UNIT – II : ALGEBRA

CHAPTER-3

MATRICES

Topic-1	Matrices and Operations
	Concepts covered: Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operation on Matrices : Addition multiplication and multiplication with a scalar, simple properties of addition, multiplication and scalar multiplication, Non commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix.

Revision Notes

- **Matrix :**

  A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as elements or the entries of the matrix. It is denoted by the uppercase letters *i.e.*, *A, B, C,* etc.

  Consider a matrix A given as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

  Here in matrix A depicted above, the horizontal lines of elements are said to constitute rows of the matrix A and vertical lines of elements are said to constitute columns of the matrix. Thus, matrix A has m **rows** and n **columns**. The array is enclosed by square brackets [ ], the parentheses ( ) or the double vertical bars || ||.

  - A matrix having m rows and n columns is called a matrix of order $m \times n$ (read as 'm by n' matrix). A matrix 'A' of order $m \times n$ is depicted as $A = [a_{ij}]_{m\times n}; i, j \in N$.
  - Also, in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.
  - No. of elements in the matrix $A = [a_{ij}]_{m\times n}$ is given as (mn).

- **Types of Matrices :**

  (a) Column matrix : A matrix having only one column is called a **column matrix** or **column vector**.

  e.g., $A = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times 1}$

  General notation : $A = [a_{ij}]_{m\times 1}$

  (b) Row matrix : A matrix having only one row is called a **row matrix** or **row vector**.

  e.g., $A = [2 \;\; 5 \;\; -4]_{1\times 3}, B = [\sqrt{2} \;\; 4]_{1\times 2}$

  General notation : $A = [a_{ij}]_{1\times n}$

  (c) Square matrix : It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $n\times n$ matrix is said to constitute a square matrix of order $n \times n$ and is known as a **square matrix of order** 'n'.

e.g., $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times3}$ is a square matrix of order 3.

General notation : $A = [a_{ij}]_{n\times n}$

(d) Diagonal matrix : A square matrix $A = [a_{ij}]_{m\times n}$ is said to be diagonal matrix if all the elements, except those in the leading diagonal are zero *i.e.,* $a_{ij} = 0$ for all $i \neq j$.

e.g., $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$ is a diagonal matrix of order 3.

- Also, there are more notation specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can also be written as diag (2 5 4) or diag [2, 5, 4]
- Note that the elements $a_{11}, a_{22}, a_{33}, ..., a_{mn}$ of a square matrix $A = [a_{ij}]_{m\times n}$ of order $m \times n$ are said to constitute the principal diagonal or simply **the diagonal of the square matrix** A. These elements are known as **diagonal elements of matrix** A.

(e) Scalar matrix : A diagonal matrix $A = [a_{ij}]_{m\times n}$ is said to be a scalar matrix if its diagonal elements are equal.

i.e., $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$

e.g., $A = \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}_{3\times3}$ is a scalar matrix of order 3.

(f) Unit or Identity matrix : A square matrix $A = [a_{ij}]_{m\times n}$ is said to be an identity matrix if $a_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

A unit matrix can also be defined as the scalar matrix if each of its diagonal elements is unity. We denote the identity matrix of order m by I_m or I.

e.g., $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

(g) Zero matrix or Null matrix : A matrix is said to be a **zero matrix** or **null matrix** if each of its elements is 'zero'.

e.g., $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$, $C = [0 \;\; 0]_{1\times2}$

(h) Triangular matrix :

(i) Lower triangular matrix : A square matrix is called a lower triangular matrix if all the entries above the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 3 & 5 \end{bmatrix}$

(ii) Upper triangular matrix : A square matrix is called a **upper triangular matrix** if all the entries below the main diagonal are zero.

e.g., $A = \begin{bmatrix} 1 & -8 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

➢ **Equality of Matrices :**

Two matrices A and B are said to be equal and written as $A = B$, if they are of the same order and their corresponding elements are identical *i.e.,* $a_{ij} = b_{ij}$ *i.e.,* $a_{11} = b_{11}$, $a_{22} = b_{22}$, $a_{32} = b_{32}$, etc.

- **Addition of Matrix :**

  If A and B are two $m \times n$ matrices, then another $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by '$A + B$'.

  Thus, if $A = [a_{ij}]$, $B = [b_{ij}] \Rightarrow A + B = [a_{ij} + b_{ij}]$.

  Properties of matrix addition :

  - Commutative property : $A + B = B + A$
  - Associative property : $A + (B + C) = (A + B) + C$
  - Cancellation law : **(i)** Left cancellation : $A + B = A + C \Rightarrow B = C$

    (ii) Right cancellation : $B + A = C + A \Rightarrow B = C$
  - **Existence of additive identity :**

    $$A + O = O + A = A$$

    where O is the $m \times n$ zero matrix or the additive identity for matrix addition.
  - **Existence of additive inverse :**

    $$A + (-A) = (-A) + A = 0$$

- **Multiplication of a Matrix by a Scalar :**

  If a $m \times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus, if $A = [a_{ij}]$, and it is multiplied by a scalar k, then $kA = [ka_{ij}]$, *i.e.*. $A = [a_{ij}]$ $\Rightarrow kA = [ka_{ij}]$

  e.g., $$A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix} \Rightarrow 3A = \begin{bmatrix} 6 & -12 \\ 15 & 18 \end{bmatrix}$$

- **Multiplication of Two Matrices :**

  Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{jk}]$ be a $n \times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m \times p$ matrix $C = [c_{ik}]$ such that $C_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB, *i.e.*, $C = AB$.

  Properties of matrix multiplication :

  - Note that the product AB is defined only when the number of columns in matrix A is equal to the number of rows in matrix B.
  - If A and B are $m \times n$ and $n \times p$ matrices, respectively, then the matrix AB will be an $m \times p$ matrix, *i.e.*, order of matrix AB will be $m \times p$.
  - In the product AB, A is called the **pre-factor** and B is called the **post-factor**.
  - If two matrices A and B are such that AB is possible, then it is not necessary that the product BA is also possible.
  - If A is an $m \times n$ matrix and both AB as well as BA are defined, then B will be an $n \times m$ matrix.
  - If A is an $n \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$.
  - Matrix multiplication is **associative**, *i.e.*, $A(BC) = (AB)C$.
  - Matrix multiplication is **distributive** over the **addition**, *i.e.*, $A.(B+C) = AB + AC$.
  - Matrix multiplication is not commutative.

- **Existence of non-zero matrices whose product is zero.**

  The product of two matrices can be zero without either factor being a zero matrix.

  e.g., Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

  Here, $A \neq 0$ and $B \neq 0$.

  Also, $$AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **Transpose of a Matrix :**

  If $A = [a_{ij}]_{m \times n}$ be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a transpose of matrix A. The transpose of A is denoted by A' or A^T, *i.e.*, if $A^T = [a_{ji}]_{n \times m}$.

  For example, if $A = \begin{bmatrix} 5 & -4 & 1 \\ 0 & \sqrt{5} & 3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 5 & 0 \\ -4 & \sqrt{5} \\ 1 & 3 \end{bmatrix}$

Properties of Transpose of Matrices :

- $(A+B)^T = A^T+B^T$
- $(A^T)^T = A$
- $(kA)^T = kA^T$, where k is any constant
- $(AB)^T = B^TA^T$
- $(ABC)^T = C^TB^TA^T$

➢ **Symmetric matrix :** A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix if $A^T = A$, *i.e.*, if $A = [a_{ij}]$, then $[a_{ji}] = [a_{ji}]$.

For example, $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $B = \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$

➢ **Skew Symmetric Matrix :**
A square matrix $A = [a_{ij}]$ is said to be **a skew symmetric matrix** if $A^T = -A$ *i.e.*, if $A = [a_{ij}]$, then $[a_{ji}] = -[a_{ij}]$.

For example, $A = \begin{bmatrix} 0 & 1 & -5 \\ -1 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Notes

- For any matrices AA^T and A^TA are symmetric matrices
- If A and B are two symmetric matrices of same order, then
  (i) AB is symmetric if and only if $AB = BA$.
  (ii) $A \pm B$, $AB + BA$ are also symmetric matrices.

Notes

- All the diagonal elements in a skew-symmetric matrix are zero.
- If A and B are two symmetric matrices, then $AB - BA$ is a skew symmetric matrix.

Key Formulae

➢ For any square matrix A, the matrix $A + A^T$ is a symmetric and $A - A^T$ is always a skew symmetric matrix.

➢ A square matrix can be expressed as the sum of a symmetric and skew symmetric matrix, *i.e.*, $A = \frac{1}{2}(P) + \frac{1}{2}(Q)$,

where $P = A + A^T$ is a symmetric matrix and $Q = A - A^T$ is a skew symmetric matrix.

Mnemonics

Concept : Types of Matrices
Mnemonics 'Remember Crist Subah Dophar Syam Nite'
Interpretation :

R	C	S	D	S	N	I	T	E
↓	↓	↓	↓	↓	↓	↓	↓	↓
Row Matrix	Column Matrix	Square Matrix	Diagonal Matrix	Scalar Matrix	Null Matrix	Identity Matrix	Triangular Matrix	Equal Matrix

Topic-2

Invertible Matrices and Martin's Rule

Concepts covered : Invertible matrices, Proof of uniqueness of inverse, if it exists, Martin's rule and related problems.

Revision Notes

➢ **Determinant :** A unique number (real or complex) can be associated to every square matrix is known as its determinant. The determinant of matrix A is denoted by det A or $|A|$.

eg. : **(i)** If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2,

then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(ii) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3,

then $|A| = (-1)^{1+1} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- **Singular Matrix & Non-Singular Matrix :**

(a) Singular matrix: A square matrix A is said to be singular if $|A| = 0$, *i.e.*, its determinant is zero.

e.g., $A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{vmatrix}$

$= 1(15 - 12) - 2(12 - 12) + 3(4 - 5)$

$= 3 - 0 - 3 = 0$

$\therefore$ A is singular matrix.

(b) Non-singular Matrix : A square matrix A is said to be non-singular if $|A| \neq 0$.

e.g., $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 0(0 - 1) - 1(0 - 1) + 1(1 - 0)$

$= 0 + 1 + 1 = 2 \neq 0$

$\therefore$ A is a non-singular matrix.

- A square matrix A is **invertible** if and only if A is **non-singular.**

- **Minors :** Minors of an element a_{ij} of a determinant (*or a determinant corresponding to matrix A*) is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence, we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, 3 × 3) determinant.
- **Co-factors :** Cofactor of an element a_{ij} denoted by A_{ij} is defined by $A=(-1)^{(i+j)}M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.
- **Adjoint of a Square Matrix :**

Let $A = [a_{ij}]$ be a square matrix. Also, assume $B = [A_{ij}]$, where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the **adjoint of matrix** A and it is denoted by ***adj* (*A*)**.

To find adjoint of a 2 × 2 matrix: If the adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of off-diagonal elements, *i.e.,* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $adj\ A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example, consider a square matrix of order 3 as $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then in order to find the adjoint matrix A, we find a matrix B (formed by the co-factors of elements of matrix A as mentioned above in the definition)

i.e., $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj\, A = B^T = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$

- **Algorithm to find A^{-1} by Determinant Method :**

Step 1 : Find $|A|$.

Step 2 : If $|A| = 0$, then, write "A is a singular matrix and hence not invertible". Else write "A is non-singular matrix and hence invertible".

Step 3 : Calculate the co-factors of elements of matrix A.

Step 4 : Write the matrix of co-factors of elements of A and then obtain its transpose to get $adj\ A$ (*i.e.*, adjoint A).

Step 5 : Find the inverse of A by using the relation : $A^{-1} = \frac{1}{|A|}(adj\, A)$

- **Properties associated with various Operations of Matrices & Determinants :**

(a) $AB = I = BA$

(b) $AA^{-1} = I$ or $A^{-1}I = A^{-1}$

(c) $(AB)^{-1} = B^{-1}A^{-1}$

(d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(e) $(A^{-1})^{-1} = A$

(f) $(A^T)^{-1} = (A^{-1})^T$

(g) $A(adj\, A) = (adj\, A)\, A = |A|\, I$

(h) $adj(AB) = adj\,(B)\, adj\,(A)$

(i) $adj\,(A^T) = (adj\, A)^T$

(j) $(adj\, A)^{-1} = (adj\, A^{-1})$

(k) $|adj\, A| = |A|^{n-1}$, *if* $|A| \neq 0$, where n is of the order of A

(l) $|AB| = |A|\,|B|$

(m) $|A\, adj\, A| = |A|^n$, where n is of the order of A

(n) $|A^{-1}| = \frac{1}{|A|}$, if matrix A is invertible

(o) $|A| = |A^T|$

- $|kA| = k^n|A|$, where n is of the order of square matrix A and k is any scalar.
- If A is non-singular matrix of order n, then $adj\,(adj\, \text{A}) = |A|^{n-2} A$.

- **Uniqueness Theorem :** Let A be an invertible square matrix of order n. Suppose B and C are the two inverse of A.

Then $AB = BA = I_n$ (by definition of inverse matrix)

$AC = CA = I_n$

Now, $B = BI_n = B(AC)$ [$\because$ Matrix multiplication is associative]

$= (BA)C$

$= I_nC$

$= C$

$\therefore$ $B = C$, *i.e.*, any two inverse of A are equal matrices.

Hence, the inverse of A is unique.

- **Solving System of Equations by Matrix Method [Martin's Rule]**

Homogeneous and Non-homogeneous system : A system of equations $AX = B$ is said to be a homogeneous system if $B = 0$. Otherwise it is called a non-homogeneous system of equations.

Let given system of equations is :

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Step 1: Assume $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Step 2 : Find $|A|$. Now there may be following situations :

(i) If $|A| \neq 0 \Rightarrow A^{-1}$ exists, then the given system of equations is consistent and therefore, the system has **unique solution.** In that case, write

$$AX = B$$

$\Rightarrow$ $X = A^{-1}B$ $\left[\text{where } A^{-1} = \frac{1}{|A|}(adj\, A)\right]$

(ii) If $|A| = 0 \Rightarrow A^{-1}$ does not exist, then the given system of equations is either inconsistent or it has infinitely many solutions. In order to check proceed as follow :

- Compute $(adj\, A)\, B$.
- If $(adj\, A)B \neq 0$, then the given system of equations is inconsistent, *i.e.*, it has no solution.
- If $(adj\, A)B = 0$, then the given system of equations is consistent with infinitely many solutions.

  [In order to find these infinitely many solutions, replace one of the variables by some real number and proceed in the same manner in the new two variables system of equations]