied?

30. If $n = 12\sqrt{3}$
$$\frac{1}{\log_2 n}+\frac{1}{\log_3 n}+\frac{1}{\log_4 n}+\frac{1}{\log_6 n}+\frac{1}{\log_8 n}+\frac{1}{\log_9 n}+\frac{1}{\log_{18} n} = ?$$

31. $(\log_2 x)^2 + 2\log_2 x - 8 = 0$,Where x is a natural number. If $x^p = 64$, then what is the value of $x + p$.

Directions for 41 and 42: $A = \sum_{i=2}^{a}\log_3(i), B = \sum_{j=2}^{b}\log_3(j)$ & $C = \sum_{k=2}^{(a-b)}\log_3\log_3 k$, where $a \geq b$. If D = A – B – C. Then answer the following questions.

32. If $a = 10$ then for what value of b, D is minimum

33. For $a = 6$, D is maximum for b =

34. If 'p' and 'q' are integers and $\log_p(-q^2 + 6q - 8) + \log_q(-2p^2 + 20p - 48) = 0$ then $p \times q$ = ?

Space for Rough Work

ANSWER KEY

Level of Difficulty (I)

1. (a)	2. (b)	3. (a)	4. (c)
5. (a)	6. (b)	7. (d)	8. (a)
9. (c)	10. (c)	11. (a)	12. (b)
13. (c)	14. (b)	15. (d)	16. (c)
17. (c)	18. (a)	19. (d)	20. (d)
21. (c)	22. (c)	23. (d)	24. (b)
25. (c)			

Level of Difficulty (II)

1. (b)	2. (b)	3. (a)	4. (a)
5. (d)	6. (c)	7. (a)	8. (c)
9. (b)	10. (a)	11. (a)	12. (d)
13. (b)	14. (a)	15. (c)	16. (d)
17. (b)	18. (c)	19. (b)	20. (a)
21. (c)	22. (b)	23. (d)	24. (c)
25. (a)	26. (d)	27. 25/48	28. (b)
29. 1	30. 4	31. 7	32. 10
33. 3	34. 15		

Solutions and Shortcuts

Level of Difficulty (I)

1. Log 32700 = log 3.27 + log 10000 = log 3.27 + 4
2. Log 0.0867 = log (8.67/100) = log 8.67 – log 100
   Log 8.67 – 2
3. $\log_{10} 125 = \log_{10}(1000/8) = \log 1000 - 3\log 2$
   $= 3 - 3 \times 0.301 = 2.097$
4. $\log_{32} 8 = \log 8/\log 32$ (By base change rule)
   $= 3 \log 2/5\log 2 = 3/5$.
5. $\log_{0.5} x = 25 \Rightarrow x = 0.5^{25} = (½)^{25} = 2^{-25}$
6. $x = 3^{1/2} = \sqrt{3}$.
7. $\log_{15}3375 \times \text{Log}_4 1024$
   $= 3 \log_{15} 15 \times 5 \log_4 4 = 3 \times 5 = 15$.
8. The given expression is:
   $\text{Log}_a (4 \times 16 \times 64 \times 256) = 10$
   i.e. $\log_a 4^{10} = 10$
   Thus, $a = 4$.
9. $1/2 \log_{625} 5 = [1/(2 \times 4)] \log_5 5 = 1/8$.
10. $\log x (x + 3) = 1 \Rightarrow 10^1 = x^2 + 3x$.
    or $x(x + 3) = 10$.
11. $\log_{10} a = b \Rightarrow 10^b = a \Rightarrow$ By definition of logs.
    Thus $10^{3b} = (10^b)^3 = a^3$.
12. 3 log 5 + 2 log 4 – log 2
    = log 125 + log 16 – log 2
    = log (125 × 16)/2 = log 1000 = 3.
13. $10^1 = 3x - 2 \Rightarrow x = 4$.
14. $10^2 = 2x - 3 \Rightarrow x = 51.5$
15. $1/10 = 12 - x \Rightarrow x = 11.9$
16. $x^2 - 6x + 6 = 10° \Rightarrow x^2 - 6x + 6 = 1$
    $\Rightarrow x^2 - 6x + 5 = 0$
    Solving gives us $x = 5$ and 1.
17. $x \log 2 = 3$
    $\log 2 = 3/x$.
    Therefore, $x = 3/\log 2$
18. $3^x = 7 \Rightarrow \log_3 7 = x$
    Hence $x = 1/\log_7 3$
19. $x = \log_5 10 = 1/\log_{10} 5 = 1/\log 5$.
20. $x = \log_{0.01} 2 = -\log 2/2$.
21. $\log x = \log (7.2/2.4) = \log 3 \Rightarrow x = 3$
22. $\log x = \log 18 \Rightarrow x = 18$
23. $\log x = \log 25 + \log 8 = \log (25 \times 8) = \log 200$.
24. $\log (x - 13) + \log 8 = \log[3x + 1]$
    $\Rightarrow \log (8x - 104) = \log (3x + 1)$
    $\Rightarrow 8x - 104 = 3x + 1$
    $5x = 105 \Rightarrow x = 21$
25. $\log (2x - 2)/(11.66 - x) = \log 30$
    $\Rightarrow (2x - 2)/(11.66 - x) = 30$
    $2x - 2 = 350 - 30x$
    Hence, $32x = 352 \Rightarrow x = 11$.

Level of Difficulty (II)

1. 2/3 log a – 5 log b – 1/2 log c.
2. 2 Log (81/100) × 2/3 log (27/10) ÷ log 9
   $= 2 [\log 3^4 - \log 100] \times 2/3 [(\log 3^3 - \log 10)] \div 2 \log 3$
   $= 2 [\log 3^4 - \log 100] \times 2/3 [(3\log 3 - 1)] \div 2 \log 3$
   Substitute log 3 = 0.4771 ⇒ – 0.0552.
3. Let the number be y.
   $y = 108^{10}$
   $\Rightarrow \log y = 10 \log 108$
   $\text{Log } y = 10 \log (27 \times 4)$
   $\text{Log } y = 10 [3\log 3 + 2 \log 2]$
   $\text{Log } y = 10 [1.43 + 0.602]$
   Hence $\log y = 10[2.03] = 20.3$
   Thus, y has 21 digits.
4. $\log y = 25 \log 125$
   $= 25 [\log 1000 - 3 \log 2] = 25 \times (2.097)$
   = 52 +
   Hence 53 digits.
5. $0.5 \log_{0.625} 128$
   $= 0.5 [\log_8 128/\log_8 0.625]$
   $= 1/2 [\log_8 128/\log_8 0.625]$
   $$\frac{\text{Log}_8 128}{2(\log_8 5 - \log_8 8)} = \frac{\text{Log}_8 128}{2[\log_8 5 - 1]} = \frac{2 + \log_8 2}{2(\log_8 5 - 1)}$$
6. $(75/35) \times (49/25) \times (x/105) \times (25/13) = 1$
   $\Rightarrow x = 13$
7. $(16/9) \times (10/x) \times (63/160) = 1$
   $\Rightarrow x = 7$
8. Solve in similar fashion.

9. $\text{Log}_{17} 275 < \log_{19} 375$

Because the value of $\text{Log}_{17}275$ is less than 2 while $\log_{19}375$ is greater then 2.

10. $\log_{11}1650 > 3$

$\text{Log}_{13}1950 < 3$

Hence, $\log_{11}1650 > \log_{13}1950$

11. $\frac{\log_2 4096}{3} = \log_8 4096$

12. $x = (16/15) \times (25^5/24^5) \times (81^3/80^3)$

None of these is correct.

13 – 15.

Solve similarly as 3 and 4.

18. $\log (a^n b^n c^n / a^n b^n c^n) = \log 1 = 0$

19. $(1/2) \log x = 2 \log_x 10$

$\Rightarrow \quad \log x = 4 \log_x 10$

$\Rightarrow \quad \log x = 4/\log_{10} x \quad \Rightarrow \quad (\log x)^2 = 4$

So $\log x = 2$ and $x = 100$.

20. $x = \log_{(21/10)} 2$

$$= \frac{\text{Log}\,2}{\text{Log}\,21 - \log 10} = \frac{\text{Log}\,2}{[\text{Log}\,3 + \log 7 - 1]}$$

21. $6x^2 + 12x + 3 = 0 \quad \text{or} \quad 2x^2 + 4x + 1 = 0$

Solving we get both the options (a) and (b) as correct. Hence, option (c) is the correct answer.

25. For $x = 0$, we have LHS

$\text{Log}_2 8 = 3$.

RHS: $10^{\log 3} = 3$.

We do not get LHS = RHS for either $x = 3$ or $x = 6$. Thus, option (a) is correct.

26. $\frac{\log x}{b - c} = \frac{\log y}{c - a} = \frac{\log z}{a - b} = k$

$\Rightarrow x = 10^{k(b-c)}, y = 10^{k(c - a)}, z = 10^{k(a - b)}$

$\therefore xyz = 10^{k(b - c + c - a + a - b)} = 10^0 = 1$

Therefore option (a) is correct.

$x^a y^b z^c = 10^{k[a(b-c) + b(c - a) + c(a - b)]}$

$= 10^{k(ab - ac + bc - ab + ca - bc)}$

$= 10^{K.0} = 1$

Therefore option (b) is correct.

$x^{b + c} y^{c + a} z^{a + b} = 10^{k[(b + c)(b - c) + (c + a)(c - a) + (a + b)(a - b)]}$

$= 10^{k.0} = 1$

Therefore option (c) is also correct.

Since all the first three options are correct, we choose option (d) as the correct answer.

27. $\log_x \left[\frac{1}{5} + \frac{1}{12} + \frac{1}{21} + \frac{1}{32} + \frac{1}{45} + \ldots + \infty \text{ terms}\right]^2$

$= 2 \log_x (\frac{1}{5} + \frac{1}{12} + \frac{1}{21} + \frac{1}{32} + \frac{1}{45} + \ldots + \infty \text{ terms})$

Let $\frac{1}{5} + \frac{1}{12} + \frac{1}{21} + \frac{1}{32} + \frac{1}{45} + \ldots + \infty \text{ terms} = P$

$P = \frac{1}{4}\left[\frac{4}{1\times 5} + \frac{4}{2\times 6} + \frac{4}{3\times 7} + \frac{4}{4\times 8} + \frac{4}{5\times 9} + \ldots \infty\right]$

$4P = \left[1 - \frac{1}{5} + \frac{1}{2} - \frac{1}{6} + \frac{1}{3} - \frac{1}{7} + \frac{1}{4} - \frac{1}{8} + \frac{1}{5} - \frac{1}{9}\right]$

$4P = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right]$

$4P = \frac{25}{12}$

$P = \frac{25}{48}$

$\log_x \frac{25}{48} = 1 \text{ or } x = 25/48$

28. $\log_4 x^2 . x\log_{27} 8 . \log_x 243 = \frac{2\log x}{\log 4} . \frac{x\log 8}{\log 27} . \frac{\log 243}{\log x}$

$= \frac{\log x}{\log 2} . \frac{3x\log 2}{3\log 3} . \frac{5\log 3}{\log x} = 5x$

29. $\log_2(\log_4 16) = \log_2 \log_4 4^2 = \log_2 2 = 1$

$\log_3 \log_6 (x^3 - 18x^2 + 108x) = 1$

$\log_6(x^3 - 18x^2 + 108x) = 3$

$x^3 - 18x^2 + 108x = 6^3$

$x^3 - 18x^2 + 108x - 216 = 0$

$(x - 6)^3 = 0$

$x = 6$ is the only value for which the above equation is true.

30. $n = 12\sqrt{3} = 2^2 \times 3^{1.5}$

$\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \frac{1}{\log_6 n}$

$+ \frac{1}{\log_8 n} + \frac{1}{\log_9 n} + \frac{1}{\log_{18} n}$

$= \log_n 2 + \log_n 3 + \log_n 4 + \log_n 6 + \log_n 8 + \log_n 9 + \log_n 18$

$= \log_n (2 \times 3 \times 4 \times 6 \times 8 \times 9 \times 18)$

$= \log_n (2^8 \times 3^6)$

$= \log_n (2^2 \times 3^{1.5})^4$

$= 4 \log_n (2^2 \times 3^{1.5})$

$= 4\log_{2^2 \times 3^{1.5}} \left(2^2 \times 3^{1.5}\right) = 4$

31. $(\log_2 x)^2 + 2\log_2 x - 8 = 0$

$(\log_2 x)^2 + 4\log_2 x - 2\log_2 x - 8 = 0$

$\log_2 x\,[\log_2 x + 4] - 2\,[\log_2 x + 4] = 0$

$[\log_2 x - 2]\,[\log_2 x + 4] = 0$

Since, x is a natural number hence $[\log_2 x + 4]$ cannot be zero. Hence, $\log_2 x - 2 = 0$

$\log_2 x = 2$

$x = 2^2 = 4$

We are given that: $x^p = 64$. Since x is 4, this means that:

$4^p = 64$

$p = 3$

Hence, the value of $(x + p) = 4 + 3 = 7$

32. $A = \sum_{i=1}^{a} \log_3 i = \log_3 1 + \log_3 2 + \log_3 4 + \ldots + \log_3 a$

$= \log_3(2.3.4\ldots.a) = \log_3(a!)$

Similarly $B = \log_3(b!)$, $C = \log_3(a - b)!$

$D = \log_3 a! - \log_3 b! - \log_3(a - b)!$

$= \log_3 \frac{a!}{b!(a-b)!}$

$= \log_3({}^aC_b)$

If $a = 10$, then D will be minimum when $b = 10$, since the smallest value of ${}^nC_r$ occurs when $n = r$.

33. If $a = 6$, then D will be maximum for $b = 3$ (Since the value of ${}^nC_r$ attains its' maximum when the value of r is half the value of n)

34. Both p, q must be greater than 0 as logarithms are not defined to negative bases. Now looking at the two parts of the expression we see that both: $-q^2 + 6q - 8 > 0$ and $-2p^2 + 20p - 48 > 0$. This leads us to the following conclusions: $q^2 - 6q + 8 < 0$. Hence, $(q - 2)(q - 4) < 0$. The only integer value of q that satisfies this is $q = 3$.

Likewise, $2p^2 - 20p + 48 < 0$ means $2(p - 4)(p - 6) < 0$

Only integer value of p which satisfies the above In equality is $p = 5$.

$\therefore p \times q = 3 \times 5 = 15$

Space for Rough Work

TRAINING GROUND FOR BLOCK V

HOW TO THINK IN PROBLEMS ON BLOCK V

1. Let $x_1, x_2, \ldots, x_{100}$ be positive integers such that $x_! + x_{!-1} + 1 = k$ for all !, where k is a constant. If $x_{10} = 1$, then the value of x_1 is

(a) k (b) $k - 2$
(c) $k + 1$ (d) 1

Solution: Using the information in the expression which defines the function, we realise that if we use x_{10} as $x_!$, the expression gives us:

$x_{10} + x_9 + 1 = k \rightarrow x_9 = k–2$; Further, using the value of x_9 to get the value of x_8 as follows:

$x_9 + x_8 + 1 = k \rightarrow k - 2 + x_8 + 1 = k \rightarrow x_8 = 1$;

Next: $x_8 + x_7 + 1 = k \rightarrow x_7 = k - 2$.

In this fashion, we can clearly see that x_6 would again be 1 and x_5 be k–2; $x_4 = 1$ and $x_3 = k$–2; $x_2 = 1$ and $x_1 = k$–2. Option (b) is correct.

2. If $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-2}\, a_{n-1} + 3$ for $n > 1$, which of the following options would be true?

(a) a_{450} is odd and a_{451} is even
(b) a_{450} is odd and a_{451} is odd
(c) a_{450} is even and a_{451} is even
(d) a_{450} is even and a_{451} is odd

Solution: In order to solve this question, you need to think about how the initial values of a_x would behave in terms of being even and odd.

The value of $a_2 = 1 \times 1 + 3 = 4$ (This is necessarily even since we have the construct as follows: Odd × Odd + Odd = Odd + Odd = Even.)

The value of $a_3 = 1 \times 4 + 3 = 7$ (This is necessarily odd since we have the construct as follows: Odd × Even + Odd = Even + Odd = Odd.)

The value of $a_4 = 4 \times 7 + 3 = 31$ (This is necessarily odd since we have the construct as follows: Odd × Even + Odd = Even + Odd = Odd.)

The value of $a_5 = 7 \times 31 + 3 = 220$ (This is necessarily even since we have the construct as follows: Odd × Odd + Odd = Odd + Odd = Even.)

The next two in the series values viz: a_6 and a_7 would be odd again since they would take the construct of Odd × Even + Odd = Even + Odd = Odd. Also, once, a_6 and a_7 turn out to be odd, it is clear that a_8 would be even and a_9 a_{10} would be odd again. Thus, we can understand that the terms a_2, a_5, a_8, a_{11}, a_{14} are even while all other terms in the series are odd. Thus, the even terms occur when we take a term whose number answers the description of $a_{3n+2.}$ If you look at the options, all the four options in this question are asking about the value of a_{450} and a_{451}. Since, neither of these terms is in the series of $a_{3n+2,}$ we can say that both of these are necessarily odd and hence, Option (b) would be the correct answer.

3. If $\dfrac{a+b}{b+c} = \dfrac{c+d}{d+a}$, then

(a) $a = c$
(b) either $a = c$ or $a + b + c + d = 0$
(c) $a + b + c + d = 0$
(d) $a = c$ and $b = d$

Solution: Such problems should be solved using values and also by doing a brief logical analysis of the algebraic equation. In this case, it is clear that if $a = c = k$ (say), the LHS of the expression would become equal to the RHS of the expression (and both would be equal to 1). Once we realise that the expression is satisfied for LHS = RHS, we have to choose between Options a, b and d. However, a closer look at Option (d) shows us that since it says $a = c$ and $b = d$ it is telling us that both of these (i.e., $a = c$ as well as $b = d$) get satisfied and we have already seen that even if only $a = c$ is true, the expression gets satisfied. Thus, there is no need to have $b = d$ as simultaneously true with $a = c$ as true. Based on this logic we can reject Option (d). To check for Option (b) we need to see whether $a + b + c + d = 0$ would necessarily be satisfied if the expression is true. In order to check this, we can take a set of values for a, b, c and d such that their sum is equal to 0 and check whether the equation is satisfied. Taking, a, b, c and d as 1, 2, 3 and –6 respectively we get the LHS of the expression as 3/(–1) = –3; the RHS of the expression would be (–3)/(–5) = 3/5 which is not equal to the LHS. Thus, we can understand that $a + b + c + d = 0$ would not necessarily satisfy the equation. Hence, Option (a) is the correct answer.

4. If a, b, c and d satisfy the equations

$a + 7b + 3c + 5d = 0$
$8a + 4b + 6c + 2d = -32$
$2a + 6b + 4c + 8d = 32$
$5a + 3b + 7c + d = -32$

Then $(a + d)\,(b + c)$ equals

(a) 64 (b) –64
(c) 0 (d) None of the above

Solution: Adding each of the four equations in the expression we get: $16(a + d) + 20(b + c) = -32$.

Also, by adding the second and the third equations we get: $a + b + c + d = 0$, which means that $(a + d) = -(b + c)$.

Then from: $16(a + d) + 20(b + c) = -32$, we have: $-16(b + c) + 20(b + c) = -32 \rightarrow 4(b + c) = -32$. Hence, $(b + c) = -8$ and $(a + d) = 8$. Hence, the multiplication of $(a + d)\,(b + c) = -64$

5. For any real number x, the function $I(x)$ denotes the integer part of x – i.e., the largest integer less than or equal to x. At the same time the function $D(x)$

denotes the fractional part of x. For arbitrary real numbers x, y and z, only one of the following statements is correct. Which one is it?

(a) $I(x + y) = I(x) + I(y)$

(b) $I(x + y + z) = I(x + y) + I(z) = I(x) + I(y + z) = I(x + z) + I(y)$

(c) $I(x + y + z) = I(x + y) + I(z + D(y + x))$

(d) $D(x + y + z) = y + z – I(y + z) + D(x)$

Solution: There are three principle themes you need to understand in order to answer such questions.

1. **Thinking in language**. The above question is garnished with a plethora of mathematical symbols. Unless you are able to convert each of the situations given in the options into clear logical language, your mind cannot make sense of what is written in the options. The best mathematical brains work this way. Absolutely nobody has the mathematical vision to solve such problems by simply reading the notations in the problem and/or the options.
2. While thinking in language terms in the case of a question such as this, in order to understand and grasp the mathematical situation confronting us, the best thing to do is to put values into the situation. This is very critical in such problems because as you can yourself see – if you are thinking about say $I(x + y + z)$ and you keep the same notation to think through the problem, you would need to carry $I(x + y + z)$ throughout the thinking inside the problem. On the other hand if you replace the $I(x + y + z)$ situation by replacing values for x, y and z, the expression would change to a single number. Thus, if you take $I(4.3 + 2.8 + 5.3) = I(12.4) = 12$. Obviously thinking further in the next steps with 12, as the handle, would be much easier than trying to think with $I(x + y + z)$.
3. Since the question here is asking us to identify the correct option which always gives us LHS = RHS, we can proceed further in the problem using the options given to us. While doing this, when you are testing an option, the approach has to be to try to think of values for the variables such that the option is rejected, i.e., we need to think of values such that LHS ≠ RHS. In this fashion, the idea is to eliminate 3 options and identify the one option that cannot be eliminated because it cannot be disproved.

Keeping these principles in mind, if we try to look at the options in this problem we have to look for the one correct statement.

Let us check Option (a) to begin with:

The LHS can be interpreted as: $I(x + y)$ means the integer part of $x + y$. Suppose we use x as 4.3 and y as 4.2 we would get $I(x + y) = I(4.3 + 4.2) = I(8.5) = 8$

The RHS in this case would be $I(4.3) + I(4.2) = 4 + 4 = 8$. This gives us LHS = RHS.

However, if you use $x = 4.3$ and $y = 4.8$ you would see that LHS = $I(4.3 + 4.8) = I(9.1) = 9$, while the RHS would be $I(4.3) + I(4.8) = 4 + 4 = 8$. This would clearly gives us LHS ≠ RHS and hence this option is incorrect.

The point to note here is that whenever you are solving a function based question through the rejection of the options route, the vision about what kind of numbers would reject the case becomes critical. My advise to you is that as you start solving questions through this route, you would need to improve your vision of what values to assume while rejecting an option. This is one key skill that differentiates the minds and the capacities of the top people from the average aspirants. Hence, if you want to compete against the best you should develop this numerical vision. To illustrate what I mean by numerical vision, think of a situation where you are faced with the expression $(a + b) > (a \times b)$. Normally this does not happen, except when you are multiplying with numbers between 0 and 1.)

Moving on with our problem. Let us look at Option (b).

$I(x + y + z) = I(x + y) + I(z) = I(x) + I(y + z) = I(x + z) + I(y)]$

In order to reject this option, the following values would be used:

$x = 4.3$, $y = 5.1$, and $z = 6.7$

LHS = $I(4.3 + 5.1 + 6.7) = I(16.1) = 16$

When we try to see whether the first expression on the RHS satisfies this we can clearly see it does not because:

$I(x + y) + I(z)$ would give us $I(4.3 + 5.1) + I(6.7) = I(9.4) + I(6.7) = 9 + 6 = 15$ in this case.

Thus, $I(x + y + z) = I(x + y) + I(z)$ is disproved and this option can be rejected at this point.

We thus move onto Option (c), which states:

$I(x + y + z) = I(x + y) + I(z + D(y + x))$

Let us try this in the case of the values we previously took, i.e., $x = 4.3$, $y = 5.1$, and $z = 6.7$

We get:

$I(4.3 + 5.1 + 6.7) = I(4.3 + 5.1) + I(6.7 + D(4.3 + 5.1))$

$\rightarrow I(16.1) = I(9.4) + I(6.7 + D(9.4))$

$\rightarrow I(16.1) = I(9.4) + I(6.7 + 0.4)$

$\rightarrow I(16.1) = I(9.4) + I(7.1)$

$\rightarrow 16 = 9 + 7.$

Suppose we try 4.3, 4.9 and 5.9 we get:

$I(4.3 + 4.9 + 5.9) = I(4.3 + 4.9) + I(5.9 + D(4.3 + 4.9)$

$\rightarrow I(15.1) = I(9.2) + I(5.9 + D(9.2))$

$\rightarrow I(15.1) = I(9.4) + I(5.9 + 0.2)$

$\rightarrow I(15.1) = I(9.4) + I(6.1)$

$\rightarrow 15 = 9 + 6.$

We can see that this option is proving to be difficult to shake off as a possible answer. However, this logic is not enough to select this as the correct answer. In order to make sure that you never err when you solve a question this way, you would need to either do one of two things at this point in the problem solving approach.

Approach 1: Try to understand and explain to yourself the mathematical reason as to why this option should be correct.

Approach 2: Try to eliminate the remaining option/s at this point of time.

Amongst these, my recommendation would be to go for Approach 2 because that is likely to be easier than Approach 1. Approach 1 is only to be used in case you have seen and understood during your checking of the option as to why the particular option is always guaranteed to be true. In case, you have not seen the mathematical logic for the same during your checking of the option, you typically should not try to search for the logic while trying to solve the problem. The quicker way to the correct answer would be to eliminate the remaining option/s.

(Of course, once you are done with solving the question, during your review of the question, you should ideally try to explain to yourself as to why one particular option worked – because that might become critical mathematical logic inside your mind for the next time you face a similar mathematical situation.)

In this case, let us try to do both. To freeze Option (c) as the correct answer, you would need to look at Option (d) and try to reject it.

Option (d) says:

$D(x + y + z) = y + z - I(y + z) + D(x)$

Say we take $x = 4.3$, $y = 5.1$, and $z = 6.7$ we can see that:

$D(4.3 + 5.1 + 6.7) = 4.3 + 5.1 - I(4.3 + 5.1) + D(4.3) \rightarrow$

$D(16.1) = 9.4 - I(9.4) + D(4.3) \rightarrow$

$0.1 = 9.4 - 9 + 0.3 \rightarrow$

$0.1 = 0.7$, which is clearly incorrect.

Hence, Option (c) is the correct answer.

If we were to look at the mathematical logic for Option (c) (for our future reference) we can think of why Option (c) would always be true as follows:

One of the problems in these greatest integer problems is what can be described as the loss of value due to the greatest integer function.

Thus $I(4.3 + 4.8) > I(4.3) + I(4.8)$ since the LHS is 9 and the RHS is only 8. What happens here is that the LHS gains by 1 unit because the .3 and the .8 in the two numbers add up to 1.1 and help the sum of the two numbers to cross 9. On the other hand if you were to look at the RHS in this situation, you would realise that the decimal values of 4.3 and 4.8 are individually both lost.

In this context, when you look at the LHS of the equation given in Option (c), you see that the value of the LHS would retain the integer values of x, y and z while the sum of the decimal values of x, y and z would get aggregated and combined into 1 number. This gives us three cases:

Case 1: When the addition of the decimal values of x, y and z is less than 1;

Case 2: When the addition of the decimal values of x, y and z is more than 1 but less than 2;

Case 3: When the addition of the decimal values of x, y and z is more than 2 but less than 3.

Each of these three cases would be further having a two way fork – viz:

Case A: When the addition of the decimal values of x and y add up to less than 1;

Case B: When the addition of the decimal values of x and y add up to more than 1 but less than 2.

I would encourage the reader to take this case from this point and move it to a point where you can explore each of these six situations and see that for all these situations, the value of the LHS of the expression is equal to the value of the RHS of the equation.

6. During the reign of the great government in the country of Riposta, the government forms committees of ministers whenever it is faced with a problem. One particular year, there are x ministers in the government and they are organised into 4 committees such that:

(i) Each minister belongs to exactly two committees.
(ii) Each pair of committees has exactly one minister in common.

Then

(a) $x = 4$
(b) $x = 6$
(c) $x = 8$
(d) x cannot be determined from the given information

Solution: In order to think about this situation, you need to think of the number of unique people you would need in order to make up the committees as defined in the problem. However, before you start to do this, a problem you need to solve is—how many members do you put in each committee?

Given the options for x, when we look at the options, it is clear that the number is unlikely to be larger than 4. Hence, suppose we try to think of committees with 4 members, we will get the following thought process:

First, we create the first two committees with exactly 1 member common between the two committees. We would get the following table at this point of time:

Committee 1	*Committee 2*	*Committee 3*	*Committee 4*
1	4		
2	5		
3	6		
4	7		

Here we have taken the individual members of the committees as 1, 2, 3, 4, 5, 6 and 7. We have obeyed the second rule for committee formation (i.e. each pair of committees has exactly one member in common) by taking only the member 4 as the common member.

From this point in the table, we need to try to fill in the remaining committees obeying the twin rules given in the problem.

So, each member should belong to exactly two committees and each pair of committees should have only 1 member in common.

When we try to do that in this table we reach the following point.

Committee 1	*Committee 2*	*Committee 3*	*Committee 4*
1	**4**	**7**	
2	5	**1**	
3	6	8	
4	**7**	9	

Here we have taken 7 common between the Committees 2 and 3, while 1 is common between Committees 1 and 3. This point freezes the individuals 1, 4 and 7 as they have been used twice (as required). However, this leaves us with 2, 3, 5, 6, 8 and 9 to be used once more and only Committee 4 left to fill in into the table. This is obviously impossible to do and hence, we are sure that each committee would not have had 4 members.

Obviously, if 4 members are too many, we cannot move to trying 5 members per committee. Thus, we should move trying to form committees with 3 members each.

When we do so, the following thought process unfolds:

We first fill in the first two committees by keeping exactly one person common between these committees. By taking the person '3' as a common member between Committees 1 and 2, we reach the following table:

Committee 1	*Committee 2*	*Committee 3*	*Committee 4*
1	3		
2	4		
3	5		

We now need to fill in Committee 3 with exactly 1 member from Committee 1 and exactly 1 member from Committee 2. Also, we cannot use the member number '3' as he has already been used twice. Thus, by repeating '5' from Committee 2 and '1' from Committee 1 we can reach the following table.

Committee 1	*Committee 2*	*Committee 3*	*Committee 4*
1	3	5	
2	4	1	
3	5		

Since the remaining person in Committee 3 would have to be unique from members of Committees 1 and 2, we would need to introduce a new member (say 6) in order to complete Committee 3. Thus, our table evolves to the following situation.

Committee 1	*Committee 2*	*Committee 3*	*Committee 4*
1	**3**	**5**	
2	4	**1**	
3	**5**	6	

At this point, the members 2, 4 and 6 have not been used a second time. Also, the committee 4 has to have its 3 members filled such that it has exactly 1 member common with committees 1, 2 and 3 respectively.

This is easily achieved using the following structure.

Committee 1	*Committee 2*	*Committee 3*	*Committee 4*
1	**3**	**5**	2
2	4	**1**	4
3	**5**	6	6

Hence, we can clearly see that the value of x is 6, i.e., the government had 6 ministers.

7. During the IPL Season 14, the Mumbai Indians captained by a certain Sachin Tendulkar who emerged out of retirement, played 60 games in the season. The team never lost three games consecutively and never won five games consecutively in that season. If N is the number of games the team won in that season, then N satisfies

(a) $24 \le N \le 48$ (b) $20 \le N \le 48$
(c) $12 \le N \le 48$ (d) $20 \le N \le 42$

Solution: In order to solve this question, we need to see the limit of the minimum and maximum number of matches that the team could have won. Let us first think about the maximum number of matches the team could have won. Since the team 'never won five games consecutively' during that season, we would get the value for the maximum number of wins by trying to make the team win 4 games consecutively—as many times as we can. This can be thought of as follows:

WWWWLWWWWLWWWWL... and so on.

From the above sequence, we can clearly see that with 4 wins consecutively, we are forming a block of 5 matches in which the team has won 4 and lost 1. Since, there are a total of 60 matches in all, there would be 12 such blocks of 5 matches each. The total number of wins in this case would amount to $12 \times 4 = 48$ (this is the highest number possible).

This eliminates Option 4 as the possible answer.

If we think about the minimum number of wins, we would need to maximise the number of losses. In order to do so, we get the following thought process:

Since the team never lost three games consecutively, for the maximum number of losses the pattern followed would be–

LLWLLWLLWLLW… and so on

Thus, there are two losses and 1 win in every block of 3 matches. Since, there would be a total of 20 such blocks, it would mean that there would be a total of $20 \times 1 = 20$ wins. This number would represent the minimum possible number of wins for the team.

Thus, N has to be between 20 and 48. Thus, Option (b) is correct.

8. If the roots of the equation $x^3 - ax^2 + bx - c = 0$ are three consecutive integers, then what is the smallest possible value of b?

(a) $-1/\sqrt{3}$ (b) -1
(c) 0 (d) 1
(e) $1/\sqrt{3}$

Solution: Since the question represents a cubic expression, and we want the smallest possible value of b—keeping the constraint of their roots being three consecutive values—a little bit of guesstimation would lead you to think of –1, 0 and 1 as the three roots for minimising the value of *b*.

Thus, the expression would be $(x + 1)(x)(x – 1) = (x^2 + x)(x – 1) = x^3 – x$. This gives us the value of *b* as –1.

It can be seen that changing the values of the roots from –1, 0 and 1 would result in increasing the coefficient of *x*– which is not what we want. Hence, the correct answer should be that the minimum value of *b* would be –1.

Note: that for trial purposes if you were to take the values of the three roots as 0, 1, and 2, the expressions would become $x(x – 1)(x – 2) = (x^2 – x)(x – 2)$ which would lead to the coefficient of *x* being 2. This would obviously increase the value of the coefficient of *x* above –1.

You could also go for changing the three consecutive integral roots in the other direction to –2, –1 and 0. In such a case the expression would become: $x(x + 2)(x + 1) = (x2 + 2x)(x + 1) \rightarrow$ which would again give us the coefficient of *x* as + 2.

The total solving time for this question would be 30 seconds if you were to hit on the right logic for taking the roots as –1, 0 and 1. In case you had to check for the value of b in different situations by altering the values of the roots (as explained above) the time would still be under 2 minutes.

9. A shop stores *x* kg of rice. The first customer buys half this amount plus half a kg of rice. The second customer buys half the remaining amount plus half a kg of rice. Then the third customer buys half the remaining amount plus half a kg of rice. Thereafter, no rice is left in the shop. Which of the following best describes the value of *x*?

(a) $2 \le x \le 6$ (b) $5 \le x \le 8$
(c) $9 \le x \le 12$ (d) $11 \le x \le 14$
(e) $13 \le x \le 18$

Solution: This question is based on odd numbers as only with an odd value of *x* would you keep getting integers if you halved the value of rice and took out another half a kg from the shop store.

From the options, let us start from the second option. (**Note**: In such questions, one should make it a rule to start from one of the middle options only as the normal realisation we would get from checking one option would have been that more than one option gets removed if we have not picked up the correct option—as we would normally know whether the correct answer needs to be increased from the value we just checked or should be decreased.)

Thus trying for $x = 7$ according to the second option, you would get

$7 \rightarrow 3 \rightarrow 1 \rightarrow 0$ (after three customers).

This means that $5 \le x \le 8$ is a valid option for this question. Also, since the question is definitive about the correct range, there cannot be two ranges. Hence, we can conclude that Option 2 is correct.

Note: The total solving time for this question should not be more than 30 seconds. Even if you are not such an experienced solver through options, and you had to check 2–3 options in order to reach the correct option, you would still need a maximum of 90 seconds.

Directions for Questions 10 and 11: Let $f(x) = ax^2 + bx + c$, where *a*, *b* and *c* are certain constants and $a \ne 0$. it is known that $f(5) = – 3 f(2)$ and that 3 is a root of $f(x) = 0$.

10. What is the other root of $f(x) = 0$?

(a) – 7 (b) – 4
(c) 2 (d) 6
(e) Can not be determined

11. What is the value of $a + b + c$?

(a) 9 (b) 14
(c) 13 (d) 37
(e) Can not be determined

Solution: Since, 3 is a root of the equation, we have $9a + 3b + c = 0$ (**Theory point**—A root of any equation $f(x) = 0$ has the property that if it is used to replace '*x*' in every part of the equation, then the equation $f(x) = 0$ should be satisfied.)

Also $f(5) = –3f(2)$ gives us that $25a + 5b + c = –3(4a + 2b + c) \rightarrow 37a + 11b + 4c = 0$

Combining both equations we can see that $37a + 11b + 4c = 4(9a + 3b + c) \rightarrow a – b = 0$. i.e., $a = b$

Now, we know that the sum of roots of a quadratic equation is given by $–b/a$. Hence, the sum of roots has to be equal to –1. Since one of the roots is 3, the other must be –4.

The answer to Question 10 would be Option (2).

For Question 11 we need the sum of $a + b + c$. We know that $a + b = 0$. Also, product of roots is –12. One of the possible equations could be $(x – 3)(x + 4) = 0 \rightarrow x^2 + x – 12 = 0$, which gives us the value of $a + b + c$ as –10. However, –10 is not in the options. This should make us realise that there is a possibility of another equation as: $(2x – 6)(2x + 8) = 0 \rightarrow 4x^{2\,+}\,4x – 48 = 0$ in which case the value of $a + b + c$ changes. Hence, the correct answer is 'cannot be determined'.

12. Suppose, the seed of any positive integer *n* is defined as follows:

Seed(*n*) = *n*, if $n < 10$

= seed(*s*(*n*)), otherwise,

Where *s*(*n*) indicates the sum of digits of *n*. For example, Seed(7) = 7, seed (248) = seed(2 + 4 + 8) = seed (14) = seed (1 + 4) = seed (5) = 5 etc. How many positive integers *n*, such that $n < 500$, will have seed $(n) = 9$?

(a) 39 (b) 72
(c) 81 (d) 108
(e) 55

Solution: The first number to have a seed of 9 would be the number 9 itself.

The next number whose seed would be 9 would be 18, then 27 and you should recognise that we are talking about numbers which are multiples of 9. Hence, the number of such numbers would be the number of numbers in the Arithmetic Progression:

9, 18, 27, 36, 45, ...495 = [(495 – 9)/9] + 1 = 55 such numbers.

13. Find the sum of $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}$ + ...

$+\sqrt{1+\frac{1}{2007^2}+\frac{1}{2008^2}}$

(a) $2008-\frac{1}{2008}$ (b) $2007-\frac{1}{2007}$

(c) $2007-\frac{1}{2008}$ (d) $2008-\frac{1}{2007}$

(e) $2008-\frac{1}{2009}$

Solution: Such questions are again solved through logical processes. If you were to try this problem by going through mathematical processes you would end up with a messy solution which is not going to yield any answer in any reasonable time frame.

Instead, look at the following process.

The first thing you should notice is that the value in the answer has got something to do with the number 2008. Suppose we were to look at only the first term of the expression, by analogy the value of the sum should have something to do with the number 2. Accordingly by looking at the value obtained we can decide on which of the options fits the given answer.

So, for the first term, we see that the value is equal to the square root of 2.25 = 1.5

By analogy that in this case the value of 2008 is 2, the value of 2007 would be 1 and 2009 would be 3. Replacing these values the options become:

(a) $2-\frac{1}{2}$ (b) $1-\frac{1}{1}$

(c) $1-\frac{1}{2}$ (d) $2-\frac{1}{1}$

(e) $2-\frac{1}{3}$

It can be easily verified that only Option (a) gives a value of 1.5. Hence, that is the only possible answer as all other values are different. In case you need greater confirmation and surety, you can solve this for the first two terms too.

14. A function $f(x)$ satisfies $f(1) = 3600$, and $f(1) + f(2) + \ldots + f(n) = n^2 f(n)$, for all positive integers $n > 1$. What is the value of $f(9)$?

(a) 80 (b) 240

(c) 200 (d) 100

(e) 120

Solution: This question is based on chain functions where the value of the function at a particular point depends on the previous values.

$f(1) + f(2) = 4f(2) \rightarrow f(1) = 3f(2) \rightarrow f(2) = 1200$

Similarly, for $f(3)$ we have the following expression:

$f(1) + f(2) + f(3) = 9f(3) \rightarrow \quad f(3) = 4800/8 = 600$

Further $f(1) + f(2) + f(3) = 15\, f(4) \rightarrow f(4) = 5400/15 = 360$

Further $f(1) + f(2) + f(3) + f(4) = 24\, f(5) \rightarrow 5760/24 = f(5) = 240$

If you were to pause a while at this point and try to look at the pattern of the numerical outcomes in the series we are getting we get:

3600, 1200, 600, 360, 240, 1200/7

A little bit of perceptive analysis about the fractions used as multipliers to convert $f(1)$ to $f(2)$ and $f(2)$ to $f(3)$ and so on will tell us that the respective multipliers themselves are following a pattern viz:

$f(1) \times 1/3 = f(2)$;

$f(2) \times 2/4 = f(3)$;

$f(3) \times 3/5 = f(4)$ and $f(4) \times 4/6 = f(5)$

Using this logic string we can move onto the next values as follows:

$f(6) = 240 \times 5/7 = 1200/7$;

$f(7) = 1200/7 \times 6/8 = 900/7$;

$f(8) = 900/7 \times 7/9 = 100$ and

$f(9) = 100 \times 8/10 = 80$.

Thus, Option (a) is the correct answer.

Directions for Questions 15 and 16: Let S be the set of all pairs (i, j) where $1 \le i < j \le n$, and $n \ge 4$. Any two distinct members of S are called "friends" if they have one constituent of the pairs in common and "enemies" otherwise. For example, if $n = 4$, then S = {(l, 2), (l, 3), (l, 4), (2, 3), (2, 4), (3, 4)}. Here, (l, 2) and (l, 3) are friends, (l, 2) and (2, 3) are also friends, but (l, 4) and (2, 3) are enemies.

15. For general n, how many enemies will each member of S have?

(a) $n - 3$ (b) $(n^2 - 3n - 2)/2$

(c) $2n - 7$ (d) $(n^2 - 5n + 6)/2$

(e) $(n^2 - 7n + 14)/2$

Solution: Solve by putting values: Suppose we have $n = 5$; The members would be {(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)}

In this case any member can be found to have 3 enemies.

Thus the answer to the above question should give us a value of 3 with $n = 5$.

Option (a): n–3 = 5–3 = 2. Hence, cannot be the answer.

Option (b): 8/2 = 4. Hence, cannot be the answer.

Option (c): 10 – 7 = 3. To be considered.

Option (d): 6/2 = 3. To be considered.

Option (e): 4/2 = 2. Hence, cannot be the answer.

We still need to choose one answer between Options (c) and (d).

It can be seen that for $n = 6$, the values of Options (c) and (d) will differ. Hence, we need to visualise how many enemies each member would have for $n = 6$.

The members would be {(1,2), (1,3), (1,4), (1,5), (1,6), (2,3),(2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)}. It can be clearly seen that the member (1,2) will have 6 enemies. Option (c) gives us a value of 5 and hence can be eliminated while Option (d) gives us a value of 6 leaving it as the only possible answer.

16. For general n, consider any two members of S that are friends. How many other members of S will be common friends of both these members?

(a) $(n^2 - 5n + 8)/2$ (b) $2n - 6$
(c) $n(n-3)/2$ (d) $n - 2$
(e) $(n^2 - 7n + 16)/2$

Solution: Again for this question consider the following situation where $n = 6$.

The members would be {(1,2), (1,3), (1,4), (1,5), (1,6), (2,3),(2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)}.

Suppose we consider the pair (1,2) and (1,3). Their common friends would be (1,4), (1,5), (1,6) and (2,3) .

Thus there are 4 common friends for any pair of friendly members. (You can verify this by taking any other pair of friend members.)

Thus for $n = 6$, the answer should be 4.

Checking the options it is clear that only option 4 gives us a value of 4.

Maximum Solving time: 60 – 90 seconds

17. In a tournament, there are n teams $T_1, T_2, \ldots, T_n$, with $n > 5$. Each team consists of k players, $k > 3$. The following pairs of teams have one players in common:

T_1 & T_2, T_2 & T_3, …, T_{n-1} & T_n, and T_n & T_1.

No other pair of teams has any player in common. How many players are participating in the tournament, considering all the n teams together?

(a) $n(k - 2)$ (b) $k(n - 2)$
(b) $(n - 1)(k - 1)$ (d) $n(k - 1)$
(e) $k(n - 1)$

Thought process:

If we take 6 teams and 4 players per team, we would get 4 players in T_1 (each one of them unique), 3 more players in T_2 (since 1 player of T_2 would be shared with T_1), 3 more players in T_3 (since 1 player of T_3 would be shared with T_2), 3 more players in T_4 (since 1 player of T_4 would be shared with T_3), 3 more players in T_5 (since 1 player of T_5 would be shared with T_4) and 2 more players in T_6 (since 1 player of T_6 would be shared with T_5 and one with T_1). Hence, there would be a total of 18 (4 + 3 + 3 + 3 + 3 + 2) players with $n = 6$ and $k = 4$. Checking from the options we see that only Option (d) gives us 18 as the solution.

Maximum solution time: 60 seconds.

18. Consider four digit numbers for which the first two digits are equal and the last two digits are also equal. How many such numbers are perfect squares?

(a) 4 (b) 0
(b) 1 (d) 3
(e) 2

Thought process:

A lot of CAT takers got stuck on this question for over 5–7 minutes in the exam, since they tried to find out the squares of all two digit numbers starting from 32. However, if you are aware of the logic of finding squares of two digit numbers, you would realise that only three two-digit numbers after 32 have the last two digits in their squares equal (38, 62 and 88). Hence, you do not need to check any other number apart from these three. Checking these you would get the square of 88 as 7744. And hence, there is only one such number.

Note: Of course, you would ignore the values of squares with the last two digits as '00'.

19. A confused bank teller transposed the rupees and paise when he cashed a cheque for Shailaja, giving her rupees instead of paise and paise instead of rupees. After buying a toffee for 50 paise, Shailaja noticed that she was left with exactly three times as much as the amount on the cheque. Which of the following is a valid statement about the cheque amount?

(a) Over Rupees 22 but less than Rupees 23
(b) Over Rupees 18 but less than Rupees 19
(c) Over Rupees 4 but less than Rupees 5
(d) Over Rupees 13 but less than Rupees 14
(e) Over Rupees 7 but less than Rupees 8

Thought Process:

Deduction 1: Question Interpretation: The solution language for this question requires you to think about what possible amount could be such that when its rupees and paise value are interchanged, the resultant value is 50 paise more than thrice the original amount.

Dedcution 2: Option checking process:

Armed with this logic, suppose we were to check for Option (a) i.e., the value is above ₹ 22 but below ₹ 23. This essentially means that the amount must be approximately between ₹ 22.66 and ₹ 22.69. (We get the paise amount to be between 66 and 69 based on the fact that the relationship between the Actual Amount, x and the transposed amount y is: y – 50 paise = $3x$.

Hence, values below 22.66 and values above 22.70 are not possible.

→ From this point onwards we just have to check whether this relationship is satisfied by any of the values between ₹ 22.66 and ₹ 22.69.

→ Also, realise the fact that in each of these cases the paise value in the value of the transposed amount y would be 22. Thus, $3x$ should give us the paise value as 72 (since we have to subtract 50 paise from the value of 'y' in order to get the value of $3x$).

→ This also means that the unit digit of the paise value of $3x$ should be 2.

→ It can be clearly seen that none of the numbers 66, 67, 68 or 69 when being multiplied by 3 give us a units digit of 2. Hence, this is not a possible answer.

Checking for Option (b) in the same fashion:

You should realise that the outer limit for the range of values when the amount is between 18 and 19 is: 18.54 to 18.57. Also, the number of paise in the value of the transposed sum 'y' would be 18. Hence, the value of $3x$ should give us a paise value as 68 paise. Again, using the units digit principle, it is clear that the only value where the units digit would be 8 would be for a value of 18.56.

Hence, we check for the check amount to be 18.56. Transposition of the rupee and paise value would give us 56.18. When you subtract 50 paise from this you would get 55.68 which also happens to be thrice 18.56. Hence, the correct answer is Option (b).

Notice here that if you can work out this logic in your reactions, the time required to check each option would be not more than 30 seconds. Hence, the net problem solving time to get the second option as correct would not be more than 1 minute. Add the reading time and this problem should still not require more than 2 minutes.

20. How many pairs of positive integers m, n satisfy $1/m + 4/n = 1/12$ where n is an odd integer less than 60?

(a) 7 (b) 5
(c) 3 (d) 6
(e) 4

Thought Process:

Deduction 1: Since two positive fractions on the LHS equals 1/12 on the right hand side, the value of both these fractions must be less than 1/12. Hence, n can take only the values 49, 51, 53, 55, 57 and 59.

Deduction 2: We now need to check which of the possible values of n would give us an integral value of m.

The equation can be transformed to: $1/12 - 4/n = 1/m$ $\rightarrow (n - 48)/12n = 1/m$. On reading this equation you should realise that for m to be an integer the LHS must be able to give you a ratio in the form of $1/x$. It can be easily seen that this occurs for $n = 49$, $n = 51$ and $n = 57$. Hence, there are only 3 pairs.

21. The price of Darjeeling tea (in rupees per kilogram) is $100 + 0.1\ n$, on the n^{th} day of 2007 (n = l, 2, ..., 100), and then remains constant. On the other hand, the price of Ooty tea (in rupees per kilogram) is $89 + 0.15n$, on the n^{th} day of 2007 (n = l, 2, .,., 365). On which date in 2007 will the prices of these two varieties of tea be equal?

(a) May 21 (b) April 11
(c) May 20 (d) April 10
(e) June 30

The gap between the two prices initially is of ₹11 or 1100 paise. The rate at which the gap closes down is 5 paise per day for the first hundred days. (The gap covered would be 500 paise which would leave a residual gap of 600 paise.) Then the price of Darjeeling tea stops rising and that of Ooty tea rises at 15 paise per day. Hence, the gap of 600 paise would get closed out in another 40 days. Hence, the prices of the two varieties would become equal on the 140^{th} day of the year. 31 + 28 + 31 + 30 + 20 = 140, means May 20^{th} is the answer.

22. Let a, b, m, n be positive real numbers, which satisfy the two conditions that

(i) If $a > b$ then $m > n$; and
(ii) If $a > m$ then $b < n$

Then one of the statements given below is a valid conclusion. Which one is it?

(a) If $a < b$ then $m < n$ (b) If $a < m$ then $b > n$
(c) If $a > b + \text{m}$ then $m < b$
(d) If $a > b + \text{m}$ then $m > b$

The best way to think about this kind of a question is to try to work out a possibility matrix of the different possibilities that exist with respect to which of the values is at what position relative to each other. While making this kind of a figure for yourself, use the convention of keeping the higher number on top and the lower number below.

If we look at Condition (i) as stated in the problem, it states that: if $a > b$ then $m > n$

This gives us multiple possibilities for the placing of the four variables in relative order of magnitude. These relative positions of the variables can be visualised as follows for the case that $a > b$:

	Possibility 1	*Possibility 2*	*Possibility 3*
Largest number	a	a	a
2^{nd} largest number	b	m	m
3^{rd} largest number	m	b	n
Smallest number	n	n	b

Looking at the options, Option (a) can be rejected because, when we use the condition If X then necessarily Y, it does not mean that If Not X, then not Y.

For instance, if I make a statement like – “If the Jan Lokpal Bill is passed, corruption will be eradicated from the country; this does not mean that if the Jan Lokpal Bill is not passed then corruption would not be eradicated from the country.”

Note: For this kind of reverse truth to exist the existing starting conditionality has to be of the form, only if X, then Y. In such a case the conclusion, if not X, then not Y is valid.

For instance, if I make a statement like – “Only if the Jan Lokpal Bill is passed, will corruption be eradicated from the country; this necessarily means the reverse – i.e. if the Jan Lokpal Bill is not passed then corruption would not be eradicated from the country.”

This exact logic helps us eliminate the first option – which says that if $a < b$ then m should be less than n (this would obviously not happen just because if $a > b$, then m is greater than n – we would need the only if condition in order for this to work in the reverse fashion.)

Option (b) has the same structure based on the Condition (ii) in the problem – it tries to reverse a “If *X*, then *Y* conditionality” into a “If not *X*, then not *Y*” conclusion – which would only have been valid in the case of ‘Only if *X*’ as explained above.

This leaves us with Options (c) and (d) to check. If you go through these options, you realise that they are basically opposite to each other.

The following thought process would help you identify which of these is the correct answer.

When we say that $a > b + m$ where a, b and m are all positive it obviously means that a must be greater than both b and m. Thus, in this situation we have $a > b$ as well as $a > m$. In this case both the Conditions (i) and (ii) would activate themselves. It is at this point that the possibility matrix for the case of $a > b$ would become usable.

The possibility matrix that exists currently for $a > b$ is built using the following thought chain:

First think of the various positions in which ‘*a*’ and ‘*b*’ can be put, with ‘*a*’ greater than ‘*b*’ given that we have to fix up 4 numbers in decreasing order. The following possibilities emerge when we do this.

	Possibility 1	*Possibility 2*	*Possibility 3*	*Possibility 4*	*Possibility 5*	*Possibility 6*
Largest number	a	a	a			
2^{nd} largest number	b			a	a	
3^{rd} largest number		b			b	a
Smallest number			b	b		b

When we add the fact that when $a > b$, then m is also greater than n to this picture, the complete possibility matrix emerges as below:

	Possibility 1	*Possibility 2*	*Possibility 3*	*Possibility 4*	*Possibility 5*	*Possibility 6*
Largest number	a	a	a	m	m	m
2^{nd} largest number	b	m	m	a	a	n
3^{rd} largest number	m	b	n	n	b	a
Smallest number	n	n	b	b	n	b

Now, for Options (c) and (d) we know that the condition to be checked for is $a > b + m$ which means that $a > b$ and $a > m$ simultaneously. We have drawn above the possibility matrix for $a > b$. We also know from Condition (ii) in the problem above, that when $a > m$, then b should be less than n. Looking at the possibility matrix we need to search for cases where simultaneously each of the following is occurring- (i) $a > b$; (ii) $a > m$ and $b < n$. We can see that the possibilities 1, 2 and 5 will get rejected because in each of these cases b is not less than n. Similarly, Possibilities 4 and 6 both do not have $a > m$ and hence can be rejected. Only Possibility 3 remains and in that case, we can see than $m > b$.

Hence, Option (d) is correct.

23. A quadratic function $f(x)$ attains a maximum of 3 at $x = 1$. The value of the function at $x = 0$ is 1. What is the value of $f(x)$ at $x = 10$?

(a) –119 (b) –159
(c) –110 (d) –180
(e) –105

Let the equation be $ax^2 + bx + c = 0$. If it gains the maximum at $x = 1$ it means that ‘*a*’ is negative.

Also $2ax + b = 0 \rightarrow x = -b/2a$. So $-b/2a$ should be 1. So the expression has to be chosen from:

$-X^2 + 2x + c$

$-2X^2 + 4x + c$

$-3X^2 + 6x + c$ ....and so on (since we have to keep the ratio of $-b/2a$ constant at 1).

Also, it is given that the value of the function at $x = 0$ is 1. This means that $c = 1$. Putting this value of c in the possible expressions we can see that at $x = 1$, the value of the function is equal to 3 in the case:

$-2X^2 + 4x + 1$

So the expression is– $2X^2 + 4x + 1$

At $x = 10$, the value would be $-200 + 40 + 1 = -159$.

Directions for Questions 24 and 25: Let $a_1 = p$ and $b_1 = q$, where p and q are positive quantities. Define

$a_n = pb_{n-1}$, $b_n = qb_{n-1}$, for even $n > 1$,

and $a_n = pa_{n-1}$, $b_n = qa_{n-1}$, for odd $n > 1$.

24. Which of the following best describes $a_n + b_n$ for even n?

(a) $q(pq)^{(n/2-1)}(p + q)$ (b) $(pq)^{(n/2-1)}(p + q)$

(c) $q^{(1/2)n}(p + q)$ (d) $q^{(1/2)n}(p + q)^{(1/2)n}$

(e) $q(pq)^{(n/2)-1}(p + q)^{(1/2)n}$

Again to solve this question, we need to use values.

Let $a_1 = p = 5$ and $b_1 = q = 7$ (any random values.)

In such a case,

$a_2 = 5 \times 7 = 35$ and $b_2 = 7 \times 7 = 49$. So the sum of $a_2 + b_2 = 84$.

Checking the values we get:

Option a: $7 \times 1 \times (12) = 84$

Option b: $1 \times (12)$

Option c: $7 \times (12)$

Option d: $7 \times (12)$

Option e: $7 \times (12)$

Obviously apart from Option 2 all other options have to be considered. So it is obvious that the question setter wants us to go at least till the value of n as 4 to move ahead.

$a_3 = 5 \times 35 = 175$, $b_3 = 7 \times 35 = 245$

$a_4 = 5 \times 245 = 1225$, $b_4 = 7 \times 245 = 1715$

Sum of $a_4 + b_4 = 2940$.

Option a: $7 \times 35 \times 12 = 2940$,

Option c: $7 \times 7 \times 12$ eliminated

Option d: $7 \times 7 \times 12 \times 12$ eliminated

Option e: $7 \times 35 \times 12 \times 12$ eliminated

Hence, only Option (a) gives us a value of 2940 for $n = 4$. Thus it has to be correct.

25. If $p = 1/3$ and $q = 2/3$, then what is the smallest odd n such that $a_{n} + b_n < 0.01$?

(a) 7 (b) 13

(c) 11 (d) 9

(e) 15

According to the question $a_1 = p = 1/3$ and $b_1 = q = 2/3$.

$a_2 = 1/3 \times 2/3 = 2/9$, $b_2 = 2/3 \times 2/3 = 4/9$

$a_3 = 1/3 \times 2/9 = 2/27$, $b_3 = 2/3 \times 2/9 = 4/27$

$a_4 = 1/3 \times 4/27 = 4/81$, $b_3 = 2/3 \times 4/27 = 8/81$

In this way, you can continue to get to the value of n at which the required sum goes below 0.01. (It would happen at $n = 9$). However, if you are already comfortably placed in the paper, you can skip this process as it would be time consuming and also there is a high possibility of silly errors being induced under pressure.

26. The number of ordered pairs of integers (x, y) satisfying the equation $x^2 + 6x + 2y^2 = 4 + y^2$ is

(a) 12 (b) 8

(c) 10 (d) 14

These kinds of questions and thinking are very common in examinations and hence you need to understand how to solve such questions.

In order to think of such questions, you need to first 'read' the equation given. What do I mean by 'reading' the equation? Let me illustrate:

The first thing we do is to simplify the equation by putting all the variables on the LHS. This would give us the equation $x^2 + 6x + y^2$. When we have an equation like $x^2 + 6x + y^2 = 4$, we should realise that the value on the RHS is fixed at 4. Also, if we take a look at the LHS of the equation we realise that the terms x^2 and y^2 would always be positive integers or 0 (given that x and y are integers). $6x$ on the other hand could be positive, zero or negative depending on the value of x (if x is positive $6x$ is also positive, if x is negative $6x$ would be negative and if x is 0, $6x$ would also be zero.)

Thus, we can think of the following structures to build a value of 4 on the LHS for the equation to get satisfied:

	Value of x^2	*Value of 6x*	*Value of y^2*
Case 1	0	0	+
Case 2	+	+	+
Case 3	+	–	+
Case 4	+	–	0

Once you have these basic structures in place, you can think of the cases one by one. Thinking in this structured fashion makes sure that you do not miss out on any possible solutions — and that, as you should realise, is critical for any situation where you have to count the number of solutions. You simply cannot get these questions correct without identifying each possible situation. Trying to do such questions without first structuring your thought process this way would lead to disastrous results in such questions!! Hence, this thinking is very critical for your development of quantitative thinking.

Let us look at **Case 1:** In Case 1, since the value of the first two components on the LHS are 0, it must mean that we are talking about the case of $x = 0$. Obviously, in this case the entire value of 4 for the LHS has to be created by using the term y^2.

Thus to make $y^2 = 4$, we can take $y = +2$ or $y = -2$. This gives us two possible solutions (0, 2) and (0, –2)

Case 2: The minimum positive value for $6x$ would be when $x = 1$. This value for $6x$ turns out to be 6– which has already made the LHS larger than 4. To this if we were to add two more positive integers for the values of x^2 and y^2 it would simply take the LHS further up from +6. Hence, in Case 2 there are no solutions.

Case 3: In this case the value of x has to be negative and y can be either positive or negative. Possible negative values of x as –1, –2, –3, –4, –5 etc. give us values for $6x$ as –6, –12, –18, –24, –30, etc.

We then need to fill in values of x^2 and y^2 and see whether it is possible to add an exact value to any of these and get + 4 as the final value of the LHS.

This thinking would go the following way:

If $6x = -6$: x must be –1 and hence $x^2 = 1$. Thus, $x^2 + 6x = -5$, which means for $x^2 + 6x + y^2 = 4$ we would need $-5 + y^2 = 4 \rightarrow y^2 = 9 \rightarrow y = +3$ and $y = -3$

Thus we have identified two more solutions as (–1, 3) and (–1, –3)

If $6x = -12$: x must be –2 and hence $x^2 = 4$. Thus, $x^2 + 6x = -8$, which means for $x^2 + 6x + y^2 = 4$ we would need $-8 + y^2 = 4 \rightarrow y^2 = 12 \rightarrow y$ is not an integer in this case and hence we get no new solutions for this case.

If $6x = -18$: x must be –3 and hence $x^2 = 9$. Thus, $x^2 + 6x = -9$, which means for $x^2 + 6x + y^2 = 4$ we would need $-9 + y^2 = 4 \rightarrow y^2 = 13 \rightarrow y$ is not an integer in this case and hence we get no new solutions for this case.

If $6x = -24$: x must be –4 and hence $x^2 = 16$. Thus, $x^2 + 6x = -8$, which means for $x^2 + 6x + y^2 = 4$ we would need $-8 + y^2 = 4 \rightarrow y^2 = 12 \rightarrow y$ is not an integer in this case and hence we get no new solutions for this case.

If $6x = -30$: x must be –5 and hence $x^2 = 25$. Thus, $x^2 + 6x = -5$, which means for $x^2 + 6x + y^2 = 4$ we would need $-5 + y^2 = 4 \rightarrow y^2 = 9 \rightarrow y = +3$ and $y = -3$

Thus we have identified two more solutions as (–5,3) and (–5, –3)

If $6x = -36$: x must be –6 and hence $x^2 = 36$. Thus, $x^2 + 6x = 0$, which means for $x^2 + 6x + y^2 = 4$ we would need $0 + y^2 = 4 \rightarrow y^2 = 4 \rightarrow y = +2$ and $y = -2$

Thus we have identified two more solutions as (–6, 2) and (–6, –2)

If $6x = -42$: x must be –7 and hence $x^2 = 49$. Thus, $x^2 + 6x = 7$, which means that $x^2 + 6x$ itself is crossing the value of 4 for the LHS. There is no scope to add any value of y^2 as a positive integer to get the LHS of the equation equal to 4. Thus, we can stop at this point.

Notice that we did not need to check for the Case 4 because if there were a solution for Case 4, we would have been able to identify it while checking for Case 3 itself.

Thus, the equation has 8 solutions.

27. The number of ordered pairs of integral solutions (m, n) which satisfy the equation $m \times n - 6(m + n) = 0$ with $m \leq n$ is:

(a) 5 (b) 10
(c) 12 (d) 9

In order to solve a question of this nature, you again first need to 'read' the equation:

The equation $m \times n - 6(m + n) = 0$ can be restructured as: $m \times n = 6(m + n)$

When we read an equation of this form, we should be able to read this as:

The LHS is the product of two numbers, while the RHS is always going to be a multiple of 6. Further, we also know that the value of $m \times n$ would normally always be higher than the value of $m + n$. Thus, we are trying to look for situations where the product of two integers is six times their sum.

While looking for such solutions, we need to look for situations where either:

The RHS is non negative – hence = 0, 6, 12, 18, 24, 30, 36…

The RHS is negative – hence = –6, –12, –18…

We would now need to use individual values of $m + n$ in order to check for whether $m \times n = 6(m + n)$

For $m + n = 0$; $6(m + n) = 0$. If we use m and n both as 0, we would get 0 = 0 in the two sides of the equation. So this is obviously one solution to this equation.

For the RHS = –6, we can visualise the product of $m \times n$ as –6, if we use m as –3 and n as 2 or vice versa. Thus, we will get two solutions as (2, –3) and (–3, 2)

For the RHS = –12, $m + n = -2$, which means that m and n would take values like (–4,2); (–5,3); (–6,4). If you look inside the factor pairs of –12, there is no factor pair which has an addition of –2.

For RHS = –18, $m + n = -3$. Looking into the factor pairs of 18 (viz: 1 × 18, 2 × 9, 3 × 6) we can easily see –6 and 3 as a pair of factors of –18 which would add up to –3 as required. Thus, we get two ordered solutions for (m,n) viz: (–6,3); (3,–6)

For RHS = –24, $m + n = -4$. If we look for factors of –24, which would give us a difference of –4, we can easily see that within the factor pairs of 24 (viz: 1 × 24, 2 × 12, 3 × 8 and 4 × 6) there is no opportunity to create a sum of $m + n = -4$.

For RHS = –30, $m + n = -5 \rightarrow$ factors of 30 are 2 × 15, 5 × 6; there is no opportunity to create a $m + n = -5$ and $m \times n = -30$ simultaneously. Hence, there are no solutions in this case.

For RHS = –36, $m + n = -6 \rightarrow$ factors of 36 are 2 × 18, 3 × 12, 4 × 9 and we can stop looking further; there is no opportunity to create $m + n = -6$ and $m \times n = -36$ simultaneously. Hence, there are no solutions in this case.

For RHS = –42, $m + n = -7 \rightarrow$ factors of 42 are 2 × 21, 3 × 14, 6 × 7 and we can stop looking further; there is no opportunity to create $m + n = -42$ and $m \times n = -7$ simultaneously. Hence, there are no solutions in this case.

For RHS = –48, $m + n = -8 \rightarrow$ factors of 48 are 2 × 24, 3 × 16, 4 × 12… 4 × 12 gives us the opportunity to create $m + n = -8$ and $m \times n = -48$ simultaneously. Hence, there are two solutions in this case viz: (4, –12) and (–12, 4)

Note: While this process seems to be extremely long and excruciating, it is important to note that there are a lot of refinements you can make in order to do this fast. The 'searching inside the factors' shown above is itself a hugely effective short cut in this case. Further, when you are looking for pairs of factors for any number, you need not look at the first few pairs because their difference would be very large. This point is illustrated below:

For RHS = –54, $m + n = –9 \rightarrow$ factors of 54 are 3×18, 6×9 and we can stop looking further; (Notice here that we did not need to start with 1×54 and 2×27 because they are what can be called as 'too far apart' from each other). There is no opportunity to create $m + n = –6$ and $m \times n = –36$ simultaneously. Hence, there are no solutions in this case.

For RHS = –60, $m + n = –10 \rightarrow$ relevant factor search for 60 are 4×15, 5×12 and we can stop looking further; there is no opportunity to create $m + n = –10$ and $m \times n = –60$ simultaneously. Hence, there are no solutions in this case.

For RHS = –66, $m + n = –11 \rightarrow$ relevant factor search for 66 is 6×11 and we can stop looking further; there is no opportunity to create $m + n = –11$ and $m \times n = –66$ simultaneously. Hence, there are no solutions in this case.

Note: While solving through this route each value check should take not more than 5 seconds at the maximum. The question that starts coming into one's mind is, how far does one need to go in order to check for values?? Luckily, the answer is not too far.

As you move to the next values beyond –66, –72 (needs $m + n = –12$, while the factors of 72 do not present this opportunity), –78 (needs $m + n = –13$, which does not happen again).

To move further you need to start working out the logic when $6(m + n)$ is positive. In such a case, the value of m and n would both need to be positive for $m \times n$ also to be positive. If $m + n = 1$, we cannot get two positive values of m and n such that their product is 6. If $m + n = 2$, $m \times n = 12$ would not happen.

A little bit of logical thought would give you that the first point at which this situation would get satisfied. That would be when $6(m + n) = 144$ which means that $m + n = 24$ and for $m \times n$ to be equal to 144, the value of each of m and n would be 12 each.

(A brief note about why it is not possible to get a value before this:

If we try $6(m + n) = 132$ (for instance), we would realise that $m + n = 22$. The highest product $m \times n$ with a limit of $m + n = 22$ would occur when each of m and n is equal and hence individually equal to 11 each. However, the value of 11 for m and n gives us a product of 121 only, which is lower than the required product of 132. This would happen in all cases where $6(m + n)$ is smaller than 144.)

Checking subsequent values of $6(m + n)$ we would get the following additional solutions to this situation:

6(m + n)	*(m + n)*	*Relevant Factor pairs for the value of 6(m + n)*	*Solutions*
150	25	10,15	10,15; 15,10
156	26	None	None
162	27	9,18	9,18; 18,9
168,174	28,29	None	None
180,186	30,31	None	None
192	32	8,24	8,24; 24,8
198,204	33,34	None	None
210,216	35,36		

Break-down of the above thought process:

Note that while checking the factor pairs for a number like 216, if you were to list the entire set of factors along with the sum of the individual factors within the pairs, you would get a list as follows:

Factor Pairs for 216:

Pair	*Sum of Factors in the Pair*
1×216	217
2×108	110
3×72	75
4×54	58
6×36	42
8×27	35
9×24	33
12×18	30

However, a little bit of introspection in the correct direction would show you that this entire exercise was not required in order to do what we were doing in this question – i.e., trying to solve for $m + n = 36$ and $m \times n = 216$.

The first five pairs where the larger number itself was greater than or equal to 36 were irrelevant as far as searching for the correct pair of factors is concerned for this question. When we saw the sixth pair of 8×27, we should have realised that since the sum of $8 + 27 = 35$, which is < 36, the subsequent pairs would also have a sum smaller than 36. Hence, you can stop looking for more factors for 216.

Thus, effectively to check whether a solution exists for $6(m + n) = 216$, in this question, all we needed to identify was the $8 \times 27 = 216$ pair and we can reject this value for $6(m + n)$ giving us an integral solution in this case.

This entire exercise can be completed in one '5 second thought' as follows:

If $6(m + n) = 216 \rightarrow m + n = 36$ then one factor pair is 6×36 itself whose sum obviously is more than 36. So, looking for the next factor pair where the smaller number is > 6, we see that 8×27 gives us a factor pair sum of $8 + 27 = 35 < 36$ and hence we reject this possibility.

Also, you should realise that we do not need to look further than 216 – as for values after 216, when we go to the next factor pair after 6 × (m + n), we would realise that the sum of the factor pair would be lower than the required value for the immediately next factor pair.

Hence, the following solutions exist for this question: 0,0; 2,–3; –3,2; 3,–6; –6,3; 4,–12; –12,4; 12,12; 15,10; 10,15; 9,18; 18,9; 8,24; 24,8

A total of 14 solutions

Author's note: Doubtless this question is very long, but if you are able to understand the thought process to adopt in such situations, you would do yourself a big favor in the commonly asked questions of finding number of integral solutions.

28. How many positive integral solutions exist for the expression $a^2 - b^2 = 666$?

In order to solve this question, we need to think of the expression $(a - b)(a + b) = 666$ Obviously, the question is based on factor pairs of 666. If we look at the list of factor pairs of 666 we get:

1	666
2	333
3	222
6	111
9	74
18	37

Since, 37 is a prime number, we will get no more factor pairs.

Now, if we look at trying to fit in the expression $(a - b)(a + b)$ for any of these values, we see the following occurring:

Example: $(a - b) = 9$ and $(a + b) = 74$. If we try to solve for a and b we get: $2a = 83$ (by adding the equations) and 'a' would not be an integer. Consequently, b would also not be an integer and we can reject this possibility as giving us a solution of $a^2 - b^2 = 666$.

Armed with this logic if we were to go back to each of the factor pairs in the table above, we realise that the sum of the two factors within a factor pair is always odd and hence none of these factor pairs would give us a solution for the equation.

Thus, the correct answer would be 0.

29. How many positive integral solutions exist for the expression $a^2 - b^2 = 672$?

In order to solve this question, we need to think of the expression $(a - b)(a + b) = 672$ Obviously, the question is based on factor pairs of 672. If we look at the list of factor pairs of 672 we get:

		Sum of factors
1	672	Odd
2	336	Even
3	224	Odd
4	168	Even
6	112	Even
8	84	Even
12	56	Even
16	42	Even
21	32	Odd
24	28	Even

Based on our understanding of the logic in the previous question, we should realise that this works for all situations where the sum of factors is even. Hence, there are 7 positive integral solutions to the equation $a^2 - b^2 = 672$.

30. The function a_n is defined as $a_n - a_{n-1} = 2n$ for all $n \geq 2$. $a_1 = 2$.

Find the value of $a_1 + a_2 + a_3 + \ldots a_{12}$

In order to solve such questions, the key is to be able to identify the pattern of the series. A little bit of thought would give you the following structure:

$$a_1 = 2$$

$$a_2 - a_1 = 4 \rightarrow a_2 = 6$$

$$a_3 - a_2 = 6 \rightarrow a_3 = 12$$

$$a_4 - a_3 = 8 \rightarrow a_4 = 20$$

$$a_5 - a_4 = 10 \rightarrow a_5 = 30$$

If we look for the pattern in these numbers we should be able to see the following:

2 + 6 + 12 + 20 + 30…

= 2(1 + 3 + 6 + 10 + 15 + ….)

Looking at it this way shows us that the numbers in the brackets are consecutive triangular numbers.

Hence, for the sum till a_{12} we can do the following:

2(1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78) = 728.

31. The brothers Binnu and Kinnu have been challenged by their father to solve a mathematical puzzle before they are allowed to go out to play. Their father has asked them "Imagine two integers a and b such that $1 \leq b \leq a \leq 10$. Can you correctly find out the value of the expression $\sum ab$?" Can you help them identify the correct value of the foregoing expression?

(a) 1155 (b) 1050
(c) 1705 (d) None of these

This question is again based on a pattern recognition principle. The best way to approach the search of the pattern is to start by working out a few values of the given expression. The following pattern would start emerging:

Value of a	*Possible values of b*	*Value of the sum of the possible products 'a × b'*	*Explanation*
1	1	1	With '*a*' as 1, the only value for *b* is 1 itself.
2	1, 2	$2 \times 1 + 2 \times 2 = 2 \times 3$	With '*a*' as 2, *b* can take the values of 1 and 2 – and $2 \times 1 + 2 \times 2$ can be written as 2×3
3	1, 2, 3	$3 \times 1 + 3 \times 2 + 3 \times 3 = 3 \times 6$	With '*a*' as 3, *b* can take the values of 1, 2 and 3 – and $3 \times 1 + 3 \times 2 + 3 \times 3$ can be written as 3×6
4	1, 2, 3, 4	$4 \times 1 + 4 \times 2 + 4 \times 3 + 4 \times 4 = 4 \times 10$	

At this point you should realise that the values are following a certain pattern – the series of values for 'a' are multiplied by a series of consecutive triangular numbers (1, 3, 6, 10 and so on.) Thus, we can expect the subsequent numbers to be got by multiplying 5×15, 6×21, 7×28, 8×36, 9×45 and 10×55.

Hence, the answer can be got by:

$1 \times 1 + 2 \times 3 + 3 \times 6 + 4 \times 10 + 5 \times 15 + 6 \times 21 + 7 \times 28 + 8 \times 36 + 9 \times 45 + 10 \times 55 =$

$1 + 6 + 18 + 40 + 75 + 126 + 196 + 288 + 405 + 550 = 1705$

32. For the above question, what would be the answer in case the inequality is expressed as: $1 \le b < a \le 10$?

In this case the solution would change as follows:

Value of a	*Possible values of b*	*Value of the sum of the possible products 'a × b'*	*Explanation*
1	No possible values for '*b*', since *b*" has to be greater than or equal to 1 but less than '*a*' at the same time	0	With '*a*' as 1, *b* has no possible values that it can take.
2	1	2×1	With '*a*' as 2, *b* can only take the value of 1.
3	1, 2	$3 \times 1 + 3 \times 2 = 3 \times 3$	With '*a*' as 3, *b* can take the values of 1 and 2 – and $3 \times 1 + 3 \times 2$ can be written as 3×3
4	1, 2, 3	$4 \times 1 + 4 \times 2 + 4 \times 3 = 4 \times 6$	With '*a*' as 4, *b* can take the values of 1,2 and 3 – and $4 \times 1 + 4 \times 2 + 4 \times 3$ can be written as 4×6

At this point you should realise that the values are following a certain pattern (just like in the previous question) – the series of values for '*a*' are multiplied by a series of consecutive triangular numbers (1, 3, 6, 10 and so on). The only difference in this case is that the multiplication is what can be described as 'one removed', i.e., it starts from the value of '*a*' as 2. Thus, we can expect the subsequent numbers to be got by multiplying 5×10, 6×15, 7×21, 8×28 and 9×36 and 10×45. Notice here that the values of '*a*' can go all the way till 10 in this case as the inequality on the rightmost side of the expression is $a \le 10$.

Hence, the answer can be got by:

$1 \times 0 + 2 \times 1 + 3 \times 3 + 4 \times 6 + 5 \times 10 + 6 \times 15 + 7 \times 21 + 8 \times 28 + 9 \times 36 + 10 \times 45 =$

$0 + 2 + 9 + 24 + 50 + 90 + 147 + 224 + 324 + 450 = 1320.$

33. Consider the equation of the form $x^2 + bx + c = 0$. The number of such equations that have real roots and have coefficients *b* and *c* in the set {1, 2, 3, 4, 5, 6,7}, is

(a) 20 (b) 25

(c) 27 (d) 29

We know that in order to have the roots of an equation to be real, we should have the values of the discriminant of the quadratic equation (defined as the value $b^2 - 4ac$ for a standard quadratic equation $ax^2 + bx + c = 0$) to be non-negative.

In the context of the given equation in this problem, since the value of the coefficient of x^2 is 1, it means that we need to have $b^2 - 4c$ to be positive or 0.

The only thing to be done from this point is to look for possible values of *a* and *b* which fit this requirement. In order to do this, assume a value for '*b*' from the set {1, 2, 3, 4, 5, 6, 7} and try to see which values of *b* and *c* satisfy $b^2 - 4c \ge 0$.

When $b = 1$, *c* can take none of the values between 1 and 6, since "$b^2 - 4c$" would end up being negative

When $b = 2$, *c* can be 1;

When $b = 3$, *c* can be 1 or 2;

When $b = 4$, *c* can be 1 or 2 or 3 or 4;

When $b = 5$, c can take any value between 1 and 6;
When $b = 6$, c can take any value between 1 and 7;
When $b = 7$, c can take any value between 1 and 7
Thus, there are a total of 27 such equations with real roots.

34. The number of polynomials of the form $x^3 + ax^2 + bx + c$ which are divisible by $x^2 + 1$ and where a, b and c belong to {1, 2, …., 8), is
(a) 1 (b) 8
(c) 9 (d) 10

For a polynomial $P(x)$ to be divisible by another polynomial $D(x)$, there needs to be a third polynomial $Q(x)$ which would represent the quotient of the expression $P(x)/D(x)$. In other words, this also means that the product of the polynomials $D(x) \times Q(x)$ should equal the polynomial $P(x)$.

In simpler words, we are looking for polynomials that would multiply $(x^2 + 1)$ and give us a polynomial in the form of $x^3 + ax^2 + bx + c$. The key point in this situation is that the expression that would multiply $(x^2 + 1)$ to give an expression of the form $x^3 + ….$ Would necessarily be of the form (x + constant). It is only in such a case that we would get an expanded polynomial starting with x^3. Further, the value of the constant has to be such that the product of 1 × constant would give a value for 'c' that would belong to the set {1, 2, 3, 4, 5, 6, 7, or 8). Clearly, there are only 8 such values possible for the constant – viz 1, 2, 3,…8 and hence the required polynomials that would be divisible by $x^2 + 1$ would be got by the expansion of the following expressions: $(x^2 + 1)(x + 1)$; $(x^2 + 1)(x + 2)$; $(x^2 + 1)(x + 3)$;… ; $(x^2 + 1)(x + 8)$. Thus, there would be a total of eight such expressions which would be divisible by $x^2 + 1$. Hence, Option (b) is the correct answer.

35. A point P with coordinates (x, y) is such that the product of the coordinates $xy = 144$. How many possible points exist on the $X–Y$ plane such that both x and y are integers?
(a) 15 (b) 16
(c) 30 (d) 32

The number of values for (x, y) such that both are integers and their product is equal to 144 is dependent on the number of factors of 144. Every factor pair would gives us 4 possible solutions for the ordered pair (x, y). For instance, if we were to consider 1 × 144 as one ordered pair, the possible values for (x, y) would be (1,144); (144,1); (–1,–144); (–144,–1). This would be true for all factor pairs except the factor pair 12 × 12. In this case, the possible pairs of (x, y) would be (12,12) and (–12,–12).

If we find out the factor pairs of 144 we will get the following list:

Factor Pair	*Number of solutions for (x, y)*
1 × 144	4 solutions – as explained above
2 × 72	4 solutions
3 × 48	4 solutions
4 × 36	4 solutions
6 × 24	4 solutions
8 × 18	4 solutions
9 × 16	4 solutions
12 × 12	2 solutions

Thus, there are a total of 30 solutions in this case.

36. Let $x_1, x_2, ……, x_{40}$ be forty nonzero numbers such that $x_i + x_{i+1} = k$ for all i, $1 \le i \le \infty$. If $x_{14} = a$, $x_{27} = b$, then $x_{30} + x_{39}$ equals
(a) $3(a + b) – 2k$
(b) $k + a$
(c) $k + b$
(d) None of the foregoing expressions

Since, $x_{14} = a$, x_{15} would equal $(k – a)$ [we get this by equating $x_{14} + x_{15} = k \to a + x_{15} = k \to x_{15} = (k – a)$]. By using the same logic on the equation $x_{15} + x_{16} = k$, we would get $x_{16} = a$. Consequently $x_{17} = k – a$, $x_{18} = a$, $x_{19} = k – a$, $x_{20} = a$. Thus, we see that every odd term is equal to '$k – a$' and every even term is equal to 'a'. Further, we can develop a similar logic for x_i in the context of 'b'. Since, $x_{27} = b$, $x_{28} = k – b$, $x_{29} = b$ and so on. This series also follows a similar logic with x_i being equal to b when 'i' is odd and being equal to $k – b$ when 'i' is even.

Thus, for every value of x_i, we have two ways of looking at its value—viz either in terms of a or in terms of b. Thus, for any x_{even}, we have for instance $x_{20} = a = k – b$.

Solving $a = k – b$ we get $a + b = k$.

Further, when we look at trying to solve for the specific value of what the question has asked us, i.e., the value of $x_{30} + x_{39}$ we realise that we can either solve it in terms of 'a' or in terms of 'b'. If we try to solve it in terms of 'a' we would see the following happening:

$X_{30} + x_{39} = a + k – a = k$

Similarly, in terms of 'b' $x_{30} + x_{39} = k–b + b = k$. Since we know that $k = a + b$ (deduced above), we can conclude $x_{30} + x_{39} = a + b$. However, if we look at the options, none of the options is directly saying that.

Options (b) and (c) can be rejected because their values are not equal to $a + b$. A closer inspection of Option (a), gives us an expression: $3(a + b)–2k$. This expression can be expressed as $3(a + b) – 2(a + b) = (a + b)$ and hence this is the correct answer.

37. The great mathematician Ramanujam, once was asked a puzzle in order to test his mathematical prowess. He was given two sets of numbers as follows:

Set X is the set of all numbers of the form: $4^n – 3n – 1$, where $n = 1, 2, 3, …..$

Set Y is the set of all numbers of the form $9n$, where $n = 0, 1, 2, 3, ……….$

Based on these two definitions of the set, can you help Ramanujam identify the correct statement from amongst the following options:

(a) Each number in Y is also in X
(b) Each number in X is also in Y
(c) Every number in X is in Y and every number in Y is in X.
(d) There are numbers in X that are not in Y and vice versa.

In order to solve such questions conveniently, you would need to first create a language representation for yourself with respect to the two sets.

Set X can be mentally thought of as:

A positive integral power of 4 – a multiple of 3 (with the multiplier being equal to the power of 4 used) – a constant value '1'.

Thus, the set of values in X can be calculated as (0, 9, 54, 243 and so on)

Similarly, Set Y can be thought of as multiples of 9, starting from $9 \times 0 = 0$.

The numbers that would belong to the set Y would be: (0, 9, 18, 27, 36 and pretty much all multiples of 9)

It can be clearly seen that while all values in X are also in Y, the reverse is not true. Hence, the statement in Option (b) is correct.

38. The number of real roots of the equation $\log_{2x}\left(\frac{2}{x}\right)(\log_2(x))^2 + (\log_2(x))^4 = 1$, for values of $x > 1$, is

(a) 0 (b) 1
(c) 2 (d) 27

The only way to handle such questions is to try to get a 'feel' of the equation by inserting a few values for x and trying to see the behaviour of the various terms in the equation.

Towards this end, let us start by trying to insert values for x in the given equation.

Because the problem tells us that $x > 1$, the first value of x which comes to mind is $x = 2$. At $x = 2$, the value of the LHS would become equal to 1. This can be thought of as follows:

LHS = $\log_{2x}(1)(\log_2(2))^2 + (\log_2(2))^4 = 0 + 1 = 1$.

As we try to take higher values of x as 3, 4, 5 and so on we realise first that for values like 3, 5 we will get terms like

$\text{Log}_6\,(2/3)(\log_2(3))^2 + (\log_2(3))^4$ which is clearly not going to be an integral value, because terms like $\log_2 3$, $\log_6 2$ and $\log_6 3$ would have irrational decimal values by themselves. In fact, we can see that for numbers of x that are not powers of 2, we would never get an integral value to the LHS of the expression.

Thus, we need to see the behaviour of the expression only for values of x like 4, 8 and so on before we can conclude about the number of real roots of the equation.

At $x = 4$, the expression becomes:

$\text{Log}_8\,(2/4)(\log_2(4))^2 + (\log_2(4))^4$

By thinking about this expression it is again clear that there are going to be decimals in the first part of the expression—although they are going to be rational numbers and not irrational—and hence we cannot summarily rule out the possibility of an integral value of this expression—without doing a couple of more calculations. However, there is another thought which can help us confirm that the equation will not get satisfied in this case because the LHS is much bigger than the RHS.

This thought goes as follows:

The LHS has two parts: The first part is $\text{Log}_8\,(2/4)(\log_2(4))^2$ while the second part is $(\log_2(4))^4$; The value of the second part is 16 while the value of the first part (though it is negative) is much smaller than the required –15 which will make the LHS = 1.

Hence, we can reject this value.

39. The number of points at which the curve $y = x^6 + x^3 - 2$ cuts the x-axis is

(a) 1 (b) 2
(c) 4 (d) 6

By replacing $x^3 = m$, the equation given in the question, can be written in the form:

$y = m^2 + m - 2 \rightarrow$

$y = (m + 2)(m - 1) \rightarrow m = -2$ and $m = 1$.

This gives $x^3 = -2$ and $x^3 = 1$. This gives us two clear values of x (these would be the roots of the equation) and hence x, the curve would cut the 'x' axis at two points exactly.

40. Number of real roots of the equation $8x^3 - 6x + 1 = 0$ lying between –1 and 1, is

(a) 0 (b) 1
(c) 2 (d) 3

In order to trace the number of real roots for any equation (cubic or larger) the only feasible way to look at it is to try to visualise how many times the graph of the parallel function (in this case: $8x^3 - 6x + 1$) cuts the X-axis.

The following thought process would help you do this:

Since the question is asking us to find out the number of real roots of $8x^3 - 6x + 1 = 0$ between the range –1 and +1, we will need to investigate only the behaviour of the curve for the function $y = 8x^3 - 6x + 1$ between the values –1 and +1

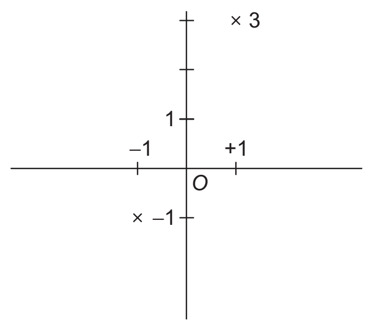
The first thing we do is to look at the values of the function at –1, 0 and +1.

At $x = -1$; the value of the function is $-8 + 6 + 1 = -1$.

At $x = 0$; the value of the function is 1.

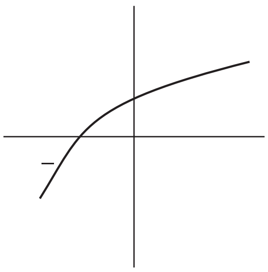
At $x = 1$: the value of the function is 3.

If plotted on a graph, we can visualise the three points as given here.

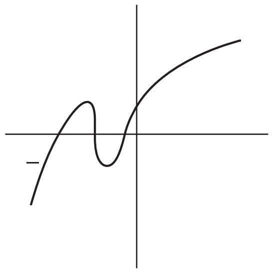


From this visualisation it is clear that the value of the function is –1 at $x = –1$ and +1 at $x = 0$ and +3 at $x = 1$. It is clear that somewhere between –1 and +1, the function would cut the x-axis at least once as it transits from a negative value to a positive value. Hence, the equation would necessarily have at least one root between –1 to 0. What we need to investigate in order to solve this question is specifically—does the function cut the x-axis more than once during this range of values on the x-axis? Also, we should realise that in case the graph will cut the X–axis more than once, it would cut it thrice—since if it goes from negative to positive once, and comes back from positive to negative, it would need to become positive again to go above the x-axis.

This can be visualised as the following possible shapes:

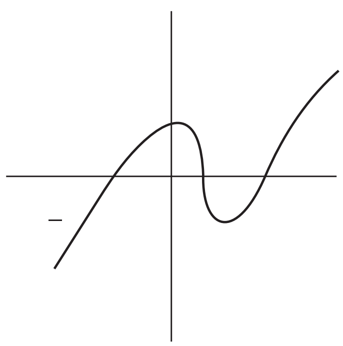


If the graph cuts the x-axis once (and it has one real root in this range)



If the graph cuts the x-axis thrice (and it has three real roots in this range)

Of course the graph can technically also cut the x-Axis twice between the values of $x = 0$ and $x = 1$. In such a case, the graph would look something like below:



Which of these graphs would be followed would depend on our analysis of the behaviour of the values of $8x^3 – 6x + 1$ between the values of x between –1 and 0 first and then between 0 and 1.

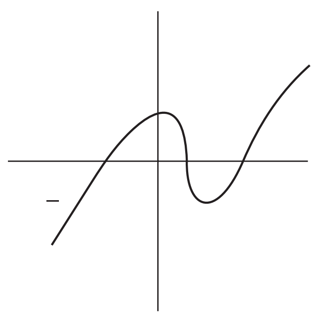
Between –1 and 0:

The value of $8x^3$ would remain negative, while $–6x$ would be positive and +1 would always remain constant. If we look that the values of $8x^3$ as we increase the value of x from –1 to –0.9 to –0.8 to –0.7 to … –0.1, the negative impact of $8x^3$ reduces as its magnitude reduces. (Please understand the difference between magnitude and value when we are talking about a negative number. For instance when we talk about increasing a negative number its magnitude is decreased).

At the same time the positive magnitude of $–6x$ also reduces but the rapidity with which the value of $–6x$ would decrease will be smaller than the rapidity with which the value of $8x^3$ would decrease. Hence, the positive parts of the expression would become 'more powerful' than the negative part of the expression – and hence the graph would not cut the x-axis more than once between –1 and 0.

The last part of our investigation, then would focus on the behaviour of the graph between the values of $x = 0$ and $x = 1$. When x moves into the positive direction (i.e., when we take $x > 0$) we realise that of the three terms $8x^3$ and + 1 would be positive, while the value of $– 6x$ would be negative.

It can be easily visualised that at $x = 1/4$, the value of the expression on the LHS of the equation becomes: $8/64 – 6/4 + 1$, which is clearly negative. Thus, after $x = 0$, when we move to the positive values of x, the value of the expression $8x^3 – 6x + 1$ becomes negative once more. Thus the correct graph would look as below:



Thus, the equation has three real roots between –1 and +1.

BLOCK REVIEW TESTS

REVIEW TEST 1

Directions for Questions 1 to 3: These are based on the functions defined below

$Q(a, b)$ = Quotient when a is divided by b

$R^2(a, b)$ = Remainder when a is divided by b

$R(a, b) = a^2/b^2$

$SQ(a, b) = \sqrt{(a-1)/(b-1)}$

1. $SQ(5, 10) - ? > 0$
   (a) $(8/3)\,R(5, 10)$ (b) $R^2(5, 10) + Q(5, 10)$
   (c) $R^2(5, 10)/2$ (d) $\frac{1}{2}\{R(2,3) + SQ(17,26)\}$
2. $SQ(a, b)$ is same as
   (a) $bQ(a, b) + R^2(a)$ (b) $\sqrt{R(a,b) - 1}$
   (c) $[R\{(a-1), (b-1)\}$ (d) $\sqrt{R(a,1) - 1}/\sqrt{R(b,1)}$
3. Which of the following relations cannot be false?
   (a) $R(a, b) = R^2(a, b) \cdot Q(a, b)$
   (b) $a^2 \cdot Q(a, b) = b^2 \cdot R^2(a, b)$
   (c) $a = R^2(a, b) + y \cdot Q(a, b)$
   (d) $SQ(a, b) = R(a, b) \cdot R^2(a, b)$

Directions for Questions 4 to 7: *Answer the questions based on the following information:*

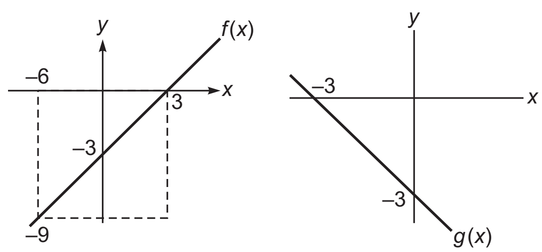
$W(a, b)$ = least of a and b

$M(a, b)$ = greatest of a and b

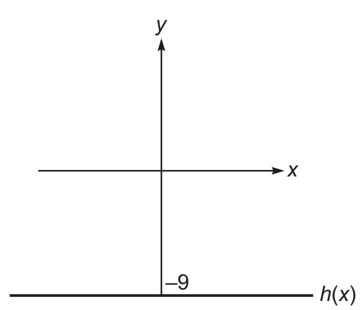
$N(a)$ = absolute value of a

4. Find the value of $1 + M[\,y + N\{-W(x, y)\}, N\{y + W(M(x, y), N(y))\}]$ given that $x = 2$ and $y = -3$.
   (a) 0 (b) 1
   (c) 2 (d) 3
5. Given that $a > b$, then the relation $M\{N(x), W(x, y)\} = W[x, N\{M(x, y)\}]$ does not hold if
   (a) $x > 0, y < 0, |x| > |y|$
   (b) $x > 0, y < 0, |y| > |x|$
   (c) $x > 0, y > 0$
   (d) $x < 0, y < 0$
6. Which of the following must be correct for $x, y < 0$
   (a) $N(W(x, y)) \le W(N(x), N(y))$
   (b) $N(M(x, y)) > W(N(x), N(y))$
   (c) $N(M(x, y)) = W(N(x), N(y))$
   (d) $N(M(x, y)) < M(N(x), N(y))$
7. For what value of x is $W(x^2 + 2x, x + 2) < 0$?
   (a) $-2 < x < 2$ (b) $-2 < x < 0$
   (c) $x < -2$ (d) Both (2) and (3)
8. It is given that, $(a^{n-3} + a^{n-5}b^2 + \ldots + b^{n-3})\,pq = 0$, where p and $q \neq 0$ and n is odd then $\dfrac{(a^n - b^n)(a+b)}{(a^n + b^n)(a-b)}$ =?
   (a) 1 (b) –1
   (c) 3/2 (d) 0
9. If $f = \dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_{4.5} \pi}$, which of the following is true?
   (a) $f > 4$ (b) $2 < f < 4$
   (c) $1 < f < 2$ (d) $0 < f < 1$
10. If $px + qy > rx + sy$, and $y, x, p, q, r, s > 0$ and if $x < y$, then which of the following must be true?
    (a) $\dfrac{p}{q} > \dfrac{q}{r}$ (b) $p - q > r - s$
    (c) $p + q > r + s$ (d) $p + q < r + s$

Directions for Questions 11 to 13: $f(x)$ *and* $g(x)$ are defined by the graphs shown below:

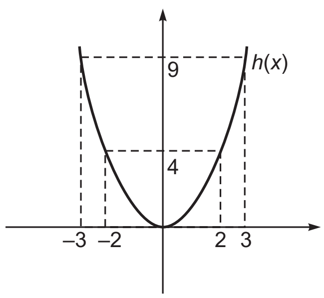


Each of the following questions has a graph of function $h(x)$ with the answer choices expressing $h(x)$ in terms of a relationship of $f(x)$ or/and $g(x)$. Choose the alternative that could represents the relationship appropriately.

11. 

    (a) $6f(x) + 6\,g(x)$ (b) $-1.5f(x) + 1.5\,g(x)$
    (c) $1.5\,f(x) + 1.5\,g(x)$ (d) None of these

12.



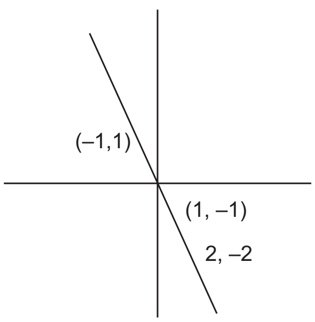
(a) $9 - f(x)\,g(x)$

(b) $[f(x) + g(x) + 4]^2 + [f(x) - 2]^2 + [g(x)]$

(c) $[f(x) - g(x) + 4]^2 - 2$

(d) $f(x)g(x) - 9$

13.



(a) $2f(x) + g(x)$

(b) $f(x) + 2g(x) - 9$

(c) $\frac{3}{2}f(x) + \frac{g(x)}{2}$

(d) None of these

14. If $p^a = q^b = r^c$ and $\frac{p}{q} = \frac{q}{r}$, $\left(\frac{1}{a} + \frac{1}{c}\right) b = ?$

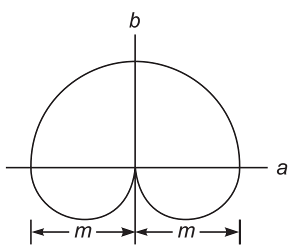
(a) 1

(b) 1/2

(c) 3/4

(d) 2

15. What could be the equation of the following curve?



(a) $(a^2 + b^2)^2 = m^2(a^2 + b^2)$

(b) $(a^2 + b^2 - mb)^2 = m^2(a^2 + b^2)$

(c) $a^2 + b^2 - mb = m^2(a^2 + b^2)$

(d) None of these

Directions for Questions 16 to 18: It is given that $f(x) = p^x$, $g(x) = (-p)^x$; $h(x) = (1/p)^x$, $k(x) = (-1/p)^x$

16. Think of a situation where a function is odd or even, if the function is odd it is given a weightage of 1; otherwise, it is given a weightage of 0. What is the result if the weightages of four functions are added?

(a) 2

(b) 0

(c) 1

(d) –1

17. If '*p*' and '*x*' are both whole numbers other than 0 and 1, which of the functions must have the highest value?

(a) $g(x)$ only

(b) $f(x)$ only

(c) $g(x)$ and $h(x)$ both

(d) $h(x)$ and $k(x)$ *both*

18. Which of the following is true if *'p'* is a positive number and *x* is a real number?

(a) $\{f(x) - h(x)\} / \{g(x) - k(x)\}$ is always positive

(b) $f(x).\ g(x)$ is always negative

(c) $f(x).\ h(x)$ is always greater than one

(d) $g(x).\ h(x)$ could exist outside the real domain

19. If x, and $y \geq 1$ and belong to set of integers then which of the following is true about the function $(xy)^n$?

(a) The function is odd if '*x*' is even and '*y*' is odd.

(b) The function is odd if '*x*' is odd and '*y*' is even.

(c) The function is odd if '*x*' and '*y*' both are odd.

(d) The function is even if '*x*' and '*y*' both are odd.

20. If $f(a,b)$ = remainder left upon division of b by a, then the maximum value for $f(f(a, b), f(a + 1, b + 1)) \times f(f(a, b), 0)$ is ….. (b and a are co-primes)

(a) $a - 1$

(b) a

(c) 0

(d) 1

Space for Rough Work

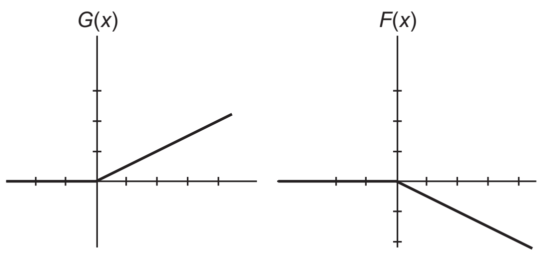
REVIEW TEST 2

1. The number of solutions of $\dfrac{\log 5 + \log(y^2+1)}{\log(y-2)} = 2$ is:
   (a) 3 (b) 2
   (c) 1 (d) None of these

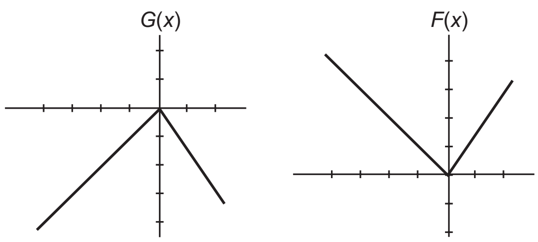
Information for Questions 2 and 3: Given below are two graphs labeled $F(x)$ and $G(x)$. Compare the graphs and give the answer in accordance to the options given below:

(a) $F(-x) = G(x) + x/3$
(b) $F(-x) = -G(x) - x/2$
(c) $F(-x) = -G(x) + x/2$
(d) None of these

2.

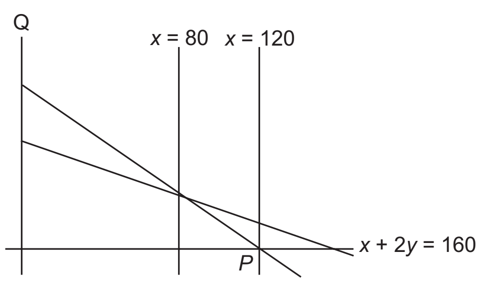


3.



4. If a is a natural number which of the following statements is always true?
   (a) $(a+1)(a^2+1)$ is odd
   (b) $9a^2 + 6a + 6$ is even
   (c) $a^2 - 2a$ is even
   (d) $a^2(a^2 + a) + 1$ is odd

5. In the figure below, equation of the line PQ is



   (a) $x + y = 120$ (b) $2x + y = 120$
   (c) $x + 2y = 120$ (d) $2x + y = 180$

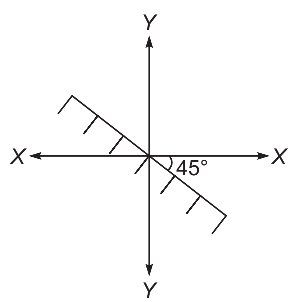
6. For which of the following functions is $\dfrac{f(a) - f(b)}{a - b}$ constant for all the numbers 'a' and 'b', where $a \neq$ b?
   (a) $f(y) = 4y + 7$ (b) $f(y) = y + y^2$
   (c) $f(y) = \cos y$ (d) $f(y) = \log_e y$

7. Given that $f(a, b, c) = \dfrac{a+b+c}{3}$ then
   (a) $f(a, b, c) \geq \dfrac{|a|+|b|+|c|}{3}$
   (b) $f(a, b, c) \geq \max(a, b, c)$
   (c) $|f(a, b, c)| \geq \dfrac{|a+b+c|}{3}$
   (d) $|f(a, b, c)| \leq \dfrac{|a|+|b|+|c|}{3}$

8. We are given two variables x and y. The values of the variables are $x = \dfrac{1}{a+b}$ and $y = \dfrac{3}{c+x}$. Find the value of the expression $\dfrac{7y}{x}$
   (a) $\dfrac{21(a+b)^2}{ca+cb+1}$ (b) $\dfrac{3(a+b)}{7ab+ac}$
   (c) $\dfrac{7}{3(ca+cb+1)}$ (d) None of these

9. If $p = \dfrac{12-|x-3|}{12+|x-3|}$ the maximum value that 'p' can attain is:
   (a) 1 (b) 2
   (c) 21 (d) 12

10.



Refer to the graph. What does the shaded portion represent?
   (a) $x + y \leq 0$ (b) $x \geq y$
   (c) $x + 1 \geq y + 1$ (d) $x + y \geq 0$

11. If a, b, c are positive numbers and it is known that $a^2 + b^2 + c^2 = 8$ then:
   (a) $a^3 + b^3 + c^3 \leq 16\sqrt{\dfrac{2}{3}}$
   (b) $a^3 + b^3 + c^3 \geq 64$
   (c) $a^3 + b^3 + c^3 \geq 16\sqrt{\dfrac{2}{3}}$
   (d) $a^3 + b^3 + c^3 \leq 64$

12. Find the value of the expression given in terms of variables 'x' and 'y'.

$$\frac{(x^2+(a-c)x-ac)\,(x^2-ax-bx+ab)\,(x+c)}{(x^2-a^2)\,(x^2-bx^2-c^2x+bc^2)}$$

(a) $\frac{(x-b)(x-c)}{(x-a)}$ (b) $\frac{(x+a)(x+c)}{(x-b)}$

(c) $\frac{(x+a)(x-b)}{(x-c)}$ (d) None of these

Directions for Questions 13 and 14: These questions are based on the relation given below:

$f^a(y) = f^{a-1}(y-1)$ where $a > 1$ (integer values only) and $f^1(y) = 2/y$ if 'y' is positive or $f^1(y) = 1/(y^2+1)$ otherwise.

13. What is $f^a(a-1)$?
(a) 0 (b) 1
(c) 2 (d) Indeterminate.

14. What is the value of $f^a(a+1)$?
(a) 1 (b) a
(c) 2a (d) 2

15. Raman derived an equation to denote distance of a Haley's comet (x) in the form of a quadratic equation. Distance is given by solution of quadratic $x^2 + Bx + c = 0$. To determine constants of the above equation for Haley's Comet, two separate series of experiments were conducted by Raman. Based on the data of first series, value of x obtained is (1, 8) and based on the second series of data, value of x obtained is (2, 10). Later on it is discovered that first series of data gave incorrect value of constant C while second series of data gave incorrect value of constant B. What is the set of actual distance of Haley's Comet found by Raman?
(a) (11, 3) (b) (6, 3)
(c) (4, 5) (d) (3,11)

16. 'a', 'b' and 'c' are three real numbers. Which of the following statements is/are always true?
(A) $(a-1)(b-1)(c-1) < abc$.
(B) $(a^2 + b^2 + c^2)/2 \geq ca + cb - ab$
(C) $a^2b\sqrt{c}$ is a real number
(a) Only A is true (b) Only B and C are true
(c) Only B is true (d) None is true

17. If we have f[g(y)] = g[f(y)], then which of the following is true?
(a) [f[f[g[g[g[g(y)]]]]]] = [f[g[g[f[f[g(y)]]]]]]
(b) [f[f[f[g[f[g(y)]]]]]] = f[f[g[g[g[f(y)]]]]]
(c) [g[f[g[g[g[f(y)]]]]]] = [f[g[g[f[f[f(y)]]]]]]
(d) [g[f[g[g[f[g[f(y)]]]]]]] = [f[g[g[g[g[g[f(y)]]]]]]]

18. $f(a) = \frac{a^8-1}{a^2+1}$ and $g(a) = \frac{a^4-3}{(a+1)^2}$, what is $f\left(\frac{1}{g(2)}\right)$?

(a) 0.652 (b) $\frac{1468}{2250}$

(c) $-\frac{734}{1625}$ (d) None of these

19. Dev was solving a question from his mathematics book when he encountered the expression $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$ then $a^a\, b^b\, c^c$ is
(a) –1 (b) 1
(c) 0.5 (d) 2

20. The number of integral solutions of $\frac{\log 5 + \log(a^2+1)}{\log(a-2)}$ = is:

(a) 3 (b) 2
(c) 1 (d) None of these

Space for Rough Work

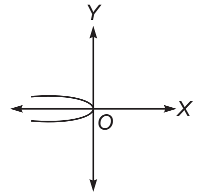
REVIEW TEST 3

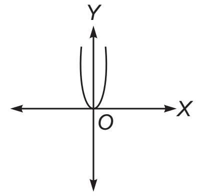
Directions for the Questions 1 to 3: Refer to the data given below and answer the questions._

Given $\frac{a}{b} = \frac{1}{2}, \frac{c}{d} = \frac{1}{3}$ and $z = \frac{a+c}{b+d}$, answer the questions below on limits of z.

1. If $y \geq 0$ and $p \geq 0$ then the limits of 'z' are:
   (a) $z \leq 0$ or $z \geq 1$ (b) $\frac{1}{3} \leq z \leq \frac{1}{2}$
   (c) $z \geq \frac{1}{2}$ or $z \leq \frac{1}{3}$ (d) $0 \leq z \leq 1$
2. $c \leq 0$ and $1/3 \leq z \leq 1/2$ only if:
   (a) $a > 1.5c$ (b) $c > -1$
   (c) $a \leq 0$ (d) $a > -1.5c$
3. If $a = -31$, which of the following value of 'd' gives the highest value of 'z'?
   (a) $d = 72$ (b) $d = 721$
   (c) $d = -31$ (d) $d = 0$
4. Find the integral solution of: $5y - 1 < (y + 1)^2 < (7y - 3)$
   (a) 2 (b) $2 < y < 4$
   (c) $1 < y < 4$ (d) 3
5. If $f(a) = \frac{a-1}{a+1}$, $x \geq 0$ and if $y = f\left(\frac{1}{a}\right)$ then
   (a) As 'a' decreases, 'y' decreases
   (b) As 'a' increases, 'y' decreases
   (c) As 'x' increases, 'y' increases
   (d) As 'x' increases, 'y' remains unchanged
6. If f and g are real functions defined by $f(a) = a + 2$ and $g(a) = 2a^2 + 5$, then fog is equal to
   (a) $2a^2 + 7$ (b) $2a^2 + 5$
   (c) $2(a + 2)^2 + 5$ (d) $2a + 5$
7. If 'p' and 'q' are the roots of the equation $x^2 - 10x + 16 = 0$, the value of $(1 - p)(1 - q)$ is
   (a) –7 (b) 7
   (c) 16 (d) –16
8. Given that 'a' and 'b' are positive real numbers such that $a + b = 1$, then what is the minimum value of $\sqrt{12 + \frac{1}{a^2}} + \sqrt{12 + \frac{1}{b^2}}$?
   (a) 8 (b) 16
   (c) 24 (d) 4
9. Let p, q and r be distinct positive integers satisfying $p < q < r$ and $p + q + r = k$. What is the smallest value of k that does *not* determine p, q, r uniquely?
   (a) 9 (b) 6
   (c) 7 (d) 8
10. Given odd positive integers p, q and r which of the following is not necessarily true?
   (a) $p^2 q^2 r^2$ is odd (b) $3(p^2 + q^3)r^2$ is even
   (c) $5p + q + r^4$ is odd (d) $r^2 (p^4 + q^4)/2$ is even
11. $f(a) = (a^2 + 1)(a^2 - 1)$ where $a = 1, 2, 3\ldots$ which of the following statement is not correct about $f(a)$?
   (a) $f(a)$ is always divisible by 5
   (b) $f(a)$ is always divisible by 3
   (c) $f(a)$ is always divisible by 30
   (d) None of these

Directions for Questions 12 and 13: The following questions are based on the graph of parabola plotted on $x - y$ axes. Answer the questions according to given conditions if applicable and deductions from the graph.

12. 

   The above graph represent the equations
   (a) $y^2 = kx, k > 0$ (b) $y^2 = kx, k < 0$
   (c) $x^2 = ky - 1, k < 0$ (d) $x^2 = ky - 1, k > 0$
13. 

   The above graph represents the equation
   (a) $x^2 = ky, k < 0$ (b) $y^2 = kx + 1, k > 0$
   (c) $y^2 = kx + 1, k < 0$ (d) None of these
14. Mala while teaching her class on functions gives her students a question.
   According to the question the functions are $f(x) = -x$, $g(x) = x$. She also provides her students with following functions also.
   $$f(x, y) = x - y \text{ and } g(x, y) = x + y$$
   Since she wants to test the grasp of her students on functions she asks them a simple question "which of the following is not true?" and provides her students with the following options. None of her students were able to answer the question in single attempt. Can you answer her question?
   (a) $f[f(g(x, y))] = g(x, y)$
   (b) $g[f[g[f(x, y)]]] = f(x, y)$
   (c) $f(x) + g(x) + f(x, y) + g(x, y) = g(x) - f(x)$
   (d) None of these

Directions for Questions 15 and 16: We are given that $f(x) = f(y)$ and $f(x, y) = x + y$, if $x, y > 0$

$f(x, y) = xy$, if $x, y = 0$

$f(x, y) = x - y$, if $x, y < 0$

$f(x, y) = 0$, otherwise

15. Find the value of the following function: $f[f(2, 0), f(-3, 2)] + f[f(-6, -3), f(2, 3)]$.
    (a) 0 (b) 2
    (c) –8 (d) None of these
16. Find the value of the following function: $\{f[f(1, 2), f(2, 3)]\} \times \{f[f(1.6, 2.9), f(-1, -3)]\}$
    (a) 12 (b) 36
    (c) 48 (d) 52
17. Given that $f(a) = a(a + 1)(a + 2)$ where $a = 1, 2, 3, \ldots$. Then find $S = f(1) + f(2) + f(3) + \ldots\ldots + f(10)$?
    (a) 4200 (b) 4290
    (c) 4400 (d) None of these
18. We have three inequalities as:
    (i) $2^a > a$ (ii) $2^a > 2a + 1$
    (iii) $2^a > a^2$

    For what natural numbers n are all the three inequalities satisfied?
    (a) $a \geq 3$ (b) $a \geq 4$
    (c) $a \geq 5$ (d) $a \geq 6$
19. For the curve $x^3 - 3xy + 2 = 0$, the set A of points on the curve at which the tangent to the curve is horizontal and the set B of points on the curve at which the tangent to the curve is vertical are respectively:
    (a) (1, 1) and (0, 0) (b) (0, 0) and (1, 1)
    (c) (1, 1) and null set (d) None of these
20. If $f(a) = a^2 - \frac{1}{a^2}$ and $g(a) = \frac{1}{\sqrt{f(a) - 4}}$, then the real domain for all values of 'a' such that $f(a)$ and $g(a)$ are both real and defined is represented by the inequality:
    (a) $a^2 - a - 1 > 0$ (b) $a^4 - 4a^2 - 1 > 0$
    (c) $a^2 - 4a - 1 > 0$ (d) None of these

Space for Rough Work

ANSWER KEY

Review Test 1

1. (d)	2. (c)	3. (c)	4. (c)
5. (d)	6. (c)	7. (d)	8. (a)
9. (c)	10. (c)	11. (c)	12. (a)
13. (d)	14. (d)	15. (b)	16. (b)
17. (b)	18. (d)	19. (b)	20. (c)

Review Test 2

1. (d)	2. (d)	3. (d)	4. (d)
5. (a)	6. (a)	7. (d)	8. (a)
9. (a)	10. (a)	11. (c)	12. (d)
13. (b)	14. (a)	15. (c)	16. (c)
17. (c)	18. (d)	19. (b)	20. (d)

Review Test 3

1. (b)	2. (c)	3. (d)	4. (b)
5. (b)	6. (a)	7. (d)	8. (a)
9. (d)	10. (d)	11. (d)	12. (b)
13. (d)	14. (b)	15. (a)	16. (d)
17. (b)	18. (c)	19. (c)	20. (b)