# 4. BINOMIAL THEOREM

## MATHEMATICAL INDUCTION

The technique of Induction is used to prove mathematical theorems. A variety of statements can be proved using this method. Mathematically, if we show that a statement is true for some integer value, say n = 0, and then we prove that the statement is true for some integer k+1 if it is true for the integer k (k is greater than or equal to 0), then we can conclude that it is true for all integers greater than or equal to 0.

The solution in mathematical induction consists of the following steps:

**Step 1:** Write the statement to be proved as P(n) where n is the variable.

**Step 2:** Show that P(n) is true for the starting value of n equal to 0(say).

**Step 3:** Assuming that P(k) is true for some k greater than the starting value of n, prove that P(k+1) is also true.

**Step 4:** Once P(k+1) has been proved to be true, we say that the statement is true for all values of the variable.

The following illustrations will help to understand the technique better.

**Illustration 1:** Prove that 1+2+3+...+n=n(n+1)/2 for all n, n is natural. **(JEE MAIN)**

**Sol:** Clearly, the statement P(n) is true for n = 1. Assuming P(k) to be true, add (k+1) on both sides of the statement.

P(n):1+2+3+...+n=n(n+1)/2

Clearly, P(1) is true as 1=1.2/2.

Let P(k) be true. That is, let 1+2+3+...+k be equal to k(k+1)/2

Now, we have to show that P(k+1) is true, or that

1+2+3+...+(k+1)=(k+1)(k+2)/2.

L.H.S = 1+2+3+...+(k+1)

= 1+2+3+...+k+(k+1) = k(k+1)/2 + (k+1) (As P(k) is true)

= (k+1) (k/2+1) = (k+1)(k+2)/2

= R.H.S

**Illustration 2:** Prove that (n+1)! >$2^n$ for all n>1. **(JEE MAIN)**

**Sol:** For n = 2, the given statement is true. Now assume the statement to be true for n = m and multiply (m+2) on both sides.

Let $(n+1)! > 2^n$ ... (i)

Putting n=2 in eq. (i), we get,

$3! > 2^2$

$3! > 4$

Since this is true,

Therefore the equation holds true for n=2.

Assume that equation holds true for n=m,

$(m+1)! > 2^m$ ... (ii)

Now, we have to prove that this equation holds true for n=m+1, i.e. $(m+2)! > 2^{m+1}$ .

From equation 2, $(m+1)! > 2^m$ .

Multiply above equation by m+2

$(m+2)! > 2^m (m+2)$

$> 2^{m+1} + 2^m.m$

$> 2^{m+1}$.

Hence proved.

**Illustration 3:** Prove that $n^2 + n$ is even for all natural numbers n. **(JEE MAIN)**

**Sol:** Consider $P(n) = n^2 + n$. It can written as a product of two consecutive natural numbers. Use this fact to prove the question.

Consider that P(n) $n^2 + n$ is even, P(1) is true as $1^2 + 1= 2$ is an even number.

Consider P(k) be true,

To prove : P(k + 1) is true.

P(k + 1) states that $(k + 1)^2 + (k + 1)$ is even.

Now, $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1 = k^2 + k + 2k + 2$

As P(k) is true, hence $k^2 + k$ is an even number and can be written as $2\lambda$, where $\lambda$ is sum of natural number.

$\therefore\ 2\lambda + 2k + 2 \Rightarrow 2(\lambda + k + 1)$ = a multiple of 2.

Thus, $(k + 1)^2 + (k + 1)$ is an even number.

Hence, P(n) is true for all n, where n is a natural number.

**Illustration 4:** Prove that exactly one among n+10, n+12 and n+14 is divisible by 3, considering n is always an natural number. **(JEE MAIN)**

**Sol:** We can observe here that

For n = 1,

n+10 = 11

n+12 = 13

n+14 = 15

Exactly one i.e 15 is divisible by 3.

Let us assume that that for n = m exactly one out of n+10, n+12, n+14 is divisible by 3

Without the loss of generality consider for n=m, m+10 was divisible by 3

Therefore, m+10 = 3k

m+12 = 3k+2

m+14 = 3k+4

We need to prove that for n=m+1 , exactly one among them is divisible by 3. Putting m+1 in place of n, we get

(m+1)+10 = m+11 = 3k + 1 (not divisible by 3)

(m+1)+12 = m+13 = 3k+3 = 3(k+1) (divisible by 3)

(m+1)+14 = m+15 = 3k+5 (not divisible by 3)

Therefore, for n=m+1 also exactly one among the three, n+10, n+12 and n+14 is divisible by 3.

Similarly we can prove that exactly one among three of these is divisible by 3 by considering cases when n+12 = 3k and n+14 = 3k.

# BINOMIAL THEOREM

## 1. INTRODUCTION TO BINOMIAL THEOREM

### 1.1 Introduction

Consider two numbers a and b, then

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = (a+b)(a+b)^2 \quad = (a+b)(a^2 + 2ab + b^2) \quad = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = (a+b)^2 (a+b)^2 \quad = (a^2 + 2ab + b^2)(a^2 + 2ab + b^2) \quad = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

As the power increases, the expansion becomes lengthy, difficult to remember and tedious to calculate. A binomial expression that has been raised to a very large power (or degree), can be easily calculated with the help of Binomial Theorem.

### 1.2 Binomial Expression

A binomial expression is an algebraic expression which contains two dissimilar terms.

For example: $x + y,\ a^2 + b^2,\ 3 - x,\ \sqrt{x^2+1} + \frac{1}{\sqrt[3]{x^3+1}}$ etc.

### 1.3 Binomial Theorem

Let n be any natural number and x, a be any real number, then

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots + {}^nC_r\, x^{n-r} a^r + \ldots\ldots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

i.e. $(x+a)^n = \sum_{0}^{n} {}^nC_r x^{n-r} a^r$ where ${}^nC_r = \frac{n!}{r!(n-r)!}$

and the co-efficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ................. and ${}^nC_n$ are known as binomial coefficient.

## CONCEPTS

**(a)** The total number of terms in the expansion of $(x + a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$, is (n + 1).

**(b)** The sum of the indices of x and a in each term is n.

**(c)** ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients and also represented by $C_0$, $C_1$, $C_2$ and so on.

(i) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$ (ii) ${}^nC_r = {}^nC_{n-r}$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (iv) ${}^nC_r = n/(n-r).{}^{n-1}C_r$

**Vaibhav Gupta (JEE 2009, AIR 22)**

**Illustration 5:** Expand the following binomials

(i) $(x - 2)^5$ (ii) $\left(1-\frac{3x^3}{2}\right)^4$ **(JEE MAIN)**

**Sol:** By using formula of binomial expansion.

(i) $(x-2)^5 = {}^5C_0x^5 + {}^5C_1x^4(-2)^1 + {}^5C_2x^3(-2)^2 + {}^5C_3x^2(-2)^3 + {}^5C_4x(-2)^4 + {}^5C_5(-2)^5$

$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

(ii) $\left(1-\frac{3x^3}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\frac{3x^3}{2}\right) + {}^4C_2\left(-\frac{3x^3}{2}\right)^2 + {}^4C_3\left(-\frac{3x^3}{2}\right)^3 + {}^4C_4\left(-\frac{3x^3}{2}\right)^4$

$= 1 - 6x^3 + \frac{27}{2}x^6 - \frac{27}{2}x^9 + \frac{81}{16}x^{12}$

# 2. DEDUCTIONS FROM BINOMIAL THEOREM

## 2.1 Results of Binomial Theorem

**D-1** On replacing a by –a, in the expansion of $(x + a)^n$, we get

$$(x-a)^n = {}^nC_0\,x^na^0 - {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}.a^2 - \ldots.. + (-1)^r\,{}^nC_rx^{n-r}a^r + \ldots\ldots + (-1)^n\,{}^nC_nx^0a^n$$

i.e. $(x-a)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_rx^{n-r}a^r$

Therefore, the terms in $(x - a)^n$ are alternatively positive and negative, and the sign of the last term is positive or negative depending on whether n is even or odd.

**D-2** Putting x = 1 and a = x in the expansion of $(x + a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_rx^r + \ldots. + {}^nC_nx^n$$

$$\Rightarrow (1+x)^n = \sum_{r=0}^{n} {}^nC_rx^r$$

This is the expansion of $(1 + x)^n$ in ascending powers of x.

**D-3** Putting a = 1 in the expansion of $(x + a)^n$ , we get

$$(x+1)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + .... + {}^nC_rx^{n-r} + ....... + {}^nC_{n-1}x + {}^nC_n \Rightarrow (1+x)^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}$$

This is the expansion of $(1 + x)^n$ in descending powers of x.

**D-4** Putting x = 1 and a = –x in the expansion of $(x + a)^n$, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2\,x^2 - {}^nC_3x^3 + ..... + (-1)^r\,{}^nC_r\,x^r + ..... + (-1)^n\,{}^nC_nx^n$$

**D-5** From the above expansions, we can also deduce the following

$$(x+a)^n + (x-a)^n = 2\left[{}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + .....\right]$$

and $(x+a)^n - (x-a)^n = 2\left[{}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}\,a^3 + .....\right]$

**CONCEPTS**

If n is odd then $\left\{(x+a)^n + (x-a)^n\right\}$ and $\left\{(x+a)^n - (x-a)^n\right\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ where as if n is even, then $\left\{(x+a)^n + (x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms.

**Nikhil Khandelwal (JEE 2009, AIR 94)**

## 2.2 Properties of Binomial Coefficients

Using binomial expansion, we have

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ....... + {}^nC_rx^r + ...... + {}^nC_nx^n$$

Also, $(1+x)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + ....... + {}^nC_rx^{n-r} + ...... + {}^nC_{n-1}x + {}^nC_n$

Let us represent the binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2, ......., {}^nC_{n-1}, {}^nC_n$ by $C_0, C_1, C_2, ......., C_{n-1}, C_n$ respectively. Then the above expansions become

$(1+x)^n = C_0 + C_1x + C_2x^2 + ...... + C_nx^n$ i.e. $(1+x)^n = \sum_{r=0}^{n} C_rx^r$

Also, $(1+x)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ...... + C_rx^{n-r} + ...... + C_{n-1}x + C_n$ i.e. $(1+x)^n = \sum_{r=0}^{n} C_rx^{n-r}$

The binomial coefficients $C_0$, $C_1$, $C_2$,........$C_{n-1}$, and $C_n$ posses the following properties:

**Property-I** In the expansion of $(1 + x)^n$, the coefficients of terms equidistant from the beginning and the end are equal.

**Property-II** The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is $2^n$.

i.e. $C_0 + C_1 + C_2 + ....... + C_n = 2^n$ or, $\sum_{r=0}^{n} {}^nC_r = 2^n$ .

**Property-III** The sum of the coefficient of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficient of the even terms and each is equal to $2^{n-1}$.

i.e. $C_0 + C_2 + C_4 + ........ = C_1 + C_3 + C_5 + ....... = 2^{n-1}$

**Property-IV** $^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$ and so on.

**Property-V** $C_0 - C_1 + C_2 - C_3 + C_4 - ...... + (-1)^n C_n = 0$

i.e. $\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0$

## CONCEPTS

(a) $^{(n+1)}C_r = {}^nC_r + {}^nC_{r-1}$ (b) $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$ (c) $\frac{^nC_r}{r+1} = \frac{^{n+1}C_{r+1}}{n+1}$

(d) When n is even, $(x+a)^n + (x-a)^n = 2\left(x^n + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 + ..... + {}^nC_n a_n\right)$

When n is odd, $(x+a)^n + (x-a)^n = 2\left(x^n + {}^nC_2 x^{n-2} a^2 + ..... + {}^nC_{n-1} x a^{n-1}\right)$

When n is even $(x+a)^n - (x-a)^n = 2\left({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + ..... + {}^nC_{n-1} x a^{n-1}\right)$

When n is odd $(x+a)^n - (x-a)^n = 2\left({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + ..... + {}^nC_n a^n\right)$

**Saurabh Gupta (JEE 2010, AIR 443)**

**Illustration 6:** If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + .......... + C_n x^n$, then show that **(JEE MAIN)**

(i) $C_0 + 4C_1 + 4^2 C_2 + .......... + 4^n C_n = 5^n$

(ii) $C_0 + 2C_1 + 3C_2 + .......... + (n+1)C_n = 2^{n-1}(n+2)$

(iii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ........ + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

**Sol:** By using properties of binomial coefficients and methods of summation, differentiation, and integration we can easily prove given equations.

(i) $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ......... + C_n x^n$

Putting x=4, we have $C_0 + 4C_1 + 4^2 C_2 + .......... + 4^n C_n = 5^n$

(ii) $C_0 + 2C_1 + 3C_2 + .......... + (n+1)C_n = 2^{n-1}(n+2)$

**Method 1: By Summations**

$r^{th}$ term in the series is given by $(r+1).{}^nC_r$

Therefore, L.H.S = $^nC_0 + 2.{}^nC_1 + 3.{}^nC_2 + ....... + (n+1).{}^nC_n = \sum_{r=0}^{n} (r+1).{}^nC_r$

$$=\sum_{r=0}^{n} r.{}^{n}C_r + \sum_{r=0}^{n} {}^{n}C_r = n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^{n}C_r \;=\; n.2^{n-1} + 2^n = 2^{n-1}(n+2) \;=\; R.H.S$$

**Method 2: By Differentiation**

$(1+x)^n = C_0 + C_1x + C_2x^2 + ....... + C_nx^n$

Multiplying x on both sides, $x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + ....... + C_nx^{n+1}$

On differentiating, we have $(1+x)^n + xn(1+x)^{n-1} = C_0 + 2.C_1x + 3.C_2x^2 + ...... + (n+1)C_nx^n$

Putting x = 1, we get $C_0 + 2.C_1 + 3.C_2 + ...... + (n+1)C_n = 2^n + n.2^{n-1}$

$C_0 + 2.C_1 + 3.C_2 + ...... + (n+1)C_n = 2^{n-1}(n+2)$

(iii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ......... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

**Method 1: By Summations**

$r^{th}$ term in the series is given by $(-1)^r.\frac{{}^{n}C_r}{r+1}$

Therefore, L.H.S. = $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ...... + (-1)^n.\frac{C_n}{n+1} \;=\sum_{r=0}^{n}(-1)^r.\frac{{}^{n}C_r}{r+1}$

$$=\frac{1}{n+1}\sum_{r=0}^{n}(-1)^r\,{}^{n+1}C_{r+1} \qquad \left\{\text{using } \frac{n+1}{r+1}.{}^{n}C_r = {}^{n+1}C_{r+1}\right\} \;=\frac{1}{n+1}\left[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - ........ + (-1)^n.{}^{n+1}C_{n+1}\right]$$

Adding and subtracting the term ${}^{n+1}C_0$, we have

$$=\frac{1}{n+1}\left[-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ........ + (-1)^n.{}^{n+1}C_{n+1} + {}^{n+1}C_0\right]$$

$$=\frac{1}{n+1} \qquad \text{as}\left[-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ........ + (-1)^n.{}^{n+1}C_{n+1} = 0\right] = R.H.S.$$

**Method 2: By Integration**

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ......... + C_nx^n.$

On integrating both sides within the limits –1 to 0, we have

$$\int_{-1}^{0}(1+x)^n\,dx = \int_{-1}^{0}\left(C_0 + C_1x + C_2x^2 + ......... + C_nx^n\right)dx$$

$$\Rightarrow \left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ...... + C_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \frac{1}{n+1} - 0 = 0 - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + ...... + (-1)^{n+1}\frac{C_n}{n+1}\right] \Rightarrow C_0 - \frac{C_1}{2} + \frac{C_2}{3} + ...... + (-1)^n\frac{C_n}{n+1} = \frac{1}{n+1}$$

**Illustration 7:** If $(1+x)^n = C_0 + C_1x + C_2x^2 + ...... + C_nx^n$, then prove that

(i) $C_0^2 + C_1^2 + C_2^2 + ...... + C_n^2 = {}^{2n}C_n$

(ii) $C_0C_2 + C_1C_3 + C_2C_4 + ...... + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1 . C_0^2 + 3. C_1^2 + 5 . C_2^2 + ...... + (2n+1) . C_n^2 . = 2n . {}^{2n-1}C_n + {}^{2n}C_n$ **(JEE ADVANCED)**

**Sol:** In the expansion of $(1+x)^{2n}$, (i) and (ii) can be proved by comparing the coefficients of $x^n$ and $x^{n-2}$ respectively. The third equation can be proved by two methods - the method of summation and the methods of differentiation.

(i) $(1+x)^n = C_0 + C_1x + C_2x^2 + ........ + C_nx^n$ ....(i)

Also, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ........ + C_nx^0$ .... (ii)

Multiplying equation (i) and (ii)

$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ...... + C_nx^n)(C_0x^n + C_1x^{n-1} + ....... + C_nx^0)$ .... (iii)

On comparing the coefficients of $x^n$ both sides, we have

$\Rightarrow {}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + ....... + C_n{}^2$ Hence, Proved.

(ii) From (iii), on comparing the coefficients of $x^{n-2}$ or $x^{n+2}$, we have

$C_0C_1 + C_1C_3 + C_2C_4 + ....... + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1 . C_0^2 + 3. C_1^2 + 5 . C_2^2 + ...... + (2n+1) . C_n^2 . = 2n . {}^{2n-1}C_n + {}^{2n}C_n$

**Method 1: By Summation**

$r^{th}$ term in the series is given by $(2r+1)\,{}^nC_r^2$

L.H.S. $= 1.C_0^2 + 3.C_1^2 + 5.C_2^2 + ...... + (2n+1)C_n^2 \qquad = \sum_{r=0}^{n}(2r+1)\,{}^nC_r^2$

$= \sum_{r=0}^{n} 2.r.({}^nC_r)^2 + \sum_{r=0}^{n}({}^nC_r)^2 = 2\sum_{r=1}^{n}.n.{}^{n-1}C_{r-1}\,{}^nC_r + {}^{2n}C_n$

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ....... + {}^nC_nx^n$ .... (i)

$(x+1)^{n-1} = {}^{n-1}C_0x^{n-1} + {}^{n-1}C_1x^{n-2} + ....... + {}^{n-1}C_{n-1}x^0$ .... (ii)

Multiplying (i) and (ii) and comparing coefficients of $x^n$, we have

${}^{2n-1}C_n = {}^{n-1}C_0.{}^nC_1 + {}^{n-1}C_1.{}^nC_2 + ....... + {}^{n-1}C_{n-1}.{}^nC_n$

i.e. $\sum_{r=1}^{n} {}^{n-1}C_{r-1}\,{}^nC_r = {}^{2n-1}C_n$

Hence, required summation is $2n.\,{}^{2n-1}C_n + {}^{2n}C_n$

**Method 2: By Differentiation**

$(1+x^2)^n = C_0 + C_1x^2 + C_2x^4 + C_3x^6 + ...... + C_nx^{2n}$

Multiplying x on both sides

$x(1+x^2)^n = C_0x + C_1x^3 + C_2x^5 + ...... + C_nx^{2n+1}$

Differentiating both sides

$$x.n\left(1+x^2\right)^{n-1}.2x+\left(1+x^2\right)^n=C_0+3.C_1x^2+5.C_2x^4+......+\left(2n+1\right)C_nx^{2n} \quad .... (i)$$

$$\left(x^2+1\right)^n=C_0x^{2n}+C_1x^{2n-2}+C_2x^{2n-4}+.......+C_n \quad .... (ii)$$

On multiplying (i) and (ii), we have

$$2nx^2\left(1+x^2\right)^{2n-1}+\left(1+x^2\right)^{2n}=\left(C_0+3C_1x^2+5C_2x^4+.....+\left(2n+1\right)C_nx^{2n}\right)\left(C_0x^{2n}+C_1x^{2n-2}+......+C_n\right)$$

Comparing coefficient of $x^{2n}$,

$$2n.^{2n-1}C_{n-1}+^{2n}C_n=C_0^2+3C_1^2+5C_2^2+.....+\left(2n+1\right)C_n^2$$

$$\therefore C_0^2+3C_1^2+5C_2^2+.....+\left(2n+1\right)C_n^2=2n.^{2n-1}C_n+^{2n}C_n$$

**Illustration 8:** If $\left(1+x\right)^n=C_0+C_1x+C_2x^2+......+C_nx^n$,

Prove that $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+........+C_{n-r}C_n=\dfrac{2n!}{\left(n-r\right)!\left(n+r\right)!}$ **(JEE MAIN)**

**Sol:** Clearly the differences of lower suffixes of binomial coefficients in each term is r.

By using properties of binomial coefficients we can easily prove given equations.

Given $\left(1+x\right)^n=C_0+C_1x+C_2x^2+........+C_{n-r}x^{n-r}+.........+C_nx^n$ .... (i)

Now $\left(x+1\right)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+........+C_rx^{n-r}+C_{r+1}x^{n-r-1}+.........+C_n$ ....(ii)

Multiplying (i) and (ii), we get

$$\left(x+1\right)^{2n}=\left(C_0+C_1x+C_2x^2+.....+C_{n-r}x^{n-r}+.........+C_nx^n\right)$$

$$\times\left(C_0x^n+C_1x^{n-1}+C_2x^{n-2}+.....+C_rx^{n-r}+C_{r+1}x^{n-r-1}+C_{r+2}x^{n-r-2}+......+C_n\right) \quad .... (iii)$$

Now coefficient of $x^{n-r}$ on L.H.S. of (iii) = $^{2n}C_{n-r}=\dfrac{2n!}{\left(n-r\right)!\left(n+r\right)!}$

and coefficient of $x^{n-r}$ on R.H.S. of (iii) = $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+........+C_{n-r}C_n$

But (iii) is an identity, therefore, of $x^{n-r}$ in R.H.S. = Coefficient of $x^{n-r}$ in L.H.S.

$$\Rightarrow C_0C_r+C_1C_{r+1}+C_2C_{r+2}+........+C_{n-r}C_n=\frac{2n!}{\left(n-r\right)!\left(n+r\right)!}$$

Hence, Proved.

**Illustration 9:** Prove that $^nC_0\cdot{}^{2n}C_r-{}^nC_1\cdot{}^{2n-2}C_r+.......=2^{2n-r}.{}^nC_{r-n}$ if r > n and 0 if r < n. **(JEE MAIN)**

**Sol:** By comparing coefficient of $x^r$ in L.H.S. and R.H.S. in the expansion of $\left[\left(1+x\right)^2-1\right]^n$ we can prove it.

$$\left[\left(1+x\right)^2-1\right]^n={}^nC_0\left(1+x\right)^{2n}-{}^nC_1\left(1+x\right)^{2n-2}+{}^nC_2\left(1+x\right)^{2n-4}+... \quad .... (i)$$

Coefficient of $x^r$ in R.H.S. = $^nC_0.{}^{2n}C_r-{}^nC_1.{}^{2n-2}C_r+............$ ... (ii)

L.H.S. = $[(1+x)^2-1]^n=[2x+x^2]^n=x^n(2+x)^n$

$\therefore$ Coefficient of $x^r$ in $x^n (2 + x)^n$

= Coefficient of $x^{r-n}$ in $(2 + x)^n = {}^nC_{r-n} 2^{2n-r}$ if $r > n$

= 0 if $r < n$ (Since lower suffix cannot be negative)

But (i) is an identity, therefore coefficient of $x^r$ in R.H.S. = coefficient of $x^r$ in L.H.S.

Hence ${}^nC_0.{}^{2n}C_r - {}^nC_1.{}^{2n-2}C_r + ......= {}^nC_{r-n}2^{2n-r}$ if $r > n$

$= 0$ if $r < n$.

**Illustration 10:** Show that $C_0.{}^{2n}C_n - C_1.{}^{2n-1}C_n + C_2.{}^{2n-2}C_n - C_3.{}^{2n-3}C_n + ...... + (-1)^n C_n.{}^nC_n = 1$ **(JEE ADVANCED)**

**Sol:** Observe the pattern in the terms on the LHS. The first term $C_0.{}^{2n}C_n$ is the co-efficient of $x^n$ in the expansion of $C_0(1+x)^{2n}$. Similarly, $C_1.{}^{2n-1}C_n$ is the co-efficient of $x^n$ in $C_0(1+x)^{2n}$ and so on. On adding all the coefficients of $x^n$ we can prove the given equation.

Note that $C_0.{}^{2n}C_n - C_1.{}^{2n-1}C_n + C_2.{}^{2n-2}C_n - C_3.{}^{2n-3}C_n + ...... + (-1)\ {}^nC_n.{}^nC_n$

= Coefficient of $x^n$ in $\left[C_0(1+x)^{2n} - C_1(1+x)^{2n-1} + C_2(1+x)^{2n-2} - C_3(1+x)^{2n-3} + ....(-1)^n C_n(1+x)^n\right]$

= Coefficient of $x^n$ in $(1+x)^n\left[C_0(1+x)^n - C_1(1+x)^{n-1} + C_2(1+x)^{n-2} - C_3(1+x)^{n-3} + .....(-1)^n C_n\right]$

= Coefficient of $x^n$ in $(1+x)^n\left[(1+x)-1\right]^n$

= Coefficient of $x^n$ in $(1+x)^n(x)^n$

= Coefficient of the constant terms in $(1 + x)^n = 1$

# 3. TERMS IN BINOMIAL EXPANSION

## 3.1 General Term in Binomial Expansion

We have, $(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} x^2 + ...... + {}^nC_r x^{n-r} a^r + ...... + {}^nC_n x^0 a^n$

$(r+1)^{th}$ term is given by ${}^nC_r\, x^{n-r} a^r$

Thus, if $T_{r+1}$ denotes the $(r+1)^{th}$ term, then $T_{r+1} = {}^nC_r x^{n-r} a^r$

This is called the general term of the binomial expansion.

**(a)** The general term in the expansion of $(x - a)^n$, is given by $T_{r+1} = (-1)^r.{}^nC_r x^{n-r} a^r$

**(b)** The general term in the expansion of $(1 + x)^n$, is given by $T_{r+1} = {}^nC_r\, x^r$

**(c)** The general term in the expansion of $(1 - x)^n$ , is given by $T_{r+1} = (-1)^r\, {}^nC_r x^r$

**(d)** In the binomial expansion of $(x + a)^n$, the $r^{th}$ term from the end is $((n + 1) - r + 1)^{th}$ term i.e. $(n - r + 2)^{th}$ term from the beginning.

**Illustration 11:** The number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is **(JEE MAIN)**

**Sol:** As we know that number of dissimilar terms in the expansion of $(1 - x)^n$ is n+1. Rewrite the given expression in the form of $(1 - x)^n$.

$(1 - 3x + 3x^2 - x)^{20} = [(1 - x)^3]^{20} = (1 - x)^{60}$

Therefore number of dissimilar terms in the expansion of $(1 - 3x + 3x^2 - x^3)^{20}$ is 61.

**Illustration 12:** Find (i) $28^{th}$ term of $(5x + 8y)^{30}$ (ii) $7^{th}$ term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$ **(JEE MAIN)**

**Sol:** Here in this problem, by using $T_{r+1} = {}^nC_r x^{n-r} a^r$ we can easily obtain $(r+1)^{th}$ term of given expansion.

(i) $28^{th}$ term of $(5x + 8y)^{30}$

$$T_{28} = T_{27+1} = {}^{30}C_{27}(5x)^{30-27}(8y)^{27} = \frac{30!}{3!.27!}(5x)^3.(8y)^{27}$$

(ii) $7^{th}$ term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

$$T_7 = T_{6+1} = {}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6 = \frac{9!}{3!6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$$

**Illustration 13:** Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$. **(JEE ADVANCED)**

**Sol:** In this problem, by using $T_{r+1} = {}^nC_r x^{n-r} a^r$ we can easily obtain $(r+1)^{th}$ term of given expansion and after that by using the conditions of rational number we can obtain number of rational terms.

The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{1/6}\right)^r = {}^{1000}C_r 3^{\frac{1000-r}{2}} 2^{\frac{r}{2}}$$

$T_{r+1}$ will be rational if the power of 3 and 2 are integers. It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers.

Therefore the possible set of values of r is {0, 2, 4... ... 1000}. Hence, number of rational terms is 501.

## 3.2 Middle Term in Binomial Expansion

**(a)** If n is even, then the number of terms in the expansion i.e. (n + 1) is odd, therefore, there will be only one middle term which is $\left(\frac{n+2}{2}\right)^{th}$ term i.e. $\left(\frac{n}{2}+1\right)^{th}$ term.

So middle term = $\left(\frac{n}{2}+1\right)^{th}$ term i.e. $T_{\left(\frac{n}{2}+1\right)} = {}^nC_{\frac{n}{2}} x^{\frac{n}{2}} a^{\frac{n}{2}}$

**(b)** If n is odd, then the number of terms in the expansion i.e. (n+1) is even, therefore there will be two middle terms which are

$= \left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ term i.e. $T_{\left(\frac{n+1}{2}\right)} = {}^nC_{\left(\frac{n-1}{2}\right)} x^{\frac{n+1}{2}} a^{\frac{n-1}{2}}$ and $T_{\left(\frac{n+3}{2}\right)} = {}^nC_{\left(\frac{n+1}{2}\right)} x^{\frac{n-1}{2}} a^{\frac{n+1}{2}}$

## CONCEPTS

- When there are two middle terms in the expansion then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest Binomial coefficient.

**Neeraj Toshniwal (JEE 2009, AIR 21)**

**Illustration 14:** Find the middle term(s) in the expansion of (i) $\left(1-\frac{x^2}{2}\right)^{14}$ (ii) $\left(3a-\frac{a^3}{6}\right)^{9}$ **(JEE MAIN)**

**Sol:** By using appropriate formula of finding middle term(s) i.e. $\left(\frac{n}{2}+1\right)^{th}$ when n is even and $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ when n is odd, we can obtain the middle terms of given expansion.

(i) $\left(1-\frac{x^2}{2}\right)^{14}$ Since, n is even, therefore middle term is $\left(\frac{14}{2}+1\right)^{th}$ term.

$\therefore T_8 = {}^{14}C_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}x^{14}$

(ii) $\left(3a-\frac{a^3}{6}\right)^{9}$

Since, n is odd therefore, the middle terms are $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+1}{2}+1\right)^{th}$.

$\therefore\ T_5 = {}^9C_4\left(3a\right)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}a^{17}$ and $T_6 = {}^9C_5\left(3a\right)^{9-5}\left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}a^{19}$.

## 3.3 Determining a Particular Term

In the expansion of $\left(x^{\alpha}\pm\frac{1}{x^{\beta}}\right)^n$, if $x^m$ occurs in $T_{r+1}$, then r is given by

$n\alpha - r(\alpha+\beta) = m \qquad \Rightarrow r = \frac{n\alpha - m}{\alpha+\beta}$

Thus in above expansion if constant term i.e. the term independent of x, occurs in $T_{r+1}$ then r is determined by

$n\alpha - r(\alpha+\beta) = 0 \qquad \Rightarrow r = \frac{n\alpha}{\alpha+\beta}$

**Illustration 15:** The term independent of x in the expansion of $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)^9$ is **(JEE MAIN)**

**Sol:** By using the result proved above i.e. $r = \frac{n\alpha}{\alpha+\beta}$, we can obtain the term independent of x. Here, $\alpha$ and $\beta$ are obtained by comparing given expansion to $\left(x^{\alpha}\pm\frac{1}{x^{\beta}}\right)^n$.

On comparing $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)^9$ with $\left(x^{\alpha}\pm\frac{1}{x^{\beta}}\right)^n$, we get $\alpha = 2, \beta = 1, n = 9$

i.e. $r = \frac{9(2)}{2+1} = 6$ $\therefore$ (6 + 1) = 7th term is independent of x.

**Illustration 16:** The ratio of the coefficient of $x^{15}$ to the term independent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$ is **(JEE MAIN)**

**Sol:** Here in this problem, by using standard formulas of finding general term and term independent of x we can obtain the required ratio.

General term in the expansion is $T_{r+1} = {}^{15}C_r \left(x^2\right)^{15-r}\left(\frac{2}{x}\right)^r$ i.e., ${}^{15}C_r x^{30-3r}.2^r$

For $x^{15}$, 30 – 3r = 15 $\Rightarrow$ 3r = 15 $\Rightarrow$ r = 5

$\therefore T_6 = T_{5+1} = {}^{15}C_5\left(x^2\right)^{15-5}\left(\frac{2}{x}\right)^5$ i.e., ${}^{15}C_5 x^{15}.2^5$

$\therefore$ Coefficient of $x^{15}$ is ${}^{15}C_5\, 2^5$ (r = 5)

For the constant term 30 – 3r = 0 $\Rightarrow$ r = 10.

$\therefore\ T_{11} = T_{10+1} = {}^{15}C_{10}\left(x^2\right)^{15-10}\left(\frac{2}{x}\right)^{10}$ i.e., ${}^{15}C_{10}2^{10}$

$\therefore$ Coefficient of constant term is ${}^{15}C_{10}2^{10}$.

Hence, the required ratio is 1 : 32.

**Illustration 17:** The term independent of x in the expansion of $\left(\sqrt[6]{x} - \frac{1}{\sqrt[3]{x}}\right)^9$ is equal to **(JEE MAIN)**

**Sol:** By using the formula $T_{r+1} = {}^nC_r x^{n-r}a^r$ we can solve it.

$$T_{r+1} = {}^9C_r\left(\sqrt[6]{x}\right)^{9-r}\left(-\frac{1}{\sqrt[3]{x}}\right)^r = {}^9C_r(-1)^r x^{\frac{9-r}{6}-\frac{r}{3}} = {}^9C_r(-1)^r x^{\left(\frac{9-3r}{6}\right)}$$

$$\Rightarrow \frac{9-3r}{6} = 0 \quad \Rightarrow r = 3 \quad \therefore T_4 = T_{3+1} = -\ {}^9C_3$$

**Illustration 18:** If the second, third and fourth terms in the expansion of $(b+a)^n$ are 135, 30 and 10/3 respectively, then n is equal to **(JEE MAIN)**

**Sol:** In this problem, by using the formula of finding general term we will get the equation of given terms and by taking ratios of these terms we can get the value of n.

$T_2 = {}^nC_1\, ab^{n-1} = 135$ ...(i)

$T_3 = {}^nC_2\, a^2b^{n-2} = 30$ ...(ii)

$T_4 = {}^nC_3\, a^3b^{n-3} = \frac{10}{3}$ ...(iii)

On dividing (i) by (ii), we get

$$\frac{{}^nC_1ab^{n-1}}{{}^nC_2a^2b^{n-2}} = \frac{135}{30} \qquad \Rightarrow \frac{n}{\frac{n(n-1)}{2}}\frac{b}{a} = \frac{9}{2} \quad ...(iv)$$

$$\therefore \frac{b}{a} = \frac{9}{4}(n-1) \quad ...(v)$$

Dividing (ii) and (iii), we get

$$\frac{\frac{n(n-1)}{2}}{\frac{n(n-1)(n-2)}{3.2}}\cdot\frac{b}{a}=\frac{30\times 3}{10}=9 \qquad \Rightarrow \frac{3}{(n-2)}\frac{b}{a}=9 \qquad \text{...(vi)}$$

Eliminating a and b from (v) and (vi) ⇒ n = 5

**Illustration 19:** If a, b, c and d are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, n being positive integer, show that $\frac{a}{a+b}+\frac{c}{c+d}=\frac{2b}{b+c}$ **(JEE MAIN)**

**Sol:** Consider four consecutive terms and use ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$.

The $(r+1)^{th}$ term is $T_{r+1}={}^nC_r x^r$

∴ The coefficient of term $T_{r+1}={}^nC_r$

∴ Now take four consecutive terms as (r – 1)th, rth, (r + 1)th and (r + 2)th

∴ We get $a={}^nC_{r-2}$, $b={}^nC_{r-1}$, $c={}^nC_r$, $d={}^nC_{r+1}$

$$a+b={}^nC_{r-2}+{}^nC_{r-1}={}^{n+1}C_{r-1}$$

$$b+c={}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$$

$$c+d={}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$$

$$\therefore \frac{a}{a+b}=\frac{{}^nC_{r-2}}{{}^{n+1}C_{r-1}}=\frac{n!}{(r-2)!(n-r+2)!}\times\frac{(r-1)!(n-r+2)!}{(n+1)!}=\frac{r-1}{n+1}$$

$$\frac{b}{b+c}=\frac{{}^nC_{r-1}}{{}^{n+1}C_r}=\frac{n!}{(r-1)!(n-r+2)!}\times\frac{r!(n-r+1)!}{(n+1)!}=\frac{r}{n+1}$$

$$\frac{c}{c+d}=\frac{{}^nC_r}{{}^{n+1}C_{r+1}}=\frac{n!}{n!(n-r)!}\times\frac{(r+1)!\,(n-r)!}{(n+1)!}=\frac{r+1}{n+1}$$

$$\therefore \frac{a}{a+b}+\frac{c}{c+d}=\frac{r-1}{n+1}+\frac{r+1}{n+1}=\frac{2r}{n+1}=2\left(\frac{r}{n+1}\right)=\frac{2b}{b+c}$$

## 3.4 Finding a Term from the End of Expansion

In the expansion of $(x+a)^n$, $(r+1)^{th}$ term from end = $(n-r+1)^{th}$ term from beginning i.e. $T_{r+1}(E)=T_{n-r+1}(B)$

$\therefore T_r(E)=T_{n-r+2}(B)$

**Illustration 20:** The $4^{th}$ term from the end in the expansion of $(2x-1/x^2)^{10}$ is **(JEE MAIN)**

**Sol:** By using $T_r(E)=T_{n-r+2}(B)$ we will get the fourth term from the end in the given expansion.

Required term = $T_{10-4+2}=T_8={}^{10}C_7(2x)^3\left(-\frac{1}{x^2}\right)^7=-960\,x^{-11}$

## 3.5 Greatest Term in the Expansion

Let $T_{r+1}$ and $T_r$ be (r+1)th and rth terms respectively in the expansion of $(x+a)^n$. Then, $T_{r+1} = {}^nC_r x^{n-r} a^r$ and $T_r = {}^nC_{r-1} x^{n-r+1} a^{r-1}$.

$$\therefore \frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^{n-r} a^r}{{}^nC_{r-1} x^{n-r+1} a^{r-1}} \quad = \frac{n!}{(n-r)!r!} x \frac{(r-1)!(n-r+1)!}{n!} . \frac{a}{x} \quad = \frac{n-r+1}{r} . \frac{a}{x}$$

Now, $T_{r+1} >=< T_r \quad \Rightarrow \frac{T_{r+1}}{T_r} >=< 1 \quad \Rightarrow \frac{n-r+1}{r} . \frac{a}{x} >=< 1 \quad \Rightarrow \left\{\left(\frac{n+1}{r}\right) - 1\right\} \frac{a}{x} >=< 1$

$$\Rightarrow \frac{n+1}{r} - 1 >=< \frac{x}{a} \quad \Rightarrow \frac{n+1}{r} >=< \left(1 + \frac{x}{a}\right) \quad \Rightarrow \frac{n+1}{1+\frac{x}{a}} >=< r$$

Thus, $T_{r+1} >=< T_r$ according as $\left(\frac{n+1}{1+\frac{x}{a}}\right) >=< r$ ...(i)

Now, two cases arise

**Case-I:** When $\frac{n+1}{1+\frac{x}{a}}$ is an integer Let $\frac{n+1}{1+\frac{x}{a}} = m$, Then, from (i), we have

$T_{r+1} > T_r$, for $r = 1, 2, 3, .....(m-1)$ ....(ii)

$T_{r+1} = T_r$, for $r = m$ ...(iii)

and, $T_{r+1} < T_r$, for $r = m+1, ....n$ ....(iv)

$\therefore T_2 > T_1, T_3 > T_2, T_4 > T_3, ......T_m > T_{m-1}$ [From (ii)]

$T_{m+1} = T_m$ [From (iii)]

and, $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, T_{n+1} < T_n$ [From (iv)]

$\Rightarrow T_1 < T_2 < ..... < T_{m-1} < T_m = T_{m+1} > T_{m+2} .... > T_n$

This shows that $m^{th}$ and $(m + 1)^{th}$ terms are greatest terms.

**Case-II:** When $\left[\frac{n+1}{1+\frac{x}{a}}\right] = m$. Then, from (i), we have

$T_{r+1} > T_r$ for $r = 1, 2, ....., m$ ....(v)

and $T_{r+1} < T_r$ for $r = m+1, m+2, ....., n$ ....(vi)

$\therefore T_2 > T_1, T_3 > T_2, ......, T_{m+1} > T_m$ [From (v)]

and, $T_{m+2} < T_{m+1}, T_{m+3} < T_{m+2}, ......, T_{n+1} < T_n$ [From (vi)]

$\Rightarrow T_1 < T_2 < T_3 < ..... < T_m < T_{m+1} > T_{m+2} > T_{m+3} ...... < T_{n+1}$

$\Rightarrow$ $(m + 1)^{th}$ term is the greatest term.

Following algorithm may be used to find the greatest term in a binomial expansion.

## 3.6 Algorithm to Find Greatest Term

**Step I:** From the given expansion, get $T_{r+1}$ and $T_r$

**Step II:** Find $\frac{T_{r+1}}{T_r}$

**Step III:** Put $\frac{T_{r+1}}{T_r} > 1$

**Step IV:** Simplify the inequality obtained in step III, and write it in the form of either r < m or r > m.

**Step V:** If m is an integer, then $m^{th}$ and $(m+1)^{th}$ terms are the greatest terms and they are equal.

If m is not an integer, then $([m]+1)^{th}$ term is the greatest term, where [m] means the integral part of m.

## 3.7 Greatest Coefficient

**Case-I** When n is even, we have

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{n!}{(n-r)!r!} \times \frac{(r+1)!(n-r-1)!}{n!} = \frac{r+1}{n-r} \quad \text{....(i)}$$

Now, for $0 \le r \le \frac{n}{2} - 1$ $\qquad \Rightarrow 1 \le r+1 \le \frac{n}{2}$ and $\frac{n}{2} + 1 < n - r \le n$

$\Rightarrow \frac{r+1}{n-r} < 1$ $\qquad$ [Using (i)] $\Rightarrow \frac{{}^nC_r}{{}^nC_{r+1}} < 1 \Rightarrow {}^nC_r < {}^nC_{r+1}$

Putting $r = 0, 1, 2, ......, \left(\frac{n}{2} - 1\right)$, we get ${}^nC_0 < {}^nC_1, {}^nC_1 < {}^nC_2, {}^nC_2 < {}^nC_3 ..... < {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}}$

$$\Rightarrow {}^nC_0 < {}^nC_1 < {}^nC_2 < ..... < {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}} \quad \text{...(ii)}$$

Since ${}^nC_{n-r} = {}^nC_r$

$$\therefore {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, {}^nC_2 = {}^nC_{n-2}, ......, {}^nC_{\frac{n}{2}-1} < {}^nC_{\frac{n}{2}}$$

Substituting these values in (ii), we get

$$ {}^nC_n < {}^nC_{n-1} < {}^nC_{n-2} < ..... < {}^nC_{\frac{n}{2}+1} < {}^nC_{\frac{n}{2}} \quad \text{...(iii)}$$

From (ii) and (iii), we refer that the maximum value of ${}^nC_r$ is ${}^nC_{n/2}$.

**Case-II** When n is odd

We have, $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$ $\qquad$ ...(i)

Now, $0 \le r < \frac{n-3}{2}$ $\qquad \Rightarrow 0 < r+1 < \frac{n-1}{2}$ and $\frac{n-1}{2} \le n-r \le n$

$\Rightarrow \frac{r+1}{n-1} < 1 \Rightarrow \frac{^nC_r}{^nC_{r+1}} < 1$ [Using (i)] $\Rightarrow$ $^nC_r < {^nC_{r+1}}$

Putting r = 0, 1, 2, ......., $\frac{n-3}{2}$

We get $^nC_0 < {^nC_1}, {^nC_1} < {^nC_2}, {^nC_2} < {^nC_3}, ....., < {^nC_{\frac{n-3}{2}}} < {^nC_{\frac{n-1}{2}}} = {^nC_{\frac{n+1}{2}}}$

$\Rightarrow {^nC_0} < {^nC_1} < {^nC_2} < {^nC_3} < ..... < {^nC_{\frac{n-3}{2}}} < {^nC_{\frac{n-1}{2}}} = {^nC_{\frac{n-1}{2}}}$ ...(ii)

Since $^nC_{n-r} = {^nC_r}$. Therefore,

$\therefore {^nC_0} = {^nC_n}, {^nC_1} = {^nC_{n-1}}, {^nC_2} = {^nC_{n-2}}, ....., {^nC_{\frac{n-1}{2}}} = {^nC_{\frac{n+1}{2}}}$ ...(iii)

From (ii) and (iii), it follows that the maximum value of $^nC_r$ is $^nC_{\frac{n-1}{2}} = {^nC_{\frac{n+1}{2}}}$

**Illustration 21:** Find the numerically greatest term in the expansion of $(3 - 4x)^{15}$, when $x = \frac{1}{4}$. **(JEE MAIN)**

**Sol:** Follow the algorithm mentioned above.

Let $r^{th}$ and $(r + 1)^{th}$ be two consecutive terms in the expansion of $(3 - 4x)^{15}$
$T_{r+1} > T_r$

$^{15}C_r 3^{15-r} \left(|-4x|\right)^r > {^{15}C_{r-1}} 3^{15-(r-1)} \left(|-4x|\right)^{r-1}$

$\frac{(15)!}{(15-r)!r!}|-4x| > \frac{3.(15!)}{(16-r)!(r-1)!}$ $\Rightarrow 5.\frac{1}{5}(16-r) > 3r$ $\Rightarrow 16 - r > 3r$

$\Rightarrow 4r < 16$ $\Rightarrow r < 4$

Hence, we have $T_1 < T_2 < T_3 < T_4$.

Similarly, if we simplify $T_{r+1} = T_r$, we get r=4.

Therefore the numerically greatest term is $T_4$ and $T_5$.

# 4. APPLICATION OF BINOMIAL THEOREM

## 4.1 Divisibility Test

**Illustration 22:** Show that $7^{2n} + 7$ is divisible by 8, where n is a positive integer. **(JEE MAIN)**

**Sol:** Write $7^{2n} + 7$ in the form of $8\lambda + c$, where c is a constant. If c = 0 then we can conclude that $7^{2n} + 7$ is divisible by 8.

$7^{2n} + 7 = (8 - 1)^{2n} + 7$ $= {^{2n}C_0} 8^{2n} - {^{2n}C_1}.8^{2n-1} + {^{2n}C_2}.8^{2n-2} - ...... + {^{2n}C_{2n}} + 7$

$= 8^{2n}.{^{2n}C_0} - 8^{2n-1}.{^{2n}C_1} + ...... - 8.{^{2n}C_{2n-1}} + 8$ $= 8\lambda$ where $\lambda$ is a positive integer

Hence, $7^{2n} + 7$ is divisible by 8.

**Illustration 23:** Prove that $13^{99} - 19^{57}$ is divisible by 162. **(JEE ADVANCED)**

**Sol:** Reduce $13^{99} - 19^{57}$ into the form of $162\lambda + C$ using binomial expansion and If C = 0 then $13^{99} - 19^{57}$ is divisible by 162.

Let the given number be called S. Hence, $S = 13^{99} - 19^{57} = (1 + 3\times4)^{99} - (1 + 9\times2)^{57}$

$$S = \left\{1 + {}^{99}C_1.(3\times4) + {}^{99}C_2.(3\times4)^2 + {}^{99}C_3.(3\times4)^3 + ...... + {}^{99}C_{99}.(3\times4)^{99}\right\}$$

$$- \left\{1 + {}^{57}C_1.(9\times2) + {}^{57}C_2.(9\times2)^2 + {}^{57}C_3.(9\times2)^3 + ...... + {}^{57}C_{57}.(9\times2)^{57}\right\}$$

$$S = \left\{1 + {}^{99}C_1.(3\times4) + (3^4\times2)k_1\right\} - \left\{1 + {}^{57}C_1.(9\times2) + (3^4\times2)k_2\right\}$$

All terms like $\left\{{}^{99}C_1.(3\times4)^2, {}^{99}C_2.(3\times4)^3, ......, {}^{99}C_{99}.(3\times4)^{99}\right\}$ and

$\left\{{}^{57}C_2.(9\times2)^2, {}^{57}C_3.(9\times2)^3, ......, {}^{57}C_{57}.(9\times2)^{57}\right\}$ have a common factor of $(3^4.2 = 162)$.

Hence they can be written as $(3^4.2)\,k_1$ and $(3^4.2)\,k_2$ respectively, where $k_1$ and $k_2$ are integers.

Therefore, $S = 1 + {}^{99}C_0.(3\times4) - 1 - {}^{57}C_1.(9\times2) + (162)(k_1 - k_2)$

$= (1188 - 1026) + \{162\times(k_1 - k_2)\}$ = (162 × some integer)

Hence the given number S is exactly divisible by 162.

## 4.2 Finding Remainder

**Illustration 24:** What is the remainder when $5^{2015}$ is divisible by 13. **(JEE MAIN)**

**Sol:** In this problem, we can obtain required remainder by reducing $5^{2015}$ into the form of $13\lambda + a$, where $\lambda$ and a are integers.

$5^{2015} = 5.5^{2014} = 5.(25)^{1007}$

$$= 5(26-1)^{1007} = 5\left[{}^{1007}C_0(26)^{1007} - {}^{1007}C_1(26)^{1006} + ...... + {}^{1007}C_{1006}(26)^1 - {}^{1007}C_{1007}(26)^0\right]$$

$$= 5\left[{}^{1007}C_0(26)^{1007} - {}^{1007}C_1(26)^{1006} + ...... + {}^{1007}C_{1006}(26)^1 - 1\right]$$

$$= 5\left[{}^{1007}C_0(26)^{1007} - {}^{1007}C_1(26)^{1006} + ...... + {}^{1007}C_{1006}(26)^1 - 13\right] + 60$$

$= 13(k) + 52 + 8$ = 13 × (some integer) + 8.

## 4.3 Finding Digits of a Number

**Illustration 25:** Find the last two digits of the number $(13)^{10}$. **(JEE MAIN)**

**Sol:** Write $(13)^{10}$ in the form of $(x-1)^n$, such that x is a multiple of 10. Then using expansion formula we will get last two digits.

$$(13)^{10} = (169)^5 = (170-1)^5 = {}^5C_0(170)^5 - {}^5C_1.(170)^4 + ....... + {}^5C_4(170)^1 - {}^5C_5(170)^0$$

$= {}^5C_0(170)^5 - {}^5C_1.(170)^4 + ... + {}^5C_3(170)^2 + 5\times170 - 1$ = A multiple of 100 + 849

Therefore, the last two digits are 49

**Illustration 26:** Find the last three digits of $13^{256}$. **(JEE MAIN)**

**Sol:** Similar to above problem..

We have $13^2$ = 169 = 170 – 1

Now, $13^2 = \left(13^2\right)^{128} = \left(170-1\right)^{128}$

$$= {}^{128}C_0(170)^{128} - {}^{128}C_1.(170)^{127} + {}^{128}C_2.(170)^{126} - ... + {}^{128}C_{126}(170)^2 - {}^{128}C_{127}(170) + 1$$

= 1000 m + (128) (170) (10794) + 1 (where m is a positive integer)

= 1000 m + 234877440 + 1 = 1000m+234877441

Thus, the last three digits of $13^{256}$ are 441.

## 4.4 Relation between Two Numbers

**Illustration 27:** Which number is smaller $(1.01)^{1000000}$ or 10,000 **(JEE MAIN)**

**Sol:** By reducing $(1.01)^{1000000}$ into the form of $\left(1+0.01\right)^n$ and solve it by using expansion formula we can obtain the value of $(1.01)^{1000000}$ .

$(1.01)^{1000000} = (1 + 0.01)^{1000000}$

$= 1 + {}^{1000000}C_1(0.01) + {}^{1000000}C_2(0.01)^2 + {}^{1000000}C_3(0.01)^3 + ...$

$= 1 + 1000000 \times (0.01) +$ some positive terms

= 1 + 10000 + some positive terms

Hence 10,000 < $(1.01)^{1000000}$.

# 5. MULTINOMIAL THEOREM

Using binomial theorem, we have

$$\left(x + a\right)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r, \quad n \in N$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r} a^r \qquad = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r, \qquad \text{where } s = n - r$$

Let us now consider the expansion of $\left(x_1 + x_2 + x_3\right)^n$

$$\left(x_1 + x_2 + x_3\right)^n = \sum_{k=0}^{n} {}^nC_k\, x_1^{n-k}\left(x_2 + x_3\right)^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} x_1^{n-k}\left(\sum_{p=0}^{k} \frac{k!}{(k-p)!\,p!} x_2^{k-p} x_3^p\right)$$

$$= \sum_{k=0}^{n}\sum_{p=0}^{k} \frac{n!}{(n-k)!\,(k-p)!\,p!} x_1^{n-k} x_2^{k-p} x_3^p \qquad = \sum_{p+q+r=n} \frac{n!}{r!\,q!\,p!} x_1^r\, x_2^q\, x_3^p \quad \text{where, } k - p = q,\ n - k = r.$$

And so on, if we want to generalize for n terms, we get

$$\left(x_1 + x_2 + ..... + x_k\right)^n = \sum_{r_1+r_2+...+r_k=n} \frac{n!}{r_1!r_2!.....r_k!} x_1^{r_1} x_2^{r_2} .......x_k^{r_k}$$

Therefore, general term in the expansion of $\left(x_1 + x_2 + ..... + x_k\right)^n$ is $\frac{n!}{r_1!r_2!r_3!.....r_k!} x_1^{r_1} x_2^{r_2} x_3^{r_3} .......x_k^{r_k}$

The number of terms is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + .........+ r_k = n$, because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$.

**Number of terms for the following expansions**

**(a)** $\left(x + y + z\right)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$ The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms.

**(b)** $\left(x + y + z + u\right)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$. There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ term in the above.

**CONCEPTS**

The greatest coefficient in the expansion of $\left(x_1 + x_2 + ...... + x_m^n\right)$ is $\frac{n!}{(q!)^{m-r}\left[(q+1)!\right]^r}$, where q and r are the quotient and remainder respectively when n is divided by m.

**Aman Gour (JEE 2012, AIR 230)**

## 6. BINOMIAL THEOREM FOR ANY INDEX

Let n be a rational number and x be a real number such that |x| < 1, then

$$\left(1+x\right)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)(n-2)...(n-r+1)}{r!}x^r + ... + \text{terms upto } \infty$$

The general term in the expansion of $(1 + x)^n$ is $\frac{n\left(n-1\right)\left(n-2\right).........\left(n-r+1\right)}{r!}x^r$ and is represented by $T_{r+1}$.

**CONCEPTS**

The above result is also true for complex x, n.

**B Rajiv Reddy (JEE 2012, AIR 11)**

**Illustration 28:** If x is very large and n is a negative integer or a proper fraction, then an approximate value of $\left(\frac{1+x}{x}\right)^n$ is equal to__________ **(JEE MAIN)**

**Sol:** Since x is very large therefore $\frac{1}{x}$ will be very small. Neglect the terms containing three and higher powers of $\frac{1}{x}$ in the expansion to obtain the approximate value of $\left(\frac{1+x}{x}\right)^n$.

$$\left(1+\frac{1}{x}\right)^n = 1+\frac{n}{x}+\frac{n(n-1)}{1.2}\left(\frac{1}{x}\right)^2 + ......$$ Since x is very large, we can ignore terms after the 2nd term.

**Illustration 29:** If $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$ is approximately equal to a + bx for small values of x, then (a, b) is equals to. **(JEE MAIN)**

**Sol:** Calculate the value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$ and equate it to a + bx.

Using the binomial expansion for any rational index, we have

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{2\left[1-\frac{x}{4}\right]^{1/2}} = \frac{\left[1+\frac{1}{2}(-3x)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2+......\right]+\left[1+\frac{5}{3}(-x)+\frac{5}{3}\frac{2}{3}\frac{1}{2}(-x)^2+......\right]}{2\left[1+\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2+........\right]}$$

$$= \frac{\left[1-\frac{19}{12}x+\frac{53}{144}x^2-..........\right]}{\left[1-\frac{x}{8}-\frac{1}{8}x^2-.......\right]} = 1-\frac{35}{24}x+.......$$

Neglecting the higher powers of x, $\Rightarrow a+bx = 1-\frac{35}{24}x \quad \Rightarrow a=1, b=-\frac{35}{24}$

**Illustration 30:** Find the coefficient of $a^3b^2c^4d$ in the expansion of $(a - b - c + d)^{10}$ **(JEE ADVANCED)**

**Sol:** Expand $(a - b - c + d)^{10}$ using multinomial theorem and by using coefficient property we can obtain the required result.

Using multinomial theorem, we have

$$(a-b-c+d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!r_2!r_3!r_4!}(a)^{r_1}(-b)^{r_2}(-c)^{r_3}(d)^{r_4}$$

We want to get coefficient of $a^3b^2c^4$, this implies that $r_1 = 3$, $r_2 = 2$, $r_3 = 4$, $r_4 = 1$

$\therefore$ Coefficient of $a^3b^2c^4d$ is $\frac{(10)!}{3!2!4!}(-1)^2(-1)^4 = 12600$

**Illustration 31:** In the expansion of $\left(1+x+\frac{5}{x}\right)^{11}$ find the term independent of x. **(JEE ADVANCED)**

**Sol:** By expanding $\left(1+x+\frac{5}{x}\right)^{11}$ using multinomial theorem and obtaining the coefficient of $x^0$ we will get the term independent of x.

$$\left(1+x+\frac{5}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \frac{(11)!}{r_1!r_2!r_3!}(1)^{r_1}(x)^{r_2}\left(\frac{5}{x}\right)^{r_3}$$

The exponent 11 is to be divided in such a way that we get $x^0$. Therefore, possible set of values of $(r_1, r_2, r_3)$ are (11, 0, 0), (9, 1, 1), (7, 2, 2) (5, 3, 3), (3, 4, 4), (1, 5, 5) Hence the required term is

$$\frac{(11)!}{(11)!}\left(5^0\right)+\frac{(11)!}{9!1!1!}5^1+\frac{(11)!}{7!2!2!}5^2+\frac{(11)!}{5!3!3!}5^3+\frac{(11)!}{3!4!4!}5^4+\frac{(11)!}{1!5!5!}5^5$$

$$=1+\frac{(11)!}{9!2!}.\frac{2!}{1!1!}5^1+\frac{(11)!}{7!4!}.\frac{4!}{2!2!}5^2+\frac{(11)!}{5!6!}.\frac{6!}{3!3!}5^3+\frac{(11)!}{3!8!}.\frac{8!}{4!4!}5^4+\frac{(11)!}{1!10!}.\frac{(10)!}{5!5!}5^5$$

$$=1+{}^{11}C_2\times{}^2C_1\times5^1+{}^{11}C_4\times{}^4C_2\times5^2+{}^{11}C_6\times{}^6C_3\times5^3+{}^{11}C_8\times{}^8C_4\times5^4+{}^{11}C_{10}\times{}^{10}C_5\times5^5$$

$$=1+\sum_{r=1}^{5}{}^{11}C_{2r}.{}^{2r}C_r\times5^r$$

# PROBLEM-SOLVING TACTICS

Summation of series involving binomial coefficients

For $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+......+{}^nC_nx^n$, the binomial coefficients are ${}^nC_0,{}^nC_1,{}^nC_2,...,{}^nC_n$. A number of series may be formed with these coefficients figuring in the terms of a series.

Some standard series of the binomial coefficients are as follows:

**(a)** By putting x = 1, we get $\quad {}^nC_0+{}^nC_1+{}^nC_2+......+{}^nC_n=2^n$ ...(i)

**(b)** By putting x =-1, we get $\quad {}^nC_0-{}^nC_1+{}^nC_2-......+(-1)^n.{}^nC_n=0$ ...(ii)

**(c)** On adding (i) and (ii), we get $\quad {}^nC_0+{}^nC_2+{}^nC_4+......=2^{n-1}$ ...(iii)

**(d)** On subtracting (ii) from (i), we get $\quad {}^nC_1+{}^nC_3+{}^nC_5+......=2^{n-1}$ ...(iv)

**(e)** ${}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+......+{}^{2n}C_{n-1}+{}^{2n}C_n=2^{2n-1}$

**Proof:** From the expansion of $(1+x)^{2n}$, we get ${}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+......+{}^{2n}C_{2n-1}+{}^{2n}C_{2n}=2^{2n}$

$$\Rightarrow 2\left({}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+.......+{}^{2n}C_{n-1}\right)+{}^{2n}C_n=2^{2n}\ [\because {}^{2n}C_0={}^{2n}C_{2n},{}^{2n}C_1={}^{2n}C_{2n-1}\text{ and so on.}]$$

**(f)** ${}^{2n+1}C_0+{}^{2n+1}C_1+{}^{2n+1}C_2+.......+{}^{2n+1}C_n=2^{2n}$

**Proof:** (as above)

**(g)** Sum of the first half of ${}^nC_0+{}^nC_1+...+{}^nC_n$ = Sum of the last half of ${}^nC_0+{}^nC_1+...+{}^nC_n=2^{n-1}$

**(h)** **Bino-geometric series:** ${}^nC_0+{}^nC_1x+{}^nC_2x^2+.......+{}^nC_nx^n=(1+x)^n$

**(i)** **Bino-arithmetic series:** $a\,{}^nC_0+(a+d)\,{}^nC_1+(a+2d)\,{}^nC_2+.......+(a+nd)\,{}^nC_n$

Consider an AP-a, (a+d), (a+2d), ... , (a+nd)

Sequence of Binomial Co-efficient - ${}^nC_0,{}^nC_1,{}^nC_2,.......,{}^nC_n$

A **bino-arithmetic** series is nothing but the sum of the products of corresponding terms of the sequences. It can be added in two ways.

**(i)** By elimination of r in the multiplier of binomial coefficient from the (r+1)th term of the series (By using $r.{}^nC_r=n\,{}^{n-1}C_{r-1}$)

**(ii)** By differentiating the expansion of $x^d(1+x^d)^n$.

**(j) Bino-harmonic series:** $\frac{^nC_0}{a}+\frac{^nC_1}{a+d}+\frac{^nC_2}{a+2d}+......+\frac{^nC_n}{a+nd}$

Consider an HP - $\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d},...,\frac{1}{a+nd}$

Sequence of Binomial Co-efficient - $^nC_0, {}^nC_1, {}^nC_2,......., {}^nC_n$

It is obtained by the sum of the products of corresponding terms of the sequences. Such series are calculated in two ways :

**(i)** By elimination of r in the multiplier of binomial coefficient from the (r + 1)$^{th}$ term of the series $\left(\text{By using} \frac{1}{r+1}{}^nC_r = \frac{1}{n+1}{}^{n+1}C_{r+1}\right)$

**(ii)** By integrating suitable expansion.

For explanation see illustration 2

**(k) Bino-binomial series:** $^nC_0.{}^nC_r + {}^nC_1.{}^nC_{r+1} + {}^nC_2.{}^nC_{r+2} + ....... + {}^nC_{n-r}.{}^nC_r$

or, $^mC_0.{}^nC_r + {}^mC_1.{}^nC_{r-1} + {}^mC_2.{}^nC_{r-2} + ..... + {}^mC_r.{}^nC_0$

As the name suggests such series are obtained by multiplying two binomial expansion, one involving the first factors as coefficient and the other involving the second factors as coefficient. They can be calculated by equating coefficients of a suitable power on both sides.

For explanation see illustration 4

# FORMULAE SHEET

**Binomial theorem for any positive integral index:**

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + ........ + {}^nC_rx^{n-r}a^r + ...... + {}^nC_na^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r$$

**(a)** General term – $T_{r+1} = {}^nC_rx^{n-r}a^r$ is the (r + 1)$^{th}$ term from beginning.

**(b)** (m + 1)$^{th}$ term from the end = (n – m + 1)$^{th}$ from beginning = $T_{n-m+1}$

**(c)** Middle term

**(i)** If n is even then middle term = $\left(\frac{n}{2}+1\right)^{th}$ term

**(ii)** If n is odd then middle term = $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$

Binomial coefficient of middle term is the greatest binomial coefficient.

**To determine a particular term in the given expansion:**

Let the given expansion be $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^{th}$, if $x^n$ occurs in $T_{r+1}$ (r + 1)$^{th\ term}$ then r is given by $n\alpha - r(\alpha+\beta) = m$ and for $x^0, n\alpha - r(\alpha+\beta) = 0$

**Properties of Binomial coefficients:**

For the sake of convenience the coefficients ${}^{n}C_{0}, {}^{n}C_{1}, {}^{n}C_{2} ...... {}^{n}C_{r} ....... {}^{n}C_{n}$ are usually denoted by $C_0, C_1, ...... C_r ....... C_n$ respectively.

$$C_0 + C_1 + C_2 + ....... + C_n = 2^n$$

$$C_0 - C_1 + C_2 - C_3 + ....... + C_n = 0$$

$$C_0 + C_2 + C_4 + ....... = C_1 + C_3 + C_5 + ......... = 2^{n-1}$$

$${}^{n}C_{r} = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\,{}^{n-2}C_{r-2} \text{ and so on........}$$

$${}^{2n}C_{n+r} = \frac{2n!}{(n-r)!(n+r)!}$$

$${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$$

$$C_1 + 2C_2 + 3C_3 + ....... + {}^{n}C_{n} = n.2^{n-1}$$

$$C_1 - 2C_2 + 3C_3 ....... = 0$$

$$C_0 + 2C_1 + 3C_2 + ....... + (n+1)C_n = (n+2)2^{n-1}$$

$$C_0^2 + C_1^2 + C_2^2 + ....... + C_n^2 = \frac{(2n)!}{(n!)^2} = {}^{2n}C_{n}$$

$$C_0^2 - C_1^2 + C_2^2 - C_3^2 + ..... = \begin{cases} 0, \text{ if n is odd} \\ (-1)^{n/2}\,{}^{n}C_{n/2}, \text{ if n is even} \end{cases}$$

**Note:** ${}^{2n+1}C_{0} + {}^{2n+1}C_{1} + ....... + {}^{2n+1}C_{n} = {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + ...... + {}^{2n+1}C_{2n+1} = 2^{2n}$

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ...... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1};\quad C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} ...... + \frac{(-1)^{n}C_n}{n+1} = \frac{1}{n+1}$$

**(a) Greatest term:**

**(i)** If $\frac{(n+1)a}{x+a} \in Z$ (integer) then the expansion has two greatest terms. These are $k^{th}$ and $(k + 1)^{th}$ where x and a are +ve real numbers.

**(ii)** If $\frac{(n+1)a}{x+a} \notin Z$ then the expansion has only one greatest term. This is $(K + 1)^{th}$ term $k = \left[\frac{(n+1)a}{x+a}\right]$ denotes greatest integer less than or equal to x}

**(b) Multinomial theorem:**

Generalized $(x_1 + x_2 + ...... + x_k)^n = \sum_{r_1+r_2+.....r_k=n} \frac{n!}{r_1!r_2!......r_k!} x_1^{r_1} x_2^{r_2} .....x_k^{r_k}$

**(c) Total no. of terms in the expansion** $(x_1 + x_2 + ......x_n)^m$ is ${}^{m+n-1}C_{n-1}$

# Solved Examples

## JEE Main/Boards

**Example 1:** Find the coefficient of $\frac{1}{y^2}$ in $\left(\frac{c^3}{y^2}+y\right)^{10}$

**Sol:** By using formula of finding general term we can easily get coefficient of $\frac{1}{y^2}$.

In the binomial expansion, (r + 1)$^{th}$ term is

$$T_{r+1} = {}^nC_r\left(y\right)^r\left(\frac{c^3}{y^2}\right)^{n-r} : n = 10$$

$$\Rightarrow T_{r+1} = {}^{10}C_r\left(y\right)^r\left(c^3\right)^{10-r}\left(\frac{1}{y^2}\right)^{10-r}$$

$$= {}^{10}C_r\, c^{30-3r}y^{3r-20} \quad \text{...(i)}$$

$\therefore 3r - 20 = -2; r = 6$

$\therefore$ 7$^{th}$ term will contain $y^{-2}$ and from (i) the coefficient of $y^{-2}$ is = 210 $c^{12}$

**Example 2:** Use Binomial theorem to find the value of $(10.1)^5$.

**Sol:** After reducing $(10.1)^5$ into the form of $(10 + 0.1)^n$ we can use binomial expansion to get required result.

$(10.1)^5 = (10 + 0.1)^5$

$$= (10)^5 + {}^5C_1(10)^4(0.1) + {}^5C_2(10)^3(0.1)^2$$

$$+ {}^5C_3(10)^2(0.1)^3 + {}^5C_4 10(0.1)^4 + (0.1)^5$$

$$= (10)^5 + 5(10^3) + 10(10)^3(0.01) + 10(10)^2$$

$$(0.001) + 5(10)(0.0001) + (0.00001)$$

$$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001$$

= 105101.00501

**Example 3:** Find the middle term(s) in the expansion of $\left(2x^2 - \frac{1}{x}\right)^7$.

**Sol:** Since n = 7 is a odd number. Therefore, find the $\frac{n+1}{2}^{th}$ and $\frac{n+3}{2}^{th}$ term.

The total number of terms in the expansion are 8. Therefore $\frac{7+1}{2}^{th}$ and $\frac{7+3}{2}^{th}$ i.e. 4$^{th}$ and 5$^{th}$ terms are the two middle terms. 4$^{th}$ term = ${}^7C_3\left(2x^2\right)^{7-3}\left(-\frac{1}{x}\right)^3$

$$= -\frac{7!}{3!4!}16x^{8-3} = -560x^5$$

and 5$^{th}$ term = ${}^7C_4\left(2x^2\right)^{7-4}\left(\frac{-1}{x}\right)^4 = 280x^2$

Hence the two middle terms are $-560x^5$ and $280x^2$.

**Example 4:** The coefficient of (r – 1)$^{th}$, r$^{th}$ and (r + 1)$^{th}$ term in the expansion of $(x + 1)^n$ are in the ratio 1:3:5. Find n and r.

**Sol:** In this problem, by using the formula of general term we will get the equation of given terms and by taking ratios of these terms we can get the value of n and r.

Coefficient of (r – 1)$^{th}$ term is ${}^nC_{r-2}$

Coefficient of r$^{th}$ term is ${}^nC_{r-1}$

Coefficient of (r + 1)$^{th}$ term is ${}^nC_r$

Coefficient are in ratio of 1 : 3 : 5

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

or $\frac{r-1}{n-r+2} = \frac{1}{3}$ and $\frac{r}{n-r+1} = \frac{3}{5}$

i.e. n – 4r + 5 = 0 and 3n – 8r + 3 = 0

Solving both we get n = 7 & r = 3

**Example 5:** Find the remainder when $27^{10} + 7^{51}$ is divided by 10

**Sol:** We can obtain the remainder by reducing $27^{10} + 7^{51}$ into the form of $10\lambda + a$, where $\lambda$ is any integer and a is an integer less than 10.

We have $27^{10} = 3^{30} = 9^{15} = (10 - 1)^{15}$

$7^{51} = 7.7^{50} = 7.(49)^{25} = 7(50 - 1)^{25}$

$27^{10} = 10m_1$ ...(i)

$7^{51} = 7(50 - 1)^{25} = 10m_2 - 7$ ...(ii)

Adding (i) and (ii)

$27^{10} + 7^{51} = (10m_1 - 1) + (10m_2 - 7) = 10m_1 + 10m_2 - 8$

$= 10m_1 + 10m_2 - 10 + 2$

Thus, the remainder is 2 when $27^{10} + 7^{51}$ is divided by 10.

**Example 6:** If A be the sum of odd numbered terms and B the sum of even numbered terms in the expansion of $(x + a)^n$ prove that $A^2 - B^2 = (x^2 - a^2)^n$

**Sol:** Do it yourself.

$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a$

$x + {}^nC_2x^{n-2}a^2 + .... + {}^nC_na^n = A + B$

When $A = {}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + ....$

$B = {}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + ....$

$\therefore (x-a)^n = A - B,\ A^2 - B^2 = (A-B)(A+B)$

$= (x-a)^n(x+a)^n = (x^2 - a^2)^n$

**Example 7:** If $C_r$ denotes the binomial coefficient ${}^nC_r$, prove that :

$$C_0^2 + C_1^2 + ...... + C_n^2 = \frac{2n!}{(n!)^2}.$$

**Sol:** Multiply the expansion of $(x+1)^n$ and $(1+x)^n$ and compare the coefficients of $x^n$ on both sides.

We know that $(1+x)^n = {}^nC_0 + {}^nC_1x$

$+{}^nC_2x^2 + .... + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n$

$(x+1)^n = {}^nC_0x^n + {}^nC_1x^{n-1}$

$+{}^nC_2x^{n-2} + ..... + {}^nC_{n-1}x + {}^nC_n$

Multiplying these equations side by side, we get

$(1+x)^n(x+1)^n = (C_0 + C_1x + C_2x^2 + .... + C_{n-1}x^{n-1} + C_nx^n)$

$\times(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + .... + C_{n-1}x + {}^nC_n)$

Coefficient of $x^n$ on R.H.S. is equal to

$C_0^2 + C_1^2 + C_2^2 + .... + C_{n-1}^2 + C_n^2$

Coefficient of $x^n$ in L.H.S. is ${}^{2n}C_n = \frac{2n!}{n!n!}$.

This proves the required identity.

**Example 8:** If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ...... + a_{2n}x^{2n}$ show that

(i) $a_0 + a_1 + a_2 + ...... + a_{2n} = 3^n$

(ii) $a_0 - a_1 + a_2 - a_3 + ...... + a_{2n} = 1$

(iii) $a_0 + a_3 + a_6 + ...... = 3^{n-1}$

**Sol:** By putting x = 1, -1, and $\omega, \omega^2$

Respectively in the expansion of $(1 + x + x^2)^n$ we will get the result.

Given $(1 + x + x^2)^n$

$= a_0 + a_1x + a_2x^2 + ...... + a_{2n}x^{2n}$ ....(i)

(i) Putting x = 1, we get

$3^n = a_0 + a_1 + a_2 + ..... + a_{2n}$ .....(A)

(ii) Putting x = –1 in (i), we get

$1 = a_0 - a_1 + a_2 - a_3 ..... + a_{2n}$

(iii) Putting $x = \omega, \omega^2$ successively in (i), we get
$0 = a_0 + a_1\omega + a_2\omega^2 + a_3$

$+a_4\omega + a_5\omega^2 + ..... + a_{2n}\omega^{2n}$ .....(B) $0 = a_0 + a_1\omega^2 + a_2\omega + a_3$

$+a_4\omega^2 + a_5\omega + a_6 + ..... + a_{2n}\omega^{4n}$ ...(C)

Adding (A), (B) and (C) we have

$3^n = 3(a_0 + a_3 + a_6 + ......)$

$\therefore a_0 + a_3 + a_6 + ....... = 3^{n-1}$

**Example 9:** If $(1+x)^n = C_0 + C_1x +$

$C_2x^2 + C_3x^3 + ....... + C_nx^n$

then prove that $C_1^2 + 2C_2^2 + 3C_3^2 + ...... + nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$

**Sol:** Expanding $(1+x)^n$ and $(x+1)^n$ and multiplying these two expansion and comparing the coefficient of $x^{n-1}$ we will prove above equation.

Given $(1+x)^n = C_0 + C_1x +$

$C_2x^2 + C_3x^3 + ..... + C_nx^n$

Differentiating both sides w. r. t. to x, we get

$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + ..... + nC_nx^{n-1}$

$\Rightarrow n(1+x)^{n-1} = C_1 + 2C_2x$

$+3C_2x^2 + ..... + nC_nx^{n-1}$ ....(i)

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2}$

$+C_3x^{n-3} + C_4x^{n-4} + ....... + C_n$ ....(ii)

Multiplying (i) and (ii), we get

$$n(1+x)^{2n-1} = \left(C_1 + 2C_2x + 3C_3x^2 + \ldots.. + nC_nx^{n-1}\right)$$

$$\times\left(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \ldots. + C_n\right) \quad \ldots.(iii)$$

Now, coefficient of $x^{n-1}$ on R.H.S.

$= C_1^2 + 2C_2^2 + 3C_3^2 + \ldots. + nC_n^2$ and coefficient of $x^{n-1}$ on

L.H.S. = $n.{}^{2n-1}C_{n-1}$

$$= n\frac{(2n-1)!}{(n-1)!n!} = \frac{(2n-1)!}{(n-1)!(n-1)!} = \frac{(2n-1)!}{\left[((n-1)!)^2\right]}$$

But (iii) is an identity, therefore the coefficient of $x^{n-1}$ in R.H.S. = coefficient of $x^{n-1}$ in R.H.S.

$$\Rightarrow C_1^2 + 2C_2^2 + 3C_3^2 + \ldots. + nC_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$$

**Example 10:** Find the numerically greatest term in the expansion of $(3 - 5x)^{15}$ when x = 1/5.

**Sol:** Follow the algorithm for the greatest term.

Using standard notations w.r.t. $(x + a)^n$

$$\frac{n+1}{1+\left|\frac{x}{a}\right|} = \frac{16}{1+\left|\frac{3}{(-1)}\right|} = 4$$

$T_4$ and $T_5$ are numerically equal to each other and are greater than any other term.

**Example 11:** If $(1 + x + x^2) = a_0 + a_1x + a_2x^2 + a_3x^3 \ldots + a_{2n}x^{2n}$

Then show that

$$a_0 + a_3 + a_6 + \ldots\ldots = a_1 + a_4 + a_7 + \ldots.. = 3^{n-1}.$$

**Sol:** By Putting x = 1 ω, ω² respectively in the given equation and adding these values we can prove it.

$$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + \ldots\ldots \quad \ldots(i)$$

$$0 = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + \ldots\ldots \quad \ldots.(ii)$$

Because $1 + \omega + \omega^2 = 0$

$$0 = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + \ldots\ldots \quad \ldots(iii)$$

Adding these

$$3^n = 3(a_0) + a_1\left(1+\omega+\omega^2\right) + a_2$$

$$\left(1+\omega^2+\omega^4\right) + a_3\left(1+\omega^3+\omega^6\right)$$

$$+\ldots. = 3(a_0 + a_3 + a_6 + \ldots.)$$

$$\therefore a_0 + a_3 + a_6 + \ldots\ldots = 3^{n-1}$$

From (i) + (ii) × ω² (iii) × ω,+we get,

$$3^n + 0\times\omega^2 + 0\times\omega$$

$$= a_0\left(1+\omega^2+\omega\right) + a_1\left(1+\omega^3+\omega^3\right)$$

$$+a_2\left(1+\omega^4+\omega^5\right) + a_3\left(1+\omega^5+\omega^7\right)$$

$$+a_4\left(1+\omega^6+\omega^9\right) + \ldots$$

$$\therefore 3^n = 3(a_1 + a_4 + a_7 + \ldots\ldots)$$

Because coefficient of each is

$1+\omega+\omega^2 = 0$, using $\omega^3 = 1$

$$\therefore a_1 + a_4 + a_7 + \ldots\ldots.. = 3^{n-1}$$

Again, from (i) + (ii) ω + (iii) × ω³, we get

$$= 3^n = a_0\left(1+\omega+\omega^2\right) + a_1\left(1+\omega^2+\omega^4\right)$$

$$+a_2\left(1+\omega^3+\omega^3\right) + \ldots\ldots = 3(a_2 + a_5 + a_8 + \ldots\ldots..)$$

**Example 12:** Sum the series

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots + \frac{C_n}{n+1}$$

**Sol:** Expanding $(1+x)^n$ integrating it from 0 to 1 or by using summation method we will get result.

$$\text{Sum} = \sum_{r=1}^{n+1}\frac{C_{r-1}}{r} = \sum_{r=1}^{n+1}\frac{1}{n+1}.{}^{n+1}C_r$$

$$= \frac{1}{n+1}\left({}^{n+1}C_0 + {}^{n+1}C_1 + \ldots\ldots + {}^{n+1}C_{r+1} - {}^{n+1}C_0\right)$$

$$= \frac{1}{n+1}\left(2^{n+1} - 1\right)$$

**Alternative method**

$$(1+x)^n = C_0 + C_1 + C_2x^2 + \ldots.. + C_nx^n$$

Integrating both sides w.r.t. x from 0 to 1

$$\int_0^1 (1+x)^n\,dx = \int_0^1\left(C_0 + C_1x + \ldots\ldots + C_nx^n\right)dx$$

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots.. + \frac{C_n}{n+1}$$

**Example 13:** Find the last three digits of $27^{26}$.

**Sol:** By reducing $27^{26}$ into the form $(730-1)^n$ and using simple binomial expansion we will get required digits.

We have $27^2 = 729$.

Now $27^{26} = (729)^{13} = (730-1)^{13}$

$$= {}^{13}C_0(730)^{13} - {}^{13}C_1(730)^{12} + {}^{13}C_2(730)^{11}$$

$$-\ldots\ldots - {}^{13}C_{10}(730)^3 - {}^{13}C_{12}(730)^2$$

$$-{}^{13}C_{12}(730)+1$$

$$=1000m+\frac{(13)(12)}{2}(14)^2-(13)(730)+1$$

Where m is a positive integer

= 1000 m + 15288 – 9490 + 1

= 1000 m + 5799

Thus, the last three digits of $17^{256}$ are 799.

## JEE Advanced/Boards

**Example 1:** Find the coefficient of $x^4$ in the expansion of

(i) $(1 + x + x^2 + x^3)^{11}$

(ii) $(2 - x + 3x^2)^6$

**Sol:** By expanding given equation using expansion formula we can get the coefficient $x^4$ .

(i) $1 + x + x^2 + x^3 = (1 + x) + x^2(1 + x) = (1 + x)(1 + x^2)$

$$\therefore (1+x+x^2+x^3)^{11} = (1+x)^{11}(1+x^2)^{11}$$

$$=\left(1+{}^{11}C_1x+{}^{11}C_2x^2+{}^{11}C_3x^3+{}^{11}C_4x^4+\ldots\ldots.\right)$$

$$\left(1+{}^{11}C_1x^2+{}^{11}C_2x^4+\ldots\ldots\right)$$

To find term in $x^4$ from the product of two brackets on the right-hand-side, consider the following products terms as

$$1\times{}^{11}C_2x^4+{}^{11}C_2x^2\times{}^{11}C_1x^2+{}^{11}C_4x^4$$

$$=\left[{}^{11}C_2+{}^{11}C_2\times{}^{11}C_1+{}^{11}C_4\right]x^4$$

$$\left[55+605+330\right]x^4 = 990x^4$$

$\therefore$ The coefficient of $x^4$ is 990.

(ii) $(2-x+3x^2)^6 = \left[2-x(1-3x)\right]^6$

$$= [2^6 - {}^6C_1\times2^5\times x(1-3x)+{}^6C_2 2^4$$

$$\times x^2(1-3x)^2 - {}^6C_3 2^3\times x^3(1-3x)^3$$

$$+{}^6C_4 2^2\times x^4(1-3x)^4 - 2\times{}^6C_5$$

$$\times x^5(1-3x)^5 + {}^6C_6 \times x^6(1-3x)^6]$$

The term in $x^4$ will come only from the three terms, viz.

(a) ${}^6C_2\times2^4\times x^2(1-3x)^2 = 15\times16x^2(1-6x+9x^2)$

$\therefore$ The term in $x^4$ is (15) (16) ($9x^4$)

(b) $-{}^6C_3 2^3\times x^3(1-3x)^3$

$$=-20\times8\times x^3\left[1-9x+27x^2-27x^3\right]$$

$\therefore$ The term in $x^4$ is $-20\times(-9)\times(8)x^4$

(c) ${}^6C_4 2^2x^4(1-3x)^4 = 15\times4x^4(1-4\times3x+\ldots\ldots.)$

$\therefore$ The term in $x^4$ is $15\times4\times x^4$

$\therefore$ The total term in $x^4$ is

$$\left[15\times16\times9+20\times8\times9+15\times4\right]\times x^4$$

$$=\left[2160+1440+60\right]x^4 = 3660x^4$$

$\therefore$ The coefficient of $x^4$ is 3660.

**Example 2:** Show that $\sum_{r=0}^{n} r(n-r)C_r^2 = n^2.{}^{2n-2}C_n$

**Sol:** By expanding and differentiating $(1+x)^n$ and $(x+1)^n$ and then multiplying these expansion we can prove given equations by comparing coefficient of $x^{n-2}$ on both side.

We have

$$(1+x)^n = C_0+C_1x+C_2x+\ldots\ldots.+C_nx^n \quad \ldots.(i)$$

Differentiating both side w.r.t x, we get

$$n(1+x)^{n-1} = C_1+2C_2x+3C_3x^2+\ldots.+{}^nC_nx^n \quad \ldots (ii)$$

(i) can also the be written as

$$(1+x)^n = (x+1)^n = C_0x^n$$

$$+C_1x^{n-1}+C_2x^{n-2}+\ldots\ldots +C_{n-1}x+C_n \quad \ldots.(iii)$$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = nC_0x^{n-1}+(n-1)$$

$$C_1x^{n-2}+(n-2)C_2x^{n-3}+\ldots..+C_{n-1} \quad \ldots.(iv)$$

Multiplying (ii) and (iv), we have

$$n^2\left(1+x\right)^{n-1}\left(x+1\right)^{n-1}=n^2\left(1+x\right)^{2n-2}$$

$$=[C_1+2C_2+3C_3x^2+.....+{}^rC_nx^{n-1}]$$

$$x\,[nC_0x^{n-1}+(n-1)C_1x^{n-2}+(n-2)$$

$$C_2x^{n-3}+......+C_{n-2}x+C_{n-1}] \quad ....(v)$$

The coefficient of $x^{n-2}$ on the LHS of (v) is

$$n^2.{}^{2n-2}C_{n-2}=n^2.{}^{2n-2}C_n$$

The coefficient of $x^{n-2}$ on the RHS of (v) is

$$1.\left(n-1\right)C_1^2+2.\left(n-2\right)C_2^2+.....+\left(n-1\right).1C_{n-1}^2$$

$$=\sum_{r=0}^{n-1}r\left(n-r\right)C_r^2=\sum_{r=0}^{n}r\left(n-r\right)C_r^2$$

Hence, $\sum_{r=0}^{n}r\left(n-r\right)C_r^2=n^2\left({}^{2n-2}C_n\right)$

**Example 3:** Prove that

(i) $C_0+\frac{C_1}{2}+\frac{C_2}{3}+......+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$

(ii) $2.C_0+2^2.\frac{C_1}{2}+2^3.\frac{C_2}{3}+......$
$+2^{n+1}.\frac{C_n}{n+1}=\frac{3^{n+1}-1}{n+1}$

(iii) $C_0-\frac{1}{2}C_1+\frac{1}{3}C_2-\frac{1}{4}C_3+......$
$+\left(-1\right)^n\frac{C_n}{n+1}=\frac{1}{n+1}$

(iv) $\frac{C_0}{1.2}+\frac{C_1}{2.3}+\frac{C_2}{3.4}+......$
$+\frac{C_n}{\left(n+1\right).\left(n+2\right)}=\frac{2^{n+2}-n-3}{\left(n+1\right)\left(n+2\right)}$

(v) $C_0+\frac{C_2}{3}+\frac{C_4}{5}+......=\frac{2^n}{n+1}$

**Sol:** Expand $\left(1+x\right)^n$ and integrate it within the limit 0 to 1, 0 to 2, -1 to 0 and -1 to 1 respectively to prove these equations

$$\left(1+x\right)^n=C_0+C_1x+C_2x^2+C_3x^3+.......+C_nx^n \quad ....(i)$$

(i) Integrating both sides of equation (i) within limits 0 to 1, we get

$$\int_0^1(1+x)^n dx=$$

$$\int_0^1(C_0+C_1x+C_2x+C_2x^2+C_3x^3+....C_nx^n)\,dx$$

$$\left(\frac{\left(1+x\right)^{n+1}}{n+1}\right)_0^1=C_0x+C_1\frac{x^2}{2}+$$

$$\left.C_2\frac{x^3}{3}+.......+C_n\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\frac{2^{n+1}-1}{n+1}=C_0+\frac{C_1}{2}+\frac{C_2}{3}+.......+\frac{C_n}{n+1}$$

(ii) Integrating both sides of equation (i) within limits 0 to 2.

$$\int_0^2\left(1+x\right)^n=\int_0^2\left(C_0+C_1x+C_2x^2+C_3x^3+.....C_nx^n\right)dx$$

$$\text{or}\left(\frac{\left(1+x\right)^{n+1}}{n+1}\right)_0^2=\left[C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+....+C_n\frac{x^{n+1}}{n+1}\right]_0^2$$

$$\text{or}\frac{3^{n+1}-1}{n+1}=C_0.2+2^2.+\frac{C_1}{2}+2^3.\frac{C_2}{3}+.......+2^{n+1}.\frac{C_n}{n+1}$$

(iii) Integrating both sides of equation (i) within limits –1 to 0,

$$\int_{-1}^0\left(1+x\right)^n dx=\int_{-1}^0\left(C_0+C_1x+C_2x^2+C_3x^3+.....C_nx^n\right)dx$$

$$\left(\frac{\left(1+x\right)^{n+1}}{n+1}\right)_{-1}^0=C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+.......+C_n\frac{x^{n+1}}{n+1}|_{-1}^0$$

$$\frac{1}{n+1}-0=0-\left[-C_0+\frac{C_1}{2}-\frac{C_2}{3}+.....+(-1)^{n+1}\frac{C_n}{n+1}\right]$$

$$\frac{1}{n+1}=C_0-\frac{C_1}{2}+\frac{C_2}{3}+.......+\left(-1\right)^n\frac{C_n}{n+1}$$

(iv) General term of L.H.S $=\frac{{}^nC_k}{(k+1)(k+2)}$

$$=\frac{{}^{n+1}C_{k+1}}{(n+1)\,(k+2)}=\left[\because\frac{{}^nC_r}{n}=\frac{{}^{n-1}C_{r-1}}{r}\right]=\frac{{}^{n+2}C_{k+2}}{\left(n+1\right)\left(n+2\right)}$$

$\therefore$ The sum of terms on L.H.S.

$$=\sum_{k=0}^{n}\frac{{}^{n+2}C_{k+2}}{\left(n+1\right)\left(n+2\right)}=\frac{1}{\left(n+1\right)\left(n+2\right)}\cdot\sum_{k=0}^{n}{}^{n+2}C_{k+2}$$

$$= \frac{1}{(n+1)(n+2)}\left[2^{n+2} - {}^{n+2}C_0 - {}^{n+2}C_1\right]$$

$$= \frac{1}{(n+1)(n+2)}\left[2^{n+2} - 1 - (n+2)\right] = \frac{2^{n+2} - n - 3}{(n+1)(n+2)}$$

(v) Integrating both sides of equation (i) within limits –1 to 1, we get

$$\int_{-1}^{1}(1+x)^n\,dx =$$

$$\int_{-1}^{1}\left(C_0 + C_1x + C_2x^2 + C_3x^3 + ...... + C_nx^n\right)dx$$

$$\left(\frac{(1+x)^{n+1}}{n+1}\right)_{-1}^{1} = C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ....... + C_n\frac{x^{n+1}}{n+1}\Bigg|_{-1}^{1}$$

$$\frac{2^{n+1} - 0}{n+1} = \left[C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ....... + \frac{C_n}{n+1}\right] - \left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + ......\right]$$

$$\frac{2^{n+1}}{n+1} = 2\left[C_0 + \frac{C_2}{3} + \frac{C_4}{5} + .......\right]$$

$$\Rightarrow \frac{2^n}{n+1} = C_0 + \frac{C_2}{3} + \frac{C_4}{5} + .....$$

**Example 4:** Prove, by binomial expansion, that

(i) $\sum_{k=1}^{n} k^2 \cdot {}^nC_k = n(n+1)2^{n-2}$

(ii) $\prod_{k=1}^{n}(C_{k-1} + C_k) = \frac{C_0C_1........C_{n-1}(n+1)^n}{n!}$

**Sol:** Expanding $(1+x)^n$ and differentiating it twice we will prove given equation (i) and by multiplying and dividing by $C_0C_1C_2.......C_{n-1}$ in L.H.S. of equation (ii) we can prove it.

(i) Now $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ....... + C_nx^n$

Differentiating twice w.r.t. x, we get

$$n(n-1)(1+x)^{n-2} = 2.C_2 + 3.2.C_3x$$
$$+4.3C_4x^2 + ...... + n(n-1)C_nx^{n-2}$$

Substituting x = 1, we get

$$n(n-1)2^{n-2} = \sum_{k=1}^{n} k(k-1)(C_k)$$

$$\therefore \sum_{k=1}^{n}(k^2)({}^nC_k) = n(n-1)2^{n-2} + n.2^{n-1}$$

$$\left[\because k.{}^nC_k = n.{}^{n-1}C_{k+1}\right]$$

$$= 2^{n-2}\left[n^2 - n + 2n\right]$$

$$\therefore \sum_{k=1}^{n} k^2\,{}^nC_k = n(n+1)2^{n-2}$$

(ii) To prove $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)......$

$$(C_{n-1} + C_n) = \frac{C_0C_1......C_{n-1}(n+1)^n}{n!}$$

Multiply and divide L.H.S. by $C_0C_1C_2.......C_{n-1}$; then,

$$\text{L.H.S.} = C_0C_1C_2.......C_{n-1}\left(1 + \frac{C_1}{C_0}\right)$$
$$\left(1 + \frac{C_2}{C_1}\right)........\left(1 + \frac{C_n}{C_{n-1}}\right)$$

On using $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ we have,

$$\text{L.H.S.} = C_0C_1C_2.....C_{n-1}$$
$$\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right)......\left(1 + \frac{C_n}{C_{n-1}}\right)$$

$$= C_0C_1C_2...C_{n-1}(1+n)\left(\frac{1+n}{2}\right)\left(\frac{1+n}{3}\right)...\left(\frac{n+1}{n}\right)$$

$$= \frac{C_0C_1...C_{n-1}(1+n)^n}{1.2.3........n} = \frac{C_0C_1.......C_{n-1}(n+1)^n}{n!} = \text{R.H.S.}$$

**Example 5:** If $(1+x)^n = C_0 + C_1x + C_2x^2$
$+C_3x^3 + ....... + C_nx^n$

Then find the value of $\sum_{0 \le i} \sum_{< j \le n}(C_i + C_j)^2$

**Sol:** By using summation and coefficients properties we can prove given equations.

$$\sum_{0 \le i} \sum_{< j \le n}(C_i + C_j)^2$$

$$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + ...... +$$

$$(C_0 + C_n)^2 + (C_1 + C_2)^2 + ..... + ..... +$$

$$(C_1 + C_n)^2 + (C_2 + C_3)^2 + .... + .... + (C_2 + C_n)^2$$

$+ \ldots\ldots + (C_{n-1} + C_n)^2$

$= n\left(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2\right) + 2\sum_{0\le i}\sum_{<j\le n} C_i \cdot C_j$

The square of the sum of n terms is given by

$= \left(C_0 + C_1 + C_2 + C_3 + \ldots\ldots + C_n\right)^2$

$= \left(C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2\right) + 2\sum_{0\le i}\sum_{<j\le n} C_i \cdot C_j$

$\therefore 2\sum_{0\le i}\sum_{<j\le n} C_i \cdot C_j$

$= \left[\left(C_0 + C_1 + C_2 + C_3 + \ldots\ldots +\right)^2 - \left(C_0^2 + C_1^2 + \ldots + C_n^2\right)\right]$

$= \left(2^n\right)^2 - {}^{2n}C_n$

$\therefore \sum_{0\le i}\sum_{<j\le n}\left(C_i + C_j\right)^2 = \left[n.{}^{2n}C_n\right] + \left[2^{2n} - {}^{2n}C_n\right],$

$= (n-1)\,{}^{2n}C_n + 2^{2n}$

**Example 6:** Show that

$$\frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \frac{C_3}{10} + \ldots + 3\frac{(-1)^n C_n}{2n+1} = \frac{3^n n!}{1.4.7\ldots\ldots(3n+1)}$$

**Sol:** By expanding $\left(1-x^3\right)^n$ using binomial expansion and integrating it within a limit 0 to 1 we will prove given equation.

$\left(1-x^3\right)^n = C_0 - C_1x^3 + C_2x^6$

$-C_3x^9 + C_4x^{12} + \ldots + (-1)^n C_nx^{3n}$

Integrating both sides between limits 0 and 1, we get

$$\int_0^1 \left(1-x^3\right)^n dx = C_0 - \frac{C_1}{4} + \frac{C_2}{7} - \frac{C_3}{10} + \ldots + \frac{(-1)^n C_n}{3n+1} \quad \ldots\text{(i)}$$

Also $I_n = \int_0^1 \left(1-x^3\right)^n dx$

$= \left[x\left(1-x^3\right)^n\right]_0^1 - \int_0^1 n\left(1-x^3\right)^{n-1}; \qquad \left(-3x^2\right).x\,dx$

$= 3n\int_0^1 x^3\left(1-x^3\right)^{n-1} dx$

$= 3n\int_0^1 \left(x^3 - 1 + 1\right)\left(1-x^3\right)^{n-1} dx$

$= 3nI_{n-1} - 3nI_n;\ (1+3n)I_n = 3nI_{n-1} \therefore\ I_n = \frac{3n}{3n+1}I_{n-1}$

Replacing n by 1, 2, 3, 4, ....... n–1 successively in the above reduction formula, we get

$$I_n = \frac{3n}{3n+1}\frac{3(n-1)}{3n-2};\ \frac{3(n-2)}{3n-5}\ldots\frac{3}{4}I_0 \quad \ldots\text{(ii)}$$

But $I_0 = \int_0^1 \left(1-x^3\right)^0 dx = \int_0^1 dx = 1$

Hence, from (ii),

$$I_n = \frac{3^n n!}{(3n+1)(3n-2)(3n-5)\ldots\ldots 7.4}$$

Using (i)

$$\frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \frac{C_3}{10} + \ldots\ldots + \frac{(-1)^n C_n}{3n+1} = \frac{3^n n!}{1.4.7\ldots\ldots(3n+1)}$$

**Example 7:** Prove that

$$\frac{1}{m!}C_0 + \frac{n}{(m+1)!}C_1\frac{n(n-1)}{(m-2)!}C_2 + \ldots + \frac{n(n-1)\ldots\ldots 2.1}{(m+n)!}C_n$$

$$= \frac{(m+n+1)(m+n+2)\ldots\ldots(m+2n)}{(m+n)!}$$

**Sol:** As $(1+x)^{m+n}.(1+x)^n = (1+x)^{m+2n}$ and expanding this by using expansion formula and equating the coefficient of $x^n$ we can prove given equation.

$\Rightarrow \left({}^{m+n}C_0 + {}^{m+n}C_1x + {}^{m+n}C_2x^2 + \ldots + {}^{m+n}C_{m+n}x^{m+n}\right)$

$\times\left(C_0 + C_1x + C_2x^2 + \ldots + C_nx^n\right) = (1+x)^{m+2n}$

Equating the coefficients of $x^n$ on both sides, we find

${}^{m+n}C_n.C_0 + {}^{m+n}C_{n-1}.C_1 + {}^{m+n}C_{n-2}.C_2$

$+ \ldots + {}^{m+n}C_0.C_n = {}^{m+2n}C_n$

$$\Rightarrow \frac{(m+n)!}{m!n!}C_0 + \frac{(m+n)!}{(n-1)!(m+1)!}C_1$$

$$+ \frac{(m+n)!}{(n-2)!(m+2)!}C_2 + \ldots + \frac{(m+n)!}{(m+n)!}C_n = \frac{(m+2n)!}{(m+n)!n!}$$

Dividing both sides by (m + n)!/n! we find

$$\frac{1}{m!}C_0 + \frac{n}{(m+1)!}C_1 + \frac{n(n-1)}{(m+2)!}C_2 + \ldots\ldots + \frac{n(n-1)\ldots\ldots 2.1}{(m+n)!}C_n$$

$$= \frac{(m+2n)!}{(m+n)!(m+n)!} = \frac{(m+n+1)(m+n+2).....(m+2n)}{(m+n)!}$$

**Example 8:** Find the sum of the following series

$$S = C_1^2 + \frac{1+2}{2}C_2^2 + \frac{1+2+3}{3}C_3^2 + ...\text{Upto n term}$$

**Sol:** In this problem, first obtain the $r^{th}$ term and then by using binomial expansion and coefficient property we can get required sum.

The $r^{th}$ term of the given series

$$= \frac{1+2+......+r}{r}C_r^2 = \frac{r(r+1)}{2r}C_r^2 = \frac{1}{2}(r+1)C_r^2$$

$$\therefore S = \frac{1}{2}(1+1)C_1^2 + \frac{1}{2}(2+1)C_2^2 + \frac{1}{2}$$

$$(3+1)C_2^3 + .... + \frac{1}{2}(n+1)C_n^2$$

We know that

$$C_0 + C_1x + C_2x^2 + .... + C_nx^n = (1+x)^n$$

$$\Rightarrow C_0x + C_1x^2 + C_2x^3 + .... + C_nx^{n+1} = x(1+x)^n$$

Differentiating both sides w.r.t. x we get

$$C_0 + 2C_1x + 3C_2x^2 + 4C_3x^3$$

$$+.... + (n+1)C_nx^n = (1+x)^n$$

$$+nx(1+x)^{n-1} \quad ...(i)$$

Also

$$C_0 + C_1\left(\frac{1}{x}\right) + C_2\left(\frac{1}{x}\right)^2 + C_3\left(\frac{1}{x}\right)^3$$

$$+.... + C_n\left(\frac{1}{x}\right)^n = \left(1+\frac{1}{x}\right)^n \quad ...(ii)$$

Now, $C_0^2 + 2C_1^2 + 3C_2^2 + 4C_3^2 + .... + (n+1)C_n^2$

= Coefficient of constant term in

$$\left[C_0 + 2C_1x + 3C_2x^2 + 4C_3x^3 + ..... + (n+1)C_nx^n\right] \times$$

$$\left[C_0 + C_1\left(\frac{1}{x}\right) + C_2\left(\frac{1}{x}\right)^2 + ..... + C_n\left(\frac{1}{x}\right)^n\right]$$

= Coefficient of constant term in

$$\left[(1+x)^n + nx(1+x)^{n-1}\right](1+1/x)^n$$

= Coefficient of $x^n$ in

$$\left[(1+x)^n + nx(1+x)^{n-1}\right](x+1)^n$$

= Coefficient of $x^n$ in

$$\left[(1+x)^{2n} + nx(1+x)^{2n-1}\right] = {}^{2n}C_n + n.{}^{2n-1}C_{n-1}$$

$$= \frac{(2n)!}{n!n!}\left(1+\frac{n}{2}\right) = {}^{2n}C_n\left(1+\frac{n}{2}\right)$$

$$\Rightarrow 2C_1^2 + 3C_2^2 + 4C_3^2 + ...... + (n+1)C_n^2$$

$$= {}^{2n}C_n\left(1+\frac{n}{2}\right) - 1 \qquad [\because C_0 = 1]$$

$$\Rightarrow S = \frac{1}{2}\left[{}^{2n}C_n\left(1+\frac{n}{2}\right) - 1\right]$$

**Example 9:** If n be a positive integer, then prove that the integral part I of $\left(5+2\sqrt{6}\right)^n$ is an odd integer. If f be the fractional part of $\left(5+2\sqrt{6}\right)^n$ prove that I = $\frac{1}{1-f}$ – f.

**Sol:** By using expansion formula we can expand the given binomial and separating its integral and fractional part we can prove given equations.

Let $P = \left(5+2\sqrt{6}\right)^n = I + f$

Or $\quad I + f = 5^n + C_1 5^{n-1}\left(2\sqrt{6}\right)$

$$+C_2 5^{n-2}\left(2\sqrt{6}\right)^2 + ...... + C_n\left(2\sqrt{6}\right)^n \quad ...(i)$$

$$0 < 5 - 2\sqrt{6} < 1 \Rightarrow 0 < \left(5-2\sqrt{6}\right)^n < 1$$

Let $\left(5-2\sqrt{6}\right)^n$ = f′, where 0 < f′ < 1.

$$f' = 5^n - C_1 5^{n-1}\left(2\sqrt{6}\right)$$

$$+C_2 5^{n-2}\left(2\sqrt{6}\right)^2 - C_3 5^{n-3}\left(2\sqrt{6}\right)^3 + .... \quad ...(ii)$$

Adding (i) and (ii) $\quad I + f + f' =$

$$2\left[5^n + {}^nC_2 5^{n-2}\left(2\sqrt{6}\right)^2 + {}^nC_4 5^{n-1}\left(2\sqrt{6}\right)^4 ....\right]$$

Or I + f + f′ = even integer

Now $0 \le f < 1$ and 0 < f′ < 1.

$$\therefore 0 < f + f' < 2$$

$\therefore f + f' = 1$ and $\therefore$ I is an odd integer

Now $I + f = \left(5 + 2\sqrt{6}\right)^n$,

$\left(5 - 2\sqrt{6}\right)^n = f' = 1 - f \Rightarrow (I + f)(1 - f) = 1$

$\therefore (I + f) = \frac{1}{1 - f}$ $\quad\quad \therefore I = \frac{1}{1 - f} - f$

**Example 10:** If $\left(1 + x + x^2\right) = a_0 + a_1 x$
$+ a_2 x^2 + a_3 x^3 .... + a_{2n} x^{2n}$

Then show that

$a_0 + a_3 + a_6 + ...... = a_1 + a_4 + a_7 + ..... = 3^{n-1}$.

**Sol:** By using properties of binomial coefficients and cube root unity $1, \omega, \omega^2$ we can prove given problem.

The $r^{th}$ term of the given series

Putting $x = 1, \omega, \omega^2$, where $\omega$ is a non real cube root of unity.

$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + ......$ ...(i)

$0 = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + ......$ ...(ii)

Because $1 + \omega + \omega^2 = 0$

$0 = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + ......$ ...(iii)

Adding these

$3^n = 3(a_0) + a_1\left(1 + \omega + \omega^2\right) + a_2$

$\left(1 + \omega^2 + \omega^4\right) + a_3\left(1 + \omega^3 + \omega^6\right)$

$+.... = 3(a_0 + a_3 + a_6 + ....)$

$\therefore a_0 + a_3 + a_6 + ...... = 3^{n-1}$

From (i) + (ii) × $\omega^2$ + (iii) × $\omega$,

we get, $\quad 3^n + 0 \times \omega^2 + 0 \times \omega$

$= a_0\left(1 + \omega^2 + \omega\right) + a_1\left(1 + \omega^2 + \omega^3\right)$

$+ a_2\left(1 + \omega^4 + \omega^5\right) + a_3\left(1 + \omega^5 + \omega^7\right)$

$+ a_4\left(1 + \omega^6 + \omega^9\right) + ...$

$\therefore 3^n = 3(a_1 + a_4 + a_7 + ......)$

Because coefficient of each is

$1 + \omega + \omega^2 = 0$, using $\omega^2 = 1$

$\therefore a_1 + a_4 + a_7 + ........ = 3^{n-1}$

Again, from (i) + (ii) $\omega$ + (iii) × $\omega^3$, we get

$= 3^n = a_0\left(1 + \omega + \omega^2\right) + a_1\left(1 + \omega^2 + \omega^4\right)$

$+ a_2\left(1 + \omega^3 + \omega^3\right) + ...... = 3(a_2 + a_5 + a_8 + ........)$

**Example 11:** Find the

(i) Last digit

(ii) Last two digits and

(iii) Last three digits of $17^{256}$.

**Sol:** By reducing $17^{256}$ into the form $(x-1)^n$ and using simple binomial expansion we will get required digits.

Since

$17^{256} = \left(17^2\right)^{128} = (289)^{128} = (290 - 1)^{128}$

$\therefore 17^{256} = {}^{128}C_0 (190)^{128} - {}^{128}C_1 (290)^{127}$

$+ {}^{128}C_2 (290)^{126} - ... - {}^{128}C_{125} (290)^3$

$+ {}^{128}C_{126} (290)^2 - {}^{128}C_{127} (290) + 1$

$[{}^{128}C_0 (290)^{128} - {}^{128}C_1 (290)^{127}$

$+ {}^{128}C_2 (290)^{126} - ...... - {}^{128}C_{125} (290)^3]$

$+ {}^{128}C_{126} (290)^2 - {}^{128}C_{127} (290) + 1$

$= 1000m + {}^{128}C_2 (290)^2 - {}^{128}C_1 (290) + 1 \ (m \in I_+)$

$= 1000m + \frac{(128)(127)}{2}(290)^2 - 128 \times 290 + 1$

= 1000 m + (128) (127) (290) (145) – 128 × 290 + 1

= 1000 m + (128) (290) (127 × 145 –1) + 1

= 1000 m + (128) (290) (18414) + 1

= 1000 (m + 683527) + 681

Hence last three digits of $17^{256}$ must be 681. As result last two digits of $17^{256}$ or 81 and last digit of $17^{256}$ is 1.

**Example 12:** If $32^{32^{32}}$ is divided by 7, then find the remainder

**Sol:** Here in this problem, we can obtain required remainder by reducing $32^{32^{32}}$ into the form of $7\lambda + a$, where $\lambda$ is any integer and a is an integer less than 7.

We have $32 = 2^5$

$\therefore (32)^{32} = (2^5)^{32} = 2^{160}$; $(32)^{32} = (3-1)^{160}$

$= {}^{160}C_0 3^{160} - {}^{160}C_1 3^{159} + .... + {}^{160}C_{159} 3 + {}^{160}C_{160} + 1$

$= 3\left(3^{159} - {}^{160}C_1 3^{158} + .... - {}^{160}C_{159}\right) + 1$

$= 3m + 1,\ \ m \in I^+$

Now, $32^{32^{32}} = 32^{3m+1} = 2^{5(3m+1)} = 2^{15m+5}$

$\therefore 32^{32^{32}} = 2^{3(5m+1)}.2^2 = 4.(8)^{5m+1}$

$= 4.(7+1)^{5m+1}$

$= 4.({}^{5m+1}C_0 (7)^{5m+1} + {}^{5m+1}C_1 (7)^{5m}$

$+ {}^{5m+1}C_2 (7)^{5m-1} + .... +$

${}^{5m+1}C_{5m} 7 + {}^{5m+1}C_{5m+1})$

$= 4[7\{{}^{5m+1}C_0 - 7^{5m} + {}^{5m+1}C_1 7^{5m-1}$

$+ {}^{5m+1}C_2 7^{5m-2} + .... + {}^{5m+1}C_{5m}\} + 1]$

$= 4[7n+1],\quad n \in I_+ \quad = 28n + 4$

This show that where $32^{32^{32}}$ is divided by 7, then remainder is 4.

# JEE Main/Boards

## Exercise 1

**Q.1** Expand $(x^2 + 2a)^5$ by binomial theorem.

**Q.2** Expand $(a+b)^6 - (a-b)^6$. Hence find the value of $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$.

**Q.3** Show that $(101)^{50} > (100)^{50} + (99)^{50}$

**Q.4** If x > 1 and the third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ is 1000, find the value of x.

**Q.5** Find the sum of rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$.

**Q.6** Find the middle term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^7$

**Q.7** Find the middle term in the expansion of $(1 - 2x + x^2)^n$.

**Q.8** Show that the greatest coefficient in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$ is $\frac{1.3.5....(2n-1).2^n}{n!}$.

**Q.9** Given that the 4th term in the expansion of $\left(px + \frac{1}{x}\right)^n$ is $\frac{5}{2}$, find n and p.

**Q. 10** If in the expansion of $(1 + x)^m (1 - x)^n$ the coefficient of x and $x^2$ are 3 and –6 respectively then find m.

**Q.11** If the coefficients of $a^{r-1}, a^r, a^{r+1}$ in the binomial expansion of $(1 + a)^n$ are in A.P., prove that $n^2 - n(4r+1) + 4r^2 - 2 = 0$.

**Q.12** If n be a positive integer, then prove that $6^{2n} - 35n - 1$ is divisible by 1225.

**Q.13** Using binomial theorem, show that $3^{4n+1} + 16n - 3$ is divisible by 256 if n is a positive integer.

**Q.14** If $a_1$, $a_2$, $a_3$ and $a_4$ be any four consecutive coefficients in the expansion of $(1 + x)^n$, prove that

$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}$$

**Q.15** If 3 consecutive coefficients in the expansion of $(1 + x)^n$ are in the ratio 6 : 33 : 110, find n and r.

**Q.16** If a, b, c be the three consecutive coefficients in the expansion of a power of (1 + x), prove that the index of the power is $\frac{2ac + b(a+c)}{b^2 - ac}$

**Q.17** Expand $\left(x-\frac{1}{y}\right)^{11}$, $y \neq 0$

**Q.18** Expand $(1 - x + x^2)^4$

**Q.19** Which number is larger, $(1.2)^{4000}$ or 800?

**Q.20** If in the expansion of $(1 + x)^n$, the coefficients of $14^{th}$, $15^{th}$ and $16^{th}$ terms are in A.P., find n.

**Q.21** If three consecutive coefficient in the expansion of $(1 + x)^n$ be 165, 330 and 462, find n and the position of the coefficient.

**Q.22** Find the greatest term in the expansion of; $(7 - 5x)^{11}$, where $x = \frac{2}{3}$

**Q.23** Find the coefficient of $x^{-1}$ in $(1 + 3x^2 + x^4)\left(1+\frac{1}{x}\right)^8$

**Q.24** Find the value of k so that the term independent of x in $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$ of 405.

**Q.25** If A be the sum of odd terms and B the sum of even terms in the expansion of $(x + a)^n$, prove that

$$2\left(A^2+B^2\right)=(x+a)^{2n}+(x-a)^{2n}$$

**Q.26** Find the coefficient of $x^{40}$ in the expansion of $\left(1+2x+x^2\right)^{27}$

**Q.27** Find the term independent of x in $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$.

**Q.28** If $(1 + ax)^n = 1 + 8x + 24x^2 + .....$ Find a and n.

## Exercise 2

### Single Correct Choice Type

**Q.1** Given that the term of the expansion $\left(x^{1/3}-x^{-1/2}\right)^{15}$ which does not contain x is 5 m where $m \in N$, then m =

(A) 1100 (B) 1010 (C) 1001 (D) None

**Q.2** If the coefficients of $x^7$ & $x^8$ in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is:

(A) 15 (B) 45 (C) 55 (D) 56

**Q.3** The coefficient of $x^{49}$ in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)..\left(x-\frac{1}{2^{49}}\right)$ is equal to

(A) $-2\left(1-\frac{1}{2^{50}}\right)$ (B) + ve coefficient of x

(C) – ve coefficient of x (D) $-2\left(1-\frac{1}{2^{49}}\right)$

**Q.4** The last digit of $(3^P + 2)$ is

(A) 1 (B) 2 (C) 4 (D) 5

Where $P = 3^{4n}$ and $n \in N$

**Q.5** The sum of the binomial coefficient of $\left[2x+\frac{1}{x}\right]^n$ is equal to 256. The constant term in the expansion is :

(A) 1120 (B) 2110 (C) 1210 (D) None

**Q.6** The coefficient of $x^4$ in $\left[\frac{x}{2}-\frac{3}{x^2}\right]^{10}$ is

(A) $\frac{405}{256}$ (B) $\frac{504}{259}$ (C) $\frac{450}{263}$ (D) $\frac{405}{512}$

**Q.7** If $(11)^{27} + (21)^{27}$ when divided by 16 leaves the remainder

(A) 0 (B) 1 (C) 2 (D) 14

**Q.8** Last three digits of the number $N = 7^{100} - 3^{100}$ are

(A) 100 (B) 300 (C) 500 (D) 000

**Q.9** The last two digits of the number $3^{400}$ are:

(A) 81 (B) 43 (C) 29 (D) 01

**Q.10** If $\left(1+x+x^2\right)^{25}=a_0+a_1x+a_2x^2+..+a_{50}.x^{50}$ then $a_0+a_2+a_4+....+a_{50}$ is:

(A) Even

(B) Odd and of the form 3n

(C) Odd and of the form (3n –1)

(D) Odd and of the form (3n+1)

**Q.11** The sum of the series

$(1^2+1).1!+(2^2+1).2!+(3^2+1).3!+....+(n^2+1).n!$ is

(A) (n + 1). (n + 2)! (B) n.(n + 1)!

(C) (n + 1). (n + 1)! (D) None of these

**Q.12** Let $P_m$ stand for $^nP_m$. Then the expression $1.P_1 + 2.P_2 + 3.P_3 + ..... + n.P_n =$

(A) (n + 1)! –1 (B) (n + 1)! + 1

(C) (n + 1)! (D) None of these

**Q.13** The expression

$$\frac{1}{\sqrt{4x+1}}\left[\left[\frac{1+\sqrt{4x+1}}{2}\right]^7-\left[\frac{1-\sqrt{4x+1}}{2}\right]^7\right]$$

is a polynomial in x of degree

(A) 7 (B) 5 (C) 4 (D) 3

**Q.14** If the second term of the expansion $\left[a^{1/13}+\frac{a}{\sqrt{a^{-1}}}\right]^n$ is $14a^{5/2}$ then the value of $\frac{^nC_3}{^nC_2}$ is

(A) 4 (B) 3 (C) 12 (D) 6

**Q.15** If $(1+x)(1+x+x^2)$

$(1+x+x^2+x^3).....(1+x+x^2+x^3+....+x^n)$

$$=a_0+a_1x+a_2x^2+a_3x^3+....+a_mx^m$$

Then $\sum_{r=0}^{m}a_r$ has the value equal to

(A) n! (B) (n + 1)!

(C) (n –1)! (D) None of these

**Q.16** In the expansion of $(1 + x)^{43}$ if the coefficient of the $(2r + 1)^{th}$ and the $(r + 2)^{th}$ terms are equal, the value of r is :

(A) 12 (B) 13 (C) 14 (D) 15

**Q.17** The positive value of a so that the coefficient of $x^5$ is equal to that of $x^{15}$ in the expansion of $\left(x^2+\frac{a}{x^3}\right)^{10}$ is

(A) $\frac{1}{2\sqrt{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C) 1 (D) $2\sqrt{3}$

**Q.18** In the expansion of $x^2+\left(\frac{9}{43}\right)^{10}$ the term which does not contain x is :

(A) $^{10}C_0$ (B) $^{10}C_7$ (C) $^{10}C_4$ (D) None of these

**Q.19** If the $6^{th}$ term in the expansion of the binomial $\left[\frac{1}{x^{8/3}}+x^2\log_{10}x\right]^8$ is 5600, then x equals to

(A) 5 (B) 8 (C) 10 (D) 100

**Q.20** $(1+x)(1+x+x^2)(1+x+x^2+x^3).....$

$(1+x+x^2+....+x^{100})$ when written in the ascending power of x then the highest exponent of x is______.

(A) 4950 (B) 5050 (C) 5150 (D) None of these

**Q.21** Let $(5+2\sqrt{6})^n=p+f$ where $n\in N$ and $p\in N$ and $0 < f < 1$ then the value of, $f^2 - f + pf - p$ is

(A) A natural number (B) A negative integer

(C) A prime number (D) Are irrational number

**Q.22** Number of rational terms in the expansion of

$(\sqrt{2}+\sqrt[4]{3})^{100}$ is-

(A) 25 (B) 26 (C) 27 (D) 28

**Q.23** The greatest value of the term independent of x in the expansion of $\left(x\sin\theta+\frac{\cos\theta}{x}\right)^{10}$ is

(A) $^{10}C_5$ (B) $2^5$ (C) $2^5.\ ^{10}C_5$ (D) $\frac{^{10}C_5}{2^5}$

**Q.24** If $(1+x-3x^2)^{2145}=a_0+a_1x+a_2x^2+....$ then $a_0-a_1+a_2-a_3+....$ end with

(A) 1 (B) 3 (C) 7 (D) 9

**Q.25** Coefficient of $x^6$ in the binomial expansion

$\left(\frac{4x^2}{3}-\frac{3}{2x}\right)^9$ is

(A) 2438 (B) 2688 (C) 2868 (D) None

**Q.26** The expression

$$\left[x+\left(x^3-1\right)^{1/2}\right]^5+\left[x-\left(x^3-1\right)^{1/2}\right]^5$$

is a polynomial of degree

(A) 5 (B) 6 (C) 7 (D) 8

**Q.27** Given $\left(1-2x+5x^2-10x^3\right)\left(1+x\right)^n = 1 + a_1x + a_2x^2$ + ..... and that $a_1^2 = 2a_2$ then the value of n is

(A) 6 (B) 2 (C) 5 (D) 3

**Q.28** The sum of the series

$$aC_0+\left(a+b\right)C_1+\left(a+2b\right)C_2+....+\left(a+nb\right)C_n$$

is where $C_r$ denotes combinatorial coefficient in the expansion of $(1 + x)^n$, $n \in N$

(A) $(a + 2nb)2^n$ (B) $(2a + nb)2^n$

(C) $(a + nb)2^{n-1}$ (D) $(2a + nb)2^{n-1}$

## Previous Years' Questions

**Q.1** Given positive integers r > 1, n > 2 and the coefficient of $(3r)^{th}$ and $(r + 2)^{th}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal. Then ***(1980)***

(A) n = 2r (B) n = 2r + 1

(C) n = 3r (D) None of these

**Q.2** If $C_r$ stands for ${}^nC_r$, then the sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\cdot\left[C_0^2-2C_1^2+3C_2^2-....+\left(-1\right)^n\left(n+1\right)C_n^2\right]$$

Where n is an even positive integer, is equal to ***(1986)***

(A) $\left(-1\right)^{n/2}\left(n+2\right)$ (B) $\left(-1\right)^n\left(n+1\right)$

(C) $\left(-1\right)^{n/2}\left(n+1\right)$ (D) None of these

**Q.3** The expression

$$\left[x+\left(x^3-1\right)^{1/2}\right]^5+\left[x-\left(x^3-1\right)^{1/2}\right]^5$$

is a polynomial of degree ***(1992)***

(A) 5 (B) 6 (C) 7 (D) 8

**Q.4** For $2 \le r \le n$, ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$

Is equal to ***(2000)***

(A) ${}^{n+1}C_{r-1}$ (B) $2\,{}^{n+1}C_{r+1}$

(C) $2\,{}^{n+2}C_r$ (D) ${}^{n+2}C_r$

**Q.5** Let $T_n$ denotes the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1}-T_n=21$, then n equals ***(2001)***

(A) 5 (B) 7 (C) 6 (D) 4

**Q.6** If ${}^{n-1}C_r = \left(k^2-3\right){}^nC_{r+1}$, then k belongs to ***(2004)***

(A) $\left(-\infty,-2\right]$ (B) $\left[-2,-\sqrt{3}\right)\cup\left(\sqrt{3},2\right]$

(C) $\left[-\sqrt{3},\sqrt{3}\right]$ (D) $\left(\sqrt{3},\infty\right]$

**Q.7** ${}^{30}C_0\,{}^{30}C_{10} - {}^{30}C_1\,{}^{30}C_{11} + ...... {}^{30}C_{20}\,{}^{30}C_{30}$ is equal to ***(2005)***

(A) ${}^{30}C_{11}$ (B) ${}^{60}C_{10}$ (C) ${}^{30}C_{10}$ (D) ${}^{65}C_{55}$

**Q.8** For r = 0, 1, ....., let $A_r$, $B_r$ and $C_r$ denote, respectively, the coefficient of $x^r$ in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10}A_r\left(B_{10}B_r-C_{10}A_r\right)$ is equal to ***(2010)***

(A) $B_{10}-C_{10}$ (B) $A_{10}\left(B_{10}^2-C_{10}A_{10}\right)$

(C) 0 (D) $C_{10}-B_{10}$

**Q.9** If the coefficients of $x^3$ and $x^4$ in the expansion of $\left(1+ax+bx^2\right)\left(1-2x\right)^{18}$ in powers of x are both zero, then (a, b) is equal to: ***(2014)***

(A) $\left(16,\frac{251}{3}\right)$ (B) $\left(14,\frac{251}{3}\right)$

(C) $\left(14,\frac{272}{3}\right)$ (D) $\left(16,\frac{272}{3}\right)$

**Q.10** The sum of coefficients of integral powers of x in the binomial expansion of $\left(1-2\sqrt{x}\right)^{50}$ is: ***(2015)***

(A) $\frac{1}{2}\left(3^{50}+1\right)$ (B) $\frac{1}{2}\left(3^{50}\right)$

(C) $\frac{1}{2}\left(3^{50}-1\right)$ (D) $\frac{1}{2}\left(2^{50}+1\right)$

**Q.11** If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n, x\neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is: ***(2016)***

(A) 64 (B) 2187 (C) 243 (D) 729

# JEE Advanced/Boards

## Exercise 1

**Q.1** Let $f(x)=1-x+x^2-x^3.....+x^{16}-x^{17}$

$=a_0+a_1(1+x)+a_2(1+x)^2+....+a_{17}(1+x)^{17}$,

Find the value of $a_2$.

**Q.2** (a) Find the term independent of x in the expansion of

(i) $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right]^{10}$ (ii) $\left[\frac{1}{2}x^{1/3}+x^{-1/5}\right]^8$

(b) Find the value of x for which the fourth term in the expansion,

$\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}}+\frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$ is 336.

**Q.3** Find the coefficients:

(i) $x^7$ in $\left(ax^2+\frac{1}{bx}\right)^{11}$

(ii) $x^{-7}$ in $\left(ax-\frac{1}{bx^2}\right)^{11}$

(iii) Find the relation between a and b, so that these coefficients are equal.

**Q.4** (a) If the coefficients of the $r^{th}$, $(r+1)^{th}$ & $(r+2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in AP, find r.

(b) If the coefficients of $2^{nd}$, $3^{rd}$ & $4^{th}$ terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2-9n+7=0$.

**Q.5** Let a and b be the coefficient of $x^3$ in $(1+x+2x^2+3x^3)^4$ and $(1+x+2x^2+3x^3+4x^4)^4$ respectively. Find the value of (a – b).

**Q.6** Prove that the ratio of the coefficient of $x^{10}$ in $(1-x^2)^{10}$ & the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

**Q.7** Find the coefficient of

(a) $x^2y^3z^4$ in the expansion of $(ax-by+cz)^9$.

(b) $a^2b^3c^4d$ in the expansion of $(a-b-c+d)^{10}$.

**Q.8** Given $S_n=1+\frac{q+1}{2}+\left(\frac{q+1}{2}\right)^2+....+\left(\frac{q+1}{2}\right)^n$, $q\neq 1$, prove that ${}^{n+1}C_1+{}^{n+1}C_2.S_1+{}^{n+1}C_3.S_2+....+{}^{n+1}C_{n+1}.S_n=2^n.S_n$.

**Q.9** Find numerically the greatest term in the expansion of

(i) $(2+3x)^9$ when $x=\frac{3}{2}$

(ii) $(3-5x)^{15}$ when $x=\frac{1}{5}$

**Q.10** Given that

$\left(1+x+x^2\right)^n=a_0+a_1x+a_2x^2+....+a_{2n}x^{2n}$,

Find the values of :

(i) $a_0+a_1+a_2+....+a_{2n}$;

(ii) $a_0-a_1+a_2-a_3....+a_{2n}$;

(iii) $a_0^2-a_1^2+a_2^2-a_3^2+....+a_{2n}^2$

**Q.11** For which positive values of x is the fourth term in the expansion of $(5 + 3x)^{10}$ is the greatest.

**Q.12** Find the index n of the binomial $\left(\frac{x}{5}+\frac{2}{5}\right)^n$ if the 9th term of the expansion has numerically the greatest coefficient $(n \in N)$.

**Q.13** Find the number of divisors of the number

$N = {}^{2000}C_1 + 2.{}^{2000}C_2 + 3.{}^{2000}C_3 + .... + 2000.{}^{2000}C_{2000}$

**Q.14** Find number of different dissimilar terms in the sum

$(1+x)^{2012} + (1+x^2)^{2011} + (1+x^3)^{2010}$

**Q.15** Find the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.

**Q.16** Let f(n) = $\sum_{r=0}^{n}\sum_{k=r}^{n}\binom{k}{r}$. Find the total number of divisors of f (11).

**Q.17** Find the sum $\sum_{j=0}^{11}\sum_{i=j}^{11}\binom{i}{j}$.

[Note : $\binom{n}{r} = {}^nC_r$ ]

**Q.18** Let $(1+x^2)^2.(1+x)^n = \sum_{k=0}^{n+4} a_k.x^k$. If $a_1$, $a_2$ and $a_3$ are in AP, find n.

**Q.19** Prove that $\sum_{K=0}^{n} {}^nC_k \sin kx.\cos(n-k)x = 2^{n-1}\sin nx$.

**Q.20** Find the sum of the roots (real or complex) of the equation $x^{2001} + \left(\frac{1}{2}-x\right)^{2001} = 0$.

**Q.21** If for $n \in N$, $\sum_{k=0}^{2n} (-1)^k ({}^{2n}C_k)^2 = A$, then what will be the value of $\sum_{k=0}^{2n} (-1)^k (k-2n) ({}^{2n}C_k)^2$ ?

**Paragraph for questions. 22 and 23**

A path of length n is a sequence of points $(x_1, y_1), (x_2, y_2), ....., (x_n, y_n)$ with integer coordinates such that for all i between 1 and n – 1 both inclusive, either $x_{i+1} = x_i + 1$ and $y_{i+1} = y_i$ (in which case we say the ith step is rightward) or $x_{i+1} = x_i$ and $y_{i+1} = y_i + 1$ (in which case we say that the ith step is upward).

This path is said to start at $(x_1, y_1)$ and end at $(x_n, y_n)$. Let P (a, b), for a and b non negative integers, denotes the number of paths that start at (0, 0) and end at (a, b)

**Q.22** The value of $\sum_{i=0}^{10} P(i, 10-i)$, is

(A) 1024 (B) 512 (C) 256 (D) 128

**Q.23** Number of ordered pairs (i, j) where $i \neq j$ for which P (i, 100 – i) = P(j, 100 – j), is

(A) 50 (B) 99 (C) 100 (D) 101

**Q.24** If $(6\sqrt{6}+14)^{2n+1}$ = N & F be the fractional part of N, prove that NF = $20^{2n+1}$ $(n \in N)$.

**Q.25** Let P = $(2+\sqrt{3})^5$ and f = P – [P], where [P] denotes the greatest integer function. Find the value of $\left(\frac{f^2}{1-f}\right)$.

**Q.26** If $C_0, C_1, C_2, .... , C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$ , $n \in N$ then prove the following:

(a) $C_1 + 2C_2 + 3C_3 + .... + n.C_n = n.2^{n-1}$

(b) $C_0 + 2C_1 + 3C_2 + .... + (n+1)C_n = (n+2)2^{n-1}$

(c) $C_0 + 3C_1 + 5C_2 + .... + (2n+1)C_n = (n+1)2^n$

(d) $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)....(C_{n-1} + C_n)$

$$= \frac{C_0.C_1.C_2....C_{n-1}(n+1)^n}{n!}$$

(e) $1.C_0^2 + 3.C_1^2 + 5.C_2^2 + .... + (2n+1)C_n^2 = \frac{(n+1)(2n)!}{n!n!}$

**Q.27** Let I denotes the integral part and F the proper fractional part of $(3+\sqrt{5})^n$ where $n \in N$ and if $\rho$ denotes the rational part and $\sigma$ the irrational part of the same, show that

$\rho = \frac{1}{2}(I+1)$ and $\sigma = \frac{1}{2}(I+2F-1)$

**Q.28** Prove that

(a) $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + .... + \frac{n.C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

(b) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + .... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(c) $2.C_0 + \frac{2^2.C_1}{2} + \frac{2^3.C_2}{3} + \frac{2^4.C_3}{4}$

$+.... + \frac{2^{n+1}\cdot C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

**Q.29** Prove the following identities using the theory of permutation where $C_0, C_1, C_2, ...., C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in N$, the prove the following :

(a) $C_0C_1 + C_1C_2 + C_2C_3 + .... +$

$C_{n-1}C_n = \frac{2n!}{(n+1)!(n-1)!}$

(b) $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + .... + C_{n-r}C_n = \frac{2n!}{(n-r)!(n+r)!}$

(c) $\sum_{r=0}^{n-2}\left({}^nC_r.{}^nC_{r+2}\right) = \frac{(2n)!}{(n-2)!(n+2)!}$

(d) ${}^{100}C_{10} + 5.{}^{100}C_{11} + 10.{}^{100}C_{12} + 10.$

${}^{100}C_{13} + 5.{}^{100}C_{14} + {}^{100}C_{15} = {}^{105}C_{90}$

**Q.30** If $a_0, a_1, a_2$..... be the coefficients in the expansion of $(1 + x + x^2)^n$ in ascending powers of x, then prove that :

(i) $a_0a_1 - a_1a_2 + a_2a_3 - .... = 0$

(ii) $a_0a_2 - a_1a_3 + a_2a_4 - .... + a_{2n-2}a_{2n} = a_{n+1}$ or $a_{n-1}$

(iii) $E_1 = E_2 = E_3 = 3_{n-1}$;

Where $E_1 = a_0 + a_3 + a_6 + ....; E_2 = a_1 + a_4 + a_7$

$+..... \& E_3 = a_2 + a_5 + a_8 + ....$

**Q.31** Let $\sum_{r=0}^{100}\sum_{s=0}^{100}\left(C_1^2 + C_s^2 + C_rC_s\right) = m\left({}^{2n}C_n\right) + 2^p$

Where m, n and p are even natural numbers and $C_r$ represents the coefficient of $x_r$ in the expansion of $(1 + x)^{100}$. Find the value of (m + n + p).

**Q.32** The expressions $1 + x$, $1 + x + x^2$, $1 + x + x^2 + x^3$, ..... $1 + x + x^2 + ..... + x^n$ are multiplied together and the terms of the product thus obtained are arranged in increasing powers of x in the form of $a_0 + a_1x + a_2x^2 + ......$, then

(a) How many terms are there in the product.

(b) Show that the coefficients of the terms in the product, equidistant from the beginning and end are equal.

(c) Show that the sum of the odd coefficients = the sum of the even coefficients = $\frac{(n+1)!}{2}$

**Q.33** Let $S_1 = \sum_{0\le i<j\le 100}\sum C_iC_j, S_2 =$

$\sum_{0\le j<i\le 100}\sum C_iC_j$ and $S_3 = \sum_{0\le i=j\le 100}\sum C_iC_j$

Where $C_r$ represents coefficient of $x^r$ in the binomial expansion of $(1 + x)^{100}$

If $S_1 + S_2 + S_3 = a^b$ where $a, b \in N$ then find the least value of (a + b).

## Exercise 2

### Single Correct Choice Type

**Q.1** In the binomial $(2^{1/3} + 3^{-1/3})$, if the ratio of the seventh term from the beginning of the expansion to the seventh term from its end is 1/6, then n =

(A) 6 (B) 9 (C) 12 (D) 15

**Q.2** The remainder, when $(15^{23} + 23^{23})$ is divided by 19, is

(A) 4 (B) 15 (C) 0 (D) 18

**Q.3** The value of $4\left\{{}^nC_1 + 4.{}^nC_2 + 4^2.{}^nC_3 + .... + 4^{n-1}\right\}$ is

(A) 0 (B) $5^n + 1$ (C) $5^n$ (D) $5^n - 1$

**Q.4** If n be a positive integer such that $n \ge 3$, then the value of the sum to n terms of the series

$1.n - \frac{(n-1)}{1!}(n-1) + \frac{(n-1)(n-2)}{2!}$

$(n-2)-\frac{(n-1)(n-2)(n-3)}{3!}(n-3)+.....$ is

(A) 0 (B) 1

(C) –1 (D) None of these

**Q.5** If the $6^{th}$ term in the expansion of the binomial $\left[\frac{1}{x^{8/3}}+x^2\log_{10}x\right]^8$ is 5600, then x equals to-

(A) 5 (B) 8 (C) 10 (D) 100

**Q.6** Coefficient of $\alpha^t$ in the expansion of,

$(\alpha+p)^{m-1}+(\alpha+p)^{m-2}(\alpha+q)+$

$(\alpha+p)^{m-3}(\alpha+q)^2+.....+(\alpha+q)^{m-1}$

Where $\alpha\neq -q$ and $p\neq q$ is:

(A) $\frac{{}^mC_t\left(p^t-q^t\right)}{p-q}$ (B) $\frac{{}^mC_t\left(p^{m-t}-q^{m-t}\right)}{p-q}$

(C) $\frac{{}^mC_t\left(p^t+q^t\right)}{p-q}$ (D) $\frac{{}^mC_t\left(p^{m-t}+q^{m-t}\right)}{p-q}$

**Q.7** If $\left(1+x-3x^2\right)^{2145}=a_0+a_1x+a_2x^2+...$ then $a_0-a_1+a_2-a_3+....$ end with

(A) 1 (B) 3 (C) 7 (D) 9

**Q.8** Coefficient of $x^6$ in the binomial expansion $\left(\frac{4x^2}{3}-\frac{3}{2x}\right)^9$ is

(A) 2438 (B) 2688

(C) 2868 (D) None of these

**Q.9** The term independent of 'x' in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$, x > 0, is $\alpha$ times the corresponding binomial coefficient. Then '$\alpha$' is:

(A) 3 (B) $\frac{1}{3}$ (C) $-\frac{1}{3}$ (D) 1

**Q.10** The expression

$\left[x+\left(x^3-1\right)^{1/2}\right]^5+\left[x-\left(x^3-1\right)^{1/2}\right]^5$

Is a polynomial of degree

(A) 5 (B) 6 (C) 7 (D) 8

**Q.11** Value of the expression $C_0^2+C_1^2+C_2^2+....+C_n^2$ is

(A) $2^{2n-1}$ (B) $2n\,({}^{2n}C_n)$

(C) ${}^{2n}C_n$ (D) None of these

**Q.12** The sum of the series

$aC_0+(a+b)C_1+(a+2b)\,C_2+.....+(a+nb)C_n$ is

where $C_r$ denotes combinatorial coefficient in the expansion of $(1+x)^n$, $n\in N$

(A) $(a+2nb)2^n$ (B) $(2a+nb)2^n$

(C) $(a+nb)2^{n-1}$ (D) $(2a+nb)2^{n-1}$

## Previous Years' Questions

**Q.1** Prove that $C_1^2-2.C_2^2+C_3^2-.....-2n.C_{2n}^2=(-1)^n n.C_n$ ***(1979)***

**Q.2** Given,

$S_n=1+q+q^2+....+q^n$

$S_n=1+\frac{q+1}{2}+\left(\frac{q+1}{2}\right)^2+....+\left(\frac{q+1}{2}\right)^n, q\neq 1$

Prove that ${}^{n+1}C_1+{}^{n+1}C_2S_1+{}^{n+1}C_3S_2$

$+....+{}^{n+1}C_{n+1}S_n=2^nS_n$ ***(1984)***

**Q.3** Find the sum of the series

$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}...\text{upto m terms}\right]$

***(1985)***

**Q.4** If $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$ and $a_k=1$ for all $k\geq n$, then show that $b_n={}^{2n+1}C_{n+1}$ ***(1992)***

**Q.5** Let n be a positive integer and

$\left(1+x+x^2\right)^n = a_0 + a_1x + .... + a_{2n}x^{2n}$.

Show that $a_0^2 - a_1^2 + .... + a_{2n}^2 = a_n$ ***(1994)***

**Q.6** Prove that ***(2003)***

$2^k\, {}^nC_0\, {}^nC_k - 2^{k-1}\, {}^nC_1\, {}^{n-1}C_{k-1} + 2^{k-2}\, {}^nC_2\, {}^{n-2}C_{k-2} - ...... + (-1)^k\, {}^nC_k\, {}^{n-k}C_0 = {}^nC_k$

**Q.7** For r = 0, 1, ..., 10, let $A_r, B_r$ and $C_r$ denote, respectively, the coefficient of $x^r$ in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then $\sum_{r=1}^{10}\left(B_{10}B_r - C_{10}A_r\right)$ is equal to ***(2010)***

(A) $B_{10}$-$C_{10}$ (B) $A10\left(B_{10}^2 - C_{10}A_{10}\right)$

(C) 0 (D) $C_{10}$-$B_{10}$

**Q.8** The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =__ ***(2013)***

**Q.9** Coefficient of $x^{11}$ in the expansion of $\left(1+x^2\right)^4\left(1+x^3\right)^7\left(1+x^4\right)^{12}$ is ***(2014)***

(A) 1051 (B) 1106 (C) 1113 (D) 1120

**Q.10** The coefficient of $x^9$ in the expansion of $(1+x)\left(1+x^2\right)\left(1+x^3\right)....\left(1+x^{100}\right)$ is ***(2015)***

**Q.11** Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$.

Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is ***(2016)***

# Questions

## JEE Main/Boards

### Exercise 1

Q. 3 Q. 16 Q. 19 Q. 23
Q. 28 Q. 32 Q. 34

### Exercise 2

Q. 7 Q. 13 Q. 15 Q. 21
Q. 22 Q. 25 Q. 29

### Previous Years' Questions

Q. 2 Q. 3 Q. 5 Q. 6
Q. 8

## JEE Advanced/Boards

### Exercise 1

Q. 14 Q. 23 Q. 26 Q. 31
Q. 34 Q. 35

### Exercise 2

Q. 2 Q. 4 Q. 12

### Previous Years' Questions

Q. 3 Q. 4

# Answer Key

## JEE Main/Boards

### Exercise 1

**Q.1** $x^{10}+10x^8a+40x^6a^2+80x^4a^3+80x^2a^4+32a^5$

**Q.2** $4ab\left[3a^4+10a^2b^2+3b^4\right]$, $140\sqrt{2}$

**Q.4** 100

**Q.5** 41

**Q.6** $280x^2$

**Q.7** $\frac{2n!}{n!n!}(-1)^n x^n$

**Q.9** n = 6, p = $\frac{1}{2}$

**Q.10** m = 12

**Q.15** n = 12, r = 1

**Q.20** 34, 23

**Q.21** 11, $T_{3+1}$, $T_{3+2}$, $T_{3+3}$

**Q.23** 232

**Q.24** k = ± 3

**Q.26** $^{54}C_{14}$

**Q.27** $\frac{7}{18}$

**Q.28** n = 4, a = 2

### Exercise 2

**Single Correct Choice Type**

| | | | | | |
|---|---|---|---|---|---|
| **Q.1** C | **Q.2** C | **Q.3** A | **Q.4** D | **Q.5** A | **Q.6** A |
| **Q.7** A | **Q.8** D | **Q.9** D | **Q.10** A | **Q.11** B | **Q.12** A |
| **Q.13** D | **Q.14** A | **Q.15** B | **Q.16** C | **Q.17** A | **Q.18** C |
| **Q.19** C | **Q.20** B | **Q.21** B | **Q.22** B | **Q.23** D | **Q.24** B |
| **Q.25** B | **Q.26** C | **Q.27** A | **Q.28** D | | |

### Previous Years' Questions

| | | | | | |
|---|---|---|---|---|---|
| **Q.1** A | **Q.2** A | **Q.3** C | **Q.4** D | **Q.5** B | **Q.6** B |
| **Q.7** C | **Q.8** D | **Q.9** D | **Q.10** A | **Q.11** D | |

## JEE Advanced/Boards

### Exercise 1

**Q.1** 816

**Q.2** (a) (i) $\frac{5}{12}$ (ii) $T_6$ = 7, (b) x = 0 or 1

**Q.3** (i) $^{11}C_5\frac{a^6}{b^5}$ (ii) $^{11}C_6\frac{a^5}{b^6}$ (iii) ab = 1

**Q.4** (a) r = 5 or 9

**Q.5** 0

**Q.7** (a) – 1260 $a^2b^3c^4$; (b) – 12600

**Q.9** (i) $T_7 = \frac{7.3^{13}}{3}$ (ii) 455 × $3^{12}$

**Q.10** (i) $3^n$ (ii) 1, (iii) $a_n$

**Q.11** $\frac{5}{8} < x < \frac{20}{21}$ **Q.12** n = 12 **Q.13** 8016 **Q.14** 4023

**Q.15** $\frac{17}{54}$ **Q.16** 24 **Q.17** 4095 **Q.18** n = 2 or 3 or 4

**Q.20** 500 **Q.22** A **Q.23** C **Q.25** 722

**Q.31** 502 **Q.32** (a) $\frac{n^2+n+2}{2}$, (b) $a_0 = a\frac{n(n+1)}{2}$, (c) $\frac{(n+1)!}{2}$ **Q.33** 66

## Exercise 2

### Single Correct Choice Type

**Q.1** B **Q.2** C **Q.3** D **Q.4** A **Q.5** C **Q.6** B

**Q.7** B **Q.8** B **Q.9** D **Q.10** C **Q.11** C **Q.12** D

## Previous Years' Questions

**Q.3** $\frac{2^{mn}-1}{2^{mn}\left(2^n-1\right)}$ **Q.9** C

# Solutions

## JEE Main/Boards

## Exercise 1

**Sol 1:** $(x^2 + 2a)^5$

$= {}^5C_0 (x^2)^5 + {}^5C_1 (x^2)^{5-1}(2a)^1 + {}^5C_2(x^2)^{5-2}$

$(2a)^3 + {}^5C_3(x^2)^{5-3}(2a)^3 + {}^5C_4(x^2)^{5-4}$

$(2a)^4 + {}^5C_5(x^2)^{5-5}(2a)^5$

$= x^{10} + 5x^8(2a) + 10x^6(2a)^2 + 10x^4(2a)^3 + 5x^2(2a)^4 + (2a)^5$

$= x^{10} + 10x^8a + 40x^6a^2 + 80x^4a^3 + 80x^2a^4 + 32a^5$

**Sol 2:** $(a+b)^6 - (a-b)^6$

${}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3$

$+ {}^6C_4a^2b^4 + {}^6C_5ab^5 + {}^6C_6b^6$

$-({}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3$

$+ {}^6C_4a^2b^4 - {}^6C_5a^4b^5 + {}^6C_0b^6)$

$= 2[6a^5b + 20a^3b^3 + 6ab^5] = 4ab[3a^4 + 10a^2b^2 + 3b^4]$

For finding the value, put $a = \sqrt{2}$ b = 1

$\therefore \sqrt{2}$ (12 + 20 + 3)

$\Rightarrow 140\sqrt{2}$

**Sol 3:** $(101)^{50} > (100)^{50} + (99)^{50}$

$(100+1)^{50} > (100)^{50} + (100-1)^{50}$

$= (100+1)^{50} - (100-1)^{50} > 100^{50}$

Both binomial will cancel every odd terms of each others rest of the even terms are.

$= 2[{}^{50}C_1(100)^{49} + {}^{50}C_3(100)^{47}] + {}^{50}C_5(100)^{45} +$

$\ldots\ldots + {}^{50}C_{49}100]$

$= 100(100)^{49} + 2[{}^{50}C_3(100)^{47} + \ldots\ldots + {}^{50}C_{49}(100)]$

$= (100)^{50} + 2[\,^{50}C_3(100)^{47} + ... + \,^{50}C_{47}(100)] > 100^{50}$

Which is always true

So $(101)^{50}-(99)^{50} > (100)^{50}$

$= (101)^{50} > 100^{50} - (99)^{50}$

**Sol 4:** $x > 1$

$\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ and $T4 = {}^7C_3\ (2x^2)^3 \cdot \left(\frac{1}{x}\right)^4 = 280y^2$

$T_3 = T_{2+1} = {}^5C_2\left(\frac{1}{x}\right)^{5-2}\left(x^{\log_{10} x}\right)^2 = 1000$

$= 10\left(\frac{1}{x^3}\right)\left(x^{2\log_{10} x}\right) = 1000 \Rightarrow x^{\log_{10} x^2} = 100x^3$

Assume $x = 10^y$

$\Rightarrow 10^{y\log_{10}(10^y)^2} = 100(10^y)^3 = 10^{2+3y}$

$\Rightarrow 10^{2y(\log_{10} 10^y)} = 10^{2y^2} = 10^{2+3y}$

$\Rightarrow 2y^2=2+3y \Rightarrow 2y^2-3y-2=0$

$\Rightarrow (y-2)(2y+1)=0$

$\Rightarrow y=2$ or $y = -\frac{1}{2}$

$\Rightarrow x=10^2$ or $x = 10^{-1/2} \Rightarrow x=100$ or $x = \frac{1}{\sqrt{10}}$

But $x>1$ so $x=100$

**Sol 5:** $(\sqrt{2} + 3^{1/5})^{10}$

For rational number

$(\sqrt{2})^y \rightarrow y = 2n, n \in N$

$(3^{1/5})^z \rightarrow z = 5n, n \in N$

Rational terms

${}^{10}C_0(\sqrt{2})^{10} + {}^{10}C_{10}(\sqrt{2})^0(3^{1/5})^{10} = 2^5+3^2=32+9=41$

**Sol 6:** $\left(2x^2 - \frac{1}{x}\right)^7$

Middle terms are $T_4 = T_{3+1}$ and $T_5 = T_{4+1}$

$T_{4+1} = {}^7C_4(2x^2)^{7-4}\left(-\frac{1}{x}\right)^4 = \frac{7\times6\times5}{1.2.3}(2x^2)^3\frac{1}{x^4}$

$= 35\times8\times\frac{x^6}{x^4} = 280x^2$

**Sol 7:** $(1-2x+x^2)^n = (1-2x+x^2)^n$

$=(-1+x)^{2n} = (x-1)^{2n}$

Middle term= $T_{n+1} = {}^{2n}C_n(x)^{2n-n}(-1)^n$

$= \frac{2n!}{n!n!}x^n(-1)^n$

**Sol 8:** $\left(x + \frac{1}{x}\right)^{2n}$

Greatest coefficient = ${}^{2n}C_n$

$= \frac{2n!}{n!(2n-n)!} = \frac{2n!}{n!n!}$

$= \frac{2n(2n-1)(2n-2)(2n-3)(2n-4).....3.2.1}{n!(n(n-1)(n-2)(n-3)..........3.2.1)}$

$= \frac{2^n[n(n-1)(n-2)(n-3)\cdots1]1.3.5.7........(2n-1)}{n!(n(n-1)......3.2.1)}$

$= \frac{2^n1.3.5.......(2n-1)}{n!}$

**Sol 9:** $= \left(px + \frac{1}{x}\right)^n$

Given = 4th term = $\frac{5}{2}$

$T_4 = T_{3+1} = {}^nC_3(px)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2}$

$\Rightarrow {}^nC_3p^{n-3}x^{n-3+3(-1)} = \frac{5}{2}x^0$

$\Rightarrow n = 6 \Rightarrow {}^6C_3p^{6-3}x^0 = \frac{5}{2}$

$\Rightarrow \frac{6\times4\times5}{1.2.3}\ p^3 = \frac{5}{2}$

$\Rightarrow p^3 = \frac{1}{8} = \left(\frac{1}{2}\right)^3 \Rightarrow p = \frac{1}{2}$ and $n = 6$

**Sol 10:** $(1+x)^m\ (1-x)^n$

$= ({}^mC_0x^0+{}^mC_1x^1+{}^mC_2x^2+.......+{}^mC_mx^m)$

$({}^nC_0+{}^nC_1(-x)+{}^nC_2(-x)^2+.... .\ {}^nC_2(-x)^n)$

terms of x= ${}^mC_0\ {}^nC_1\ (-1) + {}^nC_0\ {}^mC_1$

$= (-n)+m=m-n=3$ (given) .... (i)

terms of

$x^2= {}^mC_0\,{}^nC_2(-1)^2+ {}^mC_2\,{}^nC_0+{}^mC_1{}^nC_1(-1)$

$= 1.\frac{n\times(n-1)}{1.2}1+\frac{m(m-1)}{1.2}1+m(-n)$

$= \frac{n^2-n}{2}+\frac{m^2-m}{2}-mn=-6$ .... (ii)

In equation (i) m–n=3 $\Rightarrow$ n = (m-3)

Put the values of n in eq. (ii)

$\frac{(m-3)^2-(m-3)}{2}+\frac{m^2-m}{2}-m(m-3)=-6$

$m^2 + 3^2 - 3(2)(m) - m + 3 + m^2 - m - 2m^2 + 6m = 12$

$2m^2 - 6m + 9 + 3 - 2m - 2m^2 + 6m = -12\ 12 - 2m = -12$

$\Rightarrow 2m = 24 \Rightarrow m = 12$ and $n = 9$

**Sol 11:** Coefficient of $a^{r-1}$, $a^r$, $a^{r+1}$ in the binomial expansion of $(1+a)^n$ are in A. P. so

Terms of $a^{r-1} = T_r = {}^nC_{r-1}(a)^{r-1}$

Terms of $a^r = T_{r+1} = {}^nC_r\, a^r$

Terms of $a^{r+1} = T_{r+2} = {}^nC_{r+1}a^{r+1}$

Coefficients of $T_r, T_{r+1}, T_{r+2}$ are in A. P. so

${}^nC_{r-1} + {}^nC_{r+1} = 2\,{}^nC_r$

$\frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{(r+1)!(n-r-1)!}=2\frac{n!}{r!(n-r)!}$

$\Rightarrow \frac{1}{(r-1)!(n-r+1)(n-r)(n-r-1)!}$

$+\frac{1}{(r+1)r(r-1)!(n-r-1)!}=\frac{2}{r(r-1)!(n-r)(n-r-1)!}$

$\Rightarrow \frac{1}{(n-r)(n-r+1)}+\frac{1}{r(r+1)}=\frac{2}{r(n-r)}$

$\Rightarrow \frac{r(r+1)+(n-r)(n-r+1)}{r(r+1)(n-r)(n-r+1)}=\frac{2}{r(n-r)}$

$\Rightarrow r^2 + r + n^2 - nr + n - nr + r^2 - r = 2(r + 1)(n - r + 1)$

$\Rightarrow 2r^2 + n^2 - 2nr + n = 2rn - 2r^2 + 2r + 2n - 2r + 2$

$\Rightarrow n^2 + 4r^2 - 4rn - n - 2 = 0$

$\Rightarrow n^2 - n(4r + 1) + 4r^2 - 2 = 0$

**Sol 12:** n is a positive integer

$\Rightarrow 6^{2n}-35n-1 = (6^2)^n-35n-1=(36)^n-35n-1$

$= (35+1)^n - 35n-1$

$= {}^nC_0 35^n + {}^nC_1 35^{n-1} + ....... + {}^nC_{n-2}35^2$

$+ \cancel{{}^nC_{n-1}35} + \cancel{{}^nC_n 35^0} - \cancel{35n} - \cancel{1}$

And $1225 = 35^2$

so each term is a multiple of $35^2$ and is divisible by 1225

**Sol 13:** $3^{4n+1}$ +16n–3is divisible by 256

$256 = 2^8 = 4^4$

$= 3^{4n+1}+16n-3$

$= 3.\ 3^{4n} +16n-3$

$= 3[4-1]^{4n}+16n-3$

$= 3[\,{}^{4n}C_0 4^{4n} + {}^{4n}C_1 4^{4n-1}(-1) + ..... +$

${}^{4n}C_{4n-2}(4)^{4n-4n+2} - {}^{4n}C_{4n-1}(4)^{4n-4n+1} + {}^{4n}C_{4n}]+16n-3$

= all terms which is multiple of $4^4$ is divisible by 256. So rest of the terms

$= 3[-{}^{4n}C_{4n-2}(4)^2 - {}^{4n}C_{4n-1}(4)^1+1]+16n-3$

$3[\frac{4n(4n-1)}{1.2}\times 4^2-4n\times 4+1]+16n-3$

$= 128n^2.\ 3-128n=128(3n^2-1)$

= 128n(3n-1) and (3n–1) is always even

so $128n(3n-1)=128\times 2^{x+1}$ (assume), $x\in N$

$= 256\times 2^x$ , which is divisible by 256

**Sol 14:** $a_1$, $a_2$, $a_3$ and $a_4$ are any four consecutive coefficients in the expansion of $(1+x)^n$

$a_1 = {}^nC_r$, $a_2 = {}^nC_{r+1}$

$a_3 = {}^nC_{r+2}$, $a_4 = {}^nC_{r+3}$

L. H. S.

$= \frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}$

$= \frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}+\frac{{}^nC_{r+2}}{{}^nC_{r+2}+{}^nC_{r+3}}$

$= \frac{{}^nC_r}{{}^{n+1}C_{r+1}}+\frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}}$

$= \frac{n!(r+1)!(n-r)!}{(n+1)!(r)!(n-r)!}+\frac{n!(r+3)!(n-r-2)!}{(n+1)!(r+2)!(n-r-2)!}$

$= \frac{(r+1)}{(n+1)} + \frac{(r+3)}{(n+1)} = \frac{2r+4}{n+1} = \frac{2(r+2)}{n+1}$

$= 2\frac{n!(r+2)!(n-r-1)!}{(n+1)!(r+1)!(n-r-1)!} = 2\frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}}$

$= 2\frac{{}^nC_{r+1}}{{}^nC_{r+2}+{}^nC_{r+1}} = \frac{2a_2}{a_2+a_3}$

**Sol 15:** 3 consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio 6 : 33 : 110

$= T_{r+1} : T_{r+2} : T_{r+3}$

$= {}^nC_r : {}^nC_{r+1} : {}^nC_{r+2}$

$= \frac{n!}{r!n-r!} : \frac{n!}{r+1!n-r-1!} : \frac{n!}{r+2!n-r-2!}$

$= \frac{1}{(n-r)(n-r-1)} : \frac{1}{(r+1)(n-r-1)} : \frac{1}{(r+1)(r+2)}$

$= 6 : 33 : 110$

$\Rightarrow \frac{(r+1)(n-r-1)}{(n-r)(n-r-1)} = \frac{6}{33} = \frac{2}{11} \Rightarrow \frac{r+1}{n-r} = \frac{2}{11}$

$\Rightarrow 11r+11 = 2n-2r \Rightarrow 2n-13r-11=0$ .... (i)

And $\frac{(r+1)(r+2)}{(r+1)(n-r-1)} = \frac{33}{110} = \frac{3}{10} \Rightarrow \frac{r+2}{n-r-1} = \frac{3}{10}$

$\Rightarrow 10r + 20 = 3n - 3r - 3 \Rightarrow 3n - 13r - 23 = 0$ .... (ii)

Subtracting equation (i) from (ii), we get n = 12

Putting n = 12 in equation (i)

$13r = 2n - 11 = 2(12) - 11 = 24 - 11 = 13 \Rightarrow r = 1$

So terms are $T_{r+1}, T_{r+2}, T_{r+3}$

**Sol 16:** a, b, c are three consecutive coefficients in the expansion of power (say n) of $(1+x)^n$

So $a = {}^nC_r$, $b = {}^nC_{r+1}$, $c = {}^nC_{r+2}$

$\frac{a}{b} = \frac{(r+1)}{(n-r)} \Rightarrow an - ar = br + b$

$\Rightarrow r = \frac{an-b}{a+b}$ ......(i)

$\frac{b}{c} = \frac{(r+2)}{(n-r-1)}$

$\Rightarrow bn - br - b = cr + 2c$

$\Rightarrow r = \frac{bn-b-2c}{b+c}$ ....... (ii)

From (i) and (ii)

$\frac{an-b}{a+b} = \frac{bn-b-b-2c}{b+c}$

$\Rightarrow abn - b^2 + acn - bc = abn - 2ab - 2ac + b^2n - 2b^2 - 2bc$

$\Rightarrow n(ac - b^2) = -ab - 2ac - bc$

$\Rightarrow n = \frac{ab+2ac+bc}{b^2-ac} = \frac{2ac+b(a+c)}{b^2-ac}$

**Sol 17:** $\left(x - \frac{1}{y}\right)^{11}$, $y \neq 0$

$= {}^{11}C_0 x^{11} + {}^{11}C_1 x^{11-1}\left(-\frac{1}{y}\right) + {}^{11}C_2 x^{11-2}\left(-\frac{1}{y}\right)^2$

$+ {}^{11}C_3 x^{11-3}\left(-\frac{1}{y}\right)^3 + \cdots + {}^{11}C_{11}\left(-\frac{1}{y}\right)^{11}$

**Sol 18:** $(1-x+x^2)^4 = ((1-x)+x^2)^4$

$= {}^4C_0(1-x)^4 + {}^4C_1(1-x)^3x^2 + {}^4C_2(1-x)^2(x^2)^2$

$+ {}^4C_3(1-x)(x^2)^3 + {}^4C_4(x^2)^4$

$= (1-x)^4 + 4x^2(1-x)^3 + 6(1+x^2-2x)x^4 + 4x^6(1-x) + x^8$

$= ({}^4C_0 - {}^4C_1x + {}^4C_2x^2 - {}^4C_3x^3 + {}^4C_4x^4)$

$+ 4x^2({}^3C_0 - {}^3C_1x + {}^3C_2x^2 - {}^3C_3x^3)$

$+ (6 + 6x^2 - 12x)x^4 + 4x^6 - 4x^7 + x^8$

$= 1 - 4x + 6x^2 - 4x^3 + x^4 + 4x^2 - 12x^3$

$+ 12x^4 - 4x^5 + 6x^4 + 6x^6 - 12x^5 + 4x^6 - 4x^7 + x^8$

$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$

**Sol 19:** $(1.2)^{4000} = (1+0.2)^{4000}$

$= {}^{4000}C_0(0.2)^0 + {}^{4000}C_1(0.2)^1 + {}^{4000}C_2(0.2)^2 + .......$

$= 1 + 4000(0.2) + ......... = 1 + 800 + .... . = 801 + .......$

So $(1.2)^{4000}$ is greater than 800

**Sol 20:** For $(1+x)^n$

$T_{14}, T_{15}$ and $T_{16}$ are in A. P.

$\Rightarrow T_{14} + T_{16} = 2T_{15} \Rightarrow {}^nC_{13} + {}^nC_{15} = 2\,{}^nC_{14}$

$\Rightarrow {}^nC_{13} + {}^nC_{15} = 2\,{}^nC_{14}$

$$\frac{n!}{13!(n-13)!}+\frac{n!}{15!(n-15)!}=2\frac{n!}{14!(n-14)!}$$

$$\frac{1}{(n-13)(n-14)}+\frac{1}{15\times 14}=\frac{2}{14(n-14)}$$

$$\frac{15\times 14+(n-13)(n-14)}{15\times 14(n-13)(n-14)}=\frac{2}{14(n-14)}$$

$210 + n^2 + 182 - n(13 + 14) = 2 \times 15(n - 13)$

$n^2 + 392 - 27n = 30n - 390$

$n - 57n + 782 = 0$

$$n=\frac{57\pm\sqrt{57^2-4(1)(782)}}{2(1)}=\frac{57\pm\sqrt{121}}{2}=\frac{57\pm 11}{2}$$

$n = 34, 23$

**Sol 21:** Given

${}^nC_r=165;\ {}^nC_{r+1}=330;\ {}^nC_{r+2}=462$

$$\frac{{}^nC_r}{{}^nC_{r+1}}=\frac{165}{330}=\frac{1}{2}$$

$$\Rightarrow \frac{n!(r+1)!(n-r-1)!}{r!(n-r)!n!}=\frac{1}{2}$$

$$\Rightarrow \frac{(r+1)}{(n-r)}=\frac{1}{2}$$

$\Rightarrow 2r+2=n-r$

$\Rightarrow 3r=n-2$ .... (i)

$$\frac{{}^nC_{r+2}}{{}^nC_{r+1}}=\frac{462}{330}=\frac{7}{5}$$

$$\frac{(r+1)!n!(n-r-1)!}{n!(r+2)!(n-r-2)!}=\frac{n-r-1}{r+2}=\frac{7}{5}$$

$\Rightarrow 5n - 5r - 5 = 7r + 14$

$\Rightarrow 12r = 5n - 19$ .... (ii)

From eq (i) and (ii)

$4(n2) = 5n19 \quad \Rightarrow n = 11$

So $3r = 11 - 2 = 9 \quad \Rightarrow r = 9/3 = 3$

Position of coefficients are $T_{3+1}, T_{3+2}, T_{3+3}$

**Sol 22:** $(7-5x)^{11},\ x=\frac{2}{3}$

$$\frac{n+1}{1+\left|\frac{x}{a}\right|}=\frac{11+1}{1+\left(\frac{7\times 3}{5\times 2}\right)}=\frac{12}{1+\frac{21}{10}}=\frac{12}{3.1}=3.87$$

So greatest is $4^{+n}$.

$$|T_4|=|T_{3+1}|={}^{11}C_3\,(7)^{11-3}\left(5\times\frac{2}{3}\right)^3$$

$$=\frac{11\times 10\times 9}{1.2.3}\times 7^8\ 5^3\times\frac{2^3}{3^3}=\frac{11}{9}2^3 5^4 7^8$$

**Sol 23:** $(1+3x^2+x^4)\left(1+\frac{1}{x}\right)^8$

For Coefficient of $x^{-1}$

$$=(1)\ {}^8C_1\left(\frac{1}{x}\right)+3x^2\ {}^8C_3\frac{1}{x^3}+x^4\ {}^8C_5\frac{1}{x^5}$$

$$=\frac{8}{x}+\frac{3\times 8\times 7\times 6}{1\times 2\times 3}\times x^{-1}+\frac{8\times 7\times 6}{1.2.3}x^{-1}$$

$$=\frac{1}{x}[8+168+56]=\frac{232}{x}$$

Coefficient of $x^{-1}=232$

**Sol 24:** $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$

$$T_{r+1}={}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{k}{x^2}\right)^r={}^{10}C_r(x)^{\frac{10-r}{2}}k^r x^{-2r}$$

$T_{r+1}$ is independent of x

So $\frac{10-r}{2}-2r=0 \Rightarrow 10 - 5r = 0 \Rightarrow r=\frac{10}{5}=2$

So Coefficient is $={}^{10}C_2k^2$

$$=\frac{10\times 9}{1\times 2}\times k^2=405 \Rightarrow k^2=9 \Rightarrow k=\pm 3$$

**Sol 25:** $(x+a)^n$

A = Sum of odd terms

B = Sum of even terms

(ii) $2(A^2+B^2)=(x+a)^{2n}+(x-a)^{2n}$

$A = {}^nC_0x^n+{}^nC_2x^{n-2}a^2+\ldots\ {}^nC_nx^0a^n$

$B = {}^nC_1x^{x-1}a+{}^nC_3x^{n-3}a^3+\ldots\ldots+{}^nC_{n-1}+xa^{n-1}$

$2(A^2+B^2) = (A+B)^2 + (A-B)^2 = (x+a)^{2n}+(x-a)^{2n}$

L. H. S. = R. H. S.

**Sol 26:** $(1+2x+x^2)^{27} = ((1+x)^2)^{27} = (1+x)^{54}$

$T_{r+1}={}^{54}C_r x^r$

Coefficient of $x^{40} \Rightarrow r=40$

Coefficient = ${}^{54}C_{40}={}^{54}C_{54-40}= {}^{54}C_{14}$

**Sol 27:** $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

$$T_{r+1}={}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

For independence of x

$2(9-r)-r=18-2r-r=18-3r=0$

Coefficient $r=6$

$$T_{r+1}=T_{6+1}={}^9C_6\left(\frac{3}{2}\right)^{9-6}\left(-\frac{1}{3}\right)^6$$

$$= {}^9C_3\times\frac{3^3}{2^3}\frac{1}{3^6}=\frac{9\times8\times7}{1.2.3}\times\frac{(1)}{2^3 3^3}=\frac{7\times3}{2.3^3}=\frac{7}{18}$$

**Sol 28:** $(1+ax)^n = 1+8x+24x^2+\ldots\ldots$

${}^nC_0 + {}^nC_1ax + {}^nC_2(ax)^2 + \ldots. = 1 + 8x + 24x^2 + \ldots\ldots$

So ${}^nC_1a = 8$ and ${}^nC_2a^2 = 24$

$na=8$ and $\frac{n(n-1)}{2}a^2=24$

$a^2n^2 - na^2 = 48 \Rightarrow (8)^2 - 8a = 48 \Rightarrow 64 - 8a = 48$

$\Rightarrow a = 2 \Rightarrow n = 4$

## Exercise 2

### Single Correct Choice Type

**Sol 1: (C)** $(x^{1/3}-x^{-1/2})^{15}$

$T_{r+1}= {}^{15}C_r\,(x^{1/3})^{15-r}(-x^{-1/2})^r$

Power of $x=\frac{15-r}{3}-\frac{r}{2}=0$ for $x^0$

$2(15-r)-3r=0$

$30 - 2r - 3r = 0 \Rightarrow 5r = 30 \Rightarrow r = 30/5 = 6$

Coefficient $T_{r+1}={}^{15}C_6\times1=5005$

$5m = 5005 \Rightarrow m = 1001$

**Sol 2: (C)** In the expansion $\left(2+\frac{x}{3}\right)^n$ the coefficients of $x^7$ & $x^8$ are equal

$${}^nC_7(2)^{n-7}\left(\frac{1}{3}\right)^7={}^nC_8(2)^{n-8}\left(\frac{1}{3}\right)^8$$

$$\frac{6}{(n-7)}=\frac{1}{8}\Rightarrow n-7=48\Rightarrow n=48+7=55$$

**Sol 3: (A)** $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)\ldots\ldots\left(x-\frac{1}{2^{49}}\right)$

Max power of x = 50

Coefficient of $x^{49}$

$$= -1-\frac{1}{2}-\frac{1}{2^2}-\frac{1}{2^3}\cdots-\frac{1}{2^{49}}$$

$$=\left[\frac{1-\left(\frac{1}{2}\right)^{50}}{1-\frac{1}{2}}\right]=-2\left[1-\frac{1}{2^{50}}\right]$$

**Sol 4: (D)** $(3^P+2)$

$P=3^{4n},\ n\in N = 3^{3^{4n}}+2$

$3^0=1, 3^1=3, 3^2=9, 3^3=27, 3^4=81$

Last digit = 1,3,9,7

Last digit of $3^x$ repeat after every power of 4 so $3^{4n}$ last digit =1

$3^1=3$

$3^1+2=5$

So last digit of $3^{3^{4n}}+2$ is 5

**Sol 5: (A)** $\left(2x+\frac{1}{x}\right)^n$

Sum of binomial coefficient= $2^n=256$

$2^n=2^8 \Rightarrow n=8$

Constant term =

$${}^8C_4(2x)^4.\left(\frac{1}{x}\right)^4=\frac{8\times7\times6\times5}{1.2.3.4}\times2^4x^{4-4}=1120$$

**Sol 6: (A)** $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$

$$T_{r+1}={}^{10}C_r\left(\frac{x}{2}\right)^{10-r}\left(-\frac{3}{x^2}\right)^r$$

Power of x=10–r–2(r)=4 (given)

$\Rightarrow$ 10 – 3r = 4 $\Rightarrow$ 3r = 10 – 4 = 6 $\Rightarrow$ r = 6/3 = 2

Coefficient $T_{r+1} = {}^{10}C_2\left(\frac{1}{2}\right)^{10-2}(-3)^2$

$= \frac{10\times 9}{1.2}2^{-8}\,3^2 = \frac{5\times 9\times 9}{2^8} = \frac{405}{256}$

**Sol 7: (A)** $11^{27}+21^{27}$

$= (16-5)^{27}+(16+5)^{27}$

$= 2\left[{}^{27}C_0 16^{27} + \ldots\ldots + {}^{27}C_{26}16\right]$

= 2. 16k = 32k

Always divisible by 16

**Sol 8: (D)** $N=7^{00}-3^{100}$

$N = (5+2)^{100} - (5-2)^{200}$

$N = 2\left[{}^{100}C_1 5^{99}.2 + \ldots\ldots + {}^{100}C_{99}5.2^{99}\right]$

$N = [{}^{100}C_1 5^{97}.100 + 10^3\,{}^{100}C_3 5^{94}$

$+\ldots\ldots + {}^{100}C_{99}10.2^{99}]$

$N = 1000.[10.5^{97} + \ldots. + 2^{99}]$

Integer

Last three digits = 000

**Sol 9: (D)** $3^{400} = (3^2)^{200}$

$(9)^{200} = (10-1)^{200}$

$= {}^{200}C_0 10^{200} - {}^{200}C_1 10^{199} + \ldots.\ {}^{200}C_{199}10^1 + {}^{200}C_{200}.1$

= 10m + 1 $(m \in N)$

Last 2 digits are 01

**Sol 10: (A)** $(1+x+x^2)^{25} = a_0 + a_1x + \ldots. + a_{50}x^{50}$

x = 1

$3^{25} = a_0 + a_1 + a_2 + \ldots.. + a_{50}$ ...(i)

x = –1

$(1 - 1 + 1)^{25} = 1$

$= a_0 - a_1 + a_2 - a_3 + \ldots\ldots + a^{50}$ .... (ii)

Sum of both $eq^n$.

$3^{25} + 1 = 2(a_0 + a_2 + a_4 + \ldots.. + a_{50})$

$a_0 + a_2 + a_4 + \ldots.. + a_{50} = \frac{1}{2}(3^{25}+1)$

$= \frac{1}{2}\left[(4-1)^{25}+1\right]$

$= \frac{1}{2}\left[\left({}^{25}C_0 4^{25} - {}^{25}C_1 4^{24} + \ldots. + {}^{25}C_{24}4 - 1\right) + 1\right]$

= 2 m always even. ($\because$ divisible by 2)

**Sol 11: (B)** $(1^2+1)1!+(2^2+1)2!+(3^2+1)3!+\ldots\ldots+(n^2+1)n!$

$T_n = (n^2 + 1)\, n! = n(n + 1)! - (n - 1)\, n!$

$S_n = n(n+1)!$

**Sol 12: (A)** $P_m \rightarrow {}^nP_m$

$1P_1 + 2P_2 + 3P_3 + \ldots\ldots + n.\,P_n$

= 1. n + 2. n(n – 1) + 3n(n – 1)(n – 2)

+ 4n(n – 1)(n – 2)(n – 3) + .... + n. n!

Add (+1 and –1)

$= 1 + {}^nC_1 + 2\,{}^nC_2 2! + 3\,{}^nC_3 3! + 4\,{}^nC_4 4! + \ldots n\,{}^nC_n n! - 1$

$= -1 + 1 + \sum_{i=0}^{n} i\,{}^nC_i(i)!$

$= -1 + 1\sum_{i=0}^{n} iP_i$

When $1 + 1P_1 + 2P_2 + 3P_3 + \ldots. + nP_n = (n+1)!$

$= -1 + 1(n+1)! - 1$

$= (n+1)! - 1$

**Sol 13: (D)**

$\frac{1}{\sqrt{4x+1}}\left[\left[\frac{1+\sqrt{4x+1}}{2}\right]^7 - \left[\frac{1-\sqrt{4x+1}}{2}\right]^7\right]$

$= \frac{1.2}{2^7\sqrt{4x+1}}\left[{}^7C_1\sqrt{4x+1} + {}^7C_3(\sqrt{4x+1})^3\right.$

$\left. + \ldots\ldots + {}^7C_7(\sqrt{4x+1})^7\right]$

$= 2^{-6}\left[{}^7C_1 + {}^7C_3(4x+1) + {}^7C_5\right.$

$\left.(4x+1)^2 + {}^7C_7(4x+1)^3\right]$

$\Rightarrow$ Max. power of $x = 3$

**Sol 14: (A)**

$$\left(a^{1/13}+\frac{a}{\sqrt{a^{-1}}}\right)^n=\left(a^{1/13}+a^{1+1/2}\right)^n=\left(a^{1/13}+a^{3/2}\right)^n$$

$$T_2={}^nC_1(a^{1/13})^{n-1}+(a^{3/2})^1=14a^{5/2}$$

$$\Rightarrow n\ a^{\frac{n-1}{13}+\frac{3}{2}}=14a^{5/2}\ \Rightarrow n=14$$

$$\frac{^{14}C_3}{^{14}C_2}=\frac{14-3+1}{3}=\frac{12}{3}=4$$

**Sol 15: (B)** $(1+x)(1+x+x^2)(1+x+x^2+x^3)$

…. . $(1+x+\ldots.\ +x^n)$

$= a_0+a_1x+a_2x^2+\ldots\ldots.+a_mx^m$

$$\sum_{r=0}^{m}ar=a_0+a_1+a_2+\ldots\ldots+a_m$$

At $x=1$

$= 2.3.4.5.6.....(n+1) = (n+1)!$

**Sol 16: (C)** $(1+x)^{43}$

Given $T_{2r+1}=T_{r+2}$

$^{43}C_{2r}={}^{43}C_{r+1}={}^{43}C_{43-(r+1)}$

$$\Rightarrow 2r=43-r-1=42-r\ \Rightarrow 3r=42\Rightarrow r=\frac{42}{3}=14$$

**Sol 17: (A)** $\left(x^2+\frac{a}{x^3}\right)^{10}$

Coefficient of $x^5$ is equal to that of $x^{15}$

$$T_{r+1}={}^{10}C_r(x^2)^{10-r}\left(\frac{a}{x^3}\right)^r$$

Power of $x=2(10-r)-3r = 20–5r$

$20-5r=5\Rightarrow r=3$

$20-5r=15\Rightarrow r=1$

$T_{3+1}=T_{1+1}$

$^{10}C_3a^3={}^{10}C_1a$

$$\frac{10\times9\times8}{1.2.3}a^2=10$$

$$a^2=\frac{1}{12}\Rightarrow a=\frac{1}{\sqrt{12}}=\frac{1}{2\sqrt{3}}$$

**Sol 18: (C)** $\left(x^2+\frac{a}{x^3}\right)^{10}$

Power of x for term

$$T_{r+1}={}^{10}C_r(x^2)^{10-r}\left(\frac{a}{x^3}\right)^r$$

Power of $x = 2(10-r)-3r = 20-5r=0$

$\Rightarrow 5r=20 = r=20/5=4$

$T_{4+1}={}^{10}C_4$ binomial coefficient

**Sol 19: (C)** $\left(\frac{1}{x^{8/3}}+x^2\log_{10}x\right)^8$

$$T_6=T_{5+1}={}^8C_5\left(\frac{1}{x^{8/3}}\right)^{8-5}\left(x^2\log_{10}x\right)^5=5600$$

$$\Rightarrow\frac{8\times7\times6}{1.2.3}\times\left(x^{-8/3}\right)^3x^{10}(\log_{10}x)^5=5600$$

$x^{-8+10}(\log_{10}x)^5=100\Rightarrow x^2(\log_{10}x)^5=100$

Assume $x=10^y$

So $10^{2y}(\log_{10}10^y)^5=10^2\Rightarrow 10^{2y-2}\ y^5=1$

$\Rightarrow y=1\Rightarrow x=10$

**Sol 20: (B)** $(1+x)(1+x+x^2)(1+x+x^2+x^3)$

…. . $(1+x+\ldots.\ x^{100})$

Highest power of x $=1+2+3+\ldots.\ +100$

$$=\frac{100(100+1)}{2}=50\times101=5050$$

**Sol 21: (B)** $(5+2\sqrt{6})^n=p+f$

$p=[(5+2\sqrt{6})^n]-f$

$f^2-f+pf-p = f(f-1)+p(f-1) = (f-1)(f+p)$

$$\text{Assume } F=(5-2\sqrt{6})^n=\left(\frac{1}{5+2\sqrt{6}}\right)^n$$

$0<f<1,\ 0<F<1$

$F+f+p=(5+2\sqrt{6})^n+(5-2\sqrt{6})^n = \text{integer} = 2I$

$F+f=2I-p = \text{Integer}$

$0<F+f<2\ \Rightarrow F+f=1\Rightarrow F=1-f$

$(F)(f+p) = (5-2\sqrt{6})^n(5+2\sqrt{6})^n = -1$

**Sol 22: (B)** $(\sqrt{2}+\sqrt[4]{3})^{100} = (2^{1/2}+3^{1/4})^{100}$

L. C. M. of 2 and 4 =4

Total terms = $n+1=100+1=101$

T rational = $^{100}C_{4n}\left(2^{1/2}\right)^{100-4n}\left(3^{1/4}\right)^{4n}$

$\Rightarrow 0 \le 100-4n \le 100$

$\Rightarrow 0 \le n \le 25\ n \in N$

$n=\{0,1,2,3.....25\}$

Total number for $n=26$

**Sol 23: (D)** $\left(x\sin\theta+\dfrac{\cos\theta}{x}\right)^{10}$

$T_{r+1} = {}^{10}C_r(x\sin\theta)^{10-r}\left(\dfrac{\cos\theta}{x}\right)^r$

Power of $x=10-r+r(-1) = 10-2r=0$ (given)

$\Rightarrow r = 5$

$T_{r+1} = {}^{10}C_5(\sin\theta)^5(\cos\theta)^5 = {}^{10}C_5\ (\sin\theta\cos\theta)^5$

$= {}^{10}C_5\left(\dfrac{\sin 2\theta}{2}\right)^5$. Max value when $\sin 2\theta = 1$

$\therefore$ Max. value = $\dfrac{^{10}C_5}{2^5}$

**Sol 24: (B)** $(1+x-3x^2)^{2145}=a_0+a_1x+a_2x^2+$

At x= – 1

$(1-1-3)^{2145} = -(3)^{2145}$

$= a_0 - a_1 + a_2 - a_3 + .......$

L. H. S. = $3^{2145}$ = $3.3^{2144}$ = $3[9]^{1072}$

Even power of 9 ends with 1. Hence $3^{2145}$ ends with 3.

**Sol 25: (B)** $\left(\dfrac{4x^2}{3}-\dfrac{3}{2x}\right)^9$

$T_{r+1} = {}^9C_r\left(\dfrac{4x^2}{3}\right)^{9-r}\left(-\dfrac{3}{2x}\right)^r$

Power of $x=2(9-r)+(-1)r = 18-3r=6$

$\Rightarrow 3r=18-6=12 \Rightarrow r = 4$

Coefficient $^9C_4\left(\dfrac{4}{3}\right)^{9-4}\left(\dfrac{-3}{2}\right)^4 = \dfrac{9\times8\times7\times6}{1.2.3.4}\left(\dfrac{4}{3}\right)^5\left(\dfrac{3}{2}\right)^4$

$= 9\times2\times7\times\dfrac{2^{10}\times3^4}{3^5\times2^4} = 21\times2^7 = 2688$

**Sol 26: (C)** $[x+(x^3-1)^{1/2}]^5+[x(x^3-1)^{1/2}]^5$

$= 2[{}^5C_0x^5 + {}^5C_2x^{5-2}(x^3-1) + {}^5C_4x^{5-4}(x^3-1)^2]$

Max power of x = 7

**Sol 27: (A)** $(1-2x+5x^2-10x^3)(1+x)^n$

$= 1+a_1x-1a_2x^2+......$ and $a_1^2=2a_2$

Coefficient of $x=a_1={}^nC_1-2=n-2$

Co-efficient of $x^2=a_2=5+{}^nC_2-2{}^nC_1= 5+\dfrac{n(n-1)}{2}-2n$

$= 5+\dfrac{n^2-n-4n}{2}=5+\dfrac{n^2-5n}{2}$

$a_1^2 = 2a_2 \Rightarrow (n-2)^2 = 2\left[\dfrac{10+n^2-5n}{2}\right]$

$\Rightarrow n^2+4-4n=10+n^2-5n \Rightarrow n=6$

**Sol 28: (D)** $aC_0+(a+b)C_1+(a+2b)C_2+....+(a+nb)C_n$

$a(C_0+C_1+C_2+.....+C_n)$

$+b(C_1+2C_2+......+nC_n)$

$= a2^n+b[n2^{n-1}] = 2^{n-1}[2a+nb]$

## Previous Years' Questions

**Sol 1: (A)** In the expansion

$(1+x)^{2n}, t_{3r} = {}^{2n}C_{3r-1}(x)^{3r-1}$

$t_{r+2} = {}^{2n}C_{r+1}(x)^{r+1}$

Since, binomial coefficient of $t_{3r}$ and $t_{r+2}$ are equal.

$\Rightarrow {}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow 3r-1=r+1$ or $2n=(3r-1)+(r+1)$

$\Rightarrow 2r=2$ or $2n=4r$

$\Rightarrow r=1$ or $n=2r$

But r > 1,

$\therefore$ We take n = 2r

**Sol 2: (A)** We have $C_n^2 - 2C_1^2 + 3C_2^2 - 4C_3^2$

$+.... + (-1)^n (n+1) C_n^2$

$= \{C_0^2 - C_1^2 + C_2^2 - C_3^2 + .... + (-1)^n C_n^2\}$

$-\{C_1^2 - 2C_2^2 + 3C_3^2 - .... + (-1)^n nC_n^2\}$

$$= (-1)^{n/2} \cdot \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} - (-1)^{\frac{n}{2}-1} \frac{n}{2} \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$$

$$= (-1)^{n/2} \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\left(1+\frac{n}{2}\right)$$

$$\therefore \frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}$$

$\{C_0^2 - 2C_1^2 + 3C_2^2 - .... + (-1)^r (n+1) C_n^2\}$

$$= \frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!} (-1)^{n/2} \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \frac{(n+2)}{2} = (-1)^{n/2}(n+2)$$

**Sol 3: (C)** We know that $(a+b)^5 + (a-b)^5$

$= {}^5C_0 a^5 + {}^5C_1 a^4 b + {}^5C_2 a^3 b^2$

$+ {}^5C_3 a^2 b^3 + {}^5C_4 ab^4 + {}^5C_5 b^5 + {}^5C_0 a^5 - {}^5C_1 a^4 b$

$+ {}^5C_2 a^3 b^2 - {}^5C_3 a^2 b^3 + {}^5C_4 ab^4 - {}^5C_5 b^5$

$= 2\left[a^5 + 10a^3b^2 + 5ab^4\right]$

$$\therefore \left[x + \left(x^3 - 1\right)^{1/2}\right]^5 + \left[x - \left(x^3 - 1\right)^{1/2}\right]^5$$

$$= 2\left[x^5 + 10x^3\left(x^3 - 1\right) + 5x\left(x^3 - 1\right)^2\right]$$

Therefore, the given expression is a polynomial of degree 7.

**Sol 4: (D)** ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$

$= \left({}^nC_r + {}^nC_{r-1}\right) + \left({}^nC_{r-1} + {}^nC_{r-2}\right)$

We know that

${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$\therefore {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$

**Sol 5: (B)** $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$

$$= \left[\binom{n}{r} + \binom{n}{r-1}\right] + \left[\binom{n}{r-1} + \binom{n}{r-2}\right]$$

$$= \binom{n+1}{r} + \binom{n+1}{r-1} = \binom{n+2}{r}$$

According to given condition, $T_n = {}^nC_3$

and $T_{n+1} - T_n = 21$

$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 21$

$$\Rightarrow \frac{1}{6}(n+1)(n)(n-1) - \frac{1}{6}n(n-1)(n-2) = 21$$

$$\Rightarrow \frac{n(n-1)}{6}\left[(n+1)-(n-2)\right] = 21$$

$$\Rightarrow \frac{n(n-1)}{6} = 21 \Rightarrow n(n-1) = 42$$

$\Rightarrow n = 7$

**Sol 6: (B)** Given, ${}^{n-1}C_r = \left(k^2 - 3\right){}^nC_{r+1}$

$$\Rightarrow {}^{n-1}C_r = \left(k^2 - 3\right)\frac{n}{r+1}\,{}^{n-1}C_r$$

$$\Rightarrow k^2 - 3 = \frac{r+1}{n}$$

$$(\text{Since}, n \ge r \Rightarrow \frac{r+1}{n} \le 1 \text{ and } n, r > 0)$$

$\Rightarrow 0 < k^2 - 3 \le 1$

$\Rightarrow 3 < k^2 \le 4$

$\Rightarrow k \in \left[-2, -\sqrt{3}\right) \cup \left(\sqrt{3}, 2\right]$

**Sol 7: (C)** Let $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11}$

$$+ \binom{30}{2}\binom{30}{12} - .... + \binom{30}{20}\binom{30}{30}$$

$\therefore A = {}^{30}C_0 . {}^{30}C_{10} - {}^{30}C_1 . {}^{30}C_{11}$

$+ {}^{30}C_2 . {}^{30}C_{12} - .... + {}^{30}C_{20} . {}^{30}C_{30}$

= Coefficient of $x^{20}$ in $(1 + x)^{30}.(1 - x)^{30}$

= Coefficient of $x^{20}$ in $(1 - x^2)^{30}$

= Coefficient of $x^{20}$ in $\sum_{r=0}^{30}(-1)^r\,{}^{30}C_r\left(x^2\right)^r$

$\therefore$ For coefficient of $x^{20}$ clearly $2r = 20 \Rightarrow r = 10$

Put $(r = 10) = {}^{30}C_{10}$

**Sol 8: (D)** $A_r$ = Coefficient of $x^r$ in $(1 + x)^{10} = {}^{10}C_r$

$B_r$ = Coefficient of $x^r$ in $(1 + x)^{2n} = {}^{20}C_r$

$C_r$ = Coefficient of $x^r$ in $(1 + x)^{30} = {}^{30}C_r$

$$\therefore \sum_{r=1}^{10} A_r\left(B_{10}B_r - C_{10}A_r\right) = \sum_{r=1}^{10} A_rB_{10}B_r - \sum_{r=1}^{10} A_rC_{10}A_r$$

$$= \sum_{r=1}^{10} {}^{10}C_r\,{}^{20}C_{10}\,{}^{20}C_r - \sum_{r=1}^{10} {}^{10}C_r\,{}^{30}C_{10}\,{}^{10}C_r$$

$$= \sum_{r=1}^{10} {}^{10}C_{10-r}\cdot{}^{20}C_{10}\,{}^{20}C_r - \sum_{r=1}^{10} {}^{10}C_{10-r}\,{}^{30}C_{10}\,{}^{10}C_r$$

$$= {}^{20}C_{10}\sum_{r=1}^{10} {}^{10}C_{10-r}.{}^{20}C_r - {}^{30}C_{10}\sum_{r=1}^{10} {}^{10}C_{10-r}\,{}^{10}C_r$$

$$= {}^{20}C_{10}\left({}^{30}C_{10} - 1\right) - {}^{30}C_{10}\left({}^{20}C_{10} - 1\right)$$

$$= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$$

**Sol 9: (D)** $\left(1 + ax + bx^2\right)$

$$\left[1 - {}^{18}C_1 2x + {}^{18}C_2(2x)^2 - {}^{18}C_3(2x)^3 + {}^{18}C_4(2x)^4\right]$$

Coefficient of $x^3$ is

$$-{}^{18}C_3\left(2^3\right) + a\left({}^{18}C_2 \times 4\right) - b\left({}^{18}C_1 \times 2\right) = 0 \qquad \text{...(i)}$$

Coefficient of $x^4$ is

$$ {}^{18}C_4\left(2^4\right) + a\left(-{}^{18}C_3 \times 2^3\right) + {}^{18}C_2 b2^2 = 0 \qquad \text{....(ii)}$$

or solving both these equation

$a = 16$ and $b = 272/3$.

**Sol 10: (A)**

$$\left(1 - 2\sqrt{x}\right)^{50} = {}^{50}C_0 - {}^{50}C_1\left(2\sqrt{x}\right)^1 + {}^{50}C_2\left(2\sqrt{x}\right)^2$$
$$- {}^{50}C_3\left(2\sqrt{x}\right)^3 + {}^{50}C_4\left(2\sqrt{x}\right)^4$$

So, sum of coefficient Integral powers of x

$$S = {}^{50}C_0 + {}^{50}C_2.2^2 + {}^{50}C_4.2^4 + .... + {}^{50}C_{50}.2^{50}$$

Now,

$$(1 + x)^{50} = 1 + {}^{50}C_1x + {}^{50}C_2x^2 + {}^{50}C_3x^3 + {}^{50}C_4x^4$$
$$+..... + {}^{50}C_{50}x^{50}$$

Put x = 2, - 2

$$3^{50} = 1 + {}^{50}C_1.2 + {}^{50}C_2.2^2 + {}^{50}C_3.2^3$$
$$+ {}^{50}C_4.2^4 + .... + {}^{50}C_{50}.2^{20} \qquad \text{....(i)}$$

$$1 = 1 - {}^{50}C_1.2 + {}^{50}C_2.2^2 - {}^{50}C_3.2^3$$
$$+ {}^{50}C_4 2^4 - .... + {}^{50}C_{50}.2^{50} \qquad \text{....(ii)}$$

(i) + (ii)

$$3^{50} + 1 = 2\left[1 + {}^{50}C_2.2^2 + {}^{50}C_4.2^4 + .... + {}^{50}C_{50}.2^{50}\right]$$

$$\therefore \frac{3^{50} + 1}{2} = 1 + {}^{50}C_2.2^2 + {}^{50}C_4.2^4 + .... + {}^{50}C_{50}.2^{50}$$

**Sol 11: (D)** Number of terms $= \frac{(n+1)(n+2)}{2} = 28$

$\Rightarrow n = 6$

$$\therefore a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + .... + \frac{a_{2n}}{x^{2n}} = \left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$$

Put x = 1, n = 6,

$$a_0 + a_1 + a_2 + .... + a_{2n} = 3^6 = 729$$

# JEE Advanced/Boards

## Exercise 1

**Sol 1:** $f(x) = 1 - x + x^2 - x^3 + ......x^{16} - x^{17}$

$= a_0 + a_1(1+x) + a_2(1+x)^2 + ....... + a_{17}(1+x)^{17}$

Differentiating both sides

$-1 + 2x ... -17x^{16} = a_1 + 2a_2(1 + x) + ... + 17a_{17}(1 + x)^{16}$

Again differentiating

$2 - 6x + ... = 2a_2 + 6a_3(1 + x) + ...$

Putting $x = -1$

$\Rightarrow 2 + 6 + 12 + 20 + ....... + 17 \times 16 = 2a_2$

$2a_2 = 1.2 + 2.3 + 3.4....... + 16.17$

$T_n$ for 1. 2+2. 3+3. 4 is $T_n = n(n+1)$

$$2a_2 = \sum_{i=1}^{16} T_n = \sum_{i=1}^{16} n^2 + \sum_{i=1}^{16} n$$

$$= \frac{(2(16)+1)16(16+1)}{6} + \frac{16(16+1)}{2} = 1632$$

$\Rightarrow a_2 = 816$

**Sol 2:** (a) (i) $\left(\sqrt{\frac{x}{3}} + \frac{\sqrt{3}}{2x^2}\right)^{10}$

$$T_{r+1} = {}^{10}C_r \left(\frac{\sqrt{x}}{\sqrt{3}}\right)^{10-r} \left(\frac{\sqrt{3}}{2x^2}\right)^r$$

$$= {}^{10}C_r \left(\frac{1}{\sqrt{3}}\right)^{10-r} x^{\frac{10-r}{2}-2r} \left(\frac{\sqrt{3}}{2}\right)^r$$

For term independent of x-

$$\frac{10-r}{2} - 2r = 0 \Rightarrow 10 - r - 4r = 0 \Rightarrow r = 2$$

$$\text{So } T_3 = T_{2+1} = {}^{10}C_2 \left(\frac{1}{\sqrt{3}}\right)^{10-2} \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{10 \times 9}{2} \times \frac{1}{3^4} \times \frac{3}{4} = \frac{5}{12}$$

(ii) $\left[\frac{1}{2}x^{1/3} + x^{-1/5}\right]^8$

$$T_{r+1} = {}^8C_r \left(\frac{1}{2}x^{1/3}\right)^{8-r} (x^{-1/5})^r$$

Power of x = $\frac{8-r}{3} - \frac{r}{5} = 0$ for independence

$\Rightarrow 5(8 - r) - 3r = 0 \Rightarrow 40 - 5r - 3r = 0 \Rightarrow r = 5$

$$T_{5+1} = T_6 = {}^8C_5 \left[\frac{1}{2}(x^{1/3})\right]^{8-5} . (x^{-1/5})^5 = \frac{8 \times 7 \times 6}{1.2.3} \times \left(\frac{1}{2}\right)^3 = 7$$

(b) $\left(5^{\frac{2}{5}\log_5 \sqrt{4^x+44}} + \frac{1}{5^{\log_5 \sqrt[3]{2^{x-1}+7}}}\right)^8$

$= (a_1 + a_2)^8$ assume

$$T_4 = T_{3+1} = {}^8C_3 (a_1)^{8-3} (a_2)^3$$

$$a_1 = 5^{\log_5 (\sqrt{4^x+44})^{2/5}} = ((4^x + 44)^{1/2})^{2/5} = (4^x + 44)^{1/5}$$

$$a_2 = \frac{1}{5^{\log_5 (2^{x-1}+7)^{1/3}}} = \frac{1}{(2^{x-1} + 7)^{1/3}} = (2^{x-1} + 7)^{-1/3}$$

$$T_4 = {}^8C_3 (4^x + 44)^{5/5} (2^{x-1} + 7)^{-3/3}$$

$$= \frac{8 \times 7 \times 6}{1.2.3} \times (4^x + 44)(2^{x-1} + 7)^{-1} = 336$$

$$\Rightarrow \frac{4^x + 44}{2^{x-1} + 7} = \frac{336}{8x7} = 6$$

$$\Rightarrow 4^x + 44 = 6 \times 2^{x-1} + 6 \times 7 = 3.2^x + 42$$

$(2^x)^2 - 3(2)^x + 44 - 42 = 0$

Assume $2^x = y$

$y^2 - 3y + 2 = 0 \Rightarrow (y - 2)(y - 1) = 0$

$\Rightarrow y = 1$ or $y = 2 \Rightarrow x = 0$ or $x = 1$

**Sol 3:** $\left(ax^2 + \frac{1}{bx}\right)^{11}$

$$T_{r+1} = {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r$$

Power of $x = 2(11 - r) + r(-1) = 7$ (given)

$\Rightarrow 22 - 2r - r = 7 \Rightarrow r = 5$

Coefficient $T_{5+1} = {}^{11}C_5 (a)^{11-5} \left(\frac{1}{b}\right)^5 = {}^{11}C_5 a^6 b^{-5}$

(ii) $\left(ax - \frac{1}{bx^2}\right)^{11}$

$$T_{r+1} = {}^{11}C_r (ax)^{11-r} \left(-\frac{1}{bx^2}\right)^r$$

Power of x = $11 - r - 2r = -7$ (given)

$\Rightarrow r = 6$

Coefficient $T_{r+1} = T_{6+1} = {}^{11}C_6 a^{11-6} \left(-\frac{1}{b}\right)^6 = {}^{11}C_6 a^5 b^{-6}$

(iii) Given that both coefficient are equal

$\Rightarrow {}^{11}C_5 a^6 b^{-5} = {}^{11}C_6 a^5 b^{-6} \Rightarrow ab = 1$

**Sol 4:** (a) $(1+x)^{14}$

Coefficients $T_r = T_{(r-1)+1} = {}^{14}C_{r-1}$

$T_{r+1} = {}^{14}C_r;\ T_{(r+1)+1} = {}^{14}C_{r+1}$

Its given that they are in A. P. so. $T_r + T_{r+2} = 2T_{r+1}$

${}^{14}C_{r-1} + {}^{14}C_{r+1} = 2\ {}^{14}C_r$

$$\frac{14!}{(r-1)!(14-r+1)!} + \frac{14!}{(r+1)!(14-r-1)!} = 2\frac{14!}{r!(14-r)!}$$

$$\Rightarrow \frac{1}{(15-r)(14-r)} + \frac{1}{(r+1)r} = \frac{2}{r(14-r)}$$

$$\Rightarrow \frac{r(r+1)+(15-r)(14-r)}{r(r+1)(14-r)(15-r)} = \frac{2}{r(14-r)}$$

$\Rightarrow r^2+r+210-14r-15r+r^2=2(r+1)(15-r)$

$\Rightarrow 2r^2-28r+210=30r-2r^2+30-2r$

$\Rightarrow 4r^2-56r+180=0 \Rightarrow r^2-14r+45=0$

$\Rightarrow (r-9)(r-5)=0 \Rightarrow r=9$ or $r=5$

(b) $(1+x)^{2n}$

Coefficients $T_2=T_{1+1}={}^{2n}C_1=2n$

$$T_3 = T_{2+1} = {}^{2n}C_2 = \frac{2n(2n-1)}{1.2} = n(2n1)$$

$$T_4 = {}^{2n}C_3 = \frac{2n(2n-1)(2n-2)}{1.2.3} = \frac{n(2n-1)(2n-2)}{3}$$

They all are in A. P.

So, $T_2 + T_4 = 2T_3$

$$2n + \frac{n(2n-1)(2n-2)}{3} = 2n(2n-1)$$

$\Rightarrow 3+(n-1)(2n-1)=3(2n-1)=6n-3$

$\Rightarrow 3+2n^2-2n-n+1=6n-3$

$\Rightarrow 2n^2-9n+7=0$

**Sol 5:** a = Coefficient of $x^3$ in $(1+x+2x^2+3x^3)^4$

b = Coefficient of $x^3$ in $(1+x+2x^2+3x^3+4x^4)^4$

$4x^4$ has no effect on the coefficient of $x^3$.

Hence a = b

$\therefore$ a – b = 0

**Sol 6:** $(1-x^2)^{10}$

$T_{r+1} = {}^{10}C_r(-x^2)^r$

Given that $2r=10 \Rightarrow r=5$

So coefficient is= $(1)^r\ {}^{10}C_r = {}^{10}C_5$

And in $\left(x-\frac{2}{x}\right)^{10}$

$$T_{r+1} = {}^{10}C_r(x)^{10-r}\left(-\frac{2}{x}\right)^r$$

Power of x = $10-r-r=0$

$\Rightarrow 10-2r=0 \Rightarrow r=5$

Coefficient = ${}^{10}C_5(-2)^5 = -{}^{10}C_5 2^5$

$$\text{Ratio of both coefficients} = \frac{{}^{10}C_5}{{}^{10}C_5 2^5} = \frac{1}{2^5} = \frac{1}{32}$$

**Sol 7:** (a) $(ax-by+cz)^9$

$$\text{General term} = \frac{9!}{r_1!r_2!r_3!}(ax)^{r_1}(-by)^{r_2}(cz)^{r_3}$$

$r_1 + r_2 + r_3 = 9$

For coefficient of $x^2y^3z^4$ so $\Rightarrow r_1=2, r_2=3, r_3=4$

$$\text{So Coefficient} = \frac{9!}{2!3!4!} \times a^2 \cdot b^3 c^4$$

$= -1260\ a^2 \cdot b^3 \cdot c^4$

(b) $(a-b-c+d)^{10}$

$$\text{General Term} = \frac{10!}{r_1!r_2!r_3!r_4!}(a)^{r_1}(-b)^{r_2}(-c)^{r_3}(d)^{r_4}$$

$r_1+r_2+r_3+r_4=10$

It given that $r_1=2, r_2=3, r_3=4, r_4=1$

$$\text{Coefficient } \frac{10!}{2!3!4!1!}(-1)^3(-1)^4$$

$$= -\frac{10\times 9\times 8\times 7\times 6\times 5}{2!3!} = -12600$$

**Sol 8:** $s_n = 1 + q + q^2 + \ldots\ldots + q^n =$

$$S_n = 1 + \frac{q+1}{2} + \left(\frac{(q+1)}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n, q \neq 1$$

$= {}^{n+1}C_1 + {}^{n+1}C_2\ s_1 + {}^{n+1}C_3 s_2$

$+\ldots\ldots+\ {}^{n+1}C_{n+1}s_n$

Constant term

${}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots + {}^{n+1}C_{n+1} = 2^{n+1}-1$

In $S_n$ constant term $= 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+.....\left(\frac{1}{2}\right)^n$

$$= \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2}\right)^{n+1} = \frac{(2^{n+1}-1)}{2^n}$$

So $(2^{n+1}-1) = (2^n),\left(\frac{(2^{x+1}-1)}{2^n}\right)$

$${}^{n+1}C_1 + {}^{n+1}C_2S_1 + {}^{n+1}C_3S_n = 2^n S_n$$

We can prove this with other terms also.

**Sol 9:** (i) $(2+3x)^9, x=\frac{3}{2}$. Now we have

$$\frac{n+1}{1+\left|\frac{a}{x}\right|} = \frac{9+1}{1+\frac{2\times 2}{3\times 3}} = \frac{10}{1+\frac{4}{9}}$$

$$= \frac{10}{1+0.44} = \frac{10}{1.44} = 6.944$$

Greatest terms is

$$T_7=T_{6+1}={}^9C_6(2)^{9-6}(3x)^6= \frac{9\times 8\times 7}{1.2.3}2^3\times 3^6\left(\frac{3}{2}\right)^6$$

$$= \frac{3^2.7.3^6.3^6}{1.2.3} = \frac{7.3^{13}}{2}$$

(ii) $(3–5x)^{15}$ When $x=\frac{1}{5}$

$$\frac{n+1}{1+\left|\frac{a}{x}\right|} = \frac{15+1}{1+\left|\frac{3\times 5}{5\times 1}\right|} = \frac{16}{1+3} = \frac{16}{4} = 4$$

So $T_4$ and $T_{4+1}$ are same greatest term. $T_4={}^{15}C_4(3)^{15-4}(-5x)^4$

$$= \frac{15\times 14\times 13\times 12}{1.2.3.4}3^{11}\left(\frac{-5}{5}\right)^4 = 455.\ 3^{12}$$

**Sol 10:** (i) $(1+x+x^2)^n=a_0++a_1x+.......+a_{2n}x^{2n}$

(i) at x=1

$a_0+a_1+a_2+a_3+.......+a_{2n}=(1+1+1)^n=3^n$

(ii) at x=–1 $\Rightarrow$ $[1-1+(-1)^2]^n=1$

$\Rightarrow 1=a_0-a_1+a_2-a_3+.......+a_{2n}$

(iii) $(1+x+x^2)^n(x^2-x+1)^{2n}$

$=(a_0x^{2n}-a_1x^{2n-1}+.......)(a_0-a_1x+.....)$

Compare $x\to -x$ in

$(x^2+x+1) \to (x^2-x+1)\ a_0 - a_1x + a_2x^2 - a_3x^3 + ..... + a_{2x}\ x^{2x}$

$[(1+x+x^2)(x^2-x+1)]^n$

$= a_0^2x^{2n} - a_1^2x^{2n} - a_3\ x^{2n} + a_4\cdot x^{2n} - ..... + a_{2n}\ x^{2n}$

$\therefore$ For x = 1

$a_0^2 - a_1^2 - a_3^2 + .... + a_{2n}^2 = 3^n$

**Sol 11:** $(5+3x)^{10}$

$T_4={}^{10}C_3(5)^{10-3}(3x)^3$

$={}^{10}C_35^73^3x^3\ {}^{10}C_35^73^3x^3$ is the greatest term

So, $\frac{n+1}{1+\left|\frac{a}{x}\right|} = \frac{10+1}{1+\left|\frac{5}{3x}\right|}$

For greatest term to be $T_4$

$= 3 < \frac{10+1}{1+\frac{5}{3x}} < 4$

$3 < \frac{33x}{3x+5} < 4$

$3(3x+5)<33x<4(3x+5)$

$9x+15<33x<12x+20$

Solving each inequality separately we get

$9x + 15 < 33x$

$\Rightarrow 24x > 15$

$\Rightarrow x > \frac{15}{24}$

$\Rightarrow x > \frac{5}{8}$

Also, $12x + 20 > 334$

$\Rightarrow x < \frac{20}{21}$

$\therefore \frac{5}{8} < x < \frac{20}{21}$

**Sol 12:** In the expansion of $\left(\frac{x}{5}+\frac{2}{5}\right)^n$, we have

$$T_9 = {}^nC_8\left(\frac{x}{5}\right)^{n-8}\left(\frac{2}{5}\right)^8$$

Coefficient = ${}^nC_8(5)^{8-n}2^8 5^{-8}$ = ${}^nC_8 5^{-n}2^8$

Which is greatest coefficient

$$8 < \frac{n+1}{1+\left|\frac{x}{a}\right|} < 9$$ assume x=1 for find

$$8 < \frac{n+1}{1+\frac{1\times5}{5\times2}} < 9$$ greatest coefficient

$$8 < \frac{n+1}{1+\frac{1}{2}} < 9 = 8 < \frac{n+1}{\frac{3}{2}} < 9$$

$$8\times\frac{3}{2} < n+1 < 9\times\frac{3}{2}$$

$$12 < n+1 < \frac{27}{2}$$

$$17 < n < \frac{22}{2} - 1 = \frac{25}{2} = 12.5$$

$11 < n < 12.5$

There is only one natural no. in region i.e., 12

**Sol 13:** N=${}^{2000}C_1$+2. ${}^{2000}C_2$+3. ${}^{2000}C_3$+......+2000${}^{2000}C_{2000}$

N=n. $2^{n-1}$ here n = 2000

$\Rightarrow$ N= 2000 x$2^{n-1}$

$\Rightarrow N = 2\times(2\times5)^3\times2^{n-1} = 2^3\times5^3 2^n$

$\Rightarrow$ N=2×(2×5)$^3$×$2^{n-1}$=$2^3$×$5^3$$2^n$

N = $2^{n+3}5^3$

Number of divisors = (n + 3 + 1)(3 + 1)

= (2000 + 4)(4) = 2004 × 4 = 8016

**Sol 14:** $(1+x)^{2012}+(1+x^2)^{2011}+(1+x^3)^{2010}$

Number of different dissimilar terms

= 2012+2011+2010 (no. of terms which is common in $(1+x)^{2012}$ and $(1+x^2)^{2011}$

of terms which is similar $(1+x)^{2012}$ and $(1+x^3)^{2010}$)

(no. of terms which is similar in $(1+x^2)^{2011}+(1+x^3)^{2010}$)

$$= 2012+2011+2010-\left[\frac{2011}{2}\right]-\left[\frac{2012+1}{3}\right]-\left[\frac{1005}{3}\right]+1$$

Where (+1) for constant term. And [x] is a singularity function. [1. 35] = 1

= 6034 – 1005 – 671 -335 = 4023

**Sol 15:** $(1+x+2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

For independent terms

Coefficient of $x^0$ in $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$ = $A_0$

Coefficient of $x^{-1}$ in $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$ = $A_1$

Coefficient of $x^{-3}$ in $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$ = $A_2$

$$T_{r+1} = {}^9C_r\left(\frac{3x^2}{2}\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

Power of x = 2(9–r) –r = 18 – 2r – r = 18 – 3r

$$x^0 \Rightarrow 18-3r=0 \Rightarrow r=\frac{18}{3}=6$$

$$T_6 = T_{5+1} = A_0 = {}^9C_6\left(\frac{3}{2}\right)^{9-5}\left(-\frac{1}{3}\right)^6$$

$$A_0 = \frac{7}{18}$$

For $x^{-1}$ = $18-3r=-1$

$3r = 18+1 = 19$

r=19/3 not natural no.

So $A_1 = 0$

For $x^{-3}$

$$18-3r=-3 \Rightarrow 3r=18+3=21 \Rightarrow r=\frac{21}{3}=7$$

$$\text{So } A_2 = {}^9C_7\left(\frac{3}{2}\right)^{9-7}\left(-\frac{1}{3}\right)^7 = \frac{-1}{27}$$

So coefficient of $x^0$ in $(1+x+2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

$$= \frac{7}{18}+2\times\left(-\frac{1}{27}\right) = \frac{21-2(2)}{54} = \frac{17}{54}$$

**Sol 16:** $f(n)=\sum_{r-0}^{n}\sum_{k-r}^{n}{}^{k}C_r$

For $f(11)=\sum_{r-0}^{11}\sum_{k-r}^{11}{}^{k}C_r$

$= ({}^0C_0+{}^1C_0+{}^2C_0+.......+{}^{11}C_0)$

$+{}^1C_1+{}^2C_1+.........+{}^1C_1$

$+{}^2C_2+{}^3C_2+........+{}^{11}C_2$ :::: :::: ::::

${}^{10}C_{10}+{}^{11}C_{10}$

${}^{11}C_{11}$

$= {}^{11}C_0+.......+{}^{11}C_{11}\ \ +{}^{10}C_0+......+{}^{10}C_{10}$ ::

$+{}^1C_1\ \ +{}^0C_0$

$= 2^{11}+2^{10}+2^9+.......+2^1+2^0$

$=\dfrac{2^{11+1}-1}{2-1}=4095$ = 4095 = $5^1.3^2\ 7^1.\ 13^1$

No. of divisors = (1+1). (2+1). (1+1)(1+1)

= 2×3×4 = 24

**Sol 17:** $\sum_{j-0}^{11}\sum_{i-j}^{11}{}^{i}C_j$

$={}^0C_0+\left({}^1C_0+{}^1C_1\right)+\left({}^2C_0+....+{}^2C_2\right)$

$+\left({}^3C_0+....+{}^3C_3\right)+.....+\left({}^{11}C_0+{}^{11}C_1+....+{}^{11}C_{11}\right)$

$=2^0+2^1+....+2^{11}$

$=2^{12}-1$

**Sol 18:** $(1+x^2).(1+x)^n=\sum_{k=0}^{n+4}a_k.x^k$

$a_1$, $a_2$ and $a_3$ are in A.P

$(1+x^4+2x^2)({}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+......{}^nC_nx^n)$

$= a_0+a_1x+a_2x^2+a_3x^3+....$

Compare terms of $x^r$ in both side

$x^1\Rightarrow$ L. H. S. = ${}^nC_1=n$

R. H. S. = $a_1$

$\Rightarrow n=a_1$ .... (i)

$x^2\Rightarrow 2{}^nC_0+{}^nC_2=a_2$

$2+\dfrac{n(n-1)}{2}=a_2$ ....... (ii)

$x^3\Rightarrow\ 2\,{}^nC_1+{}^nC_3=a_3$

$2n+\dfrac{n(n-1)(n-2)}{1.2.3}=a_3$ .... (iii)

It's given that $a_1$, $a_2$, $a_3$ are in A.P

$2a_2 = a_1 + a_3$

$4 + n(n-1) = n + 2 + \dfrac{n(n-1)\ (n-2)}{6}$

$=\dfrac{6n+12n+n(n-1)\ (n-2)}{6}$

$24 + 6n(n-1) = 18n + n(n-1)\ (n-2)$

Solving this we get, n = 2 or 3 or 4

**Sol 19:** $\sum_{k=0}^{n}{}^nC_k$ sin kx. cos (n-k) x = $2^{n-1}$ sin nx

L. H. S. = $\sum_{k=0}^{n}{}^nC_k\ \ \sin kx\cos(n-k)x$

We know that 2 sin A cos B = sin(A+B) +sin (A–B)

$\because$ A + B = kx + (n – k)x = kx + nx – kx = nx

A – B = kx – (n – k )k = kx – nx + kx = 2kx – nx

So, $\sum_{k=0}^{n}\dfrac{1}{2}{}^nC_k[\sin nx+\sin(2kx-nx)]$

$=\sum_{k=0}^{n}\dfrac{1}{2}{}^nC_k\sin nx+\sum_{k=0}^{n}\dfrac{1}{2}{}^nC_k\sin(2kx-nx)$

$=\dfrac{1}{2}\sin nx\ \ \sum_{k=0}^{n}{}^nC_k+\dfrac{1}{2}$

$[{}^nC_0\sin(-nx)+......+{}^nC_n\sin(nx)]$

$=\dfrac{1}{2}\sin nx.2^n+0=(\sin nx)2^{n-1}=2^{n-1}\sin nx$

**Sol 20:** $x^{2001}+\left(\dfrac{1}{2}-x\right)^{2001}=0$

$= x^{2001}+\left[{}^{2001}C_0\left(\dfrac{1}{2}\right)^{2001}+.....+{}^{2001}C_{1999}\right.$

$\left.\left(\dfrac{1}{2}\right)^2(-x)^{1999}+{}^{2001}C_{2000}\left(\dfrac{1}{2}\right)(-x)^{2000}\right]$

$= x\ \ +{}^{2001}C_{2001}\left(\dfrac{1}{2}\right)^0(-x)^{2001}$

$= x^{2001} + ...... + {}^{2001}C_{1999}\dfrac{(-x)^{1999}}{4}$

$+ {}^{2001}C_{2000}\dfrac{(-x)^{2000}}{2} - x^{2001}$

= Now maximum pointer of x = 2000

Sum of all solution is $= \dfrac{\text{Coefficient of } x^{2001-1}}{\text{Coefficient of } x^{2000}}$

$= \dfrac{{}^{2001}C_{1999} \times \frac{1}{4}}{{}^{2001}C_{2000} \times \frac{1}{2}} = \dfrac{2001 \times 2000 \times 1}{1.2 \times 2001 \times 1} \times \dfrac{1}{2} = 500$

**Sol 21:** Let

$S = \sum_{k=0}^{2n} (-1)^k \ (k - 2n)\ ({}^{2n}C_k)^2$ ...(i)

$\Rightarrow \quad S = \sum_{k=0}^{2n} (-1)^{2n-k}\ (2n - k)\ ({}^{2n}C_{2n-k})^2$

Writing the terms in S in the reverse order, we get

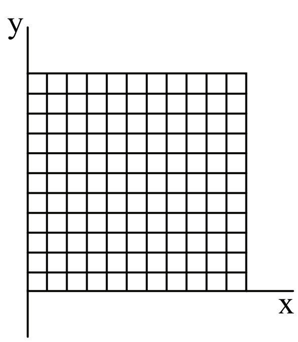
$S = \sum_{k=0}^{2n} (-1)^k\ k\ ({}^{2n}C_k)^2$ ...(ii)

Adding (i) and (ii) we get

$2S = 2n \sum_{k=0}^{2n} (-1)^k\ ({}^{2n}C_k)^2 = -2nA$

$\Rightarrow \quad S = -nA$

**Sol 22: (A)** $\left(\sum_{i=0}^{10} P(i, 10 - i)\right.$




$P(0,10) + P(1,9) + ...... + P(10,0)$

$= 1 + (9 + 1)(1) + \dfrac{10 \times 9}{2} + ....$

$= {}^{10}C_0 - {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + ...... + {}^{10}C_{10}$

$= 2^{10} = 1024$

**Sol 23: (C)** P(i,100–i) = P(j,100–j)

${}^{100}C_i = {}^{100}C_j$ and $i \neq j$

${}^{100}C_i = {}^{100}C_{100-j}$

I = 100 – j

i+j = 100

$i, j \in N, 0$

(0,100)(1,99)....... (99,1)(100,1)

Total no. of ordered pairs

(i,j) = 100

**Sol 24:** $(6\sqrt{6} + 14)^{2n+1} = I + F$

(assume)

$(6\sqrt{6} - 14) = \dfrac{(6\sqrt{6}^{\,2}) - (14)^2}{6\sqrt{6} + 14}$

$= \dfrac{20}{6\sqrt{6} + 14}$

$I = [(6\sqrt{6} + 14)^{2n+1}] = 0 < F < 1$

$e = (6\sqrt{6} - 14)^{2n+1} = 0 < e < 1$

$I + F - e = (6\sqrt{6} + 14)^{2n+1}$

$- (6\sqrt{6} - 14)^{2n+1}$

$= 2({}^nC_1 (6\sqrt{6})^{2n+1} 14$

$+ {}^nC_3 (6\sqrt{6})^{2n-11-3} 14^3 + ......)$

= 2K (K is const. integer)

$0 \le F - e < 1$

F – e=2K–I= Integer

$F - e = 0 = e = F = F = e = (6\sqrt{6} - 14)^{2n+1}$

$F = \dfrac{(20)^{2n+1}}{(6\sqrt{6} + 14)^{2n+1}}$

$(I + F)F = (6\sqrt{6} + 14)^{2n+1}$

$\dfrac{20^{2n+1}}{(6\sqrt{6} + 14)^{2n+1}}$

$(I + F)F = 20^{2n+1}$

**Sol 25:** $P = (2+\sqrt{3})^5$

$f = P - [P]$

$$2-\sqrt{3} = \frac{2^2-(\sqrt{3})^2}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}}$$

$\because 0 < 2-\sqrt{3} < 1$

$\because 0 < f < 1$

$$(2+\sqrt{3})^5 = \left(\frac{1}{2-\sqrt{3}}\right)^5 = \frac{1}{F}$$

$[P] + f + f = (2+\sqrt{3})^5 + (2-\sqrt{3})^5$

$= 2[{}^5C_0 2^5 + {}^5C_2 2^{5-2}(\sqrt{3})^2$

$+ \ldots\ldots + {}^5C_4 2^{5-4}(\sqrt{3})^4]$

$$2\left[2^5 + \frac{5\times 4}{2}\times 2^3 \times 3 + 5\times 2\times 3^2\right]$$

f+f = Integer

$0 \le f+f < 2$

$\therefore$ f+f = 1

$f = 1 - f = (2-\sqrt{3})^5$

$f = 1-(2\sqrt{3})^5$

$$\frac{f^2}{1-f} = \frac{f^2-1^2+1}{1-f} = \frac{(f-1)^{-1}(f+1)}{(1-f)} + \frac{1}{(1-f)}$$

$$= -(f+1) + \frac{1}{f}$$

$f = 1 - f = f - 2 = -(f+1)$

$$= f - 2 + \frac{1}{f}$$

$= (2+\sqrt{3})^5 - 2 + (2-\sqrt{3})^5$

$= 2[32+15\times 2^4+5\times 2\times 3^2]-2$

= 724 – 2 = 722

**Sol 26:** $(1+x)^n = {}^nC_0 + C_1 +$

$C_2x^2 + C_3x^3 + \ldots . + C_nx^n$ ... (i)

(a) Differentiating at both sides

$n(1+x)^{n-1} = C_1 + 2C_2x + \ldots + {}^3C_3x^{2n}$

$x = 1$

Put $n\cdot 2^{n-1} = C_1 + 2C_2 + \ldots\ldots + nC_n$ ... (ii)

(b) Sum of eq. (i) and (ii)

$n2^{n-1} + (1+x)^n = C_1 + {}^2C_2 + \ldots . + {}^nC_n$

$+ C_0 + C_1x + C_2x^2 + \ldots . + C_nx^n$

At $x = 1$

$2^{n-1}(n+2) = C_0 + 2C_1$

$+ 3C_2 + \ldots\ldots + (n+1)C_n$

(c) Eq. (1) + 2 × eq. (2)

At $x = 1$

$2^n + n2^n = C_0 + C_1(1\times 2+1)$

$+ C_2(2\times 2+1) + \ldots\ldots + C_n(2n+1)$

$2^{n-1}(n+2) = C_0 + 2C_1$

$+ 7C_3 + \ldots\ldots + (2n+1)C_n$

(d) $(C_0+C_1)(C_1+C_2)\ldots\ldots(C_{n-1}+C_n) =$

$$\frac{C_0C_1C_2.C_{n-1}(n+1)^{3n}}{n!}$$

Multiply and divide L. H. S. by

$C_0C_1C_2C_3\ldots\ldots C_{n-1}$

$$= C_0C_1C_2\ldots\ldots C_{n-1}\left(1+\frac{C_1}{C_0}\right)\ldots\ldots\left(1+\frac{C_n}{C_{n-1}}\right)$$

On using $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$ = L. H. S.

$$C_0C_1C_2\ldots C_{n-1}(1+n)\left(\frac{1+n}{2}\right)\left(\frac{1+n}{3}\right)\ldots\left(\frac{1+n}{n}\right)$$

$$= \frac{C_0C_1C_2\ldots C_{n-1}(n+1)^n}{n!}$$

(e) $1C_0^2 + 3C_1^2 + 5.C_2^2 + \ldots . +$

$$(2n+1)C_n^2 = \frac{(n+1)(2n)!}{n!n!}$$

We know that (part (C))

$C_0 + 3C_1 + 5C_2 + \ldots\ldots +$

$(2n+1)C_n + (n+1)2^n$

$C_0 + 3C_1 + 5C_2x^2 + \ldots\ldots$

$C_0 + {}^3C_1 + {}^5C_2x^2 + \ldots\ldots$

$+(2n+1)C_n x^n$

$= (n+1)(1+x)^n = (n+1)(1+x)^n$

Multiply with

$C_0x^n + C_1x^{n-1} + ...... + C_n = (x+1)^n$

= and compare $x^n$ and coefficient

$C_0 + 3C_1^2 + 5C_2^2 + ...... + (2n+1)C_n^2$

= Coefficient of $x^n$ in $(n+1)\ (1+x)^{n+n}$

$= (n+1)^{2n}C_n =$

L. H. S. = R. H. S.

**Sol 27:** $I = [(3\sqrt{5})^n]$

$I + F = (3+\sqrt{5})^n$

P = rational part

$\sigma$ = irrational part

$3-\sqrt{5} = \frac{9-(\sqrt{5})^2}{3+\sqrt{5}} = \frac{4}{3+\sqrt{5}}$

$0 < 3-\sqrt{5} < 1$

$F = (3-\sqrt{5})^n$

$I + F + F = (3+\sqrt{5})^n + (3-\sqrt{5})^n$

$= 2\left({}^nC_0 3^n + {}^nC_2 3^{n-2}(\sqrt{5})^2 + .....\right)$

Rational part

0<F+F<2

F+F is 1 only integer between0and 2

I + 1 = 2F = $P = \frac{1}{2}(I+1)$

$I + F - F = (3+\sqrt{5})^n - (3-\sqrt{5})^n$

$I + F + F - F - F = 2({}^nC_1 3^{n-1}$

$(\sqrt{5})^1 + {}^nC_3 3^{n-3}(\sqrt{5})^3 + ........)$

$I + 2F - (F+F) = 2\sigma$

$I + 2F - 1 = 2\sigma$

$\sigma = \frac{1}{2}(I + 2F - 1)$

**Sol 28:**

(a) $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ...... + \frac{nC_n}{C_{n-1}} = \frac{n(n+1)}{2}$

We know that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

$\Rightarrow \frac{r\,{}^nC_r}{{}^nC_{r-1}} = (n-r+1)$

L. H. S. = $(n-1+1) + (n-2+1)$

$+(n-3+1) + .......(n-n+1)$

$n^2 + n - (1+2+3+....n)$

$= n^2 + n - \frac{n(n+1)}{2}$

$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

(b) $2C_0 + \frac{2^2C_1}{2} + \frac{2^3C_2}{3}$

$+\frac{2^{n+1}C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(c) In equation (i) from above que.

$x = 2$

$2C_0 + \frac{2^2C_1}{2} + ....... + \frac{2^{n+1}C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(d) In eq. (i) $x = -1$

$\frac{(0)^{n+1}-1}{n+1} = C_0(-1) + \frac{C_1}{2} - \frac{C_2}{3}$

$+...... + (-1)^{n+1}\frac{C_n}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} + .......$

$+(-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

**Sol 29:** (a) In equation (ii) compare coefficient of $x^{n-1}$

${}^{2n}C_{n-1} = C_0C_1 + C_1C_2 + ....... + C_{n-1}C_n$

${}^{2n}C_{n-1} = \frac{2n!}{(n-1)!(n+1)!}$

$\because$ 2n–(n–1) = n+1

L. H. S. = R. H. S.

(b) In some equ. (ii) compare coefficient of $x^{n-r}$

$$^{2n}C_{n-r} = C_0C_r + ........ + C_{n-r}C_n \quad .... (ii)$$

$$^{2n}C_{n-r} = \frac{2n!}{(n+r)!(n-r)!}$$

L. H. S. = R. H. S.

(c) $\sum_{r=0}^{n-2} (^nC_r \, ^nC_{r+2}) = \frac{2n!}{(n-2)!(n+2)!}$

In equ. (iii) if $r = 2$

$$= \; ^{2n}C_{n-2} = C_0C_2 + C_1C_3 + ...... + C_{n-2}C_n$$

$$= \; ^{2n}C_{n-2} = \frac{2n!}{(n-2)!(n+2)!} \quad ....(iii)$$

L. H. S. = R. H. S.

(d) $^{100}C_{10} + 5.\,^{100}C_{11} + 10.\,^{100}C_{12}$

$+10.\,^{100}C_{13} + 5.\,^{100}C_{14} + ^{100}C_{18}$

$$= \; ^{105}C_{90} = ^{105}C_{105-90} = ^{105}C_{15} = \frac{105!}{90!15!}$$

$$\text{L. H. S.} = \frac{100!}{90!10!} + \frac{5100!}{11!89!} + \frac{10\times 100!}{12!88!}$$

$$+\frac{10\times 1001}{13!87!} + \frac{5\times 100!}{14!80!} + \frac{100!}{15!85!}$$

$$100!\left[\frac{15\times 14\times 13\times 12\times 11}{90!15!} + \right.$$

$$\frac{5\times 90\times 15\times 14\times 13\times 12}{15!90} + \frac{10\times 90\times 89\times 15\times 14\times 13}{15!90!}$$

$$+ \frac{10\times 15\times 14\times 90\times 89\times 88}{70!15!} + \frac{5.90\times 89.8887}{15!90!}$$

$$\left. + \frac{90.89.88.87.86}{90!15!}\right]$$

$$= \frac{100\times 101\times 102\times 103\times 104\times 105}{90!15!} = \frac{105!}{90!15!}$$

$$= ^{105}C_{15} = ^{105}C_{90}$$

**Sol 30:** (i) $(1+x+x^2)^n = a_0+a_1x + a_2x^2+......+a_{2n}x^{2n}$

$(1+x+x^2)^n = (x^2+x+1)^n$

So $a_0 = a_{2n}$

$a_1 = a_{2n-1}$

$a_{n-1} = a_{n+1}$

So, $a_0a_1+a_2a_3+a_4a_5+.... +$

$= a_{2n}a_{2n-1} + a_{2n-2}a_{2n-3}$

$+....... + a_1a_2+a_3a_4$

$a_0a_1-a_1a_2+a_2a_3......=0$

(ii) $(1-x+x^2)^n$

$= a_0-a_1x+a_2x^2-a_3x^3 +....$

$(1+x+x^2)^n(1-x+x^2)^n = (a_0 - a_1x + ...)$

$(a_0x^{2n} + a_1x^{2n-1} + ......)$

$(1+x^2+x^4)^n = a_0a_2x^{2n-2} - a_1a_3x^{2n-2} + ....$

Compare $x^{2n-2}$ coefficient

$a_{n+1} = a_{n-1} = a_0a_2 - a_1a_3 + ...$

$(\because$ in $(1+x+x^2)^n \;\; x \to x^2$

So Coefficient of $x^{2(n-1)} = a_{n-1} = a_{n+1})$

(iii) $(1+x+x^2)^n = a_0 + a_1x + ..... + a_{2n}x^{2n}$

Put x = 1

$3^n = a_a + a_1 + a_2 + .... + a_{2n}$ .... 1

$x = \omega$

$0 = a_0 + a_1\omega + a_2\omega^2 + a_3 + ....a_{2n}\omega^{2n}$ . . 2

$x = \omega^2$

$0 = a_0 + a_1\omega^2 + a_2\omega + .... + a_{2n}\omega^{4n}$ ...3

$A + B + C =$

$3^n = 3(a_0 + a_3 + a_6 + ....)$

$a_0 + a_3 + a_6 + ......3^{n-1}$ ... (i)

$x(1+x+x^2)^n = a_0x + a_1x^2 + ...... + a_{2n}x^{2n+1}$ .... A

$x = \omega = \; 0 = a_0\omega + a_1\omega^2 + a_2 + ..... + a_{2n}\omega^{2n+1}$ ...B

$x = \omega^2 = \; 0 = a_0\omega^2 + a_1\omega + a_2 + ..... + a_{2n}\omega^{4n+2}$ . . C

A,B,C,= $3^n = 3(a_2 + a_5 + a_8 + .....)$

$a_2 + a_5 + a_8 = 3^{n-1}$ ... (ii)

Sum as above

$x^2(1+x+x^2)^n = a_0x^2$

$x^2(1+x+x^2)^n = a_0x^2 + a_1x^3 + ..... + a_2nx^{2n+1}$

x = w =

$0 = a_0\omega^2 + a_1\omega^3 + ..... + a_{2n}\omega^{2n+1}$ .... $B_2$

$x = \omega^2 = 0 = a_0\omega^4 + a_1 + ..... + a_{2n}$ .... $C_2$

$A + B_2 + C_2$ $(a_1 + a_4 + a_7 + .....)$

$= \frac{3^n}{3} = 3^{n-1}$ ... (iii)

From (i), (ii) and (iii)

$E = E_2 = E_3 = 3^{n-1}$

**Sol 31:** $\sum_{r=0}^{100}\sum_{s=0}^{100}(C_r^2 + C_s^2 + C_rC_s) = m({}^{2n}C_n) + 2P$

M, n and p are even natural number

$(1+x)^{100}$

$C_r$ = coefficients of $x^r$ in $(1+x)^{100}$

$= \sum_{r=0}^{100}[C_r^2 \times 101 + C_0^2 + C_1^2 + ..... + C_{100}^2$

$+C_r(C_0 + C_1 + ..... + C_{100})]$

$= 101\left(\sum C_r^2\right) + 101({}^{2n}C_n) + \sum Cr(2^n)$

$= 101\ {}^{2n}C_n + 101\ {}^{2n}C_n + 2^n(2^n)$

$= 202\ {}^{2n}C_n + 2^{100+100}$

$= 202\ {}^{2n}C_n + 2^{200} = m({}^{2n}C_n) + 2^P$

n=100, m=202, P=200

Hence, n+m+p = 200+100+202=502

**Sol 32:** $(1+x)(1+x+x^2)......(1+x+x^2+.... . +x^n)$

Max. power of x

$= 1+2+3+....... n = \frac{n(n+1)}{2}$

(a) Total terms = $\frac{1+n(n+1)}{2} = \frac{n^2+n+2}{2}$

(b) 1+x=x+1

$1+x+x^2 = x^2+x+7$, So now product is

$= (x+1)\ (x^1+x+1) = .......$

$(x^n + x^{n-1} + ..... + x^2 + x + 1)$

So, $a_0 = \frac{a_{n(n+1)}}{2}$

Or if $x = \frac{1}{y}$

$(y^-)\frac{n(n+1)}{2}$

$(y+1)(y^2+y+1).... . (y^n+y^2+y+1)$

$= a_0y\frac{n(n+1)}{2} + ....... + a\frac{n(n+1)}{2}$

$a_0 = a\frac{n(n+1)}{2}$

(c) Odd coefficient = $a_1+a_3+a_5+......$

At $x = 1$

2. 3. 4. . . .... . (n+1) = (n+1)!

$= a_0a_1 + a_2 + ....... + a_{\frac{n(n+1)}{2}}$

$x = -1 = a_0-a_1+a_2-a_3=0$

$= a_0 + a_2 + a_4 + ..... = a_1 + a_3 + a_5 + ......$

Assume P = Q

$P + Q = 2P = 2Q = (n+i)! = P = Q = \frac{(n+1)!}{2}$

**Sol 33:** $S_1 = \sum_{0\le i}< \sum_{j\le 100} C_iC_j.$

$S_2 = \sum_{0\le j}< \sum_{i\le 100} C_iC_j$

$S_3 = \sum_{0\le i}= \sum_{j\le 100} C_iC_j$

$(1+x)^{100} \Rightarrow n = 100$

$S_1 + S_2 + S_3 = a^b$

$S_1 + S_2 + S_3 =$

$\sum_{0\le i}< \sum_{j\le 100} C_iC_j + \sum_{0\le j}< \sum_{i\le 100} C_iC_j + \sum_{0\le i}\sum_{j\le 100} C_iC_j$

$S_1 = S_2 \because \sum_{0\le i}\sum_{j\le 100} C_iC_j < \sum_{0\le i}\sum_{j\le 100} C_jC$

$S_3 = S_1 + C_0C_0 + C_1C_1 + C_2^2 + ...... + C_{100}^2$

$S_3 = S_1 + {}^{2n}C_n$

$S_1 + S_2 + S_3 = 2S_1 + 2^nC_n$

$= [C_0(C_1 + C_2 + ...... + C_{100}) + C_1(C_2 + ........ + C_{100}) + ....... ]$

$= \ {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + ....... + {}^{2n}C_{2n}$

When $C_1 + C_2 + ...... + C_{100} = 2^n$

$= 2^{2n} = 2^{200} = 4^{100} = 16^{50} = a^b$

$a + b = 16 + 50 = 66$

## Exercise 2

**Sol 1: (B)** Given binomial is $(2^{1/3}+3^{-1/3})^n$

$\therefore T_7 = T_{6+1} = {}^nC_6(2^{1/3})^{n-6}(3^{-1/3})^6$

$T_7'$ from end $= {}^nC_{n-6}(3^{-1/3})^{n-6}(2^{1/3})^6$

$$\Rightarrow \frac{T_7}{T_7'} = \frac{1}{6} = \frac{{}^nC_6\, 2^{n/3}2^{-2}3^{-2}}{{}^nC_6\, 3^{-n/3}3^2 2^2} = \frac{(2.3)^{n/3}}{(6)^{2+2}} = (6)^{(n/3)-4}$$

$$6^{(n/3)-4} = \frac{1}{6} \Rightarrow \frac{n}{3} - 4 = -1 \Rightarrow n=9$$

**Sol 2: (C)** We have $15^{23}+23^{23} = (19-4)^{23}+(19+4)^{23}$

$$= 2\left[{}^{23}C_0 19^{23} + {}^{23}C_2 19^{21} + ... + {}^{23}C_{22}19\right]$$

= 2. 19K always divisible by 19

So the remainder is zero

**Sol 3: (D)** $4\{{}^nC_1 + 4\,{}^nC_2 + 4^2.{}^nC_3 + ...... + 4^{n-1}\}$

$= \{4\,{}^nC_1 + 4^2\,{}^nC_2 + 4^3\,{}^nC_3 + ..... + 4^n\,{}^nC_n\}$

$= (1+x)^n = C_0 + C_1x + C_2x^2 + ..... + x^n\,{}^nC_n$

At $x = 4$

$5^n = 1 + 4C_1 + 4\,{}^2C_2 + ...$

So ${}^{4n}C_1 + 4^2\,{}^nC_2 + 4^3\,{}^nC_3 + ..... + 4^n\,{}^nC_n = 5^n - 1$

**Sol 4: (A)** $n \ge 3$

$$n - \frac{(n-1)}{1!}(n-1) + \frac{(n-1)(n-2)}{2!}(n-2)$$

$$-\frac{(n-1)(2-n)(n-3)}{3!}(n-3) + ......$$

At $n = 3$

$$= \frac{1.3-(3-1)}{1}(3-1) + \frac{(3-1)(3-2)}{2!}(3-2) - 0$$

$$= 3 - 2\times 2 + \frac{2\times 1}{2!} = 3 + 1 - 4 = 0$$

Or

$= n - {}^{n-1}C_1(n-1) + {}^{n-1}C_2(n-2) + ......$

$+3\,{}^{n-1}C_{n-3}(-1)^{n-3} + 2\,{}^{n-1}C_{n-2}(-1)^{n-2}$

$= n - {}^{n-1}C_0 + 2\,{}^{n-1}C_1 + 3\,{}^{n-1}C_2 + .......$

$+(-1)^{n-1}\,{}^{n-1}C_{n-1}$

$= n - ({}^{n-1}C_1 + {}^{n-1}C_0) + 0 = n - (n-1+1) = 0$

**Sol 5: (C)** $t_6$ in $\left[x^{-8/3} + x^2\log_{10}^{x}\right]^8 = 5600$

$${}^8C_5\left(x^{-8/3}\right)^3\left(x^2\log_{10}^{x}\right) = 5600$$

$$\Rightarrow x^2\left(\log_{10}^{x}\right)^5 = 100$$

$$\Rightarrow x = 10$$

**Sol 6: (B)** $(\alpha+p)^{m-1} + (\alpha+p)^{m-2}(\alpha+q)$

$+(\alpha+p)^3(\alpha+q)^2 + ...... + (\alpha+q)^{m-1}$

Coefficient of $^t$

$$= (\alpha+p)^{m-1}\left[1 + \left(\frac{\alpha+q}{\alpha+p}\right) + ...... + \left(\frac{\alpha+q}{\alpha+p}\right)^{m-1}\right]$$

$$= (\alpha+p)^{m-1}\left[\frac{1-\left(\frac{\alpha+q}{\alpha+p}\right)^m}{1-\left(\frac{\alpha+q}{\alpha+p}\right)}\right]$$

$$= (\alpha+p)^{m-1}\frac{\left[1-\frac{(\alpha+q)}{\alpha+p}\right]}{\alpha+p-\alpha-q}(\alpha+p)$$

$$= (\alpha+p)^m\left[\frac{1-\left(\frac{\alpha+q}{\alpha+p}\right)^m}{p-q}\right] = \left[\frac{(\alpha+p)^m - (\alpha+q)^m}{p-q}\right]$$

$$\text{Coefficient of } \alpha^t = \frac{{}^mC_t[p^{m-t} - q^{m-t}]}{p-q}$$

**Sol 7: (B)** $\left(1 + x - 3x^2\right)^{2145} = a_0 + a_1x + a_2x^2 + ...$

Put x = - 1

$\Rightarrow (1-1-3)^{2145} = a_0 - a_1 + a_2 - a_3 + ....$

$\Rightarrow a_0 - a_1 + a_2 - a_3 + ....(-3)^{2145}$

Last digit of $(-3)^{2145}$ is 3.

**Sol 8: (B)** $\left(\frac{4x^2}{3} - \frac{3}{2x}\right)^9$

$T_{r+1} = {}^9C_r \left(\frac{4x^2}{3}\right)^{9-r} \left(-\frac{3}{2x}\right)^r$

Power of $x = 2(9-r) + (-1)r$

$\Rightarrow 18 - 2r - r = 18 - 3r = 6$ (given)

$\Rightarrow 3r = 18 - 6 = 12 \Rightarrow r = 12/3 = 4$

Coefficient ${}^9C_4 \left(\frac{4}{3}\right)^{9-4} \left(\frac{-3}{2}\right)^4 = \frac{9\times 8\times 7\times 6}{1.2.3.4}\left(\frac{4}{3}\right)^5 \left(\frac{3}{2}\right)^4$

$= 9\times 2\times 7\times \frac{2^{10}\times 3^4}{3^5\times 2^4} = 21\times 2^7 = 2688$

**Sol 9: (D)** $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x > 0$

$T_{r+1} = {}^{18}C_r (9x)^{18-r} \left(\frac{1}{\sqrt{9x}}\right)^r$

Power of $x = 18 - r - \frac{r}{2} = 18 - \frac{3r}{2} = 0$ (given)

$\alpha = 9^{18-r}\left(\frac{1}{\sqrt{9}}\right)^r = (9)^{18-r-\frac{r}{2}} = (9)^{18-\frac{3r}{2}} = 9^0 = 1$

**Sol 10: (C)** $\left[x + \sqrt{x^3-1}\right]^5 + \left[x - \sqrt{x^3-1}\right]^5$

$= 2\left[{}^5C_0 x^5 + {}^5C_2 x^3 (x^3-1) + {}^5C_4 x (x^3-1)^2\right]$

$\Rightarrow$ Highest power is 7.

**Sol 11: (C)** We have

$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = \sum_{r=0}^{n} ({}^nC_r)({}^nC_r)$

$= \sum_{r=0}^{n} ({}^nC_r)({}^nC_{n-r}) \quad [\because {}^nC_r = nC_{n-r}]$

= Number of ways of choosing n persons out of n men and n women

= Number of ways of choosing n person out of 2n persons

$= {}^{2n}C_n$

**Sol 12: (D)** $aC_0 + (a+b)C_1 + .... + (a+nb)Cn$

$= [C_0 + C_1 + .... + C_n]$

$+ b[0\times C_0 + 1\times C_1 + 2\times C_2 + .... + nC_n]$

$= a2^n + bn2^{n-1}$

$= (2a+nb)2^{n-1}$

## Previous Years' Questions

**Sol 1:** We know, $(1 + x)^{2n} = C_0 + C_1 x + C_2 x^2 + ..... + C_{2n}x^{2n}$

On differentiating both sides w.r.t. x, we get

$2n(1+x)^{2n-1} = C_1 + 2.C_2 x$

$+3.C_3 x^2 + .... + 2nC_{2n}x^{2n-1}$ ... (i)

And

$\left(1-\frac{1}{x}\right)^{2n} = C_0 - C_1.\frac{1}{x} + C_2.\frac{1}{x^2}$

$-C_3.\frac{1}{x^3} + .... + C_{2n}.\frac{1}{x^{2n}}$ ... (ii)

On multiplying Eqs. (i) and (ii), we get

$2n(1+x)^{2n-1}\left(1-\frac{1}{x}\right)^{2n}$

$= \left[C_1 + 2.C_2 x + 3.C_3 x^2 + .... + 2n.C_{2n}x^{2n-1}\right]$

$\times\left[C_0 - C_1\left(\frac{1}{x}\right) + C_2\left(\frac{1}{x^2}\right) - .... + C_{2n}\left(\frac{1}{x^{2n}}\right)\right]$

The coefficient of $\left(\frac{1}{x}\right)$ on the LHS

= Coefficient of $\frac{1}{x}$ in $2n\left(\frac{1}{x^{2n}}\right)(1+x)^{2n-1}(x-1)^{2n}$

= Coefficient of $x^{2n-1}$ in $2n(1-x^2)^{2n-1}(1-x)$

$= 2n(-1)^{n-1}.^{(2n-1)}C_{n-1}$

$= (-1)^n (2n)\dfrac{(2n-1)!}{(n-1)!n!}$

$= -(-1)^n n\cdot\dfrac{(2n)!}{(n!)^2}\cdot n$ ... (iii)

$= -(-1)^n\, n\cdot C_n$

Again, the coefficient of $\left(\dfrac{1}{x}\right)$ on the RHS

$= -\left(C_1^2 - 2.C_2^2 + 3.C_3^2 + .... + -2nC_{2n}^2\right)$ ... (iv)

From Eqs. (iii) and (iv), we get

$C_1^2 - 2.C_2^2 + 3.C_3^2 - .... - 2n.C_{2n}^2 = (-1)^n n.C_n$

**Sol 2:** $^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2$

$+.... + {}^{n+1}C_{n+1}s_n = \sum_{r=1}^{n+1} {}^{n+1}C_r s_{r-1}$

Where $s_n = 1 + q + q^2 + .... + q^n = \dfrac{1-q^{n+1}}{1-q}$

$\therefore \sum_{r=1}^{n+1} {}^{n+1}C_r\left(\dfrac{1-q^r}{1-q}\right)$

$= \dfrac{1}{1-q}\left(\sum_{r=1}^{n+1} {}^{n+1}C_r - \sum_{r=1}^{n+1} {}^{n+1}C_r q^r\right)$

$= \dfrac{1}{1-q}\left[(1+1)^{n+1} - (1+q)^{n+1}\right]$

$= \dfrac{1}{1-q}\left[2^{n+1} - (1+q)^{n+1}\right]$ ... (i)

Also, $S_n = 1\left(\dfrac{q+1}{2}\right) + \left(\dfrac{q+1}{2}\right)^2 + ..... + \left(\dfrac{q+1}{2}\right)^n$

$= \dfrac{1-\left(\dfrac{q+1}{2}\right)^{n+1}}{1-\left(\dfrac{q+1}{2}\right)} = \dfrac{2^{n+1} - (q+1)^{n+1}}{2^n(1-q)}$ ... (ii)

From eqs. (i) and (ii), we get

$^{n+1}C_r + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + .... + {}^{n+1}C_{n+1}s_n = 2^n s_n$

**Sol 3:** $\sum_{r=0}^{n} (-1)^r\, {}^nC_r$

$\left[\dfrac{1}{2^r} + \dfrac{3^r}{2^{2r}} + \dfrac{7^r}{2^{3r}} + \dfrac{15^r}{2^{4r}} + \text{....upto m terms}\right]$

$\sum_{r=0}^{n} (-1)^r\, {}^nC_r\left(\dfrac{1}{2}\right)^r +$

$\sum_{r=0}^{n} (-1)^r\, {}^nC_r\left(\dfrac{3}{4}\right)^r + \sum_{r=0}^{n} (-1)^r\, {}^nC_r\left(\dfrac{7}{8}\right)^r + ...$

Upto m terms

$\left\{\text{using} \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^r = (1-x)^n\right\}$

$= \left(1-\dfrac{1}{2}\right)^n + \left(1-\dfrac{3}{4}\right)^n + \left(1-\dfrac{7}{8}\right)^n + .....$

Upto m terms

$= \left(\dfrac{1}{2}\right)^n + \left(\dfrac{1}{4}\right)^n + \left(\dfrac{1}{8}\right)^n + .....$

Upto m terms

$= \left(\dfrac{1}{2}\right)^n \left[\dfrac{1-\left(\dfrac{1}{2^n}\right)^m}{1-\dfrac{1}{2^n}}\right] = \dfrac{2^{mn}-1}{2^{mn}\left(2^n-1\right)}$

**Sol 4:** Let y = (x – a)$^m$, where m is a positive integer, $r \le m$,

Now, $\dfrac{dy}{dx} = m(x-a)^{m-1}$

$\Rightarrow \dfrac{d^2y}{dx^2} = m(m-1)(x-a)^{m-2}$

$\Rightarrow \dfrac{d^4y}{dx^4} = m(m-1)(m-2)(m-3)(x-a)^{m-4}$

.......................................

On differentiating r times, we get

$\dfrac{d^ry}{dx^r} = m(m-1)....(m-r+1)(x-a)^{m-r}$

$= \dfrac{m!}{(m-r)!}(x-a)^{m-r} = r!\left({}^mC_r\right)(x-a)^{m-r}$

And for r > m, $\dfrac{d^ry}{dx^r} = 0$

Now,

$$\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r \text{ (given)}$$

On differentiating both sides n times w.r.t. x, we get

$$\sum_{r=n}^{2n} a_r (n!)^r C_n (x-2)^{r-n}$$

$$= \sum_{r=n}^{2n} b_r (n!)^r C_n (x-3)^{r-n}$$

On putting x = 3, we get

$$\sum_{r=n}^{2n} a_r (n!)^r C_n = (b_n) n!$$

$$\Rightarrow b_r = \frac{1}{n!} \sum_{r=n}^{2n} a_r (n!)^r C_n$$

$$= {}^{2n+1}C_n$$

$$= {}^{2n+1}C_{n+1}$$

**Sol 5:** $\left(1+x+x^2\right)^n = a_0 + a_1 x + \dots + a_{2n} x^{2n}$ ... (i)

Replacing x by –1/x, we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n$$

$$= a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \frac{a_3}{x^3} + \dots + \frac{a_{2n}}{x^{2n}} \quad \text{... (ii)}$$

Now, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2$ = coefficient of the term independent of x in

$$\left[a_0 + a_1 x + a_2 x^2 + \dots + a_{2n} x^{2n}\right]$$

$$\times \left[a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \dots + \frac{a_{2n}}{x^{2n}}\right]$$

= Coefficient of the term independent of x in

$$\left(1+x+x^2\right)^n \left(1 - \frac{1}{x} + \frac{2}{x^2}\right)^n$$

Now, RHS $= \left(1+x+x^2\right)^n \left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^n$

$$= \frac{\left(1+x+x^2\right)^n \left(x^2-x+1\right)^n}{x^{2n}}$$

$$= \frac{\left[\left(x^2+1\right)^2 - x^2\right]^n}{x^{2n}} = \frac{(1+2x^2+x^4-x^2)^n}{x^{2n}}$$

$$= \frac{\left(1+x^2+x^4\right)^n}{x^{2n}}$$

Thus, $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2$

= Coefficient of the term independent of x in

$$\frac{1}{x^{2n}}\left(1+x^2+x^4\right)^n$$

= Coefficient of $x^{2n}$ in $\left(1+x^2+x^4\right)^n$

= Coefficient of $t^n$ in $\left(1+t+t^2\right)^n = a_n$

**Sol 6:** To show that

$$2^k.{}^nC_0.{}^nC_k - 2^{k-1}.{}^nC_1.{}^{n-1}C_{k-1} + 2^{k-2}.{}^nC_2.{}^{n-2}C_{k-2} - \dots +$$

$$(-1)^k\, {}^nC_k\, {}^{n-k}C_0 = {}^nC_k$$

Taking LHS

$$2^k.{}^nC_0.{}^nC_k - 2^{k-1}.{}^nC_1.{}^{n-1}C_{k-1} + \dots + (-1)^k.{}^nC_k = {}^nC_k$$

$$= \sum_{r=0}^{k} (-1)^r .2^{k-r}.{}^nC_r.{}^{n-r}C_{k-r}$$

$$= \sum_{r=0}^{k} (-1)^r .2^{k-r}.\frac{n!}{r!(n-r)!}.\frac{(n-r)!}{(k-r)!(n-k)!}$$

$$= \sum_{r=0}^{k} (-1)^r .2^{k-r}.\frac{n!}{(n-k)!k!}.\frac{k!}{r!(k-r)!}$$

$$= \sum_{r=0}^{k} (-1)^r .2^{k-r}.{}^nC_k.{}^kC_r$$

$$= 2^k.{}^nC_k \left\{\sum_{r=0}^{k} (-1)^r .\frac{1}{2^r}.{}^kC_r\right\}$$

$$= 2^k.{}^nC_k \left(1 - \frac{1}{2}\right)^k = {}^nC_k = \text{RHS}$$

**Sol 7:** Let $y = \sum_{r=1}^{10} A_r (B_{10} Br - C_{10} A_r)$

$\sum_{r=1}^{10} A_r B_r$ = coefficient of $x^{20}$ in $((1 + x)^{10} (x + 1)^{20}) - 1$

$= C_{20} - 1 = C_{10} - 1$ and $\sum_{r=1}^{10} \left(A_r\right)^2$ = coefficient of $x^{10}$ in $((1 + x)^{10} (x + 1)^{10}) - 1 = B_{10} - 1$

$\Rightarrow y = B_{10}(C_{10} - 1) - C_{10}(B_{10} - 1) = C_{10} - B_{10}$.

**Sol 8:** Let $T_{r-1}$, $T_r$, $T_{r+1}$ are three consecutive terms of $(1 + x)^{n+5}$

$T_{r-1} = {}^{n+5}C_{r-2}\,(x)^{r-2}$, $T_r = {}^{n+5}C_{r-1}x^{r-1}$, $T_{r+1} = {}^{n+5}C_r x^r$

Where, ${}^{n+5}C_{r-2} : {}^{n+5}C_{r-1} : {}^{n+5}C_r = 5 : 10 : 14$.

So $\dfrac{{}^{n+5}C_{r-2}}{5} = \dfrac{{}^{n+5}C_{r-1}}{10} \Rightarrow n - 3r = -3$ ... (i)

$\dfrac{{}^{n+5}C_{r-1}}{10} = \dfrac{{}^{n+5}C_r}{14} \Rightarrow 5n - 12r = -30$ ... (ii)

From equation (i) and (ii) n = 6

**Sol 9:** $2x_1 + 3x_2 + 4x_3 = 11$

Possibilities are (0, 1, 2); (1, 3, 0); (2, 1, 1); (4, 1, 0).

∴ Required coefficients

$= ({}^4C_0 \times {}^7C_1 \times {}^{12}C_2) + ({}^4C_1 \times {}^7C_3 \times {}^{12}C_0) + ({}^4C_2 \times {}^7C_1 \times {}^{12}C_1) + ({}^4C_4 \times {}^7C_1 \times 1)$

$= (1 \times 7 \times 66) + (4 \times 35 \times 1) + (6 \times 7 \times 12) + (1 \times 7)$

$= 462 + 140 + 504 + 7 = 1113$.

**Sol 10:** $x^9$ can be formed in 8 ways

ii.e. $x^9$, $x^{1+8}$, $x^{2+7}$, $x^{3+6}$, $x^{4+5}$, $x^{1+2}$ + 6, $x^{1+3+5}$, $x^{2+3+4}$ and coefficient in each case is 1.

$\Rightarrow$ Coefficient of $x^9 = \underbrace{1 + 1 + 1 + \ldots\ldots + 1}_{8\text{ times}} = 8$

**Sol 11:** $Z = \dfrac{-1 + i\sqrt{3}}{2} = \omega$

$$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$P^2 = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}\begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$= \begin{bmatrix} (-\omega)^{2r} + (\omega^{2s})^2 & \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} \\ \omega^{2s}(-\omega) + \omega^r\omega^{2s} & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{4s} + \omega^{2r} & \omega^{2s}(\omega^r + (-\omega)^r \\ \omega^{2s}(\omega^r + (-\omega)^r) & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

= – I (Given)

$\omega^{4s} + \omega^{2r} = -1$ and $\omega^{2s}(\omega^r + (-\omega)^r) = 0$

$\omega^r + (-\omega)^r = 0$

| $r$ | $s$ | | $r$ | $s$ |
|---|---|---|---|---|
| 1 | 1 | | 1 | 1 |
| 2 | 2 | | 3 | 3 |

Total no. pairs = 1