11. ELLIPSE

1. INTRODUCTION

An ellipse is defined as the locus of a point which moves such that the ratio of its distance from a fixed point (called focus) to its distance from a fixed straight line (called directrix, not passing through fixed point) is always constant and less than unity. The constant ratio is denoted by e and is known as the eccentricity of the ellipse.

Ellipse can also be defined as the locus of a point such that the sum of distances from two fixed points (foci) is constant. i.e. SP + S'P = constant where S_1S' are foci (two fixed points), P being a point on it. It has a lot of applications in various fields. One of the most commonly known applications is Kepler's first law of planetary motion, which says that the path of each planet is an ellipse with the sun at one focus.

Illustration 1: Find the equation of the ellipse whose focus is (1, 0) and the directrix x + y + 1 = 0 and eccentricity is equal to $\frac{1}{\sqrt{2}}$. **(JEE MAIN)**

Sol: Using the definition of ellipse we can easily get the equation of ellipse.

Let S (1, 0) be the focus and ZZ' be the directrix. Let P(x, y) be any point on the ellipse and PM be the perpendicular drawn from P on the directrix. Then by definition

SP = e. PM, where e = $\frac{1}{\sqrt{2}}$.

$\Rightarrow SP^2 = e^2PM^2 \qquad \Rightarrow (x-1)^2 + (y-0)^2 = \frac{1}{2}\left\{\frac{x+y+1}{\sqrt{1+1}}\right\}^2$

$\Rightarrow 4\{(x-1)^2 + y^2\} = (x+y+1)^2$

$\Rightarrow 4x^2 + 4y^2 - 8x + 4 = x^2 + y^2 + 1 + 2xy + 2x + 2y$

$\Rightarrow 3x^2 + 3y^2 - 2xy - 10x - 2y + 3 = 0$

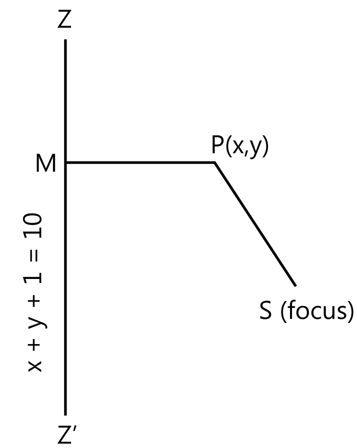


Figure 11.1

Note: The general equation of a conic can be taken as $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

This equation represents ellipse if it is non degenerate (i.e. eq. cannot be written into two linear factors)

Condition: $\Delta \neq 0$, $h^2 < ab$. Where $\left(\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\right)$

PLANCESS CONCEPTS

- The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2gx + 2fy + c = 0 \text{ and } \begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

  Degeneracy condition depends on determinant of 3x3 matrix and the type of conic depends on determinant of 2x2 matrix.

- Also the equation can be taken as the intersection of $z = ax^2 + 2hxy + by^2$ and the plane $z = -(2gx + 2fy + c)$

Vaibhav Gupta (JEE 2009, AIR 54)

2. STANDARD EQUATION OF ELLIPSE

Let the origin be the centre of the ellipse and the major and minor axis be on the x-axis and y-axis respectively. It means foci lies on x-axis and the coordinates of F_1 are (-c, o) and F_2 be (c, o). Let P be any point (x, y) on the ellipse. By the definition of the ellipse, the sum of the distances from any point P(x, y) to foci F_1 and F_2 = constant.

Let us consider this constant to be 2a for the sake of simplicity.

$$PF_1 + PF_2 = 2a \quad \text{...(i)}$$

$$PF_1^{\ 2} = (x + c)^2 + (y - 0)^2$$

$$\Rightarrow \quad PF_1 = \sqrt{(x+c)^2 + y^2} \quad \text{...(ii)}$$

Similarly, $PF_2 = \sqrt{(x-c)^2 + y^2}$ ...(iii)

Putting the value of PF_1 and PF_2 in (i) from (ii) and (iii), we get

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \quad \Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

On squaring, we get

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

$$\Rightarrow 4cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow 4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx$$

$$\Rightarrow \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x$$

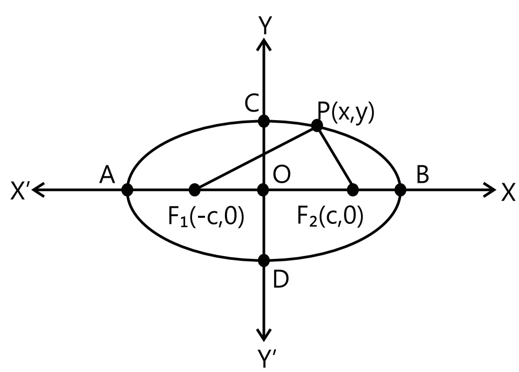


Figure 11.2

Squaring both sides, we get

$$(x-c)^2 + y^2 = a^2 - 2cx + \left(\frac{c}{a}x\right)^2$$

$$\Rightarrow \quad x^2 - 2cx + c^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$$

$$\Rightarrow \left(1-\frac{c^2}{a^2}\right)x^2+y^2=a^2-c^2$$

$$\Rightarrow \left(\frac{a^2-c^2}{a^2}\right)x^2+y^2=a^2-c^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \;\; \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(iv)}$$

[taking $b^2=a^2-c^2$]

This is the standard form of the equation of an ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, Where $b^2=a^2(1-e^2)$ i.e. b > a

PLANCESS CONCEPTS

Domain and range of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are [–a, a] and [–b, b] respectively.

Vaibhav Krishnan (JEE 2009, AIR 22)

3. TERMS RELATED TO AN ELLIPSE

Vertices: The points A and A′, in the figure where the curve meets the line joining the foci S and S′, are called the vertices of the ellipse. The coordinates of A and A′ are (a, 0) and (–a, 0) respectively.

Major and Minor Axes: In the figure, the distance AA′= 2a and BB′= 2b are called the major and minor axes of the ellipse. Since $e<1$ and $b^2=a^2(1-e^2)$. Therefore a > b $\Rightarrow$ AA′ > BB′.

Foci: In figure, the points S (ae, 0) and S′ (–ae, 0) are the foci of the ellipse.

Directrix: ZK and Z′K′ are two directrix of the ellipse and their equations are x = a/e and x=– a/e respectively.

Centre: Since the centre of a conic section is a point which bisects every chord passing through it. In case of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ every chord is bisected at C (0, 0). Therefore, C is the centre of the ellipse in the figure and C is the mid-point of AA′.

Eccentricity of the Ellipse: The eccentricity of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b is $e=\sqrt{1-\left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}=\sqrt{1-\left(\frac{b}{a}\right)^2}$

Ordinate and Double Ordinate: Let P be a point on the ellipse and let PN be perpendicular to the major axis AA′ such that PN produced meets the ellipse at P′. Then PN is called the ordinate of P and PNP′ the double ordinate of P.

Latus Rectum: It is a double ordinate passing through the focus. In Fig. 3, LL′ is the latus rectum and LS is called the semi-latus rectum. MSM′ is also a latus rectum. The length of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is $\frac{2b^2}{a}=2a(1-e^2)$.

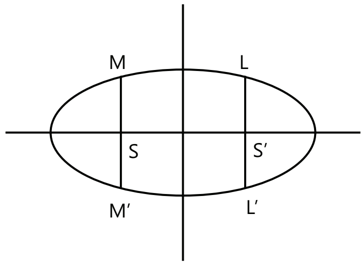


Figure 11.3

Focal Distances of a Point on the Ellipse: Let P(x, y) be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in Fig. 11.4. Then, by definition, we have

SP= ePN and S'P = e PN'

$\Rightarrow$ SP = eP'Q and S'P = e (P'Q')

$\Rightarrow$ SP = e (CQ – CP) and S'P = e (CQ' + CP')

$\Rightarrow$ $SP = e\left(\frac{a}{e}-x\right)$ $S'P = e\left(\frac{a}{e}+x\right]$

$\Rightarrow$ SP = a – ex and S'P = a + ex

Thus, the focal distances of a point P(x, y) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are a –ex and a + ex.

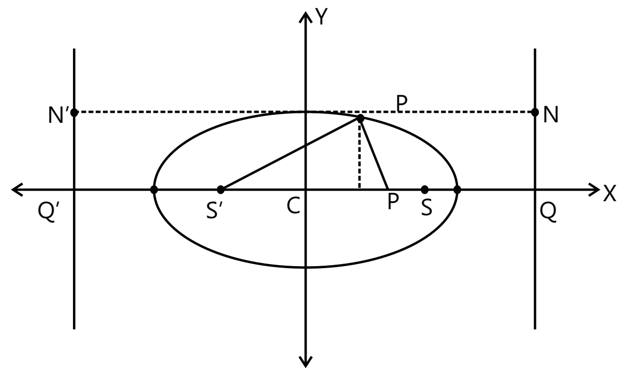


Figure 11.4

Also, SP + S'P= a – ex + a + ex = 2a = Major axis (constant)

Hence, the sum of the focal distances of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse.

PLANCESS CONCEPTS

The above property of an ellipse gives us a mechanical method of tracing an ellipse as explained below:

Take an inextensible string of a certain length and fasten its ends to two fixed knobs. Now put a pencil on the string and turn it round in such a way that the two portions of the string between it and the fixed knobs are always tight. The curve so traced will be an ellipse having its foci at the fixed knobs.

Shrikant Nagori (JEE 2009, AIR 30)

4. PROPERTIES OF ELLIPSE

Important Terms \ Ellipse	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For a > b	For b > a
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	(±ae, 0)	(0, ± be)
Equation of directrices	x = ±a/e	y = ±b/e
Relation in a, b and e	$b^2 = a^2 (1 - e^2)$	$a^2=b^2 (1-e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

Ends of latus rectum	$\left(\pm ae, \pm \frac{b^2}{a}\right)$	$\left(\pm \frac{a^2}{b}, \pm be\right)$
Parametric equations (Discussed later)	(a cos ϕ, b sinϕ) $(0 \le \phi < 2\pi)$	(a cos ϕ, b sin ϕ) $(0 \le \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ $S'P = a + ex_1$	$SP = b - ey_1$ $S'P = b + ey_1$
Sum of focal radii SP + S′P =	2a	2b
Distance between foci	2ae	2be
Distance between directrices	2a/e	2b/e
Tangents at the vertices	x = –a, x = a	y = b, y = –b

PLANCESS CONCEPTS

The vertex divides the join of the focus and the point of intersection of directrix with the axis internally and externally in the ratio e: 1

Misconceptions: If a>b it is a horizontal ellipse, if b > a it is a vertical ellipse unlike hyperbola.

Nitish Jhawar (JEE 2009, AIR 7)

Illustration 2: Find the equation of the ellipse whose foci are (4, 0) and (–4, 0) and whose eccentricity is 1/3. **(JEE MAIN)**

Sol: Use the property of the centre of an ellipse and the foci to find the equation.

Clearly, the foci are on the x-axis and the centre is (0, 0), being midway between the foci. So the equation will be in the standard form.

Let it be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Foci are $(a\cos\theta, b\sin\theta)$. Here they are $(\pm 4, 0)$.

$\therefore$ ae = 4

Given $e = \frac{1}{3}$

$\therefore$ $a.\frac{1}{3} = 4$, i.e., a = 12

Again, $b^2 = a^2(1 - e^2)$

$\Rightarrow b^2 = 12^2.\left(1 - \frac{1}{3^2}\right) = 12^2.\frac{8}{3^2} = 128$

$P(a\cos\theta, b\sin\theta)$. The equation of the ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$

Illustration 3: From a point Q on the circle $x^2 + y^2 = a^2$ perpendicular QM is drawn to x-axis, find the locus of point 'P' dividing QM in ratio 2 : 1. **(JEE MAIN)**

Sol: Starting from a point on the circle find the foot of the perpendicular on the X-axis and hence find the locus.

Let $by\sec\phi + ax\,\text{cosec}\,\phi + (a^2 + b^2) = 0$, $M \equiv (a\cos\theta, 0)$ and $P \equiv (h, k)$

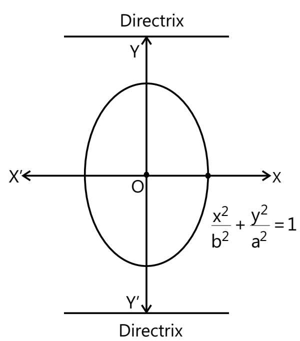
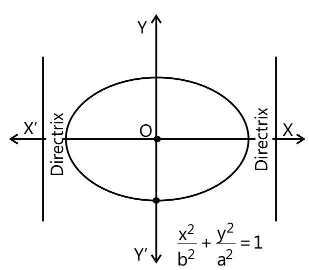
$\therefore\ h = a\cos\theta,\ k = \dfrac{a\sin\theta}{3} \Rightarrow \left(\dfrac{3k}{a}\right)^2 + \left(\dfrac{h}{a}\right)^2 = 1$

$\Rightarrow$ Locus of P is $\dfrac{x^2}{a^2} + \dfrac{y^2}{(a/3)^2} = 1$.

Illustration 4: Draw the shape of the given ellipse and find their major axis, minor axis, value of c, vertices, directrix, foci, eccentricity and the length of the latus rectum. **(JEE MAIN)**

(i) $36x^2 + 4y^2 = 144$ (ii) $4x^2 + 9y^2 = 36$

Sol: Using the standard form and basic concepts of curve tracing, sketch the two ellipses.

1.	Ellipse	$36x^2 + 4y^2 = 144$ or $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$	$4x^2 + 9y^2 = 36$ or $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
2.	Shape	Since the denominator of $\dfrac{y^2}{36}$ is larger then the denominator of $\dfrac{x^2}{4}$, so the major axis lies along y-axis	Since the denominator of $\dfrac{x^2}{9}$ is greater than the denominator of $\dfrac{y^2}{4}$, so the major axis lies along x-axis
		 **Figure 11.5**	 **Figure 11.6**
3.	Major axis	$2a = 2 \times 6 = 12$	$2a = 2 \times 3 = 6$
4.	Minor axis	$2b = 2 \times 2 = 4$	$2b = 2 \times 2 = 4$
5.	Value of c	$a^2 = 36,\ b^2 = 4$ $c = \sqrt{a^2 - b^2} = \sqrt{36 - 4} = 4\sqrt{2}$	$a^2 = 9,\ b^2 = 4$ $c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$
6.	Vertices	(0, –a) and (0, a) (0, –6) and (0, 6)	(–a, 0) and (a, 0) (–3, 0) and (3, 0)
7.	Directrices	$y = \pm\dfrac{a^2}{c} = \pm\dfrac{36}{4\sqrt{2}} = \pm\dfrac{9}{\sqrt{2}}$	$x = \pm\dfrac{a^2}{c} = \pm\dfrac{9}{\sqrt{5}}$

8.	Foci	(0, –c), (0, c) $(0, -4\sqrt{2}), (0, 4\sqrt{2})$	(–c, 0) and (c, 0) $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$
9.	Eccentricity	$e = \frac{c}{a} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$	$e = \frac{c}{a} = \frac{\sqrt{5}}{3}$
10.	Length of latus rectum	$2l = \frac{2b^2}{a} = \frac{2 \times 4}{6} = \frac{4}{3}$	$2l = \frac{2b^2}{a} = \frac{2 \times 4}{3} = \frac{8}{3}$

PLANCESS CONCEPTS

The semi-latus rectum of an ellipse is the harmonic mean of the segments of its focal chord.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 5: Show that $x^2 + 4y^2 + 2x + 16y + 13 = 0$ is the equation of an ellipse. Find its eccentricity, vertices, foci, directrices, length of the latus rectum and the equation of the latus rectum. **(JEE ADVANCED)**

Sol: Represent the equation given in the standard form and compare it with the standard form to get the eccentricity, vertices etc.

We have,

$x^2 + 4y^2 + 2x + 16y + 13 = 0 \qquad \Rightarrow (x^2 + 2x + 1) + 4(y^2 + 4y + 4) = 4$

$\Rightarrow (x+1)^2 + 4(y+2)^2 = 4 \qquad \Rightarrow \frac{(x+1)^2}{2^2} + \frac{(y+2)^2}{1^2} = 1$ ... (i)

Shifting the origin at (–1, –2) without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y,

we have $x = X - 1$ and $y = Y - 2$ ... (ii)

Using these relations, equation (i) reduces to

$\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1$, where ... (iii)

This is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where a = 2 and b = 1.

Thus, the given equation represents an ellipse.

Clearly a > b, so, the given equation represents an ellipse whose major and minor axes are along the X and Y axes respectively.

Eccentricity: The eccentricity e is given by $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

Vertices: The vertices of the ellipse with respect to the new axes are $(X = \pm a, Y = 0)$ i.e. $(X = \pm 2, Y = 0)$.

So, the vertices with respect to the old axes are given by

$(\pm 2 - 1, -2)$ i.e., (–3, –2) and (1, –2) [Using (ii)]

Foci: The coordinates of the foci with respect to the axes are given by

$(X = \pm ae, Y = 0)$ i.e. $(X = \pm\sqrt{3}, Y = 0)$.

So, the coordinates of the foci with respect to the old axes are given by

$(\pm\sqrt{3}-1, -2)$ [Putting $X=\pm\sqrt{3}, Y=0$ in (ii)]

Directrices: The equations of the directrices with respect to the new axes are

$$X = \pm\frac{a}{e} \text{ i.e. } ea^2\left(1-\frac{b^2}{d^2}\right) = 4a^2e^2\cos^2\theta$$

So, the equations of the directrices with respect to the old axes are

$$x = +\frac{4}{\sqrt{3}}-1 \text{ i.e. } x=\frac{4}{\sqrt{3}}-1 \text{ and } x=-\frac{4}{\sqrt{3}}-1 \quad \left[\text{Putting } X=\pm\frac{4}{\sqrt{3}} \text{ in (ii)}\right]$$

Length of the latus rectum: The length of the latus rectum = $\frac{2b^2}{a}=\frac{2}{2}=1$.

Equation of latus rectum: The equations of the latus rectum with respect to the new axes are

$X=\pm ae$ i.e. $X=\pm\sqrt{3}$

So, the equations of the latus rectum with respect to the old axes are

$x=\pm\sqrt{3}-1$ [Putting $X=\pm\sqrt{3}$ in (ii)]

i.e., $x=\sqrt{3}-1$ and $x=-\sqrt{3}-1$.

Illustration 6: A straight rod of given length slides between two fixed bars which include an angle of 90º. Show that the locus of a point on the rod which divides it in a given ratio is an ellipse. If this ratio is 1/2, show that the eccentricity of the ellipse is $\sqrt{3}/2$. **(JEE ADVANCED)**

Sol: Consider a rod of particular length and write the coordinates of the point in terms of the parameter. Eliminate the parameters to get eccentricity equal to $\sqrt{3}/2$.

Let the two lines be along the coordinate axes. Let PQ be the rod of length a such that $\angle OPQ=\theta$. Then, the coordinates of P and Q are $(a\cos\theta, 0)$ and $(0, a\sin\theta)$ respectively. Let R(h,k) be the point dividing PQ in the ratio $\lambda:1$. Then, $h=\frac{a\cos\theta}{\lambda+1}$ and $k=\frac{\lambda a\sin\theta}{\lambda+1}$.

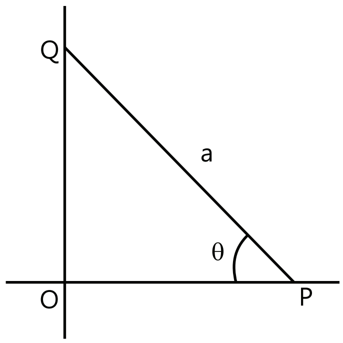


Figure 11.7

$$\Rightarrow \cos\theta=\frac{h}{a}(\lambda+1) \text{ and } \sin\theta=\frac{k}{a\lambda}(\lambda+1)$$

$$\Rightarrow \cos^2\theta+\sin^2\theta=\frac{h^2}{a^2}(\lambda+1)^2+\frac{k^2}{a^2\lambda^2}(\lambda+1)^2$$

$$\Rightarrow \frac{h^2}{\left((a/(\lambda+1)\right)^2}+\frac{k^2}{\left(a\lambda/(\lambda+1)\right)^2}=1.$$

Hence, the locus of R (h, k) is $\frac{x^2}{\left(a/(\lambda+1)\right)^2}+\frac{y^2}{\left(a\lambda/(\lambda+1)\right)^2}=1$

which is an ellipse of eccentricity given by $e=\begin{cases}\sqrt{1-\lambda^2} & \text{if } \lambda<1\\ \sqrt{1-(1/\lambda^2)} & \text{if } \lambda>1\end{cases}$

When $\lambda=\frac{1}{2}$, we have $e=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$.

Illustration 7: A man running a race course notes that the sum of the distances from the two flag posts from him is always 10 metres and the distances between the flag posts is 8 metres. Find the equation of the path traced by the man. **(JEE ADVANCED)**

Sol: Use the basic definition of an ellipse. Clearly, the path traced by the man is an ellipse having its foci at two flag posts. Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } b^2=a^2(1-e^2)$$

It is given that the sum of the distances of the man from the two flag posts is 10 metres. This means that the sum of the focal distances of a point on the ellipse is 10 m.

$\Rightarrow$ PS + PS′ = 2a = 10 $\Rightarrow$ a = 5 ...(i)

It is also given that the distance between the flag posts is 8 metres.

$\therefore$ 2ae = 8 $\Rightarrow$ ae = 4 ...(ii)

Now, $b^2=a^2(1-e^2)=a^2-a^2e^2=25-16$

$\Rightarrow$ $b^2=9$ $\Rightarrow$ b = 3 [Using (i) and (ii)]

Hence, the equation of the path is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

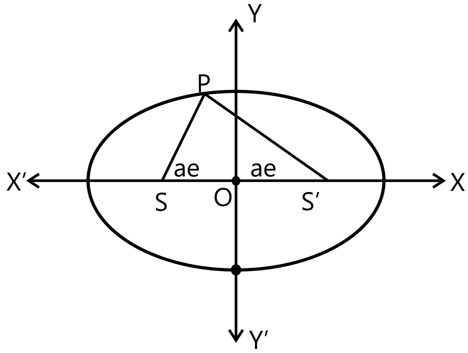


Figure 11.8

5. AUXILIARY CIRCLE

A circle with its centre on the major axis, passing through the vertices of the ellipse is called an auxiliary circle.

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse, then its auxiliary circle is $x^2+y^2=a^2$.

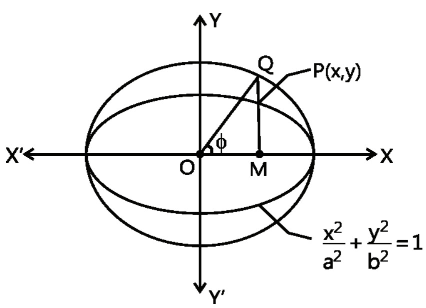


Figure 11.9

Eccentric angle of a point: Let P be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to meet the auxiliary circle in Q. Join OQ. The angle $\angle QOM=\phi$ is called the eccentric angle of the point P on the ellipse.

Note that the angle $\angle XOP$ is not the eccentric angle of point P.

PLANCESS CONCEPTS

A circle defined on the minor axis of an ellipse as diameter $x^2+y^2=b^2$ is called a minor auxiliary circle.

Ravi Vooda (JEE 2009, AIR 71)

6. PARAMETRIC FORM

6.1 Parametric Co-Ordinates of a Point on an Ellipse

Let P(x, y) be a point on an ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and Q be the corresponding point on the auxiliary circle $x^2+y^2=a^2$.

Let the eccentric angle of P be ϕ. Then $\angle XCQ = \phi$.

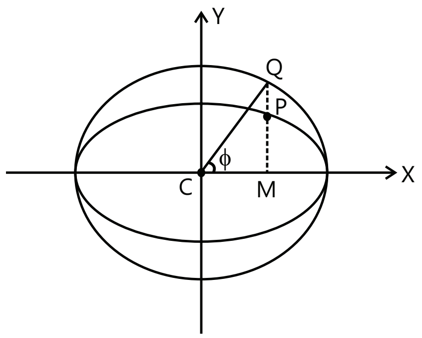


Figure 11.10

Now, x = CM

$\Rightarrow$ x = CQ cos ϕ [$\because$ CQ = radius of $x^2+y^2=a^2$]

Since P(x, y) lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\Rightarrow$ $\frac{a^2\cos^2\phi}{a^2}+\frac{y^2}{b^2}=1$.

$\Rightarrow$ $y^2=b^2(1-\cos^2\phi)=b^2\sin^2\phi$ $\Rightarrow$ $y=b\sin\phi$.

Thus, the coordinates of point P having eccentric angle ϕ can be written as (a cosϕ, b sin ϕ) and are known as the parametric coordinates of an ellipse.

6.2 Parametric Equation of an Ellipse

The equations x = acosϕ, y = bsinϕ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where ϕ is the parameter.

PLANCESS CONCEPTS

Always remember that θ is not the angle of P with x-axis. It is the angle of corresponding point Q.

Rohit Kumar (JEE 2012, AIR 79)

Illustration 8: Find the distance from the centre to the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which makes an angle α with x-axis. **(JEE MAIN)**

Sol: Establish a relation between the angle α and the eccentric angle. Use parametric coordinates of an ellipse and the distance formula to find the distance.

Let $P\equiv(a\cos\theta, b\sin\theta)$ $\therefore$ $(b/a)\tan\theta=\tan\alpha \Rightarrow$ $\tan\theta=(a/b)\tan\alpha$

$$OP = \sqrt{a^2\cos^2\theta+b^2\sin^2\theta} = \sqrt{\frac{a^2\cos^2\theta+b^2\sin^2\theta}{\sin^2\theta+\cos^2\theta}} \qquad \because \sin^2\theta+\cos^2\theta=1$$

$$=\sqrt{\frac{a^2+b^2\tan^2\theta}{1+\tan^2\theta}}=\sqrt{\frac{a^2+b^2\times(a^2/b^2)\tan^2\alpha}{1+(a^2/b^2)\tan^2\alpha}}$$

7. SPECIAL FORMS OF AN ELLIPSE

(a) If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

(b) If the equation of the curve is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$, where $x^2+2y^2-6x-12y+23=0$ and $mx-ly+p=0$ are perpendicular lines, then we substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X$, $\frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

Illustration 9: Find the equation to the ellipse whose axes are of lengths 6 and $e^2\cos^2\phi+\cos\phi-1=0$ and their equation are $x-3y+3=0$ and $3x+y-1=0$ respectively. **(JEE MAIN)**

Sol: Given the equation of the axis, we can find the centre. Use the length of the axes of the ellipse to find the required equation of the ellipse. Let P (x, y) be any point on the ellipse and let p_1 and p_2 be the lengths of perpendiculars drawn from P on the major and minor axes of the ellipse.

Then, $p_1=\frac{x-3y+3}{\sqrt{1+9}}$ and $p_2=\frac{3x+y-1}{\sqrt{9+1}}$.

Let 2a and 2b be the lengths of major and minor axes of the ellipse respectively. We have, 2a = 6 and 2b= $2\sqrt{6}$.

$\Rightarrow$ a = 3 and b = $\sqrt{6}$. The equation of the ellipse is $\frac{p_1^2}{b^2}+\frac{p_2^2}{a^2}=1$

$\Rightarrow \frac{(x-3y+3)^2}{60}+\frac{(3x+y-1)^2}{90}=1 \Rightarrow (x-3y+3)^2+2(3x+y-1)^2$

$\Rightarrow 21x^2-6xy+29y^2+6x-58y-151=0$

8. EQUATION OF A CHORD

Let P ($a\cos\alpha$, $b\sin\alpha$), Q ($a\cos\beta$, $b\sin\beta$) be any two points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then, the equation of the chord joining these two points is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$.

Illustration 10: Find the angle between two diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose extremities have eccentric angles α and $\beta=\alpha+\frac{\pi}{2}$. **(JEE MAIN)**

Sol: Find the slope of the two diameters and then use the relation between the given angles.

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Slope of OP = $m_1=\frac{b\sin\alpha}{a\cos\alpha}=\frac{b}{a}\tan\alpha$; Slope of OQ = $m_2=\frac{b\sin\beta}{a\cos\beta}=\frac{-b}{a}\cot\alpha$ given $\beta=\alpha+\frac{\pi}{2}$

$\therefore \tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{(b/a)(\tan\alpha+\cot\alpha)}{1-(b^2/a^2)}\right|=\left|\frac{2ab}{(a^2-b^2)\sin 2\alpha}\right|$

Illustration 11: If the chord joining the two points whose eccentric angles are α and β, cut the major axis of an ellipse at a distance c from the centre, show that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}$. **(JEE ADVANCED)**

Sol: Use the fact that the point (c, 0) lies on the chord joining points whose eccentric angles are α and β. The equation of the chord joining points whose eccentric angles are α and β on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\text{, is }\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

This will cut the major axis at the point (c, 0) if

$$\frac{c}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)\Rightarrow\frac{\cos\left((\alpha+\beta)/2\right)}{\cos\left((\alpha-\beta)/2\right)}=\frac{a}{c}\Rightarrow\frac{\cos\left((\alpha+\beta)/2\right)+\cos\left((\alpha-\beta)/2\right)}{\cos\left((\alpha+\beta)/2\right)-\cos\left((\alpha-\beta)/2\right)}=\frac{a+c}{a-c}$$

$$\Rightarrow\frac{2\cos(\alpha/2)\cos(\beta/2)}{-2\sin(\alpha/2)\sin(\beta/2)}=\frac{a+c}{a-c}\Rightarrow\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{c-a}{c+a}.$$

Illustration 12: The eccentric angle of any point P on the ellipse is ϕ. If S is the focus nearest to the end A of the major axis A'A such that $\angle ASP=\theta$. Prove that $\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\phi}{2}$. **(JEE ADVANCED)**

Sol: Find the distance of the point P from the X-axis and the horizontal distance of the point from nearest focus. Use trigonometry to get the desired result.

In ΔPSL, we have

PL = $b\sin\phi$ and SL = $a\cos\phi - ae$

$$\therefore\ \tan\theta=\frac{b\sin\phi}{a\cos\phi-ae}\Rightarrow\frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}=\frac{2\sqrt{1-e^2}\tan(\phi/2)}{(1-e)-(1+e)\tan^2(\phi/2)}$$

$$\Rightarrow\frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}=\frac{2\sqrt{(1+e)/(1-e)}\tan(\phi/2)}{1-\left(\sqrt{(1+e)/(1-e)}\right)}\Rightarrow\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\phi}{2}$$

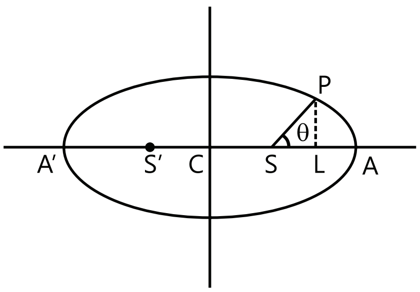


Figure 11.11

9. POSITION OF A POINT W.R.T. AN ELLIPSE

The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according to $\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}-1>0$, $=0$ or <0 respectively. $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

Illustration 13: Find the set of value(s) of 'α' for which the point $P(\alpha,-\alpha)$ lies inside the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. **(JEE MAIN)**

Sol: Apply the concept of position of a point w.r.t. the ellipse.

If $P(\alpha,-\alpha)$ lies inside the ellipse $2a^2\ S_1<0$

$$\Rightarrow\frac{\alpha^2}{16}+\frac{\alpha^2}{9}-1<0\Rightarrow\frac{25}{144},\ \alpha^2<1\Rightarrow\alpha^2<\frac{144}{25}\qquad\therefore\alpha\in\left(-\frac{12}{5},\frac{12}{5}\right).$$

10. LINE AND AN ELLIPSE

Consider a straight line of the form $y = mx + c$ and ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

By solving these two equations we get, $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$

$\Rightarrow \quad (b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$

For this equation

$\Rightarrow \quad D = 4\left(a^4m^2c^2 - (b^2 + a^2m^2)a^2(c^2 - b^2)\right)$

$\Rightarrow \quad D = 4a^2b^2(b^2 - c^2 + a^2m^2)$

$\therefore$ The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two distinct points if $a^2m^2 + b^2 > c^2$, in one point if $c^2 = a^2m^2 + b^2$ and does not intersect if $a^2m^2 + b^2 < c^2$.

Illustration 14: Find the condition for the line $lx + my + n = 0$ to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: Use the theory of equations or the standard form of the tangent. The equation of the line is $lx + my + n = 0$

$\Rightarrow \quad y = \left(-\frac{l}{m}\right)x + \left(-\frac{n}{m}\right)$. We know that the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

$\Rightarrow \quad \left(\frac{-n}{m}\right)^2 = a^2\left(-\frac{l}{m}\right)^2 + b^2 \Rightarrow n^2 = a^2l^2 + b^2m^2$.

Illustration 15: Find the condition for the line $x\cos\alpha + y\sin\alpha = p$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: Use the theory of equations or the standard form of the tangent.

The equation of the given line ℓ is $x\cos\alpha + y\sin\alpha = p \Rightarrow y = (-\cot\alpha)x - p\,\text{cosec}\,\alpha$

This will touch $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, If $(-p\text{cosec}^2\alpha)^2 = a^2\cot^2\alpha + b^2$ [Using: $c^2 = a^2m^2 + b^2$]

$\Rightarrow \quad p^2\text{cosec}^2\alpha = \frac{a^2\cos^2\alpha + b^2\sin^2\alpha}{\sin^2\alpha} \Rightarrow p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$

Illustration 16: Find the set of value(s) of 'λ' for which the line $3x - 4y + \lambda = 0$ intersect the ellipse $\frac{x^2}{16} + \frac{y^2}{16} = 1$ at two distinct points. **(JEE ADVANCED)**

Sol: Same as previous illustration.

Solving the given line with ellipse, we get $\frac{(4y-\lambda)^2}{9\times16} + \frac{y^2}{9} = 1 \Rightarrow 32y^2 - 8\lambda + (\lambda^2 - 144) = 0$

Since the line intersects the parabola at two distinct points:

$\therefore$ Roots of above equation are real & distinct $\therefore$ D > 0

$\Rightarrow \quad (8\lambda)^2 - 4.32(\lambda^2 - 144) > 0 \Rightarrow -12\sqrt{2} < \lambda < 12\sqrt{2}$

11. TANGENT TO AN ELLIPSE

11.1 Equation of Tangent

(a) Point form: The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1,y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(b) Slope form: If the line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2=a^2m^2+b^2$.

Hence, the straight line $y=mx\pm\sqrt{a^2m^2+b^2}$ always represents the tangents to the ellipse.

(i) Point of contact: Line $\left(\frac{0-b}{ae-0}\right)\left(\frac{0-b}{-ae-0}\right)=-1$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at

$$\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}},\frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right).$$

(c) Parametric form: The equation of tangent at any point $(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

Remark: The equation of the tangents to the ellipse at points $p(a\cos\theta_1, b\sin\theta_1)$ and $Q(a\cos\theta_2, b\sin\theta_2)$ are

$$\frac{x}{a}\cos\theta_1+\frac{y}{b}\sin\theta_1=1 \text{ and } \frac{x}{a}\cos\theta_2+\frac{y}{b}\sin\theta_2=1$$

And these two intersect at the point $\left(\frac{a\cos\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)},\frac{b\sin\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}\right)$

11.2 Equation of Pair of Tangents

Pair of tangents: The equation of a pair of tangents PA and PB is $SS_1 = T^2$.

Where $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$$S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

$$T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

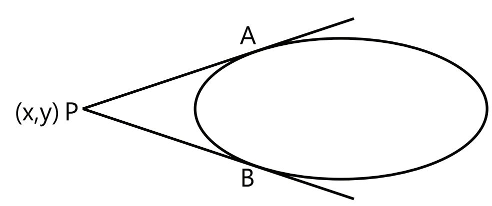


Figure 11.12

PLANCESS CONCEPTS

The portion of the tangent to an ellipse intercepted between the curve and the directrix subtends a right angle at the corresponding focus.

B Rajiv Reddy (JEE 2012, AIR 11)

11.3 Director Circle

Definition The locus of the point of intersection of the perpendicular tangents to an ellipse is known as its director circle.

Equation of the director circle the equation of the director circle, is $(x \pm ae)^2 = y^2 - 4a^2$. Clearly, it is a circle concentric to the ellipse and radius equal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

It follows from the definition of the director circle that the tangents drawn from any point on the director circle of a given ellipse to the ellipse are always at right angles.

PLANCESS CONCEPTS

Director circle is the circumcircle of ellipse's circumrectangle whose sides are parallel to the major and minor axis.

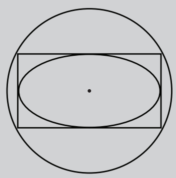


Figure 11.13

Anvit Tanwar (JEE 2009, AIR 9)

Illustration 17: A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. **(JEE ADVANCED)**

Sol: Use the condition of tangency and the standard equation of tangent. The equations of the two ellipses are

$$\frac{x^2}{4^2} + \frac{y^2}{1^2} = 1 \quad \text{...(i)}$$

and $$\frac{x^2}{6} + \frac{y^2}{3} = 1 \quad \text{...(ii) respectively.}$$

Suppose the tangents P and Q to ellipse (ii) intersect at R(h, k). PQ is the chord of contact of tangents drawn from R(h, k) to ellipse (ii). So, the equation of PQ is

$$\frac{hx}{6} + \frac{ky}{3} = 1 \quad \text{...(ii)}$$

$\Rightarrow \frac{ky}{3} = \frac{-hx}{6} + 1 \Rightarrow y = -\frac{hx}{2k} + \frac{3}{k}$. This touches the ellipse given in (i). Therefore,

$$\frac{9}{k^2} = 4\left(\frac{-h}{2k}\right)^2 + 1 \qquad [\text{Using}: c^2 = a^2m^2 + b^2]$$

$\Rightarrow h^2 + k^2 = 9 \Rightarrow (h,k)$ lies on the circle $x^2 + y^2 = 9$.

Clearly, $x^2 + y^2 = 9$ is the director circle of the ellipse (ii). Hence, the angle between the tangents at P and Q to the ellipse is a right angle.

11.4 Chord of Contact

If PQ and PR are the tangents through point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact QR is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or T = 0 at (x_1, y_1).

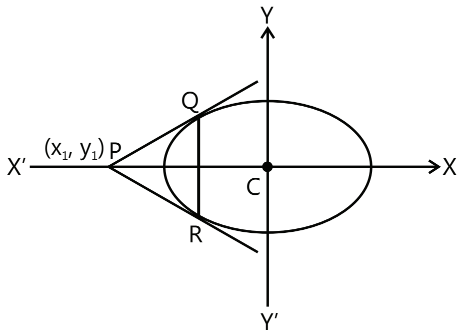


Figure 11.14

Illustration 18: Prove that the chord of contact of tangents drawn from the point (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at the centre, if $\frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$. **(JEE ADVANCED)**

Sol: Make the equation of the ellipse homogeneous using the chord and then apply the condition for the pair of straight lines to be perpendicular.

The equation of the chord of contact of tangents drawn from (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \qquad \text{...(i)}$$

The equation of the straight lines joining the centre of the ellipse i.e. the origin, to the points of intersection of the ellipse and (i) is obtained by making a homogeneous equation with the help of (i) and the ellipse and is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)^2=0 \text{ or } x^2\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+y^2\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)-\frac{2hk}{a^2b^2}xy=0 \qquad \text{...(ii)}$$

If the chord of contact of tangents subtends a right angle at the centre, then the lines represented by (ii) should be at right angles.

$$\Rightarrow \left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)=0 \quad \Rightarrow \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$

Illustration 19: Find the equations of the tangents to the ellipse $3x^2+4y^2=12$ which are perpendicular to the line $y+2x=4$. **(JEE MAIN)**

Sol: Use the slope form of the tangent. Let m be the slope of the tangent. Since the tangent is perpendicular to the line $y+2x=4$

$$\therefore\ m(-2)=-1 \Rightarrow m=\frac{1}{2};\ \text{Now, } 3x^2+4y^2=12 \ \Rightarrow \frac{x^2}{4}+\frac{y^2}{3}=1$$

Comparing this with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a^2=4$ and $b^2=3$.

So, the equations of the tangents are $y=mx\pm\sqrt{a^2m^2+b^2}$

i.e. $\quad y=\frac{1}{2}x\pm\sqrt{4(1/4)+3} \Rightarrow y=\frac{x}{2}\pm 2 \Rightarrow 2y=x\pm 4$.

Illustration 20: Find the equations of the tangents to the ellipse $9x^2+16y^2=144$ which pass through the point (2, 3). **(JEE MAIN)**

Sol: Put the given point in the standard equation of the tangent and find the value of m.

The equation of the ellipse is $9x^2+16y^2=144 \Rightarrow \frac{x^2}{4^2}+\frac{y^2}{3^2}=1$

This if of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2=4^2$ and $b^2=3^2$.

The equation of any tangents to this ellipse is $y=mx\pm\sqrt{a^2m^2+b^2}$ i.e. $y=mx\pm\sqrt{16m^2+9}$ ...(i)

If it passes through (2, 3) then $3=2m+\sqrt{16m^2+9}$

$\Rightarrow (3-2m)^2=16m^2+9\ x^2+y^2=4\ \Rightarrow m=0,-1$

Substituting these values of m in (i), we obtain y = 3 and y = –x + 5 as the equations of the required tangents.

Note: If the question was asked to find combined eq. of a pair of tangents then use $SS_1=T^2$.

Illustration 21: The locus of the points of intersection of the tangents at the extremities of the chords of the ellipse $x^2+2y^2=6$ which touch the ellipse $x^2+4y^2=4$ is. **(JEE MAIN)**

(A) $x^2+y^2=4$ (B) $x^2+y^2=6$ (C) $x^2+y^2=9$ (D) None of these.

Sol: Find the equation of the tangents for the two ellipses and compare the two equations.

We can write $x^2+4y^2=4$ as $\frac{x^2}{4}+\frac{y^2}{1}=1$ ...(i)

Equation of a tangent to the ellipse (i) is $\frac{x}{2}\cos\theta+y\sin\theta=1$ ...(ii)

Equation of the ellipse $x^2+2y^2=6$ can be written as $\frac{x^2}{6}+\frac{y^2}{3}=1$ ...(iii)

Suppose (ii) meets the ellipse (iii) at P and Q and the tangents at P and Q to the ellipse (iii) intersect at (h, k), then (ii) is the chord of contact of (h, k) with respect to the ellipse (iii) and thus its equation is $\frac{hx}{6}+\frac{ky}{3}=1$ ...(iv)
Since (ii) and (iv) represent the same line

$DA=CA-CD=\frac{a^2}{x_1}-x_1$ $\Rightarrow$ h = 3 cos θ, k = 3 sin θ and the locus of (h, k) is $x^2+y^2=9$.

Illustration 22: Show that the locus point of intersection of the tangents at two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose eccentric angles differ by a right angle is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$. **(JEE MAIN)**

Sol: Solve the equation of the tangents at the two points whose eccentric angles differ by $\frac{\pi}{2}$.

Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be two points on the ellipse such that $\theta-\phi=\frac{\pi}{2}$. The equations of tangents at P and Q are

$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ ...(i)

and, $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$ ...(ii) respectively.

Since $\frac{\sqrt{2}}{3}$, so (i) can be written as $-\frac{x}{a}\sin\phi+\frac{y}{b}\cos\phi=1$ ...(iii)

Let (h, k) be the point of intersection of (i) and (ii). Then,

$\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta=1$ and $-\frac{h}{a}\sin\theta+\frac{k}{b}\cos\theta=1$

$$\Rightarrow \left(\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta\right)^2+\left(-\frac{h}{a}\sin\theta+\frac{k}{b}\cos\theta\right)^2=1+1 \qquad \Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}=2$$

Hence, the locus of (h, k) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

Illustration 23: Prove that the locus of the mid-points of the portion of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between the axes $a^2y^2+b^2x^2=4x^2y^2$. **(JEE ADVANCED)**

Sol: Starting from the equation of the tangent, find the mid point of the tangent intercepted between the axes. Eliminate the parameter to get the locus.

The equation of the tangent at any point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

This cuts the coordinates axes at $A\left(\frac{a}{\cos\theta}, 0\right)$ and $B\left(0, \frac{b}{\sin\theta}\right)$

Let P(h, k) be the mid-point of AB. Then, $\frac{a}{2\cos\theta}=h$ and $\frac{b}{2\sin\theta}=k$

$$\Rightarrow \cos\theta=\frac{a}{2h} \text{ and } \sin\theta=\frac{b}{2k} \Rightarrow \cos^2\theta+\sin^2\theta=\frac{a^2}{4h^2}+\frac{b^2}{4k^2} \Rightarrow \frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$$

Hence, the locus of P (h, k) is $\frac{1}{52}$, $a^2y^2+b^2x^2=4x^2y^2$.

Illustration 24: Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are two foci of the ellipse, then show that $(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$. **(JEE ADVANCED)**

Sol: Use the fact that focal distances of a point (x, y) on the ellipse are a+ex and a–ex.

Let the coordinates of P be $(a\cos\theta, b\sin\theta)$, where θ is a parameter. The coordinates of F_1 and F_2 are (ae, 0) and (–ae, 0) respectively. We know that.

Therefore, $PF_1 = a + ae\cos\theta$ and $PF_2 = a - ae\cos\theta$

i.e., $PF_1 = a(1+e\cos\theta)$ and $PF_2 = a(1-e\cos\theta)$

$\therefore$ $(PF_1-PF_2)^2=\{a(1+e\cos\theta)-a(1-e\cos\theta)\}^2=4a^2e^2\cos^2\theta$ ...(i)

The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ ...(ii)

$\therefore$ d = Length of the perpendicular from (0, 0) on (ii)

$$\Rightarrow \quad d = \left|\frac{(0/a)\cos\theta + (0/b)\sin\theta - 1}{\sqrt{\cos^2\theta/a^2 + \sin^2\theta/b^2}}\right|$$

$$\Rightarrow \quad \frac{1}{d^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \Rightarrow \frac{b^2}{d^2} = \frac{b^2}{a^2}\cos^2\theta + \sin^2\theta \Rightarrow \quad 1 - \frac{b^2}{d^2} = 1 - \frac{b^2}{a^2}\cos^2\theta - \sin^2\theta$$

$$\Rightarrow \quad 1 - \frac{b^2}{d^2} = \cos^2\theta - \frac{b^2}{a^2}\cos^2\theta = \cos^2\theta\left(1 - \frac{b^2}{a^2}\right) = e^2\cos^2\theta$$

$$\Rightarrow \quad 4a^2\left(1 - \frac{b^2}{d^2}\right) = 4a^2e^2\cos^2\theta \qquad \text{...(iii)}$$

Hence, from (i) and (iii), we have $(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$.

Illustration 25: The tangent at point P(cosθ, bsinθ) of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, meets its auxiliary circle on two points, the chord joining which subtends a right angle at the centre. Show that the eccentricity of the ellipse is $(1 + \sin^2\theta)^{-1/2}$. **(JEE ADVANCED)**

Sol: Homogenize the equation of the ellipse using the equation of the tangent and then use the condition for the pair of straight lines to be perpendicular.

The equation of the tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad \text{...(i)}$$

The equation of the auxiliary circle is $x^2 + y^2 = a^2$ ...(ii)

The combined equation of the lines joining the origin with the points of intersection of (i) and (ii) is obtained by making (ii) homogeneous w.r.to (i)

$$\therefore \quad x^2 + y^2 = a^2\left(\frac{x}{a}\cos\theta + \frac{y}{b}\cos\theta\right)^2$$

$$\Rightarrow \quad x^2(1 - \cos^2\theta) + y^2\left(1 - \frac{a^2}{b^2}\sin^2\theta\right) - 2xy\frac{a}{b}\sin\theta\cos\theta = 0$$

These two lines are mutually perpendicular. Therefore, coefficient of x^2 + Coefficient of y^2 = 0

$$\Rightarrow \quad \sin^2\theta + 1 - \frac{a^2}{b^2}\sin^2\theta \Rightarrow \sin^2\theta\left(1 - \frac{a^2}{b^2}\right) + 1 = 0 \Rightarrow \frac{a^2 - b^2}{b^2}\sin^2\theta = 1$$

$$\Rightarrow \quad \frac{a^2e^2\sin^2\theta}{a^2(1 - e^2)} = 1 \Rightarrow e^2\sin^2\theta = 1 - e^2 \Rightarrow e = (1 + \sin^2\theta)^{-1/2}.$$

Illustration 26: If the tangent at (h, k) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the auxiliary circle $x^2 + y^2 = r^2$ at points whose ordinates are y_1 and y_2, show that $\frac{1}{y_1} + \frac{1}{y_2} = \frac{2}{k}$. **(JEE ADVANCED)**

Sol: Form a quadratic in y using the equation of the tangent and the ellipse and then use the sum and product of the roots to prove the above result.

The equation of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and point (h, k) is $\frac{hx}{a^2}+\frac{ky}{b^2}=1$. The ordinates of the points of intersection of (i) and the auxiliary circle are the roots of the equation

$$\frac{a^4}{h^2}\left(\frac{b^2-ky}{b^2}\right)^2+y^2=a^2$$

$$\Rightarrow \quad y^2\left(a^4k^2+b^4h^2\right)-2a^4b^2ky+a^4b^4-a^2b^4h^2=0$$

Since y_1 and y_2 are the roots of this equation.

Therefore, $y_1+y_2=\frac{2a^4b^2k}{a^4k^2+b^4h^2}$ and $y_1 y_2=\frac{a^4b^4-a^2b^4h^2}{a^4k^2+b^4h^2}$

$$\Rightarrow \quad \frac{1}{y_1}+\frac{1}{y_2}=\frac{2a^4b^2k}{a^4b^4-a^2b^4h^2}\Rightarrow\frac{1}{y_1}+\frac{1}{y_2}=\frac{2ka^2}{(a^2-h^2)b^2} \quad \Rightarrow \frac{1}{y_1}+\frac{1}{y_2}=\frac{2a^2k}{a^2k^2} \quad \Rightarrow \quad \frac{1}{y_1}+\frac{1}{y_2}=\frac{2}{k}$$

Illustration 27: Find the locus of the foot of the perpendicular drawn from the centre on any tangent to the ellipse. **(JEE ADVANCED)**

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

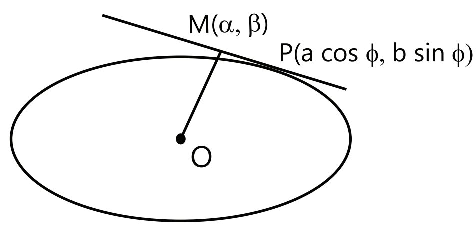


Figure 11.15

Sol: Follow the procedure for finding the locus starting from the parametric equation of the tangent.

The equation of the tangent at any point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1 \quad \text{....(i)}$$

Let $M(\alpha,\beta)$ be the foot of the perpendicular drawn from the centre (0, 0) to the tangent (i).

$x^2+y^2=C^2$ M is on the tangent, $\frac{\alpha\cos\phi}{a}+\frac{\beta\sin\phi}{b}=1$ ...(ii)

$x^2+y^2=C^2$ CM $\perp$ PM, $\frac{\beta}{\alpha}\left(-\frac{b\cos\phi}{a\sin\phi}\right)=-1$

or $\quad b\beta\cos\phi=a\sin\phi\,\alpha \qquad \therefore \quad \frac{\cos\phi}{a\alpha}=\frac{\sin\phi}{b\beta}=\frac{1}{\sqrt{a^2\alpha^2+b^2\beta^2}}$.

Putting in (ii), $\frac{\alpha}{a}.\frac{a\alpha}{\sqrt{a^2\alpha^2+b^2\beta^2}}+\frac{\beta}{b}.\frac{b\beta}{\sqrt{a^2\alpha^2+b^2\beta^2}}=1$

or $\quad \alpha^2+\beta^2=\sqrt{a^2\alpha^2+b^2\beta^2} \qquad \therefore \ (\alpha^2+\beta^2)^2=a^2\alpha^2+b^2\beta^2$

$\therefore$ The equation of the required locus is $(x^2+y^2)^2=a^2x^2+b^2y^2$.

12. NORMAL TO AN ELLIPSE

12.1 Equation of Normal in Different Forms

Following are the various forms of equations of the normal to an ellipse.

(a) **Point form:** The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

(b) **Parametric form:** The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\phi, b\sin\phi)$ is $x-2y+4=0$.

(c) **Slope form:** If m is the slope of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of normal is $x+2y+a=0$.

The co-ordinates of the point of contact are. $\left(\frac{\pm a^2}{\sqrt{a^2+b^2m^2}}, \frac{\mp mb^2}{\sqrt{a^2+b^2m^2}}\right)$.

12.2 Number of Normal and Co-normal Points

On a given ellipse exactly one normal can be drawn from a point lying on ellipse. If the point is not lying on the given ellipse, at most 4 lines which are normal to the ellipse at the points where they cut the ellipse. Such points on the ellipse are called co-normal points. In this section, we shall learn about the co-normal points and various relations between their eccentric angles.

Conormal points are the points on ellipse, whose normals to the ellipse pass through a given point are called co-normal points.

12.3 Properties of Eccentric Angles of Conormal Points

Property 1: The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.

Property 2: If θ_1, θ_2 and θ_3 are eccentric angles of three co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$

Property 3: Co-normal points lie on a fixed curve called Apollonian Rectangular Hyperbola $\left(a^2-b^2\right)xy+b^2kx-a^2hy=0$

Property 4: If the normal at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then $(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$.

Illustration 28: If the normal at an end of the latus rectum of an ellipse passes through one extremity of the minor, show that the eccentricity of the ellipse is given by $e^4 + e^2 - 1 = 0$. **JEE MAIN)**

Sol: Subtitute the point (0, ±b) in the equation of the normal and simplify it.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse. The coordinates of an end of the latus rectum are $(ae, b^2/a)$.

The equation of normal at $(ae, b^2/a)$ is $\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$

It passes through one extremity of the minor axis whose coordinates are (0, ±b).

$\therefore \quad \pm ab = a^2 - b^2$

$\Rightarrow \quad a^2b^2 = (a^2 - b^2)^2 \Rightarrow a^2.a^2(1 - e^2) = (a^2e^2)^2 \Rightarrow 1 - e^2 = e^4 \Rightarrow e^4 + e^2 - 1 = 0$

Illustration 29: Any ordinate MP of an ellipse meets the auxiliary circle in Q. Prove that the locus of the point of intersection of the normal P and Q is the circle $x^2 + y^2 = (a+b)^2$. **(JEE MAIN)**

Sol: Consider a point on the ellipse and find the intersection of the ordinate with the circle. Next find the intersection of the normal at P and Q and eliminate the parameter θ.

Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and let $Q(a\cos\theta, a\sin\theta)$ be the corresponding point on the auxiliary circle $x^2 + y^2 = a^2$. The equation of the normal at $P(a\cos\theta, b\sin\theta)$ to the ellipse is

$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$ ...(i)

The equation of the normal at $Q(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = a^2$ is

$y = x\tan\theta$ ...(ii)

Let (h , k) be the point of intersection of (i) and (ii). Then,

$ah\sec\theta - bk\,\text{cosec}\,\theta = a^2 - b^2$ ...(iii)

and, $P(a\cos\theta, b\sin\theta)$ ...(iv)

Eliminating θ from (iii) and (iv), we get

$$ah\sqrt{1 + \frac{k^2}{h^2}} - bk\sqrt{1 + \frac{h^2}{k^2}} = a^2 - b^2$$

$\Rightarrow \quad (a - b)\sqrt{h^2 + k^2} = a^2 - b^2 \Rightarrow h^2 + k^2 = (a+b)^2$

Hence, the locus of (h, k) is $x^2 + y^2 = (a+b)^2$.

Illustration 30: If the length of the major axis intercepted between the tangent and normal at a point $\begin{vmatrix} \sec\theta & \text{cosec}\,\theta & 1 \\ \sec\left(\theta + \frac{2\pi}{3}\right) & \text{cosec}\left(\theta + \frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta - \frac{2\pi}{3}\right) & \text{cosec}\left(\theta - \frac{2\pi}{3}\right) & 1 \end{vmatrix}$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the semi-major axis, prove that the eccentricity of the ellipse is given by $e = \{\sec\theta(\sec\theta - 1)\}^{1/2}$. **(JEE MAIN)**

Sol: Obtain the points of intersection of the tangent and the normal and then use the distance formula.

The equation of the tangent and normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point P

$$\begin{vmatrix} \sec\theta & \text{cosec}\,\theta & 1 \\ \sec\left(\theta+\frac{2\pi}{3}\right) & \text{cosec}\left(\theta+\frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta-\frac{2\pi}{3}\right) & \text{cosec}\left(\theta-\frac{2\pi}{3}\right) & 1 \end{vmatrix}$$ are given by

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \qquad \text{...(i)}$$

and, $ax\sec\theta-by\,\text{cosec}\theta=(a^2-b^2)$ ...(i) respectively.

Suppose (i) and (ii) meet the major axis i.e. y = 0 at Q and R respectively. Then, the coordinates of Q and R are given by

$Q(a\sec\theta,0)$ and $R\left(\frac{a^2-b^2}{a}\cos\theta,0\right)$ $\quad\therefore\quad$ QR = a [Given]

$$\Rightarrow \quad a\sec\theta-\frac{a^2-b^2}{a}\cos\theta=a \Rightarrow a^2-(a^2-b^2)\cos^2\theta=a^2\cos\theta \Rightarrow a^2-a^2e^2\cos^2\theta=a^2\cos\theta$$

$$\Rightarrow \quad 1-e^2\cos^2\theta=\cos\theta \Rightarrow e^2\cos^2\theta=1-\cos\theta \Rightarrow e^2=\sec\theta(\sec\theta-1) \Rightarrow e=\{\sec\theta(\sec\theta-1)\}^{1/2}$$

Illustration 31: If ω is one of the angles between the normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are θ and $\frac{\pi}{2}+\theta$, then prove that $\frac{2\cot\omega}{\sin 2\theta}=\frac{e^2}{\sqrt{1-e^2}}$. **(JEE ADVANCED)**

Sol: Evaluate the equation of the normal at the two points and then use the formula of the angle between two lines.

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are θ and $\frac{\pi}{2}+\theta$ are $ax\sec\theta-by\,\text{cosec}\theta=a^2-b^2$ and, $-ax\,\text{cosec}\theta-by\sec\theta=a^2b^2$ respectively. Since ω is the angle between these two normals, therefore, $\tan\omega=\left|\frac{(a/b)\tan\theta+(a/b)\cot\theta}{1-(a^2/b^2)}\right|$

$$\Rightarrow \quad \tan\omega=\left|\frac{ab(\tan\theta+\cot\theta)}{b^2-a^2}\right| \Rightarrow \tan\omega=\left|\frac{2ab}{(\sin 2\theta)(b^2-a^2)}\right| \Rightarrow \tan\omega=\frac{2ab}{(a^2-b^2)\sin 2\theta}$$

$$\Rightarrow \quad \tan\omega=\frac{2a^2\sqrt{1-e^2}}{a^2e^2\sin 2\theta} \Rightarrow \frac{2\cot\omega}{\sin 2\theta}=-\frac{e^2}{\sqrt{1-e^2}}$$

Illustration 32: If the tangent drawn at point $(t^2,2t)$ on the parabola $y^2=4x$ is the same as the normal drawn at a point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2+5y^2=20$, find the values of t and θ. **(JEE ADVANCED)**

Sol: Write the equation for the tangent and normal in terms of the parameter. Compare the two equations to get the values of t and θ.

The equation of the tangent at (t^2, 2t) to the parabola $y^2 = 4x$ is

$2ty = 2(x + t) \quad \Rightarrow ty = x + t^2 \left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ ... (i)

The equation of the normal at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$ is

$\Rightarrow \quad (\sqrt{5}\sec\theta)x - (2\operatorname{cosec}\theta)y - 1 = 0$ ... (ii)

It is given that (i) and (ii) represent the same line. Therefore, $\frac{\sqrt{5}\sec\theta}{1} = \frac{-2\operatorname{cosec}\theta}{-t} = \frac{-1}{t^2}$

$$\Rightarrow \quad t = \frac{2\operatorname{cosec}\theta}{\sqrt{5}\sec\theta} \text{ and } t = -\frac{1}{2\operatorname{cosec}\theta} \Rightarrow t = \frac{2}{\sqrt{5}}\cot\theta \text{ and } t = -\frac{1}{2}\sin\theta$$

$$\Rightarrow \quad \frac{2}{\sqrt{5}}\cot\theta = -\frac{1}{2}\sin\theta \Rightarrow 4\cos\theta = -\sqrt{5}\sin^2\theta \Rightarrow 4\cos\theta = -\sqrt{5}(1-\cos^2\theta)$$

$$\Rightarrow \quad \sqrt{5}\cos^2\theta - 4\cos\theta - \sqrt{5} = 0 \Rightarrow \sqrt{5}\cos^2\theta - 5\cos\theta + \cos\theta - \sqrt{5} = 0$$

$$\Rightarrow \quad \sqrt{5}\cos\theta(\cos\theta - \sqrt{5}) + (\cos\theta - \sqrt{5}) = 0 \Rightarrow (\cos\theta - \sqrt{5})(\sqrt{5}\cos\theta + 1) = 0$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \qquad [\because \cos\theta \neq -\sqrt{5}]$$

Putting $\cos\theta = -\frac{1}{\sqrt{5}}$ in $t = -\frac{1}{2}\sin\theta$ we get $t = -\frac{1}{2}\sqrt{1-\frac{1}{5}} = -\frac{1}{\sqrt{5}}$

Hence, $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ and $t = -\frac{1}{\sqrt{5}}$.

Illustration 33: The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points.

(JEE ADVANCED)

(a) $\left(\pm\frac{3\sqrt{5}}{7}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$ (c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

Sol: Put y = 0 in the equation of the normal to get the point Q in terms of θ. Get the locus of the mid-point as required. In the last step solve the equation of the locus and the latus rectum.

Equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$

Equation of the normal at $P(4\cos\theta, 2\sin\theta)$ to the ellipse is

$4x\sec\theta - 2y\operatorname{cosec}\theta = 4^2 - 2^2 \quad \Rightarrow 2x\sec\theta - y\operatorname{cosec}\theta = 6$

It meets x-axis at $Q(3\cos\theta, 0)$. If (h, k) are the coordinates of M, then

$$h = \frac{4\cos\theta + 3\cos\theta}{2}, k = \frac{2\sin\theta + 0}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{2h}{7}, \sin\theta = k$$

$$\Rightarrow \frac{4h^2}{49} + k^2 = 1 \text{ Locus of M is } \Rightarrow \frac{x^2}{(7/2)^2} + \frac{y^2}{1} = 1.$$

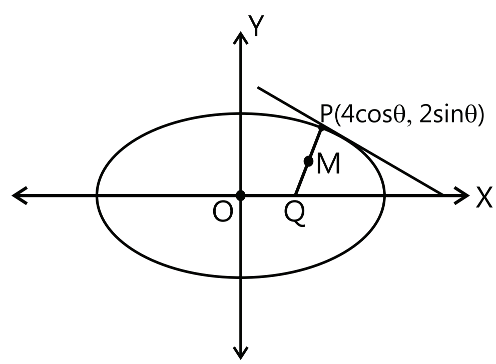


Figure 11.16

Latus rectum of the given ellipse is $x = \pm ae = \pm\sqrt{16-4} = \pm 2\sqrt{3}$

So locus of M meets the latus rectum at points for which $y^2 = 1 - \frac{12\times 4}{49} = \frac{1}{49} \Rightarrow y = \pm\frac{1}{7}$

And hence the required points are $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$.

13. CHORD BISECTED AT A GIVEN POINT

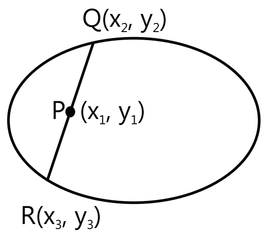


Figure 11.17

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

whose mid point is (x_1, y_1) is $T = S_1$

where $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$.

Illustration 34: Find the locus of the midpoint of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(JEE MAIN)**

Sol: In the equation T = S_1, substitute x = ae and y = 0.

Let (h, k) be the midpoint of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, the equation of the chord is

$\frac{hx}{a^2} + \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$ [Using : T = S_1]

or, $\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ It passes through the focus (ae, 0) of the ellipse.

$\therefore \frac{hae}{a^2} + 0 = \frac{h^2}{a^2} + \frac{k^2}{b^2}$. Hence, the locus of (h, k) is $\frac{xe}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

Illustration 35: Find the locus of the mid-point of the normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **(JEE ADVANCED)**

Sol: Similar to the previous question.

Let (h, k) be the mid point of a normal chord of the given ellipse. Then, its equation is

$\frac{hx}{a^2} + \frac{ky}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$ [Using: T = S_1]

or $\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ ...(i)

If (i) is a normal chord, then it must be of the form

$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$ ...(ii)

$$\therefore \frac{h}{a^3\sec\theta} = \frac{k}{-b^3\,\mathrm{cosec}\,\theta} = \frac{\frac{h^2}{a^2} + \frac{k^2}{b^2}}{a^2 - b^2}$$

$$\Rightarrow \quad \cos\theta = \frac{a^3}{h(a^2-b^2)}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right),\ \sin\theta = \frac{-b^3}{k(a^2-b^2)}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

Eliminating θ from the above relations, we get

$$\frac{a^6}{h^2(a^2-b^2)^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2+\frac{b^6}{k^2(a^2-b^2)^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2=1 \qquad \Rightarrow \left(\frac{a^6}{h^2}+\frac{b^6}{k^2}\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2=(a^2-b^2)$$

Hence, the locus of (h, k) is $\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=(a^2-b^2)^2$.

14. DIAMETERS

Definition: A chord through the centre of an ellipse is called a diameter of the ellipse.

The equation of the diameter bisecting the chords $(y = mx + c)$ of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2}{a^2m}x$, which is passing through (0, 0)

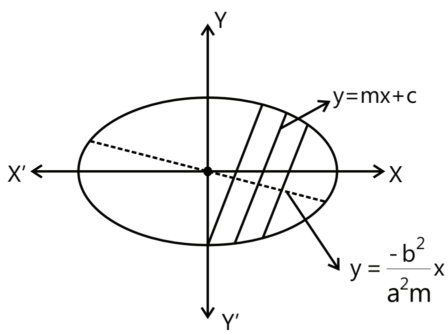


Figure 11.18

Conjugate diameter: Two diameters of an ellipse are said to be conjugate diameters if each bisects all chords parallel to the other. The coordinates of the four extremities of two conjugate diameters are

$P(a\cos\phi, b\sin\phi)$; $P'(-a\cos\phi, -b\sin\phi)$

$Q(-a\sin\phi, b\cos\phi)$; $Q'(-a\cos\phi, -b\sin\phi)$

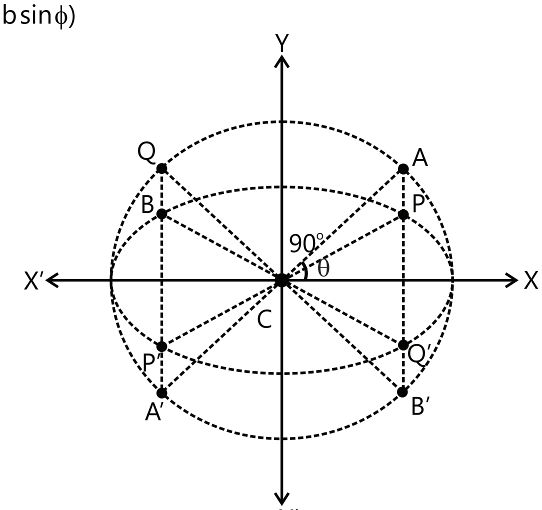


Figure 11.19

If $y = m_1 x$ and $y = m_2 x$ are two conjugate diameters of an ellipse, then $m_1 m_2 = \frac{-b^2}{a^2}$.

(a) Properties of diameters:

(i) The tangent at the extremity of any diameter is parallel to the chords it bisects or parallel to the conjugate diameter.

(ii) The tangents at the ends of any chord meets on the diameter which bisects the chord.

(b) Properties of conjugate diameters:

(i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle, i.e., $\phi - \phi' = \frac{\pi}{2}$

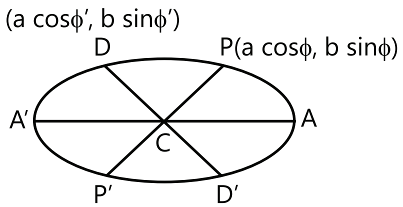


Figure 11.20

(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi axes of the ellipse i.e., $CP^2 + CD^2 = a^2 + b^2$.

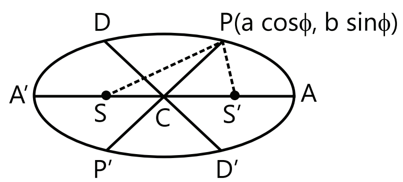


Figure 11.21

(iii) The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point i.e., $SP.S'P = CD^2$.

(iv) The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to the product of the axes i.e., Area of parallelogram = (2a)(2b) = Area of rectangle contained under major and minor axes.

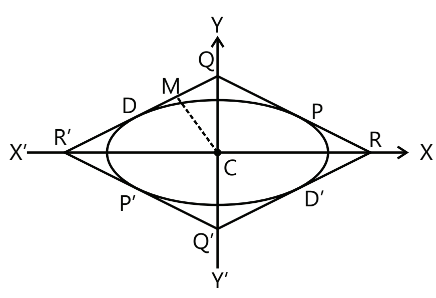


Figure 11.22

(v) The polar of any point with respect to an ellipse is parallel to the diameter to the one on which the point lies. Hence obtain the equation of the chord whose mid point is (x_1, y_1), i.e., chord is $T = S_1$.

(vi) Major and minor axes of ellipse is also a pair of conjugate diameters.

(c) Equi-conjugate diameters: Two conjugate diameters are called equi-conjugate, if their lengths are equal i.e., $(CP)^2 = (CD)^2$.

$\therefore (CP) = (CD) = \sqrt{\frac{(a^2+b^2)}{2}}$ for equi-conjugate diameters.

Illustration 36: If PCP′ and DCD′ form a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and R is any point on the circle $x^2+y^2=c^2$, then prove that $PR^2+DR^2+P'R^2+D'R^2=2(a^2+b^2+2c^2)$. **(JEE MAIN)**

Sol: Using the definition of conjugate diameters, get the coordinates of the point P, P′, Q and Q′. Starting from the L.H.S. prove the R.H.S.

Let R(h, k) be any point on the circle $x^2+y^2=c^2$. Then $h^2+k^2=c^2$ ...(i)

Since PCP′ and DCD′ form a pair of conjugate diameters, the coordinates of the extremities are:

$P(a\cos\theta, b\sin\theta)$, $P'(-a\cos\theta, -b\sin\theta)$ $D(-a\sin\theta, b\cos\theta)$, $D'(a\sin\theta, -b\cos\theta)$

$$\therefore \quad PR^2+DR^2+P'R^2+D'R^2 = (h-a\cos\theta)^2+(k-b\sin\theta)^2+(h+a\sin\theta)^2+(k-b\cos\theta)^2$$
$$+(h+a\cos\theta)^2+(k+b\sin\theta)^2+(h-a\sin\theta)^2+(k+b\cos\theta)^2$$

$= 4(h^2+k^2)+2a^2+2b^2$

$= 2a^2+2b^2+4c^2$ [Using (i)]

$= 2(a^2+b^2+2c^2)$

Illustration 37: CP and CD are conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Show that the locus of the mid-point of PD is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$. **(JEE MAIN)**

Sol: Consider two points which lie on two conjugate diameters. Find the mid point of these two points and eliminate the parameter θ to get the locus of the mid point.

Let P $(a\cos\theta, b\sin\theta)$, D $(-a\sin\theta, b\cos\theta)$ and (h, k) be the mid-point of PD. Then,

$2h = a\cos\theta - a\sin\theta$ and $2k = b\sin\theta + b\cos\theta$

$$\Rightarrow \frac{2h}{a} = \cos\theta-\sin\theta \text{ and } \frac{2k}{b} = \sin\theta+\cos\theta \Rightarrow \frac{4h^2}{a^2}+\frac{4k^2}{b^2} = (\cos\theta-\sin\theta)^2+(\sin\theta+\cos\theta)^2$$

$\Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{1}{2}$. Hence, the locus of (h, k) is $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

Illustration 38: If $y = x$ and $3y+2x=0$ are the equations of a pair of conjugate diameters of an ellipse, then the eccentricity of the ellipse is **(JEE MAIN)**

(a) $\sqrt{\frac{2}{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{2}{\sqrt{5}}$

Sol: Use the condition of conjugacy of diameters in an ellipse to find the eccentricity.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Slope of the given diameters are $m_1=1$, $m_2=-\frac{2}{3}$.

$\Rightarrow m_1m_2 = -\frac{2}{3} = -\frac{b^2}{a^2}$ [Using the condition of conjugacy of two diameters]

$3b^2 = 2a^2 \Rightarrow 3a^2(1-e^2) = 2a^2 \Rightarrow \quad 1-e^2 = \frac{2}{3} \quad \Rightarrow e^2 = \frac{1}{3} \Rightarrow e = \frac{1}{\sqrt{3}}$

Illustration 39: Show that the locus of the point of intersection of tangents at the end-point of the conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is another ellipse of the same eccentricity. **(JEE ADVANCED)**

Sol: Using two points at the end points of the conjugate diameters of an ellipse, write the equation of the tangent. Solve the two equations to eliminate the parameter θ.

Let CP and CD be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, the eccentric angles of P and D are θ and $\frac{\pi}{2}+\theta$ respectively. So, the coordinates of P and D are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ respectively. The equation of the tangents at P and D are

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad ...(i)$$

and $$\frac{-x}{a}\sin\theta+\frac{y}{b}\cos\theta=1 \quad ...(ii)$$

Let (h, k) be the point of intersection (i) and (ii). Then, $\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta=1$ and $\frac{-h}{a}\sin\theta+\frac{k}{b}\cos\theta=1$

$$\Rightarrow \left(\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta\right)^2+\left(-\frac{h}{a}\sin\theta+\frac{k}{b}\cos\theta\right)^2=1+1 \Rightarrow \frac{h^2}{a^2}+\frac{k^2}{b^2}=2$$

Hence, the locus of (h, k) is $\frac{h^2}{a^2}+\frac{k^2}{b^2}=2$ which represents an ellipse of eccentricity e, given by

$$e_1 = \sqrt{1-\frac{2b^2}{2a^2}} = \sqrt{1-\frac{b^2}{a^2}}$$

Clearly, it is same as the eccentricity of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Illustration 40: If α and β are the angles subtended by the major axis of an ellipse at the extremities of a pair of conjugate diameters, prove that $\cot^2\alpha+\cot^2\beta$ = constant. **(JEE MAIN)**

Sol: Using the co-ordinates of the co-ordinates of the end points of a diameter, find the angle subtended by the major axis. Repeat the same process for the other end of the diameter. Then find the value of $\cot^2\alpha+\cot^2\beta$ and prove that it is independent of the parameter.

Let CP and CD be a pair of conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, the coordinates of P and D are $(a\cos\theta, b\sin\theta)$ and $(-a\sin\theta, b\cos\theta)$ respectively.

$$m_1 = \text{Slope of AP} = \frac{b\sin\theta}{a\cos\theta-a} = -\frac{b}{a}\cot\frac{\theta}{2}$$

$$m_2 = \text{Slope of A'P} = \frac{b\sin\theta}{a\cos\theta+a} = \frac{b}{a}\tan\frac{\theta}{2}$$

$\therefore \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \qquad \Rightarrow \tan\alpha = \left|\frac{-(b/a)\cot(\theta/2) - (b/a)\tan(\theta/2)}{1-(b^2/a^2)}\right|$

$\Rightarrow \tan\alpha = \frac{ab}{a^2-b^2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right) \Rightarrow \tan\alpha = \left(\frac{2ab}{a^2-b^2}\right)\frac{1}{\sin\theta}$

Replacing θ by $\left(\frac{\pi}{2}+\theta\right)$, we get $\tan\beta = \left(\frac{2ab}{a^2-b^2}\right)\frac{1}{\cos\theta}$

$\therefore \quad \cot^2\alpha + \cot^2\beta = \left(\frac{a^2-b^2}{2ab}\right)^2(\sin^2\theta + \cos^2\theta) = \left(\frac{a^2-b^2}{2ab}\right)^2$ = Constant.

Illustration 41: Find the locus of the points of intersection of normals at two points on an ellipse which are extremities of conjugate diameters. **(JEE MAIN)**

Sol: Solve the equation of the normal at the extremities of conjugate diameters.

Let PP′ and QQ′ be two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the eccentric angle of the point P be 'ϕ'. Then the eccentric angle of Q is '$\phi + \frac{\pi}{2}$'.

$\therefore \quad P = (a\cos\phi, b\sin\phi)$

$Q = \left\{a\cos\left(\phi+\frac{\pi}{2}\right), b\sin\left(\phi+\frac{\pi}{2}\right)\right\}$

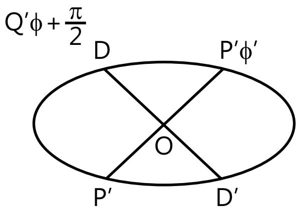


Figure 11.23

The equation of the normal at $P = (a\cos\phi, b\sin\phi)$ is $\frac{x - a\cos\phi}{(a\cos\phi)/a^2} = \frac{y - b\sin\phi}{(b\sin\phi)/b^2}$

or $\quad ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2$ ...(i)

Similarly, the equation of the normal at Q is

$ax\sec\left(\phi+\frac{\pi}{2}\right) - by\,\mathrm{cosec}\left(\phi+\frac{\pi}{2}\right) = a^2 - b^2$

or $\quad -ax\,\mathrm{cosec}\,\phi - by\sec\phi = a^2 - b^2$ ...(ii)

The locus of the point of intersection of (i) and (ii) is obtained by eliminating ϕ from them. Now we have

$ax\sec\phi - by\,\mathrm{cosec}\,\phi - (a^2-b^2) = 0$

$by\sec\phi + ax\,\mathrm{cosec}\,\phi + (a^2-b^2) = 0$

By cross multiplication,

$\frac{\sec\phi}{-by+ax} = \frac{\mathrm{cosec}\,\phi}{-by-ax} = \frac{a^2-b^2}{a^2x^2+b^2y^2}$

$\therefore \quad \cos\phi = \frac{a^2x^2+b^2y^2}{ax-by}\cdot\frac{1}{a^2-b^2}$

$\sin\phi = \frac{a^2x^2+b^2y^2}{-(ax+by)}\cdot\frac{1}{a^2-b^2}$

Squaring and adding,

$$1=\frac{(a^2x^2+b^2y^2)^2}{(a^2-b^2)^2}\left\{\frac{1}{(ax-by)^2}+\frac{1}{(ax+by)^2}\right\}=\left(\frac{a^2x^2+b^2y^2}{a^2-b^2}\right)^2\cdot\frac{2(a^2x^2+b^2y^2)}{(a^2x^2-b^2y^2)^2}$$

$$\Rightarrow\quad 2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2.(a^2x^2-b^2y^2)^2.$$

15. POLE AND POLAR

Let $P(x_1, y_1)$ be any point inside or outside the ellipse. A chord through P intersects the ellipse at A and B respectively. If tangents to the ellipse at A and B meet at Q(h, k) then locus of Q is called polar of P with respect to the ellipse and point P is called the pole.

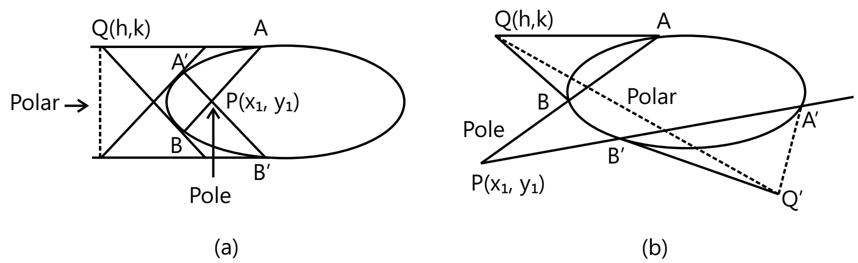


Figure 11.24

Note: If the pole lies outside the ellipse then the polar passes through the ellipse. If the pole lies inside the ellipse then the polar lies completely outside the ellipse. If the pole lies on the ellipse then the polar becomes the same as the tangent.

Equation of polar: Equation of polar of the point (x_1, y_1) with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, i.e., T = 0

Coordinates of Pole: The pole of the line $lx+my+n=0$ with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n},\frac{-b^2m}{n}\right)$.

Properties of pole and polar:

(a) If the polar of $P(x_1,y_1)$ passes through $Q(x_2,y_2)$, then the polar of $Q(x_2,y_2)$ goes through $P(x_1,y_1)$ and such points are said to be the conjugate points. Condition for conjugate points is $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$.

(b) If the pole of line $l_1x+m_1y+n_1=0$ lies on another line $l_2x+m_2y+n_2=0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

(c) Pole of a given line is the same as the point of intersection of tangents at its extremities.

(d) Polar of focus is directrix.

Illustration 42: Obtain the locus of poles of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to concentric ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. **(JEE MAIN)**

Sol: Taking a point (h , k), write the equation of the polar w.r.t. the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$. In the next step put the condition for polar to be the tangent to the other given ellipse.

The equation of the polar is $\frac{hx}{\alpha^2}+\frac{ky}{\beta^2}=1 \Rightarrow y=-\left(\frac{\beta^2 h}{\alpha^2 k}\right)x+\frac{\beta^2}{k}$

This touches $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Therefore, $\left(\frac{\beta^2}{k}\right)^2=a^2\left(\frac{-\beta^2 h}{\alpha^2 k}\right)^2+b^2$

$$\Rightarrow \quad \frac{\beta^4}{k^2}=a^2\frac{\beta^4 h^2}{\alpha^4 k^2}+b^2 \Rightarrow \frac{a^2h^2}{\alpha^4}+\frac{b^2k^2}{\beta^4}=1$$

Hence, the locus of (h, k) is $\Rightarrow \frac{a^2x^2}{\alpha^4}+\frac{b^2y^2}{\beta^4}=1$.

Illustration 43: Find the locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose poles are on the auxiliary circle or the tangents at the extremities of which intersect on the auxiliary circle. **(JEE ADVANCED)**

Sol: Compare the equation of the chord and the tangent to get the point which lies on the auxiliary circle. Substitute the point in the equation of the circle to get the required locus.

Let (h, k) be the mid-point of a chord of the ellipse. Then, its equation is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \qquad ...(i)$$

Let (x_1, y_1) be its pole with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, the equation of the polar is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad ...(ii)$$

Clearly, (i) and (ii) represent the same line. Therefore,

$$\frac{x_1}{h}=\frac{y_1}{k}=\frac{1}{(h^2/a^2)+(k^2/b^2)} \Rightarrow x_1=\frac{h}{(h^2/a^2)+(k^2/b^2)},\ y_1=\frac{k}{(h^2/a^2)+(k^2/b^2)}$$

It is given that (x_1, y_1) lies on auxiliary circle. Therefore $x_1^2+y_1^2=a^2 \Rightarrow h^2+k^2=a^2\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2$.

Hence the locus of (h, k) is $x^2+y^2=a^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$.

16. SUBTANGENT AND SUBNORMAL

Let the tangent and normal at $P(x_1, y_1)$ meet the x-axis at A and B respectively. Length of subtangent at $P(x_1, y_1)$ to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $DA=CA-CD=\frac{a^2}{x_1}-x_1$

Length of sub-normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$BD=CD-CB=x_1-\left(x_1-\frac{b^2}{a^2}x_1\right)=\frac{b^2}{a^2}x_1=(1-e^2)x_1.$$

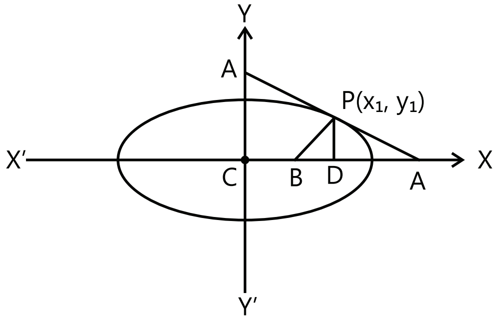


Figure 11.25

PLANCESS CONCEPTS

Misconception: As there is no y_1 term involved in the above results, don't think that the lengths are independent of y_1. Always remember that $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

Vaibhav Krishnan (JEE 2009, AIR 22)

PROBLEM-SOLVING TACTICS

- If the line $y=mx+c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $c^2=\frac{m^2(a^2-b^2)^2}{a^2+b^2m^2}$ is the condition of normality of the line to the ellipse.
- The tangent and normal at any point of an ellipse bisect the external and angles between the focal radii to the point. It follows from the above property that if an incident light ray passing through the focus (S) strikes the concave side of the ellipse, then the reflected ray will pass through the other focus (S').
- If SM and S'M' are perpendicular from the foci upon the tangent at any point of the ellipse, then SM. S'M' = b^2 and M, M' lie on the auxiliary circle.
- If the tangent at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major axis in T and minor axis in T', then CN. CT = a^2, CN'. CT' = b^2

  Where N and N' are the feet of the perpendicular from P on the respective axis.
- If SM and S' M' are perpendicular from the foci S and S' respectively upon a tangent to the ellipse, then CM and CM' are parallel to S'P and SP respectively.

FORMULAE SHEET

1. The general equation of second order $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents an ellipse

if $\Delta \neq 0$, $h^2 < ab$. where $\left(\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}\right)$

2. The sum of the focal distance of any point on an ellipse is a constant and is equal to the length of the major axis of the ellipse i.e. SP + S'P = 2a.

3. Standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Where a = length of semi-major axis,

b = length of semi-minor axis

4.

Imp. Terms \ Ellipse	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For a > b	For b > a
Centre	(0, 0)	(0, 0)
Vertices	(±a, 0)	(0, ±b)
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	(±ae , 0)	(0 , ± be)
Equation of directrices	x = ±a/e	y = ±b/e
Relation in a, b and e	$b^2 = a^2(1-e^2)$	$a^2 = b^2(1-e^2)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latus rectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric equations	(a cos ϕ, b sinϕ)	(a cos ϕ, b sin ϕ) $(0 \le \phi < 2\pi)$
Focal radii	$SP = a - ex_1$ $S'P = a + ex_1$	$SP = b - ey_1$ $S'P = b + ey_1$
Sum of focal radii (SP + S′P =)	2a	2b
Distance between foci	2ae	2be
Distance between directrices	2a/e	2b/e
Tangents at the vertices	x = –a, x = a	y = b, y = –b

5. The equations $x = a\cos\phi,\ y = b\sin\phi$ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where ϕ is the parameter.

6. **(i)** If the centre of the ellipse is at (h, k) and the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

(ii) If the equation of the ellipse is $\frac{(lx+my+n)^2}{a^2}+\frac{(mx-ly+p)^2}{b^2}=1$, where $lx+my+n=0$ and $mx-ly+p=0$ are perpendicular lines. Substitute $\frac{lx+my+n}{\sqrt{l^2+m^2}}=X$ and $\frac{mx-ly+p}{\sqrt{l^2+m^2}}=Y$, to put the equation in the standard form.

7. If $P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ are any two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of a chord joining these two points is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$.

8. The point $P(x_1, y_1)$ lies outside, on, or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according to $\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}-1>0,=0$ or <0 respectively.

9. The line $y=mx+c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on two distinct points if $a^2m^2+b^2>c^2$, on one point if $c^2=a^2m^2+b^2$ and does not intersect if $a^2m^2+b^2<c^2$. For an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the auxiliary circle is $x^2+y^2=a^2$.

10. The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. The equation of tangent to the ellipse having its slope equal to m is $y=mx\pm\sqrt{a^2m^2+b^2}$ and the point of contact is $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$. The equation of the tangent at any point $(a\cos\phi, b\sin\phi)$ is $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$.

Point of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points $P(a\cos\theta_1, b\sin\theta_1)$, and $Q(a\cos\theta_2, b\sin\theta_2)$ is $\left(\frac{a\cos\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}, \frac{b\sin\left((\theta_1+\theta_2)/2\right)}{\cos\left((\theta_1-\theta_2)/2\right)}\right)$.

11. Equation of pair of tangents drawn from an outside point $P(x_1, y_1)$ is $SS_1=T^2$.

12. For an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of director circle is $x^2+y^2=a^2+b^2$.

13. The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$. The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point $(a\cos\phi, b\sin\phi)$ is $(ax\sec\phi-by\text{cosec}\phi)=a^2-b^2$.

14. If m is the slope of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the normal is $y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$. The co-ordinates of the point of contact are $\left(\frac{\pm a^2}{\sqrt{a^2+b^2m^2}}, \frac{\pm mb^2}{\sqrt{a^2+b^2m^2}}\right)$.

15. The properties of conormal points are

(i) **Property 1:** The sum of the eccentric angles of the co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.

(ii) **Property 2:** If θ_1, θ_2 and θ_3 are eccentric angles of three co-normal points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$.

(iii) **Property 3:** Co-normal points lie on a fixed curve called an Apollonian Rectangular Hyperbola $\left(a^2-b^2\right)xy+b^2kx-a^2hy=0$

(iv) **Property 4:** If the normal at four points $P(x_1y_1)$, $Q(x_2y_2)$, $R(x_3y_3)$ and $S(x_4y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then $(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$.

16. If SM and S′M′ are perpendiculars from the foci upon the tangent at any point of the ellipse, then $SM \times S'M' = b^2$ and M, M′ lie on the auxiliary circle.

17. If the tangent at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major axis at T and minor axis at T′, then CN × CT = a^2, CN′ × CT′ = b^2. Where N and N′ are the feet of the perpendiculars from P on the respectively axis.

18. The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid point is (x_1, y_1), is $T = S_1$.

19. The chord of contact from a point $P(x_1, y_1)$ to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is T = 0 is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

20. The equation of the diameter bisecting the chords $(y = mx + c)$ of slope m of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = -\frac{b^2}{a^2m}x$.

21. If m_1 and m_2 are the slopes of two conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1m_2 = \frac{-b^2}{a^2}$.

22. The eccentric angle of the ends of a pair of conjugate diameters of an ellipse differ by a right angle, i.e., $\phi - \phi' = \frac{\pi}{2}$.

23. The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and is equal to the sum of the squares of the semi axes of the ellipse i.e., $CP^2 + CD^2 = a^2 + b^2$.

24. The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point i.e., $SP \times S'P = CD^2$.

25. The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to the product of the axes.

i.e. Area of the parallelogram = (2a)(2b) = Area of the rectangle contained under major and minor axes.

26. Two conjugate diameters are called equi-conjugate, if their lengths are equal i.e., $(CP)^2 = (CD)^2$

$\therefore (CP) = (CD) = \sqrt{\frac{(a^2+b^2)}{2}}$ for equi-conjugate diameters.

27. Equation of the polar of the point (x_1, y_1) w.r.t. an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$.

28. The pole of the line $lx + my + n = 0$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$.

29. Condition for a conjugate point is $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$.

30. The length of a sub tangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2}{x_1} - x_1$.

31. The length of a sub normal at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{b^2}{a^2}x_1 = (1-e^2)x_1$.

Solved Examples

JEE Main/Boards

Example 1: Find the centre, the eccentricity, the foci, the directrices and the lengths and the equations of the axes of the ellipse

$$5x^2 + 9y^2 + 10x - 36y - 4 = 0$$

Sol: Rewrite the equation in the standard form and compare them to get the centre, eccentricity etc.

$5x^2 + 9y^2 + 10x - 36y - 4 = 0$, the given equation can be written as

$$5(x^2 + 2x) + 9(y^2 - 4y) = 4$$

$$5(x+1)^2 + 9(y-2)^2 = 45$$

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{5} = 1$$

Shift the origin to $O' \equiv (-1, 2)$

$\therefore \quad X = x + 1;\ Y = y - 2$

$$\therefore \quad \frac{X^2}{9} + \frac{Y^2}{5} = 1 \qquad \text{...(i)}$$

This is in standard form

$\therefore \quad a = 3,\ b = \sqrt{5}$

$$\therefore \quad e^2 = \frac{a^2 - b^2}{a^2} = \frac{4}{9} \Rightarrow e = \frac{2}{3}$$

Also $ae = 3 \cdot \frac{2}{3} = 2$ and $\frac{a}{e} = \frac{9}{2}$.

Now for an ellipse in the standard form we have Centre $\equiv$ (0, 0); foci $\equiv$ ($\pm$ae, 0); directrices $x = \pm\frac{a}{e}$; axes x = 0, y = 0, length of major axis = 2a, length of minor axis = 2b.

Now for (i) the centre is given by X = 0, Y = 0

$\Rightarrow \quad x + 1 = 0,\ y - 2 = 0$

i.e. Centre $\frac{ax}{3} + \frac{by}{4} = c$

Foci are given by $X = \pm ae,\ Y = 0$

i.e. $\quad x + 1 = \pm 2$ and $y - 2 = 0$

i.e. $\quad x = 1,\ y = 2$ and $x = -3,\ y = 2$

$\therefore \quad$ Foci $\equiv (1, 2); (-3, 2)$

The equation of directrices are given by $X = \pm\frac{a}{e}$

i.e. $\quad x + 1 = \pm\frac{9}{2}$

i.e. $\quad x = \frac{7}{2},\ x = -\frac{11}{2}$

The equation of the axes are given by

X = 0, Y = 0

i.e. $\quad x + 1 = 0,\ y - 2 = 0$

i.e. $\quad x = -1,\ y = 2$

Length of the axes being 2a, 2b

i.e, $\quad 6,\ 2\sqrt{5}$.

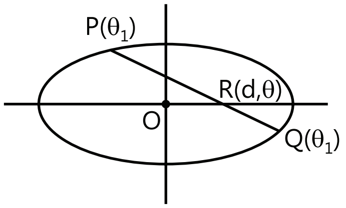
Example 2: If the chord through point θ_1 and θ_2 on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersects the major axis at (d, 0) prove that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{d-a}{d+a}$.

Sol: Substitute the point (d , 0) in the equation of the chord to prove the given result.

Equation of the chord joining the points θ_1 and θ_2 is

$$\frac{x}{a}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 - \theta_2}{2}\right)$$

Since (d, 0) lies on it



$$\therefore \quad \frac{d}{a}\cos\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 - \theta_2}{2}\right)$$

$$\frac{\cos\left((\theta_1 - \theta_2)/2\right)}{\cos\left((\theta_1 + \theta_2)/2\right)} = \frac{d}{a}$$

Applying componendo and dividendo, we get

$$\frac{d-a}{d+a} = \frac{\cos\left((\theta_1 - \theta_2)/2\right) - \cos\left((\theta_1 + \theta_2)/2\right)}{\cos\left((\theta_1 - \theta_2)/2\right) + \cos\left((\theta_1 + \theta_2)/2\right)}$$

$$= \frac{2\sin(\theta_1/2)\sin(\theta_2/2)}{2\cos(\theta_1/2)\cos(\theta_2/2)} = \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$$

Example 3: A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches it at the point P in the first quadrant and meets the x and y axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent at P.

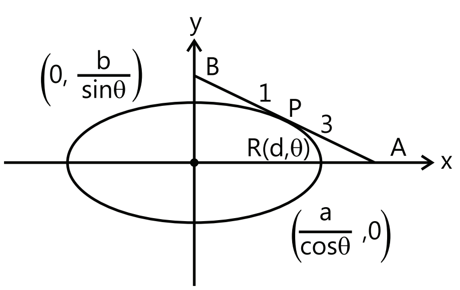
Sol: Consider a point in the parametric form and obtain the points A and B. Now use the condition that the point P divides AB in the ratio 3:1.

Let $P \equiv (a\cos\theta, b\sin\theta)$:

$$0 < \theta < \frac{\pi}{2} \quad \text{...(i)}$$

Equation of the tangent at $P(\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$



$\therefore A \equiv \left(\frac{a}{\cos\theta}, 0\right)$ and $B \equiv \left(0, \frac{b}{\sin\theta}\right)$

Now P divides segment AB in the ratio 3 : 1

$$\therefore \quad P \equiv \left(\frac{a}{4\cos\theta}, \frac{3b}{4\sin\theta}\right) \quad \text{...(ii)}$$

By (i) and (ii), we have

$$\therefore \quad \cos\theta = \frac{1}{2};\ \sin\theta = \frac{\sqrt{3}}{2}$$

$\therefore$ Equation of tangent at P is $bx + a\sqrt{3}y = 2ab$.

Example 4: If the tangent drawn at a point $(t^2, 2t); t \neq 0$ on the parabola $y^2 = 4x$ is the same as the normal drawn at a point $\left(\sqrt{5}\cos\phi, 2\sin\phi\right)$ on the ellipse $4x^2 + 5y^2 = 20$, find the value of t and ϕ.

Sol: Write the equation of the tangent and the normal using 't' and 'ϕ' and compare.

Equation of the tangent at $P(t^2, 2t)$ to $y^2 = 4x$ is

$$yt = x + t^2 \quad \text{...(i)}$$

Equation of normal at $Q\left(\sqrt{5}\cos\phi, 2\sin\phi\right)$ is

$$2y\cos\phi = x\sqrt{5}\sin\phi - \sin\phi\cos\phi \quad \text{...(ii)}$$

Equation (i) and (ii) represent the same line. Comparing the coefficients in equations (i) and (ii).

$$\frac{t}{2\cos\phi} = \frac{1}{\sqrt{5}\sin\phi} = \frac{t^2}{-\sin\phi.\cos\phi}$$

$$\Rightarrow \quad t = \frac{2}{\sqrt{5}}\cot\phi,\ t^2 = -\frac{\cos\phi}{\sqrt{5}}$$

$$\frac{4}{5}\cot^2\phi = -\frac{\cos\phi}{\sqrt{5}}$$

$$\Rightarrow \quad \cos\phi\left(\frac{4\cos\phi}{\sin^2\phi} + \sqrt{5}\right) = 0$$

$(\cos\phi \neq 0 \because t \neq 0)$

$$\Rightarrow \quad \frac{(x-5)^2}{9} + \frac{y^2}{25} = 1$$

$$\Rightarrow \quad \sqrt{5}\left(1 - \cos^2\phi\right) + 4\cos\phi = 0$$

$$\Rightarrow \quad \cos\phi = -\frac{1}{\sqrt{5}}$$

$\therefore \phi = 2n\pi \pm \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ where $0 \le \phi \le 2\pi$.

$\therefore$ Corresponding values of t are given by

$$t^2 = -\frac{\cos\phi}{\sqrt{5}} = \frac{1}{5}.$$

$$\therefore \quad t = \pm\frac{1}{\sqrt{5}}$$

Example 5: Show that the sum of the squares of the perpendiculars on any tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from 2 points on the minor axis, each of which is at a distance $\sqrt{a^2 - b^2}$ from the centre, is $2a^2$.

Sol: Use the standard equation of a tangent in terms of m and then proceed accordingly,

The general equation of a tangent to the ellipse is

$$y = mx \pm \sqrt{a^2m^2 + b^2} \quad \text{...(i)}$$

Let the points on the minor axis be $P(0, ae)$ and $Q(0, -ae)$ as $b^2 = a^2(1 - e^2)$

Length of the perpendicular from P on (i) is

$$P_1 = \frac{\left|ae \pm \sqrt{a^2m^2 + b^2}\right|}{\sqrt{1+m^2}}$$

Similarly, $P_2 = \frac{\left|-ae \pm \sqrt{a^2m^2 + b^2}\right|}{\sqrt{1+m^2}}$

Hence, $P_1^2 + P_2^2 = \frac{2}{1+m^2}\{a^2e^2 + (a^2m^2 + b^2)\}$

$= \frac{2}{1+m^2}\{(a^2 - b^2) + a^2m^2 + b^2\} = 2a^2$

Example 6: Find the equation of the ellipse having its centre at the point (2,–3), one focus at (3,–3) and one vertex at (4, –3).

Sol: Use the basic knowledge of the major axis, centre and focus to get the equation of the ellipse.

$C \equiv (2,-3)$, $S \equiv (3,-3)$ and $A \equiv (4,-3)$

Now, CA = 2

$\therefore \quad a = 2$

Again CS = 1

$\therefore \quad ae = 1 \qquad \Rightarrow e = \frac{1}{a} = \frac{1}{2}$

We know that

$b^2 = a^2 - a^2e^2$

$\Rightarrow \quad b = \sqrt{3}$

$\therefore$ Equation of ellipse is

$$\frac{(x-2)^2}{2^2} + \frac{(y+3)^2}{(\sqrt{3})^2} = 1$$

$\Rightarrow \quad 3(x-2)^2 + 4(y+3)^2 = 12$

$\Rightarrow \quad 3x^2 + 4y^2 - 12x + 24y + 36 = 0$

Example 7: Show that the angle between pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2) is $\tan^{-1}\left(-\frac{12}{\sqrt{5}}\right)$.

Sol: Starting from the standard equation of a tangent in terms of m, satisfy the point (1 ,2) and get the values of m. Using the value of m, find the angle between the two tangents. Let the equation of the tangents be

$y = mx \pm \sqrt{a^2m^2 + b^2}$. It passes through (1, 2)

$\therefore \quad (2-m)^2 = \frac{5}{3}m^2 + \frac{5}{2}$

$\Rightarrow \quad 4m^2 + 24m - 9 = 0$

Angle between the tangents is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{4\sqrt{36+9}}{-5} = -\frac{12}{\sqrt{5}}$$

$\therefore \quad \theta = \tan^{-1}\left(-\frac{12}{\sqrt{5}}\right)$

Example 8: The locus of the foot of the perpendicular drawn from the centre to any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(A) A circle (B) An ellipse

(C) A hyperbola (D) None of these.

Sol: Find the foot of the perpendicular from the centre to any tangent and eliminate the parameter.

Equation of a tangent to the ellipse is

$y = mx \pm \sqrt{a^2m^2 + b^2}$ ...(i)

Equation of the line through the centre (0, 0) perpendicular to (i) is

$y = \left(\frac{-1}{m}\right)x$ ...(ii)

Eliminating m from (i) and (ii) we get the required locus of the foot of the perpendicular as

$$y = -\frac{x^2}{y} \pm \sqrt{a^2\frac{x^2}{y^2} + b^2}$$

$\Rightarrow \quad (x^2 + y^2)^2 = a^2x^2 + b^2y^2$

which does not represent a circle, an ellipse or a hyperbola.

Example 9: The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). The equation of the ellipse is

(A) $4x^2 + 48y^2 = 48$ (B) $4x^2 + 6y^2 = 48$

(C) $x^2 + 16y^2 = 16$ (D) $x^2 + 12y^2 = 16$

Sol: Consider the standard equation of the ellipse. Use the two points given in the question to find the value of ‘a’ and ‘b’.

Let the equation of the required ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Given that it passes through (4, 0)

$\Rightarrow$ a = 4

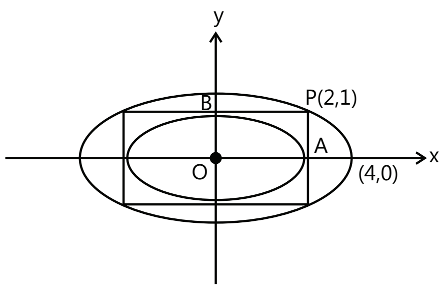
It also passes through (2, 1), one of the vertex of rectangle.

$\Rightarrow \quad \frac{4}{4^2}+\frac{1}{b^2}=1$

$\Rightarrow \quad b^2=\frac{4}{3}$ and the required equation is

$$\frac{x^2}{16}+\frac{3y^2}{4}=1$$

$\Rightarrow \quad x^2+12y^2=16$



Example 10: Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at point A and B. The equation of the locus of the point whose distances from the point P and the line AB are equal is

(A) $9x^2+y^2-6xy-54x-62y+241=0$

(B) $x^2+9y^2+6xy-54x+62y-241=0$

(C) $9x^2+9y^2-6xy-54x-62y-241=0$

(D) $x^2+y^2-2xy+27x+31y-120=0$

Sol: Write the equation of the chord of contact w.r.t. point P. Then follow the standard procedure to find the locus.

AB being the chord of contact of the ellipse from P(3, 4) has its equation

$$\frac{3x}{9}+\frac{4y}{4}=1 \Rightarrow x+3y=3$$

If (h, k) is any point on the locus, then

$$\sqrt{(h-3)^2+(k-4)^2}=\left|\frac{h+3k-3}{\sqrt{1+9}}\right|$$

$\Rightarrow 10(h^2+k^2-6h-8k+25)=(h+3k-3)^2$

Locus of (h, k) is

$9x^2+y^2-6xy-54x-62y+241=0$.

Example 11: If an ellipse slides between two perpendicular straight lines, then the locus of its centre is

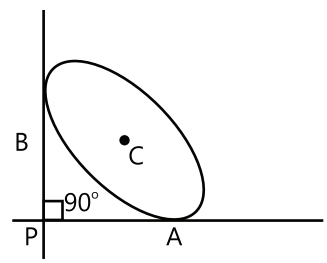
(A) A parabola (B) An ellipse

(C) A hyperbola (D) A circle

Sol: Use the concept of a Director Circle.

Let 2a, 2b be the length of the major and minor axes respectively of the ellipse. If the ellipse slides between two perpendicular lines, the point of intersection P of these lines being the point of intersection of perpendicular tangents lies on the director circle of the ellipse. This means that the centre C of the ellipse is always at a constant distance $\sqrt{a^2+b^2}$ from P. Hence the locus of C is a circle

$\frac{x^2}{9}+\frac{y^2}{4}=1$



Example 12: If α, β are the eccentric angles of the extremities of a focal chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, then $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)=$

Sol: Equate the slope of the line joining the focus and the two points.

The eccentricity $e=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$.

Let P $(4\cos\alpha, 3\sin\alpha)$ and $Q(4\cos\beta, 3\sin\beta)$ be a focal chord of the ellipse passing through the focus at $(\sqrt{7},0)$.

Then $\frac{3\sin\beta}{4\cos\beta-\sqrt{7}}=\frac{3\sin\alpha}{4\cos\alpha-\sqrt{7}}$

$\Rightarrow \quad \frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta}=\frac{\sqrt{7}}{4}$

$$\Rightarrow \quad \frac{\cos[(\alpha-\beta)/2)]}{\cos[(\alpha+\beta)/2]} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \quad \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{\sqrt{7}-4}{\sqrt{7}+4} = \frac{8\sqrt{7}-23}{9}.$$

JEE Advanced/Boards

Example 1: Common tangents are drawn to the parabola $y^2 = 4x$ and the ellipse $3x^2 + 8y^2 = 48$ touching the parabola A and B and the ellipse at C and D. Find the area of the quadrilateral ABCD.

Sol: Write the standard equation of the parabola in the slope. Use the condition for the line to be a tangent and obtain the value of m. We then find the points of contact with the ellipse and parabola and then find the area.

Let $y = mx + \frac{1}{m}$ be a tangent to the parabola $y^2 = 4x$.

It will touch the ellipse $\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{6})^2} = 1$, if $\frac{1}{m^2} = 16m^2 + 6$

[Using: $c^2 = a^2m^2 + b^2$]

$$\Rightarrow \quad 16m^4 + 6m^2 - 1 = 0$$

$$\Rightarrow \quad (8m^2-1)(2m^2+1) = 0 \Rightarrow m = \pm\frac{1}{\sqrt{8}}$$

We know that a tangent of slope m touches the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. So, the coordinates of the points of contact of the common tangents of slope $m = \pm\frac{1}{2\sqrt{2}}$ to the parabola $y^2 = 4x$ are A(8, $4\sqrt{2}$) and B(8, $-4\sqrt{2}$).

We also know that a tangent of slope m touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(\mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

Therefore, the coordinates of the points of contact of common tangents of slope $m\pm\frac{1}{2\sqrt{2}}$ to the ellipse are $C\left(-2, \frac{3}{\sqrt{2}}\right)$ and $D\left(-2, -\frac{3}{\sqrt{2}}\right)$.

Clearly AB || CD. So, the quadrilateral ABCD is a trapezium.

We have, AB $= 8\sqrt{2}$, CD $= 3\sqrt{2}$ and the distance between AB and CD is

PQ = 8 + 2 = 10

∴ Area of quadrilateral ABCD

$$= \frac{1}{2}(AB+CD)PQ = \frac{1}{2}\left(8\sqrt{2}+3\sqrt{2}\right)10 = 55\sqrt{2} \text{ sq. units.}$$

Example 2: Show that the angle between the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b$), and the circle $x^2 + y^2 = ab$ at their points of intersection in the first quadrant is $\tan^{-1}\left(\frac{(a-b)}{\sqrt{ab}}\right)$.

Sol: We find the point of intersection of the ellipse and the circle. Then we find the slope of the tangents to the circle and the ellipse and hence the angle.

At the points of intersection of ellipse and circle,

$$\frac{ab-y^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad y^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = 1 - \frac{b}{a} \Rightarrow y^2 = \frac{ab^2}{a+b}$$

$$\therefore \quad y = \pm b\sqrt{\frac{a}{a+b}} \text{ and } x = \pm a\sqrt{\frac{b}{a+b}}$$

$$P\left(\frac{a\sqrt{b}}{\sqrt{a+b}}, \frac{b\sqrt{a}}{\sqrt{a+b}}\right)$$

lies in first quadrant

Equation of tangent at P to the circle is

$$\frac{xa\sqrt{b}}{\sqrt{a+b}} + \frac{yb\sqrt{a}}{\sqrt{a+b}} = ab$$

Its slope is: $m_1 = -\frac{\sqrt{a}}{\sqrt{b}}$

Equation of the tangent at P to the ellipse is

$$\frac{xa\sqrt{b}}{a^2\sqrt{a+b}} + \frac{yb\sqrt{a}}{b^2\sqrt{a+b}} = 1$$

Its slope in $m_2 = -\frac{b^{3/2}}{a^{3/2}}$

If α is the angle between these tangents, then

$$\tan\alpha = \frac{|m_2 - m_1|}{|1+m_1m_2|} = \left|\frac{-(b^{3/2}/a^{3/2}) + (a^{1/2}/b^{1/2})}{1+(b^{3/2}/a^{3/2})(a^{1/2}/b^{1/2})}\right|$$

$$\frac{a^2-b^2}{a^{1/2}b^{1/2}(a+b)} = \frac{a-b}{\sqrt{ab}}.$$

Example 3: Any tangent to an ellipse is cut by the tangents at the extremities of the major axis at T and T′. Prove that the circle on TT′ as the diameter passes through the foci.

Sol: We find out the point of intersection of the tangent with the axis and then use these points to find the equation of the circle.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The extremities A and A′ of the major axis are A (a, 0), A′ (–a, 0). Equations of tangents A and A′ are x = a and x = –a. Any tangent to the ellipse is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

The points of intersection are

$$T\left(a,\frac{b(1-\cos\theta)}{\sin\theta}\right), T'\left(-a,\frac{b(1+\cos\theta)}{\sin\theta}\right)$$

The equation of the circle on TT′ as diameter is

$$\left(x^2-a^2\right)+\left(\left(y-\frac{b}{\sin\theta}\right)^2-\left(\frac{b\cos\theta}{\sin\theta}\right)^2\right)=0$$

$$\Rightarrow x^2-a^2+y^2-\frac{b}{\sin\theta}2y+\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}=0$$

$$\Rightarrow \quad x^2+y^2-\frac{2by}{\sin\theta}+b^2-a^2=0$$

$$\Rightarrow \quad x^2+y^2-\frac{2by}{\sin\theta}=a^2e^2$$

Foci S (ae, 0) and S′ (–ae, 0) lie on this circle.

Example 4: Let ABC be an equilateral triangle inscribed in the circle $x^2+y^2=a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a > b), meets the ellipse at P, Q, R respectively so that P, Q, R lie on the same side of the major axis as are the corresponding points A, B, C. Prove that the normals to the ellipse drawn at the points P, Q, R are concurrent.

Sol: Find the points of intersection of the perpendicular and the ellipse. Then apply the condition for the normals at these three points to be concurrent.

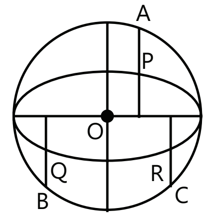
Let A, B, C have coordinates $(a\cos\theta,\ a\sin\theta)$,

$$\left[a\cos\left(\theta+\frac{2\pi}{3}\right),\ a\sin\left(\theta+\frac{2\pi}{3}\right)\right],$$

$$\left[a\cos\left(\theta+\frac{4\pi}{3}\right),\ a\sin\left(\theta+\frac{4\pi}{3}\right)\right]$$ respectively.

Then P, Q, and R have coordinates given by:

$P(a\cos\theta,\ b\sin\theta)$ $Q\left[a\cos\left(\theta+\frac{2\pi}{3}\right),\ b\sin\left(\theta+\frac{2\pi}{3}\right)\right]$ and



$R\left[a\cos\left(\theta+\frac{4\pi}{3}\right),\ b\sin\left(\theta+\frac{4\pi}{3}\right)\right]$ respectively.

Normals at P, Q, R to ellipse are concurrent, if the determinants of the coefficients is zero. i.e., if

$$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0$$

$$\therefore \sin\left(2\theta+\frac{2\pi}{3}\right)+\sin\left(2\theta+\frac{6\pi}{3}\right)+\sin\left(2\theta+\frac{4\pi}{3}\right)$$

$$=\sin(2\theta)+\sin\left(2\theta+\frac{2\pi}{3}\right)+\sin\left(2\theta+\frac{4\pi}{3}\right)$$

= 0 for all values of θ

∴ The normals are concurrent.

Example 5: Prove that the sum of the eccentric angles of the extremities of a chord which is drawn in a given direction is constant and equal to twice the eccentric angle of the point, at which the tangent is parallel to the given direction.

Sol: Consider two points on the ellipse and evaluate the slope of the chord. If the slope is constant prove that the sum of the angles is constant.

Slope of chord AB = m

$$=-\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$$

$$=\frac{2b\cos((\alpha+\beta)/2).\ \sin((\alpha-\beta)/2)}{2a\sin((\alpha+\beta)/2).\sin((\beta-\alpha)/2)}=-\frac{b}{a}\cot\left(\frac{\alpha+\beta}{2}\right).$$

$$\therefore \quad \frac{\alpha+\beta}{2}=\text{constant if m is constant}$$

Eq. of a tangent is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Slope of this tangent is $-\frac{b}{a}\cot\theta$.

Now if $m=-\frac{b}{a}\cot\theta$, then $\theta=\frac{\alpha+\beta}{2}$

So, the slopes are equal. They are parallel to each other. Hence proved.

Example 6: P and Q are two points of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ such that sum of their ordinates is 3. Prove that the locus of the intersection of the tangents at P and Q is $9x^2+25y^2=150y$.

Sol: Find the relation between the ordinate and use it to find the locus.

If (h, k) is the point of intersection of tangents at θ and ϕ, then

$$\frac{h}{a}=\frac{\cos\left((\theta+\phi)/2\right)}{\cos\left((\theta-\phi)/2\right)};\ \frac{k}{b}=\frac{\sin\left((\theta+\phi)/2\right)}{\cos\left((\theta-\phi)/2\right)}$$

$$\therefore\quad \frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{1}{\cos^2\left((\theta-\phi)/2\right)} \qquad \text{...(i)}$$

We are given that sum of ordinates is 3.

$$\therefore\quad b(\sin\theta+\sin\phi)=3$$

$$\Rightarrow\quad 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=1 \qquad \text{...(ii)}$$

$$\text{Now, } \frac{k}{b}=\frac{\sin\left((\theta+\phi)/2\right)}{\cos\left((\theta-\phi)/2\right)}=\frac{1}{2\cos^2\left((\theta-\phi)/2\right)}$$

$$\therefore\quad \frac{2k}{b}=\frac{1}{\cos^2\left((\theta-\phi)/2\right)} \qquad \text{...(iii)}$$

Hence from (i) and (iii) we get $\frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{2k}{b}$

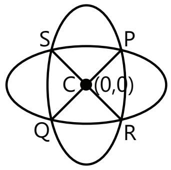
$\therefore$ Locus of (h, k) is $\frac{x^2}{25}+\frac{y^2}{9}=\frac{2y}{3}$

$$\Rightarrow\quad 9x^2+25y^2=150y$$

Example 7: If the points of intersection of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$ are the extremities of the conjugate diameters of the first ellipse, then prove that $\frac{a^2}{p^2}+\frac{b^2}{q^2}=2$.

Sol: Use the condition for the pair of lines to represent conjugate diameters.

Subtracting in order to find points of intersection, we get $x^2\left(\frac{1}{a^2}-\frac{1}{p^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{q^2}\right)=0$



Above equation will represent a pair of conjugate diameters of the first ellipse if

$$m_1m_2=-\frac{b^2}{a^2}$$

$$\therefore\quad \frac{\left((1/a^2)-(1/p^2)\right)}{\left((1/b^2)-(1/q^2)\right)}=-\frac{b^2}{a^2}$$

$$\Rightarrow\quad a^2\left(\frac{1}{a^2}-\frac{1}{p^2}\right)+b^2\left(\frac{1}{b^2}-\frac{1}{q^2}\right)=0$$

$$\Rightarrow\quad \frac{a^2}{p^2}+\frac{b^2}{q^2}=2$$

Example 8: The points of intersection of the two ellipses $x^2+2y^2-6x-12y+23=0$ and $4x^2+2y^2-20x-12y+35=0$.

(A) Lie on a circle centred at $\left(\frac{8}{3},3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$.

(B) Lie on a circle centred at $\left(-\frac{8}{3},3\right)$ and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$.

(C) Lie on a circle centred at (8, 9) and of radius $\frac{1}{3}\sqrt{\frac{47}{2}}$.

(D) Are not cyclic.

Sol: Use the concept of the curve passing through the intersection of two ellipses.

Equation of any curve passing through the intersection of the given ellipse is

$$4x^2+2y^2-20x-12y+35+$$
$$\lambda(x^2+2y^2-6x-12y+23)=0$$

Which represents a circle is

$$4+\lambda=2+2\lambda\Rightarrow\lambda=2$$

and the equation of the circle is thus,

$$6x^2+6y^2-32x-36y+81=0$$

$$\Rightarrow\quad x^2+y^2-\left(\frac{16}{3}\right)x-6y+\frac{81}{6}=0$$

Centre of the circle is $\left(\frac{8}{3},3\right)$ and the radius is

$$\sqrt{\left(\frac{8}{3}\right)^2+(3)^2-\frac{81}{6}}$$

$$=\sqrt{\frac{128+162-243}{18}}=\frac{1}{3}\sqrt{\frac{47}{2}}.$$

Paragraph for Questions 9 to 12

$C: x^2+y^2=9$, $E: \frac{x^2}{9}+\frac{y^2}{4}=1$, $L: y=2x$

Example 9: P is a point on the circle C, the perpendicular PQ to the major axis of the ellipse E meets the ellipse at M, then $\frac{MQ}{PQ}$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $\frac{1}{2}$ (D) None of these

Sol: Proceed accordingly using parametric coordinates..

Let the coordinates of P be $(3\cos\theta, 3\sin\theta)$ then the eccentric angle of M, the point where the ordinate PQ through P meets the ellipse is θ and the coordinates of M are $(3\cos\theta, 2\sin\theta)$, $\frac{MQ}{PQ}=\frac{2\sin\theta}{3\sin\theta}=\frac{2}{3}$.

Example 10: If L represents the line joining the point P and C to its centre O and intersects E at M, then the equation of the tangent at M to the ellipse E is

(A) $x+3y=3\sqrt{5}$ (B) $4x+3y=\sqrt{5}$

(C) $x+3y+3\sqrt{5}=0$ (D) $4x+3+\sqrt{5}=0$

Sol: Find the point of intersection of the line L and E. Write the equation of the tangent at M.

Line L: y = 2x meets the circle $C: x^2+y^2=9$ at points for which $x^2+4x^2=9 \Rightarrow x=\pm\frac{3}{\sqrt{5}}$.

Coordinates of P are $(\pm\frac{3}{\sqrt{5}},\pm\frac{6}{\sqrt{5}})$

$\Rightarrow$ Coordinates of M are $(\pm\frac{3}{\sqrt{5}},\pm\frac{4}{\sqrt{5}})$

Equation of the tangent at M to the ellipse E is

$\frac{x(\pm3)}{9\sqrt{5}}+\frac{y(\pm4)}{4\sqrt{5}}=1$; $x+3y=\pm3\sqrt{5}$.

Example 11: Equation of the diameter of the ellipse E conjugate to the diameter represented by L is

(A) $9x+2y=0$ (B) $2x+9y=0$

(C) $4x+9y=0$ (D) $4x-9y=0$

Sol: Use the condition of conjugate diameters to find the slope and hence write the equation of the line.

Let y = mx be the diameter conjugate to the diameter $L: y=2x$ of the ellipse E, then

$$2m=-\frac{4}{9}\left(mm'=-\frac{b^2}{a^2}\right)$$

$\Rightarrow$ $m=-\frac{2}{9}$ and the equation of the conjugate diameter is $y=\left(-\frac{2}{9}\right)x$ or $2x+9y=0$.

Example 12: If R is the point of intersection of the line L with the line x = 1, then

(A) R lies inside both C and E

(B) R lies outside both C and E

(C) R lies on both C and E

(D) R lies inside C but outside E

Sol: Use the position of a point w.r.t a circle.

Coordinates of R are (1, 2)

C (1, 2) = $1+2^2-9<0$

$\Rightarrow$ R lies inside C; E (1, 2) = $\frac{1}{9}+1-1>0$

$\Rightarrow$ R lies outside E.

Example 13: If CF is perpendicular from the centre C of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the tangent at any point P, and G is the point where the normal at P meets the minor axis, then $(CF\times PG)^2$ is equal to

Sol: Consider a parametric point on the ellipse and proceed to find CF and PG.

Equation of the tangent at $P(7\cos\theta, 5\sin\theta)$ on the ellipse is $\frac{x}{7}\cos\theta+\frac{y}{5}\sin\theta=1$, then

$$(CF)^2=\frac{7^2\times5^2}{5^2\cos^2\theta+7^2\sin^2\theta}=\frac{25\times49}{25\cos^2\theta+49\sin^2\theta}$$

Equation of the normal at P is

$$\frac{7x}{\cos\theta} - \frac{5y}{\sin\theta} = 7^2 - 5^2$$

Coordinates of G are $\left(0, \frac{-24\sin\theta}{5}\right)$

$$(PG)^2 = (7\cos\theta)^2 + \left(5\sin\theta + \frac{24\sin\theta}{5}\right)^2$$

$$= \frac{49}{25}(25\cos^2\theta + 49\sin^2\theta)$$

So, $(CF.PG)^2 = (49)^2 = 2401$.

JEE Main/Boards

Exercise 1

Q.1 Find the equation of the ellipse whose vertices are (5, 0) and (–5, 0) and foci are (4, 0) and (–4, 0).

Q.2 Find the eccentricity of the ellipse $9x^2 + 4y^2 - 30y = 0$.

Q.3 Find the equations of the tangents drawn from the point (2, 3) to the ellipse $9x^2 + 16y^2 = 144$.

Q.4 Find the eccentric angle of a point on the ellipse $\frac{x^2}{5} + \frac{y^2}{4} = 2$ at a distance 3 from the centre.

Q.5 Obtain equation of chord of the ellipse $4x^2 + 6y^2 = 24$ which has (0, 0) as its midpoint.

Q.6 Find the foci of the ellipse

$25(x+1)^2 + 9(y+2)^2 = 225$.

Q.7 Find the eccentricity of the ellipse if

(a) Length of latus rectum = half of major axis

(b) Length of latus rectum = half of minor axis.

Q.8 Find the condition so that the line $\ell x + my + n = 0$ may be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Q.9 If the normal at the point P(θ) to the ellipse $5x^2 + 14y^2 = 70$ intersects it again at the point Q(2θ), show that $\cos\theta = -\frac{2}{3}$.

Q.10 The common tangent of $\frac{x^2}{25} + \frac{y^2}{4} = 1$ and α lies in 1st quadrant. Find the slope of the common tangent and length of the tangent intercepted between the axis.

Q.11 Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum.

Q.12 Find the equations to the normals at the ends of the latus recta and prove that each passes through an end of the minor axis if $e^4 + e^2 = 1$.

Q.13 Find the co-ordinates of those points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, tangent at which make equal angles with the axes. Also prove that the length of the perpendicular from the centre on either of these is $\sqrt{\frac{1}{2}(a^2 + b^2)}$.

Q.14 Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix.

Q.15 The tangent and normal at any point A of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut its major axis in points P and Q respectively. If PQ = a, prove that the eccentric angle of the point P is given by $e^2\cos^2\phi + \cos\phi - 1 = 0$.

Q.16 A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$.

Q.17 Show that the locus of the middle points of those chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which are drawn through the positive end of the minor axis is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{y}{b}$.

Q.18 Tangents are drawn from a point P to the circle $x^2+y^2=r^2$ so that the chords of contact are tangent to the ellipse $a^2x^2+b^2y^2=r^2$. Find the locus of P.

Q.19 Show that the tangents at the extremities of all chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which subtend a right angle at the centre intersect on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$.

Q.20 Find the length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ whose middle point is $\left(\frac{1}{2},\frac{2}{5}\right)$.

Q.21 Prove that the circle on any focal distance as diameter touches the auxiliary circle.

Q.22 Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $0<b<a$. Let the line parallel to y-axis passing thorugh P meet the circle $x^2+y^2=a^2$ at the point Q such that P and Q are on the same side of the x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that PR : RQ = r : s as P varies over the ellipse.

Q.23 Consider the family of circles $x^2+y^2=r^2$, $2<r<5$. In the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2+25y^2=100$ meets the coordinate axes at A and B, then find the equation of the locus of the mid-point of AB.

Q.24 A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=a+b$ in the points P and Q. Prove that the tangents at P and Q are at right angles.

Q.25 The co-ordinates of the mid-point of the variable chord $y=\frac{1}{2}(x+c)$ of the ellipse $4x^2+9y^2=36$ are

Q.26 A triangle ABC right angled at 'A' moves so that it walsys circumscribes the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The locus of the point 'A' is

Exercise 2

Single Correct Choice Type

Q.1 The equation $\frac{x^2}{2-r}+\frac{y^2}{r-5}+1=0$ represents an ellipse, if

(A) r > 2 (B) $2<r<5$ (C) r > 5 (D) $r\in\{2,5\}$

Q.2 The eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(A) $\frac{5}{6}$ (B) $\frac{3}{5}$ (C) $\frac{\sqrt{2}}{3}$ (D) $\frac{\sqrt{5}}{3}$

Q.3 If $\tan\theta_1.\tan\theta_2=-\frac{a^2}{b^2}$ then the chord joining two points θ_1 and θ_2 on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will subtend a right angle at:

(A) Focus (B) Centre

(C) End of the major axis (D) End of the minor axis

Q.4 If the line $y=2x+c$ be a tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$, then c is equal to

(A) ±4 (B) ±6 (C) ±1 (D) ±8

Q.5 If the line $3x+4y=-\sqrt{7}$ touches the ellipse $3x^2+4y^2=1$ then, the point of contact is

(A) $\left(\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}}\right)$ (B) $\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$

(C) $\left(\frac{1}{\sqrt{7}},\frac{-1}{\sqrt{7}}\right)$ (D) $\left(\frac{-1}{\sqrt{7}},\frac{-1}{\sqrt{7}}\right)$

Q.6 The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding point Q on the auxiliary circle meet on the line:

(A) $x = \frac{a}{e}$ (B) $x = 0$ (C) $y = 0$ (D) None of these

Q.7 The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the positive and of latus rectum

(A) $x + ey + e^2a = 0$ (B) $x - ey - e^3a = 0$

(C) $x - ey - e^2a = 0$ (D) None of these

Q.8 The normal at an end of a latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through an end of the minor axis, if:

(A) $e^4 + e^2 = 1$ (B) $e^3 + e^2 = 1$

(C) $e^2 + e = 1$ (D) $e^3 + e = 1$

Q.9 If CF is perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent at P and G is the point where the normal at P meets the major axis, then the product CF.PG is:

(A) a^2 (B) $2b^2$ (C) b^2 (D) $a^2 - b^2$

Q.10 $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ and $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$. Then the value of b and the other common tangent are given by:

(A) $b = \sqrt{3}$; $x + 2y + a = 0$ (B) $b = 3$; $x + 2y + 4 = 0$

(C) $b = \sqrt{3}$; $x + 2y - 4 = 0$ (D) $b = \sqrt{3}$; $x - 2y - 4 = 0$

Q.11 An ellipse is such that the length of the latus rectum is equal to the sum of the lengths of its semi principal axes. Then:

(A) Ellipse bulges to a circle

(B) Ellipse becomes a line segment between the two foci

(C) Ellipse becomes a parabola

(D) None of these

Q.12 Which of the following is the common tangent to the ellipses, $\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{a^2 + b^2} = 1$?

(A) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$

(B) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(C) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$

(D) $by = ax + \sqrt{a^4 + a^2b^2 + b^4}$

Q.13 In the ellipse the distance between its foci is 6 and its minor axis is 8. Then its eccentricity is

(A) $\frac{4}{5}$ (B) $\frac{1}{52}$ (C) $\frac{3}{5}$ (D) None of these

Q.14 Equation of a tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which cuts off equal intercepts on the axes is-

(A) $x + y - \sqrt{41} = 0$ (B) $x - y + 9 = 0$

(C) $x + y - 9 = 0$ (D) None of these

Q.15 An ellipse has OB as a semi minor axis. FBF' are its foci, and the angle FPF' is a right angle. Then the eccentricity of the ellipse, is

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{\sqrt{2}}$

(C) $\frac{1}{2}$ (D) None of these

Q.16 The length of the latus rectum of the ellipse $9x^2 + 4y^2 = 1$, is

(A) $\frac{3}{2}$ (B) $\frac{8}{3}$ (C) $\frac{4}{9}$ (D) $\frac{8}{9}$

Q.17 If the distance between a focus and corresponding directrix of an ellipse be 8 and the eccentricity be $\frac{1}{2}$, then length of the minor axis is

(A) 3 (B) $4\sqrt{2}$ (C) 6 (D) $\frac{16}{\sqrt{3}}$

Q.18 Let 'E' be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ & 'C' be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then:

(A) Q lies inside C but outside E

(B) Q lies outside both C and E

(C) P lies inside both C and E

(D) P lies inside C but outside E

Q.19 The line, $\ell x+my+n=0$ will cut the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in poins whose eccentric angle differe by $\frac{\pi}{2}$ if:

(A) $x^2\ell^2+b^2n^2=2m^2$ (B) $a^2m^2+b^2\ell^2=2n^2$

(C) $a^2\ell^2+b^2m^2=2n^2$ (D) $a^2n^2+b^2m^2=2\ell^2$

Q.20 The locus of point of intersection of tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at two points the sum of whose eccentric angles is constant is:

(A) A hyperbola (B) An ellipse

(C) A circle (D) A straight line

Q.21 Q is a point on the auxiliary circle of an ellipse. P is the corresponding point on ellipse. N is the foot of perpendicular from focus S, to the tangent of auxiliary circle at Q. Then

(A) SP = SN (B) SP = PQ

(C) PN = SP (D) NQ = SP

Q.22 A tangent to the ellipse $4x^2+9y^2=36$ is cut by the tangent at the extremities of the major axis at T and T′. The circle on TT′ as diameter passes thorugh the point

(A) (0, 0) (B) $(\pm 5,0)$ (C) $(\pm\sqrt{5},0)$ (D) $(\pm 3,0)$

Q.23 Q is a point on the auxiliary circle corresponding to the point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If T is the foot of the perpendicular dropped from the focus S onto the tangent to the auxiliary circle at Q then the ΔSPT is:

(A) Isosceles (B) Equilateral

(C) Right angled (D) Right isosceles

Q.24 $y=mx+c$ is a normal to the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if c^2 is equal to

(A) $\frac{(a^2-b^2)^2}{a^2m^2+b^2}$ (B) $\frac{(a^2-b^2)^2}{a^2m^2}$

(C) $\frac{(a^2-b^2)^2m^2}{a^2+b^2m^2}$ (D) $\frac{(a^2-b^2)^2m^2}{a^2m^2+b^2}$

Q.25 The equation $2x^2 + 3y^2 - 8x - 18y + 35 = K$ represents:

(A) A point if K = 0

(B) An ellipse if K < 0

(C) A hyperbola if K < 0

(D) A hyperbola if K > 0

Previous Years' Questions

Q.1 If P = (x, y), F_1 = (3, 0), F_2 = (−3, 0) and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals ***(1998)***

(A) 8 (B) 6 (C) 10 (D) 12

Q.2 The number of values of c such that the straight line y = 4x + c touches the curve $\frac{x^2}{4}+y^2=1$ is ***(1998)***

(A) 0 (B) 2 (C) 1 (D) ∞

Q.3 The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is ***(2009)***

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Q.4 Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then the maximum value of A is................ ***(1994)***

Q.5 An ellipse has OB as a semi minor axis. F and F′ are its foci and the angle FBF′ is a right angle. Then, the eccentricity of the ellipse is............ ***(1997)***

Q.6 Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1, 0<b<a. Let the line parallel to y-axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same side of x-axis. For two positive real numbers r ans s, find the locus of the point R on PQ such that PR : RQ = r : s as P varies over the ellipse. ***(2001)***

Q.7 Find the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes. ***(2005)***

Q.8 A focus of an ellipse is at the origin. The directrix is the line x = 4 and the eccentricity is 1/2. Then the length of the semi−major axis is ***(2008)***

(A) $\frac{8}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{3}$

Q.9 The ellipse $x^2 + 16y^2 = 16$ is inscribed in a rectangle aligned with the coordinate axes, which in turn in inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is ***(2009)***

(A) $x^2 + 16y^2 = 16$ (B) $x^2 + 12y^2 = 16$

(C) $4x^2 + 48y^2 = 48$ (D) $4x^2 + 64y^2 = 48$

Q.10 Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (-3, 1) and has eccentricity is ***(2011)***

(A) $5x^2 + 3y^2 - 48 = 0$ (B) $3x^2 + 5y^2 - 15 = 0$

(C) $5x^2 + 3y^2 - 32 = 0$ (D) $3x^2 + 5y^2 - 32 = 0$

Q.11 An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semiminor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$as its semi-major axis. If the centre of the ellipse is the origin and its axes are the coordinate axes, then the equation of the ellipse is ***(2012)***

(A) $4x^2 + y^2 = 4$ (B) $x^2 + 4y^2 = 8$

(C) $4x^2 + y^2 = 8$ (D) $x^2 + 4y^2 = 16$

Q.12 The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having centre at (0, 3) is ***(2013)***

(A) $x^2 + y^2 - 6y - 7 = 0$ (B) $x^2 + y^2 - 6y + 7 = 0$

(C) $x^2 + y^2 - 6y - 5 = 0$ (D) $x^2 + y^2 - 6y + 5 = 0$

Q.13 The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is ***(2014)***

(A) $\left(x^2 + y^2\right)^2 = 6x^2 + 2y^2$

(B) $\left(x^2 + y^2\right)^2 = 6x^2 - 2y^2$

(C) $\left(x^2 - y^2\right)^2 = 6x^2 + 2y^2$

(D) $\left(x^2 - y^2\right)^2 = 6x^2 - 2y^2$

Q.14 The area (in sq.units) of the quadrilateral formed by the tangents at the end points of the latera recta to the Ellipse ***(2015)***

(A) $\frac{27}{4}$ (B) 18 (C) $\frac{27}{2}$ (D) 27

JEE Advanced/Boards

Exercise 1

Q.1 Find the equation of the ellipse with its centre (1, 2), focus at (6, 2) and containing the point (4, 6).

Q.2 The tangent at any point P of a circle $x^2 + y^2 = a^2$ meets the tangent at a fixed point A (a, 0) in T and T is joined to B, the other end of the diameter through A, prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$.

Q.3 The tangent at the point α on a standard ellipse meets the auxiliary circle in two points which subtends a right angle at the centre. Show that the eccentricity of the ellipse is $(1 + \sin^2 \alpha)^{-1/2}$.

Q.4 An ellipse passes through the points (–3, 1) and (2, –2) and its principal axis are along the coordinate axes in order. Find its equation.

Q.5 If any two chords be drawn through two points on the major axis of an ellipse equidistant from the centre, show that $\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}.\tan\frac{\gamma}{2}.\tan\frac{\delta}{2} = 1$ where $\alpha, \beta, \gamma, \delta$ are the eccentric angles of the extremities of the chords.

Q.6 (a) Obtain the equations of the tangents to the ellipse $5x^2 + 9y^2 = 45$, perpendicular to $3x + 4y = 11$.

(b) Prove that the straight line $\frac{ax}{3} + \frac{by}{4} = c$ will be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $5c = a^2e^2$.

Q.7 Prove that the equation to the circle having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of a latus rectum, is $x^2+y^2-2ae^3x=a^2(1-e^2-e^4)$.

Q.8 Find the equations of the lines with equal intercepts on the axis & which touch the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Q.9 The tangent at $P\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2+11y^2=256$ is also a tangent to the circle $x^2+y^2-2x-15=0$. Find the θ. Find also the equation to the common tangent.

Q.10 A tangent having slope $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$, intersects the axis of x and y in points A and B respectively. If O is the origin, find the area of triangle OAB.

Q.11 'O' is the origin & also the centre of two concentric circles having radii of the inner & the outer circle as 'a' and 'b' respectively. A line OPQ is drawn to cut the inner circle in P & the outer circle in Q. PR is drawn parallel to the y-axis & QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the two circles. If the foci of this ellipse lie on the inner circle, find the ratio of inner outer radii & find also the eccentricity of the ellipse.

Q.12 ABC is an isosceles triangle with its base BC twice its altitude. A point P moves within the triangle such that the square of its distance from BC is half the rectangle contained by its distances from the two sides. Show that the locus of P is an ellipse with eccentricity $\sqrt{\frac{2}{3}}$ passing through B & C.

Q.13 Find the equations of the tangents drawn from the point (2, 3) to the ellipse, $9x^2+16y^2=144$.

Q.14 Common tangents are drawn to the parabola $y^2=4x$ & the ellipse $3x^2+8y^2=48$ touching the parabola at A and B and the ellipse at C & D. Find the area of the quadrilateral.

Q.15 If the normal at a point P on the ellipse of semi axes a, b & centre C cuts the major & minor axes at G and g, show that $a^2(CG)^2+b^2.(Cg)^2=(a^2-b^2)^2$. Also prove that $CG=e^2CN$, where PN is the ordinate of P.

Q.16 Prove that the length of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is inclined to the major axis at angle θ is $\frac{2ab^2}{a^2+\sin^2\theta+b^2\cos^2\theta}$.

Q.17 The tangent at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersects the major axis in T & N is the foot of the perpendicular from P to the same axis. Show that the circle on NT as diameter intersects the auxiliary circle orthogonally.

Q.18 The tangents from (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect at right angles. Show that the normals at the points of contact meet on the line $\frac{y}{y_1}=\frac{x}{x_1}$.

Q.19 Find the locus of the point the chord of contact of the tangent drawn from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=C^2$, where $c<b<a$.

Q.20 Find the equation of the common tangents to the ellipse $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$.

Q.21 P and Q are the corresponding point on a standard ellipse and its auxiliary circle. The tangent at P to the ellipse meets the major axis in T. prove that QT touches the auxiliary circle.

Q.22 If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$, intersects it again at the point $Q(2\theta)$, show that $\cos\theta=-\left(\frac{2}{3}\right)$.

Q.23 A straight line AB touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the circle $x^2+y^2=r^2$; where $a>r>b$. PQ is a focal chord of the ellipse. If PQ be parallel to AB and cuts the circle in P & Q, find the length of the perpendicular drawn from the centre of the ellipse to PQ. Hence show that PQ = 2b.

Q.24 If the tangent at any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact then show that the eccentricity of the ellipse is given by: $e=\frac{\cos\beta}{\cos\alpha}$.

Q.25 An ellipse is drawn with major and minor axes of lengths 10 and 8 respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. The radius of the circle is________.

Q.26 Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. 1 f OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then the product (AB) (CD) = ________________________.

Exercise 2

Single Correct Choice Type

Q.1 The equation to the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ included between the co-ordinate axes is the curve.

(A) $9x^2+16y^2=4x^2y^2$ (B) $16x^2+9y^2=4x^2y^2$

(C) $3x^2+4y^2=4x^2y^2$ (D) $9x^2+16y^2=x^2y^2$

Q.2 P & Q are corresponding points on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and the auxiliary circle respectively. The normal at P to the ellipse meets CQ in R where C is centre of the ellipse. Then $\ell(CR)$ is

(A) 5 units (B) 6 units (C) 7 units (D) 8 units

Q.3 The equation of the ellipse with its centre at (1, 2), focus at (6, 2) and passing through the point (4, 6) is

(A) $\frac{(x-1)^2}{45}+\frac{(y-2)^2}{20}=1$ (B) $\frac{(x-1)^2}{20}+\frac{(y-2)^2}{45}=1$

(C) $\frac{(x-1)^2}{25}+\frac{(y-2)^2}{16}=1$ (D) $\frac{(x-1)^2}{16}+\frac{(y-2)^2}{25}=1$

Q.4 A line of fixed length (a+b) moves so that its ends are always on two fixed perpendicular straight lines. The locus of the point which divided this line into portions of lengths a and b is:

(A) An ellipse (B) An hyperbola

(C) A circle (D) None of these

Q.5 An ellipse is described by using an endless string which passes over two pins. If the axes are 6 cm and 4 cm, the necessary length of the string and the distance between the pins respectively in cm, are

(A) $6, 2\sqrt{5}$ (B) $6, \sqrt{5}$

(C) $4, 2\sqrt{5}$ (D) None of these

Q.6 If F_1 & F_2 are the feet of the perpendiculars from the foci S_1 and S_2 of an ellipse $\frac{x^2}{5}+\frac{y^2}{3}=1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to:

(A) 2 (B) 3 (C) 4 (D) 5

Q.7 a & b are positive real numbers, such that $a>b$. If the area of the ellipse $ax^2+by^2=3$ equals area of the ellipse $(a+b)x^2+(a-b)y^2=3$, then a/b is equal to

(A) $\frac{\sqrt{5}+1}{4}$ (B) $\frac{\sqrt{5}+1}{2}$ (C) $\frac{\sqrt{6}-1}{2}$ (D) $\frac{\sqrt{5}-1}{4}$

Q.8 The locus of image of the focus of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b)$ with respect to any of the tangent to the ellipse is

(A) $(x\pm ae)^2=y^2+4a^2$ (B) $(x\pm ae)^2=4a^2-y^2$

(C) $(x\pm ae)^2=y^2-4a^2$ (D) $(x\pm ae)^2=4a^2$

Q.9 The normal at a variable point P on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of eccentricity e meets the axes of the ellipse in Q and R then the locus of the mid-point of QR is a conic with an eccentricity e' such that:

(A) e' is independent of e

(B) e ' = 1

(C) e ' = e

(D) e ' = 1/e

Q.10 A circle has the same centre as an ellipse & passes through the foci F_1 and F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 and the area of the triangle PF_1F_2 is 30, then the distance between the foci is:

(A) 11 (B) 12

(C) 13 (D) None of these.

Q.11 The arc of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$ is:

(A) $\frac{(a^2-b^2)ab}{a^2+b^2}$ (B) $\frac{(a^2+b^2)ab}{a^2-b^2}$

(C) $\frac{(a^2-b^2)}{ab(a^2+b^2)}$ (D) $\frac{(a^2+b^2)}{(a^2-b^2)ab}$

Q.12 Co-ordinates of the vertices B and C of a triangle ABC are (2, 0) and (8, 0) respectively. The vertex A is varying in such a way that $4\tan\frac{B}{2}\tan\frac{C}{2}=1$. Then locus of A is

(A) $\frac{(x-5)^2}{25}+\frac{y^2}{16}=1$ (B) $\frac{(x-5)^2}{16}+\frac{y^2}{25}=1$

(C) $\frac{(x-5)^2}{25}+\frac{y^2}{9}=1$ (D) $\frac{(x-5)^2}{9}+\frac{y^2}{25}=1$

Multiple Correct Choice Type

Q.13 Identify the statement which are True

(A) The equation of the director circle of the ellipse, $5x^2+9y^2=45$ is $x^2+y^2=14$

(B) The sum of the focal distances of the point (0, 6) on the ellipse $\frac{x^2}{25}+\frac{y^2}{36}=1$ is 10

(C) The point of intersection of anytangent to a parabola and the perpendicular to it from the focus lies on the tangent at the vertex

(D) The line through focus and $(at_1^2, 2at_1)$ $y^2=4ax$, meets it again in on the point $(at_2^2, 2at_2)$, if $t_1t_2=-1$.

Q.14 The angle between pair of tangents drawn to the ellipse $3x^2+2y^2=5$ from the point (1, 2) is

(A) $\tan^{-1}\frac{12}{5}$ (B) $\tan^{-1}\frac{6}{\sqrt{5}}$

(C) $\tan^{-1}\frac{12}{\sqrt{5}}$ (D) $\pi-\cot^{-1}\left(-\frac{\sqrt{5}}{12}\right)$

Q.15 If P is a point of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose foci are S and S′. Let $\angle PSS'=\alpha$ and $\angle PS'S=\beta$, then

(A) PS + PS′ = 2a, if $a>b$

(B) PS + PS′ = 2b, if $a<b$

(C) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}$

(D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{\sqrt{a^2-b^2}}{b^2}[a-\sqrt{a^2-b^2}]$ when $a>b$

Q.16 The equation of the common tangents to the ellipse $x^2+4y^2=8$ and the parabola $y^2=4x$ are

(A) $2y-x=4$ (B) $2y+x=4$

(C) $2y+x+4=0$ (D) $2y+x=0$

Q.17 The distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$, from its centre is 2. Then the eccentric angle is:

(A) $\pi/4$ (B) $3\pi/4$ (C) $5\pi/4$ (D) $7\pi/4$

Q.18 The tangents at any point F on the standard ellipse with foci as S and S′ meets the tangents at the vartices A and A′ in the points V and V′, then:

(A) $\ell(AV).\ell(AV')=b^2$

(B) $\ell(AV).\ell(A'V')=a^2$

(C) $\angle V'SV=90^0$

(D) V′S′ VS is a cyclic quadrilateral

Previous Years' Questions

Q.1 If $a>2b>0$, then positive value of m for which $y=mx-b\sqrt{a+m^2}$ is a common tangent to $x^2+y^2=b^2$ and $(x-a)^2+y^2=b^2$ is ***(2002)***

(A) $\frac{2b}{\sqrt{a^2-4b^2}}$ (B) $\frac{\sqrt{a^2-4b^2}}{2b}$

(C) $\frac{2b}{a-2b}$ (D) $\frac{b}{a-2b}$

Q.2 Tangent is drawn to ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where $\theta\in(0,\pi/2)$). Then the value of θ such that the sum of intercepts on axes made by this tangent is minimum, is ***(2003)***

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{8}$ (D) $\frac{\pi}{4}$

Q.3 The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. if M is the mid point of the line segment PQ, then the locus of M intersects the latusrectum of the given ellipse at the points. ***(2009)***

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$

(C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

Q.4 An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse, in the standard form is............... ***(1996)***

Paragraph Based Questions 5 to 7

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B. ***(2010)***

Q.5 The coordinates of A and B are

(A) (3, 0) and (0, 2)

(B) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(C) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(D) (3,0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

Q.6 The orthocenter of the triangle PAB is

(A) $\left(5, \frac{8}{7}\right)$ (B) $\left(\frac{7}{5}, \frac{25}{8}\right)$ (C) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

Q.7 The equation of the locus of the point whose distances from the point P and the line AB are equal is

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Q.8 Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that ***(1995)***

$$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$$

Q.9 A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipses $x^2 + 2y^2 = 6$ at P and Q. Prove that tangents at P and Q of ellipse $x^2 + 2y^2 = 6$ are at right angles. ***(1997)***

Q.10 Find the coordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area fo the triangle PON is maximum, where O denotes the origin and N be the foot of the perpendicular from O to the tangent at P. ***(1999)***

Q.11 Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ meets the ellipse respectively at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that, the normals to the ellipse drawn at the points P, Q and R are concurrent. ***(2000)***

Q.12 Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse of the point of contact meet on the corresponding directrix. ***(2002)***

Q.13 Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1<0, y_2<0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are ***(2008)***

(A) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

Q.14 An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then ***(2009)***

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

(B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$

(D) The foci of ellipse are $\left(\pm\sqrt{2},0\right)$

Q.15 The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2+9y^2=9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is ***(2009)***

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Q.16 Match the conics in column I with the statements/ expressions in column II. ***(2009)***

Column I	Column II
(A) Circle	(p) The locus of the point (h, k) for which the line hx + ky = 1 touches the circle $x^2+y^2=4$
(B) Parabola	(q) Points z in the complex plane satisfying $\left\|z+2\right\|-\left\|z-2\right\|=\pm3$
(C) Ellipse	(r) Points of the conic have parametric representation $x=\sqrt{3}\left(\frac{1-t^2}{1+t^2}\right), y=\frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1\le x<\infty$
	(t) Points z in the complex plane satisfying $\text{Re}\left(z+1\right)^2=\left\|z\right\|^2+1$

Q.17 Equation of a common tangent with positive slope to the circle as well as to the hyperbola is ***(2010)***

(A) $2x-\sqrt{5}y-20=0$ (B) $2x-\sqrt{5}y+4=0$

(C) $3x-4y+8=0$ (D) $4x-3y+4=0$

Q.18 The ellipse $E_1:\frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes therectangle R. The eccentricity of the ellipse E_2 is ***(2012)***

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Q.19 A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)=$ area of the triangle PQR, $\Delta_1=\max\limits_{1/2\le h\le 1}\Delta(h)$ and $\Delta_2=\min\limits_{1/2\le h\le 1}=\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=$ ____ ***(2013)***

Q.20 Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$ where, $f_1 > 0$ and $f_1 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. Then m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m^2}+m_2^2\right)$ is ***(2015)***

Q.21 Consider the hyperbola $H:x^2-y^2=1$ and a circle S with center $N\left(x_2,0\right)$. Suppose that H and S touch each other at a point $P\left(x_1,y_1\right)$ with $x_1>1$ and $y_1>0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle Δ PMN, then the correct expression(s) is(are) ***(2015)***

(A) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1>1$

(B) $\frac{dm}{dx_1}=\frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1>1$

(C) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1>1$

(D) $\frac{dm}{dx_1}=\frac{1}{3}$ for $y_1>0$

Q.22 If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is ***(2016)***

(A) 3 : 1 (B) 4 : 5

(C) 5 : 8 (D) 2 : 3

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.6	Q.13	Q.16
Q.19	Q.21	Q.22
Q.23	Q.25	Q.27

Exercise 2

Q.6	Q.8	Q.11
Q.14	Q.15	Q.21
Q.23	Q.25	

Previous Years' Questions

Q.2 Q.5 Q.6

JEE Advanced/Boards

Exercise 1

Q.3	Q.5	Q.7
Q.9	Q.12	Q.15
Q.17	Q.26	

Exercise 2

Q.1	Q.4	Q.6
Q.10	Q.13	

Previous Years' Questions

Q.1	Q.4	Q.11
Q.12		

Answer Key

JEE Main/Boards

Exercise 1

Q.1 $9x^2 + 25y^2 = 225$

Q.2 $\sqrt{5}/3$

Q.3 $y - 3 = 0,\ x + y = 5$

Q.4 $\pi/4,\ 3\pi/4$

Q.5 All lines passing through origin

Q.6 $(-1, 2)$ and $(-1, -6)$

Q.7 (a) $e = \frac{1}{\sqrt{2}}$ (b) $e = \frac{\sqrt{3}}{2}$

Q.8 $\frac{a^2}{\ell^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$

Q.10 slope $= \pm\frac{2}{\sqrt{3}}$; length $= \frac{14}{\sqrt{3}}$

Q.11 (2, 1)

Q.12 $\frac{a^2x}{ae} - \frac{b^2y}{\pm\left(\frac{b^2}{a}\right)} = a^2 - b^2$ or $\frac{a^2x}{-ae} - \frac{b^2y}{\pm\left(\frac{b^2}{a}\right)} = a^2 - b^2$

Q. 13 $\left(\pm\frac{a^2}{a^2+b^2}, \pm\frac{b^2}{a^2+b^2}\right)$

Q.18 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$

Q.20 $\frac{7}{5}\sqrt{41}$

Q.22 $\frac{x^2}{a^2}+\frac{y^2(r+s)^2}{(ra+sb)^2}=1$

Q.23 $4x^2y^2=25y^2+4x^2$

Q.25 $-9c/25, 8c/25$

Q.26 $x^2+y^2=a^2+b^2$, a director circle

Exercise 2

Single Correct Choice Type

Q.1 B **Q.2** D **Q.3** B **Q.4** B **Q.5** D **Q.6** C

Q.7 B **Q.8** A **Q.9** C **Q.10** A **Q.11** A **Q.12** B

Q.13 C **Q.14** A **Q.15** B **Q.16** C **Q.17** D **Q.18** D

Q.19 C **Q.20** B **Q.21** A **Q.22** C **Q.23** A **Q.24** C

Q.25 A

Previous Years' Questions

Q.1 C **Q.2** B **Q.3** D **Q.4** $b\sqrt{a^2-b^2}$ **Q.5** $\frac{1}{\sqrt{2}}$

Q.6 $\frac{x^2}{a^2}+\frac{y^2(r+s)^2}{(ar+bs)^2}=1$ **Q.7** $\frac{14}{\sqrt{3}}$ **Q.8** A **Q.9** B

Q.10 D **Q.11** D **Q.12** A **Q.13** A **Q.14** D

JEE Advanced/Boards

Exercise 1

Q.1 $4x^2+9y^2-8x-36y-175=0$

Q.4 $3x^2+5y^2=32$

Q.6 (a) $4x-3y+3\sqrt{21}=0$; $4x-3y-3\sqrt{21}=0$

Q.8 $x+y-5=0,\ x+y+5=0$

Q.9 $\theta=\frac{\pi}{3}$ or $\frac{5\pi}{3}$; $4x\pm\sqrt{33}y-32=0$

Q.10 24 sq. units

Q.11 $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Q.13 $y-3=0$ & $x+y=5$

Q.14 $55\sqrt{2}$ sq. units

Q.19 $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$

Q.20 $by=\pm ax\sqrt{a^4+a^2b^2+b^4}$

Q.23 $\sqrt{r^2-b^2}$

Q.25 2 units

Q.26 65

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** A	**Q.5** A	**Q.6** B
Q.7 B	**Q.8** B	**Q.9** C	**Q.10** C	**Q.11** A	**Q.12** A

Multiple Correct Choice Type

Q.13 A, C, D	**Q.14** C, D	**Q.15** A, B, C	**Q.16** A, C	**Q.17** A, B, C, D	**Q.18** A, C, D

Previous Years' Questions

Q.1 A **Q.2** B **Q.3** C **Q.4** $\frac{(x-(1/3))^2}{1/9}+\frac{(y-1)^2}{1/12}=1$

Q.5 D **Q.6** C **Q.7** A **Q.10** $\left(\frac{\pm a^2}{\sqrt{a^2+b^2}},\frac{\pm b^2}{\sqrt{a^2+b^2}}\right)$

Q.13 B, C **Q.14** A, B **Q.15** D **Q.16** A $\rightarrow$ p; B $\rightarrow$ s, t; C $\rightarrow$ r; D $\rightarrow$ q, s

Q.17 B **Q.18** C **Q.20** D **Q.21** A, B, D **Q.22** C

Solutions

JEE Main/Boards

Exercise 1

Sol 1: $2a = 10$

$\Rightarrow a = 5$

$ae = 4 \Rightarrow e = \frac{4}{a}=\frac{4}{5}$

$\therefore b^2 = a^2(1 - e^2)$

$b^2 = 25\left(1-\frac{16}{25}\right) = 9$

$\therefore b = \pm 3$

Equation of ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$ or $9x^2 + 25y^2 = 225$

Sol 2: $9x^2 + 4y^2 - 30y = 0$

$9x^2 + 4\left(y^2-\frac{15}{2}y\right) = 0$

$9x^2 + 4\left(y-\frac{15}{4}\right)^2-\frac{225}{4}=0$

or $9x^2 + 4\left(y-\frac{15}{4}\right)^2=\left(\frac{15}{2}\right)^2$

or $\frac{x^2}{\left(\frac{5}{2}\right)^2}+\frac{\left(y-\frac{15}{4}\right)^2}{\left(\frac{15}{4}\right)^2}=1$

$\therefore a = \frac{15}{4}, b = \frac{5}{2}$

$e^2 = 1-\frac{b^2}{a^2} = 1 - \frac{25}{4} \times \frac{16}{225} = 1 - \frac{4}{9}=\frac{5}{9}$

$\therefore e=\frac{\sqrt{5}}{3}$

Sol 3: $\frac{x^2}{16}+\frac{y^2}{9}=1$

P(2, 3)

$\therefore a^2 = 16$ or $a = \pm 4$

$b^2 = 9$ or $b = \pm 3$

$\therefore$ Equation is $\frac{x^2}{(4)^2}+\frac{y^2}{(3)^2}=1$

$\therefore$ Equation tangent will be

$y = mx + \sqrt{a^2m^2+b^2}$

$y = mx + \sqrt{16m^2+9}$

As this line passes through (2, 3)

$\therefore 3 - 2m = \sqrt{16m^2+9}$

$\Rightarrow 9 + 4m^2 - 12m = 16m^2 + 9$

or $12m^2 + 12m = 0$

$12(m + 1)m = 0$

$\Rightarrow m = 0, -1$

$\therefore y = 3$ or $y = -x + 5$

i.e. $y - 3 = 0$ or $x + y = 5$

Sol 4: $\frac{x^2}{5}+\frac{y^2}{4}=2 \Rightarrow \frac{x^2}{10}+\frac{y^2}{8}=1$

let ϕ be eccentric angle

$\therefore$ Any point on ellipse will be $(a\cos\phi, b\sin\phi)$

$\therefore P = \left(\sqrt{10}\cos\phi, \sqrt{8}\sin\phi\right)$

$\therefore \sqrt{(\sqrt{10}\cos\phi)^2+(\sqrt{8}\sin\phi)^2} = 3$

$\Rightarrow 10\cos^2\phi + 8\sin^2\phi = 9$

or $2\cos^2\phi = 1$

or $\cos^2\phi = \frac{1}{2}$

$\therefore \cos\phi = \pm\frac{1}{\sqrt{2}}$ or $\phi = \frac{\pi}{4}, \frac{3\pi}{4}$

Sol 5: $\frac{x^2}{4}+\frac{y^2}{6}=1$

Since midpoint of chord is (0, 0)

$\therefore$ Take one point as $(a\cos\alpha, b\sin\alpha)$

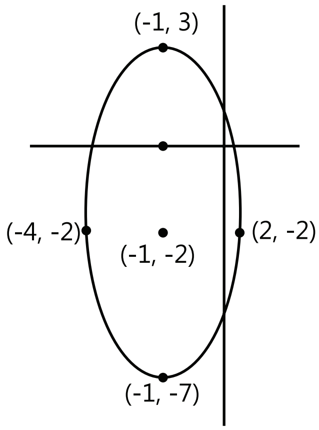
and another point as$(-a\cos\alpha, -b\sin\alpha)$

$\therefore -a\cos\alpha = a\cos\beta \Rightarrow \beta = \pi +$

So there are infinite value of α which will satisfy this condition therefore all line passing through origin will be chord to the given ellipse.

Sol 6:

$\frac{(x+1)^2}{9}+\frac{(y+2)^2}{25}=1$



$a^2 = 25$

$b^2 = 9$

$b^2 = a^2(1 - e^2)$

$9 = 25(1 - e^2)$

$\Rightarrow e = \frac{4}{5}$

y coordinate of foci

$= -2 \pm \left(5\times\frac{4}{5}\right)$

$= -2 \pm 4 = (2, 6)$

$\therefore$ foci is at $(-1, 2), (-1, -6)$

Sol 7: Length of latus rectum $= \frac{2b^2}{a}$

(a) If $\frac{2b^2}{a}=a$

$\Rightarrow 2b^2 = a^2$

$\therefore \frac{b^2}{a^2}=\frac{1}{2}$ or $e^2 = 1-\frac{1}{2}=\frac{1}{2}$

$\therefore e=\frac{1}{\sqrt{2}}$

(b) if $\frac{2b^2}{a}=b$

$\Rightarrow 2b = a$

$\therefore 4b^2 = a^2$

or $\frac{b^2}{a^2}-\frac{1}{4}$

$\therefore e^2 = 1-\frac{1}{4}=\frac{3}{4}$

$\therefore e=\frac{\sqrt{3}}{2}$

Sol 8: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\lambda x + my + n = 0$

$y = -\frac{\ell}{m}x - \frac{n}{m}$

Equation of normal

$\Rightarrow y = m'x \mp \frac{(a^2-b^2)m'}{\sqrt{a^2+b^2m'^2}}$

$\therefore \frac{-l}{m} = m'; \mp \frac{(a^2-b^2)m'}{\sqrt{a^2+b^2(m')^2}} = -\frac{n}{m}$

$\therefore \frac{n}{m} = \mp \frac{(a^2-b^2)\frac{\ell}{m}}{\sqrt{a^2+b^2\times\frac{\ell^2}{m^2}}}$

$\Rightarrow \frac{\sqrt{a^2m^2+b^2\ell^2}}{m} = \mp (a^2-b^2)\frac{\ell}{n}$

$\therefore a^2m^2 + b^2\lambda^2 = (a^2-b^2)^2\frac{\ell^2m^2}{n^2}$

$\Rightarrow \frac{a^2}{\ell^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$

Sol 9: Normal of P(a cosθ, b sinθ)

$\frac{x^2}{14} + \frac{y^2}{5} = 1$

$\frac{a^2x}{a\cos\theta} - \frac{b^2y}{b\sin\theta} = a^2 - b^2$

$\frac{\sqrt{14}x}{\cos\theta} - \frac{\sqrt{5}y}{\sin\theta} = 14 - 5 = 9$

As this passes through (a cos2θ, b sin2θ)

$\therefore \frac{14\cos 2\theta}{\cos\theta} - \frac{5\sin 2\theta}{\sin\theta} = 9$

$\therefore 14[2\cos^2\theta - 1] - 5 \times 2\cos^2\theta = 9\cos\theta$

$\therefore 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\cos\theta = \frac{9 \pm \sqrt{81+4\times14\times18}}{36}$

$= \frac{9 \pm .33}{36} = -\frac{24}{36}, \frac{42}{36}$

$\therefore \cos\theta = -\frac{2}{3}$ is only possible solution

Sol 10: Equation of tangent for $\frac{x^2}{25} + \frac{y^2}{4} = 1$

$y = mx \pm \sqrt{a^2m^2+b^2}$

$y = mx \pm \sqrt{25m^2+4}$

equation of tangent for $\frac{x^2}{16} + \frac{y^2}{16} = 1$ is

$y = mx \pm 4\sqrt{m^2+1}$

$\therefore 25m^2 + 4 = 16m^2 + 16$

or $9m^2 = 12$ or $m = \pm\frac{2}{\sqrt{3}}$

equation tangent is $y = \pm\frac{2}{\sqrt{3}}x \pm 4\sqrt{\frac{4}{3}+1}$

$y = \pm\frac{2}{\sqrt{3}}x \pm \frac{4\sqrt{7}}{\sqrt{3}}$

y intercept $= \frac{4\sqrt{7}}{\sqrt{3}}$

x intercept $= 2\sqrt{7}$

$\therefore$ Length $= \sqrt{4\times7+\frac{16}{3}\times7}$

$= \frac{\sqrt{7}\times\sqrt{28}}{\sqrt{3}} = \frac{7\times2}{\sqrt{3}} = \frac{14}{\sqrt{3}}$

Sol 11: Point on curve $\frac{x^2}{6} + \frac{y^2}{3} = 1$

is $(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$

$\therefore$ Distance of point from line x + y – 7 = 0 is

$\frac{\sqrt{6}\cos\theta + \sqrt{3}\sin\theta - 7}{\sqrt{2}} = f(\theta)$

$\therefore f'(\theta) = -\sqrt{3}\sin\theta + \frac{\sqrt{3}}{\sqrt{2}}\cos\theta = 0$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{2}}$

or $\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$

$\therefore$ Point is $\left(\frac{\sqrt{2}.\sqrt{6}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{3}}\right)$

P = (2, 1)

Sol 12: End of latus rectum is $\left(ae, \pm\frac{b^2}{a}\right)$

or $\left(-ae, \pm\frac{b^2}{a}\right)$

∴ Equation of normal is

$$\frac{a^2x}{-ae} - \frac{b^2y}{\pm\left(\frac{b^2}{a}\right)} = a^2 - b^2$$

or $\dfrac{a^2x}{-ae} - \dfrac{b^2y}{\pm\left(\frac{b^2}{a}\right)} = a^2 - b^2$

if they passed through (0, b)

$\Rightarrow \mp ab = a^2 - b^2$

$\Rightarrow \left(\dfrac{b^2}{a^2} - 1\right)^2 = \dfrac{b^2}{a^2}$

$\Rightarrow (-e^2) = 1 - e^2$

$\Rightarrow e^4 + e^2 = 1$

Sol 13: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Let point be (a cosϕ, b sinϕ)

i.e. (a cosϕ, b sin ϕ)

∴ Equation of tangent is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

Tangent is inclined at equal angle to axis

$\because$ m = ± 1 or $\left|-\dfrac{b}{a}\cot\theta\right| = |\pm 1|$

or $\tan\theta = \dfrac{b}{a}$

$\therefore \cos\theta = \pm\dfrac{a}{\sqrt{a^2+b^2}}$,

$\sin\theta = \pm\dfrac{b}{\sqrt{a^2+b^2}}$

∴ Points are $\left(\dfrac{\pm a^2}{a^2+b^2}, \pm\dfrac{b^2}{a^2+b^2}\right)$

And equation of tangent is

$$\frac{x}{a^2}\left(\pm\frac{a^2}{a^2+b^2}\right) + \frac{y}{b^2}\left(\pm\frac{b^2}{a^2+b^2}\right) = 1$$

or $\dfrac{x}{a^2+b^2} + \dfrac{y}{a^2+b^2} \pm 1 = 0$

∴ Distance of tangent from origin is

$$\left|\frac{\pm 1}{\sqrt{\left(\frac{1}{a^2+b^2}\right) + \left(\frac{1}{a^2+b^2}\right)}}\right|$$

$$= \sqrt{\frac{1}{2}(a^2+b^2)}$$

Hence proved.

Sol 14: Let P = (acosθ, bsinθ)

Slope of tangent = $-\dfrac{b}{a\tan\theta}$

∴ Slope of normal to tangent

$= \dfrac{a\tan\theta}{b}$

∴ Equation of line

FN = (y) = $\dfrac{a}{b}\tan\theta(x - ae)$ ...(i)

And equation of CP

$y = \dfrac{b}{a}\tan\theta x$ ...(ii)

$\therefore \dfrac{bx}{a} = \dfrac{a}{b}(x - ae)$

$\therefore \left(\dfrac{b^2 - a^2}{ab}\right)x = -\dfrac{a^2e}{b}$

$\therefore x = \dfrac{a}{e}$

∴ The two lines intersect on directrix

Sol 15: Normal : axsecθ – by cosecθ = $a^2 - b^2$

Tangent : $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$

for y = 0

point of intersection of normal is

$x = \dfrac{a^2 - b^2}{a\sec\theta} = \dfrac{ae^2}{\sec\theta} = ae^2\cos\theta$

Point of intersection of tangent is

$x = a\sec\theta = \dfrac{a}{\cos\theta}$

$-\dfrac{ae^2}{\sec\theta} + a\sec\theta = a$

or $-e^2 + \sec^2\theta = \sec\theta$

or $e^2\cos^2\theta - 1 = -\cos\theta$

$\Rightarrow e^2\cos^2\theta + \cos\theta - 1 = 0$

Sol 16: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$x^2 + y^2 = r^2$

equation of tangent at circle $\Rightarrow x\cos\theta + y\sin\theta = r$

or equation of tangent at ellipse is $y = mx + \sqrt{a^2m^2 + b^2}$

if it is a tangent to circle, then perpendicular from (0, 0) is equal to r.

$$\frac{\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}} = |r|$$

or $a^2m^2 + b^2 = m^2r^2 + r^2$

or $(a^2 - r^2)m^2 = r^2 - b^2$

$$\therefore m = \sqrt{\frac{r^2-b^2}{a^2-r^2}}$$

$$\therefore \tan\theta = \sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)} \text{ or } \theta = \tan^{-1}\sqrt{\frac{(r^2-b^2)}{(a^2-r^2)}}$$

Sol 17: Let (h, k) be midpoints of chords,

$\therefore$ Equation of chord with midpoint (h. , k) is

$$\frac{xh}{a^2}+\frac{4k}{b^2}-1 = \frac{h^2}{a^2}+\frac{k^2}{b^2}-1$$

It passes through (0, b)

$$\therefore \text{Equation is: } \frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{k}{b} \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{y}{b}$$

Sol 18: Let P be (h, k)

$\therefore$ Chord of contact is

$xh + yk - r^2 = 0$

or $y = -\frac{hx}{k}+\frac{r^2}{k}$

$c^2 = a^2m^2 + b^2$

or $\frac{r^4}{k^2}=\frac{r^2}{a^2}\times\frac{b^2}{k^2}+\frac{r^2}{b^2}$

or $r^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ is locus of P

Sol 19: We have to find the locus of P(h, k) such that the chord to contact subtends 90° at centre the equation of chord of contact is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \qquad \text{...(i)}$$

the equation of the straight line joining the centre of ellipse,. To the points of intersection of ellipse and (i) is obtained be making homogenous equation of (i) and then ellipse & is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} - \left(\frac{hx}{a^2}+\frac{by}{b^2}\right)^2 = 0$$

$$\therefore x^2\left(\frac{1}{a^2}-\frac{h^2}{a^4}\right)+y^2\left(\frac{1}{b^2}-\frac{k^2}{b^4}\right)\frac{-2hk}{a^2b^2}xy = 0 \qquad \text{...(ii)}$$

It chord of contact subtends 90° at origin then the lines separated by (ii) should be $\perp$

$$\Rightarrow \frac{1}{a^2}-\frac{h^2}{a^4}+\frac{1}{b^2}-\frac{k^2}{b^4}=0$$

or $\frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$

Sol 20: Equation of chord whose middle point is $\left(\frac{1}{2},\frac{2}{5}\right)$ is

$$\frac{x\times\frac{1}{2}}{25} + \frac{y\times\frac{2}{5}}{16} = \frac{\left(\frac{1}{2}\right)^2}{25} + \frac{\left(\frac{2}{5}\right)^2}{16}$$

$$\therefore \frac{x}{50}+\frac{y}{40}=\frac{1}{50}$$

$$y = -\frac{4}{5}(x-1)$$

$$\therefore \frac{x^2}{25}+\frac{\left(\frac{4}{5}\right)^2(x-1)^2}{16}=1$$

$\Rightarrow x^2 + x - 12 = 0$

$\therefore x_1 = -4$ and $x_2 = 3$

$\therefore$ Length of chord

$$L = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$= \left|(x_1-x_2)\sqrt{1+m^2}\right| = 7\sqrt{1+\frac{16}{25}} = \frac{7}{5}\sqrt{41}$$

Sol 21: Let F = (ae, 0) & p = (acosθ, bsinθ)

Radius of circle

$$= \frac{1}{2}\sqrt{b^2\sin^2\theta+a^2(e-\cos\theta)^2}$$

$$= \frac{1}{2}\sqrt{b^2\sin^2\theta+a^2e^2-2ae\cos\theta+a^2\cos^2\theta}$$

$$= \frac{1}{2}\sqrt{a^2-b^2\cos^2\theta+a^2\cos^2\theta-2ae\cos\theta}$$

$= \frac{1}{2}\sqrt{a^2e^2\cos^2\theta - 2ae\cos\theta + a^2}$

$= \frac{1}{2}\sqrt{(a - ae\cos\theta)^2} = \frac{1}{2}(a - ae\cos\theta)$

Radius of auxillary circle = a

$\therefore r_1 - r_2 = \frac{1}{2}$ (a + ae cosθ)

Centre of circle with FP as diameter

$= C = \left(\frac{ae + \cos\theta}{2}, \frac{b\sin\theta}{2}\right)$

Distance between centre

$= \sqrt{\frac{a(e+\cos\theta)^2}{4} + \frac{b^2\sin b^2\theta}{4}}$

$= \frac{1}{2}\sqrt{a^2(e+\cos\theta)^2 + b^2\sin^2\theta}$

$= \frac{1}{2}\sqrt{(a + ae\cos\theta)^2}$

$= \frac{1}{2}(a + ae\cos\theta)$

∴ The two circle touch each other internally.

Sol 22: Let the co-ordinate of P be (a cos θ, bsinθ) the coordinates of Q are (acosθ, bsinθ)

Let R(h, k) be a point on PQ such that PR : RQ = r : s

then h = acosθ and

$R = \frac{r a\sin\theta + s b\sin\theta}{r+s}$

$\Rightarrow \cos\theta = \frac{h}{a}$ & $\sin\theta = \frac{(r+s)k}{ra+sb}$

$\Rightarrow \left(\frac{h}{a}\right)^2 + \frac{(r+s)^2k^2}{(ra+sb)^2} = 1$

or $\frac{x^2}{a^2} + \frac{(r+s)^2y^2}{(ra+sb)^2} = 1$

Sol 23: Equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{4} = 1$

Tangent to ellipse is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

or $mx - y \pm \sqrt{a^2m^2 + b^2} = 0$

it is tangent to circle

$\pm \frac{\sqrt{a^2m^2 + b^2}}{\sqrt{m^2+1}} = r$

or $a^2m^2 + b^2 = r^2(m^2 + 1)$

$m^2(25 - r^2) = r^2 - 4$

$m = -\sqrt{\frac{r^2-4}{2r - r^2}}$

since tangent lies in first quadrant m < 0

$\therefore m^2 = \frac{(r^2-4)}{25-r^2}$

equation of tangent is

$y = -\sqrt{\frac{(r^2-4)}{25-r^2}}x + \sqrt{\frac{25(r^2-4)}{25-r^2} + 4}$

$y = -\sqrt{\frac{r^2-4}{25-r^2}}\,x + \sqrt{\frac{2/r^2}{25-r^2}}$

∴ Midpoint is

$\left(\frac{1}{2}\sqrt{\frac{21r^2}{r^2-4}}, \frac{1}{2}\sqrt{\frac{21r^2}{25-r^2}}\right)$

$\therefore 2x = \sqrt{\frac{21r^2}{r^2-4}}$; $2y = \sqrt{\frac{21r^2}{25-r^2}}$

$\therefore 4x^2 = \frac{21r^2}{r^2-4}$; $4y^2 = \frac{21r^2}{25-r^2}$

$\therefore \frac{25}{4x^2} + \frac{4}{4y^2} = 1$

or $25y^2 + 4x^2 = 4x^2y^2$

Sol 24: The given ellipses are

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$

chord of contact of $P(x_1, y_1)$ w. r. t. ellipse is

$\frac{xx_1}{a(a+b)} + \frac{yy_1}{b(a+b)} = 1$

or λx + my = n

It is touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a^2\lambda^2 + b^2m^2 = n^2$

$$\frac{a^2 \times x_1^2}{a^2(a+b)^2} + \frac{b^2 x y_1 2}{b^2(a+b)^2} = 1$$

$x_1^2 + y_1^2 = (a + b)^2$

Locus of P is

$x^2 + y^2 = (a + b)^2$

$= a(a + b) + b(a + b)$

Which is the director circle.

Sol 25: Let mid-point = (h, k)

$\therefore$ Equation of ellipse is

$4xh + 9ky - 36 = 4h^2 + 9k^2 - 36$

And equation of chord is

$x - 2y + c = 0$

$$\therefore \frac{4h}{1} = -\frac{ak}{2} = -\left(\frac{4h^2 + 9k^2}{c}\right) \Rightarrow h = -\frac{9}{8}k$$

$$-\frac{9k}{2} = -\frac{4 \times \frac{8}{64}k^2 + 9k^2}{c} \Rightarrow +\frac{1}{2} = \frac{\frac{9k}{16} + k}{c}$$

$\therefore 25K = 8c$

$\therefore k = \frac{8c}{25}$; $h = -\frac{9c}{25}$

Sol 26: $\therefore$ From A two $\perp$ tangents can be drawn to ellipse

$\therefore$ A is the director circle

i.e. $x^2 + y^2 = a^2 + b^2$

Exercise 2

Single Correct Choice Type

Sol 1: (B) $\frac{x}{r-2} + \frac{y^2}{5-r} = 1$

$\therefore r - 2 > 0$ and $5 - r > 0$

$\therefore r \in (2, 5)$

Sol 2: (D) $4x^2 + 8x + 9y^2 + 36y + 4 = 0$

$\Rightarrow (2x + 2)^2 + (3y + 6)^2 = 36$

$$\Rightarrow \frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

$$\therefore e = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

Sol 3: (B) Let P be (θ_1) and Q = (θ_2)

Slope OP = $\frac{b}{a}\tan\theta_1$

and slope OQ = $\frac{b}{a}\tan\theta_2$

$$\therefore M_{OP} \times M_{OQ} = \frac{b^2}{a^2} \times \left(\frac{-a^2}{b^2}\right) = -1$$

$\therefore$ It subtends 90° at centre

Sol 4: (B) $c^2 = a^2m^2 + b^2$

$\therefore c^2 = 8 \times 4 + 4$

$\therefore c = \pm\sqrt{36} = \pm 6$

Sol 5: (D) Let (x_1, y_1) be point of contact to ellipse

$\therefore 3xx_1 + 4yy_1 = 1$ is equation of tangent at (x_1, y_1)

$$\therefore \frac{3x_1}{3} = \frac{4y_1}{4} = -\frac{1}{\sqrt{7}}$$

$$\therefore (x_1, y_1) = \left(-\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$$

Sol 6: (C) x-axis

Sol 7: (B) The positive end of latus rectum is $\left(ae, \frac{b^2}{a}\right)$

$a\cos\theta = ae \therefore \cos\theta = e$

$b\sin\theta = \frac{b^2}{a^2}$ & $\sin\theta = \frac{b}{a}$

equation of normal is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$\therefore \frac{ax}{e} - \frac{by}{b} \times a = a^2(e^2)$

or $x - ey = ae^3$

$\therefore x - ey - e^3a = 0$

Sol 8: (A) Consider normal at positive end of latus rectum from above equation of normal is $x - ey - e^3a = 0$

It passes through (0, –b)

$\therefore be - e^3a = 0$

$\Rightarrow b - e^2a = 0$

$\therefore\ \frac{b}{a} = e^2$

$\Rightarrow 1 - e^2 = e^4$ or $e^2 + e^4 = 1$

Sol 9: (C) Tangent at P is

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} - 7 = 0$

$\therefore\ CF = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$

$P = (a\cos\theta, b\sin\theta)$

Equation of normal at

$P = \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$\therefore\ G = \left(\frac{(a^2 - b^2)\cos\theta}{a}, 0\right)$

$\therefore PG = \sqrt{b^2\sin^2\theta + \frac{b^4}{a^2}\cos^2\theta} = b^2\sqrt{\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2}}$

$\therefore$ PG. CF = b^2

Sol 10: (A) $y = \frac{x}{2} + 2$

$\therefore\ c^2 = a^2m^2 + b^2$

$\Rightarrow 4 = 4 \times \frac{1}{4} + b^2 \Rightarrow b = \sqrt{3}$

$\therefore$ The other common tangent has slope – m

$\therefore\ c = \frac{1}{-\frac{1}{2}} = -2$

$\therefore$ Equation is $y = -\frac{1}{2}x - 2$

or $x + 2y + 2 = 0$

Sol 11: (A) $\frac{2b^2}{a} = a + b$

or $2b^2 = a^2 + ab$

$a^2 + ab - 2b^2 = 0$

$a = -\frac{b \pm \sqrt{b^2 + 8b^2}}{2}$

$a = -\frac{b + 3b}{2} = b$

$\therefore$ Ellipse bulges to circle

Sol 12: (B) $\Rightarrow y = \pm\frac{a}{b}x \pm \sqrt{a^2 \cdot \frac{a^2}{b^2} + a^2 + b^2}$

$\Rightarrow y = \pm\frac{a}{b}x \pm \sqrt{\frac{a^4 + a^2b^2 + b^4}{b^2}}$

$\Rightarrow y = \pm\frac{a}{b}x \pm \frac{1}{b}\sqrt{a^4 + a^2b^2 + b^4}$

$\Rightarrow yb = \pm ax \pm \sqrt{a^4 + a^2b^2 + b^4}$

Sol 13: (C) 2ae = 6

2b = 8

$\therefore$ ae = 3 & b = 4

$\therefore a^2 - b^2 = 9$ & $b^2 = 16$

$\therefore a^2 = 25$

$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

Sol 14: (A) Slope of ellipse = – 1

$\therefore$ Equation is $y = -x \pm \sqrt{25 + 16}$

$\therefore\ x + y = \pm\sqrt{41}$

Sol 15: (B) FBF′ is 90°

We know that BF = BF′ = a

$\therefore\ 2a^2 = 4a^2e^2$

$\therefore\ e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$

Sol 16: (C) Ellipse is

$\frac{x^2}{\left(\frac{1}{3}\right)^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$

since a < 0

$\therefore$ Latus rectum is $\frac{2a^2}{b} = \frac{2 \times 1}{9 \times \frac{1}{2}} = \frac{4}{9}$

Sol 17: (D) $\frac{a}{e} - ae = 8$

$e = \frac{1}{2}$

$\therefore\ \frac{3a}{2} = 8\ ;\ a = \frac{16}{3}$

$\frac{1}{4} = 1 - \frac{b^2}{a^2}$

$\therefore b^2 = a^2(1 - l^2) = \frac{16}{3} \times \frac{16}{3} \times \frac{3}{4}$

$b = \frac{8}{\sqrt{3}}$

$\therefore$ Length of minor axis = 2b = $\frac{16}{\sqrt{3}}$

Sol 18: (D) E(P) > 0 , E(Q) < 0

C(P) < 0 and C(Q) < 0

$\therefore$ P lies inside C but outside E

Sol 19: (C) Let P = (acosθ, bsinθ)

$Q = \left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)$

= (– asinθ, bcosθ)

aλcosθ +mbsinθ + n = 0

And ⇒ aλcosθ + mbsinθ = – n → 1 – aλsinθ + mbcosθ + n = 0 & –aλsinθ + mbcosθ = – n(2)

squaring and adding 1 and 2

we get $a^2l^2 + m^2b^2 = 2n^2$

Sol 20: (B) $Q_1 + Q_2 = C$

Point of intersection of tangent at (θ_1) and (θ_2) is

$$(x, y) = \left(\frac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right)$$

$$\therefore \frac{x}{a\cos\left(\frac{c}{2}\right)} = \frac{y}{b\sin\left(\frac{C}{2}\right)}$$

$\therefore$ Locus of P is a straight line

Sol 21: (A) P = (acosθ, bsinθ)

Q = (acosθ, asinθ)

equation of tangent at Q

is (y – asinθ) = – 1tanθ (x – acosθ)

$x + y\tan\theta - \frac{a}{\cos\theta} = 0$ & (ae, 0)

$$\therefore SN = \frac{ae - \frac{a}{\cos\theta}}{\sqrt{1+\tan^2\theta}} = |ae\cos\theta - a|$$

$SP = \sqrt{(ae - a\cos\theta)^2 + b^2\sin^2\theta}$

$= \sqrt{a^2(\cos^2\theta + a^2 - b^2\cos^2\theta - 2a^2e\cos\theta)}$

$= \sqrt{(a\ell\cos\theta)^2 + a^2 - 2a^2e\cos\theta} = |ae\cos\theta - a|$

$\therefore$ SP = SN

Sol 22: (C) Equation of tangent is $\frac{x}{3}\cos\theta + \frac{y}{2}\sin\theta = 1$

T is x = 3 and T′ x = – 3

$\therefore$ Point of intersection of tangent & T let say

$P = \left(3, -\frac{2(1-\cos\theta)}{\sin\theta}\right) = \left(3, 2\tan\frac{\theta}{2}\right)$

$P' = \left(-3, \frac{2(1+\cos\theta)}{\sin\theta}\right) = \left(-3, 2\cot\frac{\theta}{2}\right)$

$\therefore$ Equation of circle is

$(x + 3)(x - 3) + \left(y - 2\tan\frac{\theta}{2}\right)\left(y - 2\cot\frac{\theta}{2}\right) = 0$

$\therefore$ When y = 0

$x^2 - 5 = 0$

$\therefore x = \pm\sqrt{5}$

$\therefore$ It always passes through $(\pm\sqrt{5}, 0)$

i.e. it always passes through focus.

Sol 23: (A) Q = (acosθ, asinθ)

P = (acosθ, bsinθ)

Δ SPT is an isosceles triangle.

Sol 24: (C) Equation of normal in slope form is

$$y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$$

$$\therefore c^2 = \frac{m^2(a^2 - b^2)^2}{a^2 + b^2m^2}$$

Sol 25: (A) $(\sqrt{2}x - 2\sqrt{2})^2 + (\sqrt{3}y - 3\sqrt{3})^2 = k$

$$\therefore \frac{2(x-2)^2}{k} + \frac{3(y-3)^2}{k} = 1$$

For ellipse k > 0

For a point k = 0

Previous Years' Questions

Sol 1: (C) Given, $16x^2 + 25y^2 = 400$

$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

Here, $a^2 = 25,\ b^2 = 16$

But $a^2(1-e^2)$

$\Rightarrow 16 = 25(1-e^2)$

$\Rightarrow \frac{16}{25} = 1 - e^2$

$\Rightarrow e^2 = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow e = \frac{3}{5}$

Now, foci of the ellipse are $(\pm ae,\ 0) \equiv (\pm 3, 0)$

We have $3 = a.\frac{3}{5}$

$\Rightarrow a = 5$

Now, $PF_1 + PF_2 =$ major axis = 2a

$= 2 \times 5 = 10$

Sol 2: (B) For ellipse, condition of tangency is $c^2 = a^2m + b^2$

Given line is $y = 4x + c$ and curve $\frac{x^2}{4} + y^2 = 1$

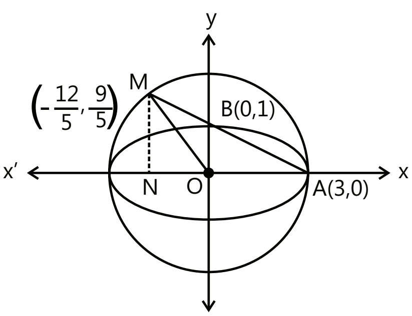
$\Rightarrow \left| ma - b\sqrt{1+m^2} \right| = \left| 1 - b\sqrt{1+m^2} \right|$

$\Rightarrow c = \pm\sqrt{65} = \sqrt{65}$ or $-\sqrt{65}$

So, number of values are 2.

Sol 3: (D) Equation of auxiliary circle is $x^2 + y^2 = 9$ ...(i)

Equation of AM is $\frac{x}{3} + \frac{y}{1} = 1$ ...(ii)

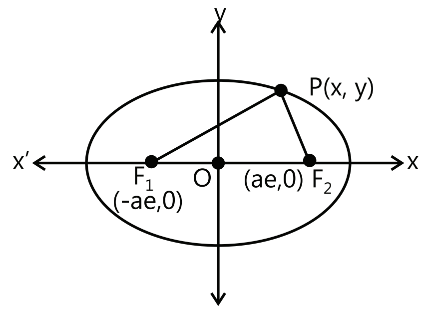


On solving equation (i) and (ii), we get $M\left(-\frac{12}{5}, \frac{9}{5}\right)$

Now, area of $\Delta AOM = \frac{1}{2}.OA \times MN = \frac{27}{10}$ sq. unit

Sol 4: Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Foci F_1 and F_2 are (—ae, 0) and (ae, 0) respectively. Let P(x, y) be any variable point on the ellipse. The area A of the triangle PF_1F_2 is given by



$\theta = \frac{\pi}{6}$

$= \frac{1}{2}(-y)(-ae \times 1 - ae \times 1)$

$= -\frac{1}{2}y(-2ae) = aey$

$= ae.b\sqrt{1 - \frac{x^2}{a^2}}$

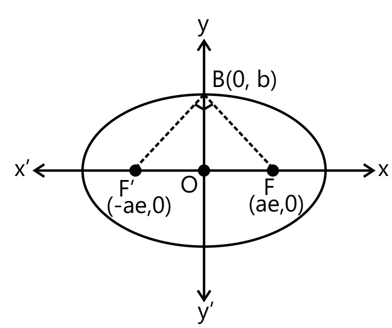
So, A is maximum when x = 0

$\Rightarrow$ Maximum of A

$= abe = ab\sqrt{1 - \frac{b^2}{a^2}} = ab\sqrt{\frac{a^2 - b^2}{a^2}} = b\sqrt{a^2 - b^2}$

Sol 5: Since, angle FBF' is right angled

$\therefore$ (slope of FB). (slope of F'B) = –1



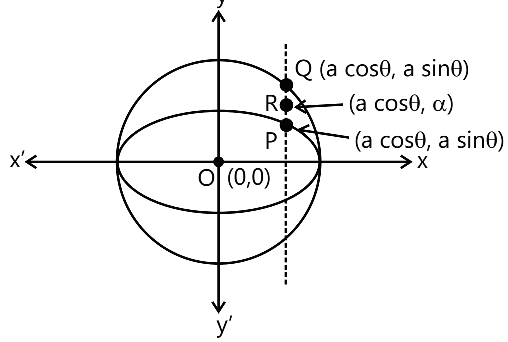
$\Rightarrow \left(\frac{0-b}{ae-0}\right)\left(\frac{0-b}{-ae-0}\right) = -1$

$\Rightarrow \frac{b^2}{-a^2e^2} = -1 \Rightarrow b^2 = a^2e^2$

$\Rightarrow a^2(1-e^2) = a^2e^2 \Rightarrow e^2 = \frac{1}{2}$

$\Rightarrow e = \frac{1}{\sqrt{2}}$

Sol 6: Given, $\frac{PR}{RQ} = \frac{r}{s}$



$\Rightarrow \frac{\alpha - b\sin\theta}{a\sin\theta - \alpha} = \frac{r}{s}$

$\Rightarrow \alpha s - b\sin\theta.s = ra\sin\theta - \alpha r$

$\Rightarrow \alpha s + \alpha r = ra\sin\theta + b\sin\theta.s$

$\Rightarrow \alpha(s+r) = \sin\theta(ra+bs)$

$\Rightarrow \alpha = \frac{\sin\theta(ra+bs)}{r+s}$

Let the coordinate of R be (h, k)

$\Rightarrow h = a\cos\theta$

and $k = \alpha = \frac{(ar+bs)\sin\theta}{r+s}$

$\Rightarrow \cos\theta = \frac{h}{a}, \sin\theta = \frac{k(r+s)}{ar+bs}$

On squaring and adding, we get

$\sin^2\theta + \cos^2\theta = \frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(ar+bs)^2}$

$\Rightarrow 1 = \frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(ar+bs)^2}$

Hence, locus of R is $\frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(ar+bs)^2} = 1$.

Sol 7: In 1st quadrat eq. of target will be of fly from

$\frac{x}{a} + \frac{y}{b} = 1 \quad \begin{cases} \text{if x intercept} = q \\ \text{y intercept} = b \end{cases}$

$\because$ (1) is tangent to circle $x^2 + y^2 = 16$

$\Rightarrow x^2 + \frac{(ab-bx)^2}{a} = 16$

$\Rightarrow x^2 + \frac{a^2b^2 + b^2x^2 - 2ab^2x}{a^2} = 16$

$\Rightarrow x^2 + b^2 + \frac{b^2}{a^2}x^2 - \frac{2b^2}{a}.x = 16$

$\Rightarrow x^2\left(1 + \frac{b^2}{a^2}\right) - \frac{2b^2}{a}x + b^2 - 16 = 0$

For unique solution

$\Rightarrow \frac{4b^2}{a^2} - 4\left(1 + \frac{b^2}{a^2}\right)\left(1^2 - 16\right) = 0$

$\Rightarrow \frac{b^4}{a^2} = b^2 - 16 + \frac{b^4}{a^2} - \frac{16b^2}{a^2}$

$\Rightarrow b^2 - 16 = 16\frac{b^2}{a^2}$

$\Rightarrow a^2b^2 - 16a^2 = 16b^2$

$\Rightarrow a^2b^2 = 16\left(a^2 + b^2\right)$ ... (ii)

Similarly (i) is tamest to ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ are will get the relatiesn

$a^2b^2 = 4a^2 + 25b^2$ ... (iii)

Solving (i) (ii) we get $a = 2\sqrt{7}$

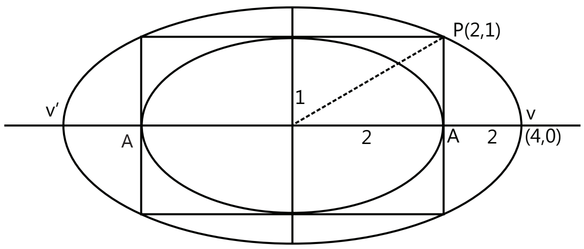
$b = 4\sqrt{\frac{7}{3}}$

$\Rightarrow$ Eq. of tangent $\frac{x}{2\sqrt{7}} + \frac{y}{4\sqrt{\frac{7}{3}}} = 1$

Distances $= \sqrt{a^2 + b^2} = \frac{14}{\sqrt{3}}$

Focus (S = 6, 2)

Sol 8: (A) Major axis is along x-axis.



$$\frac{a}{e}-ae=4$$

$$a\left(2-\frac{1}{2}\right)=4$$

$$a=\frac{8}{3}.$$

Sol 9:

$$x^2+4y^2=4\Rightarrow\frac{x^2}{4}+\frac{y^2}{1}=1\Rightarrow a=2,b=1\Rightarrow P=(2,1)$$

Required Ellipse is $\frac{x}{a^2}+\frac{y^2}{b^2}=1\Rightarrow\frac{x^2}{4^2}+\frac{y^2}{b^2}=1$

(2, 1) lies on it

$$\Rightarrow\frac{4}{16}+\frac{1}{b^2}=1\Rightarrow\frac{1}{b^2}=1-\frac{1}{4}=\frac{3}{4}\Rightarrow b^2=\frac{4}{3}$$

$$\therefore\frac{x^2}{16}+\frac{y^2}{\left(\frac{4}{3}\right)}=1\Rightarrow\frac{x^2}{16}+\frac{3y^2}{4}=1\Rightarrow x^2+12y^2=16$$

Sol 10: (D) $b^2=a^2\left(1-e^2\right)=a^2\left(1-\frac{2}{5}\right)=a^2\frac{3}{5}=\frac{3a^2}{5}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\Rightarrow\frac{9}{a^2}+\frac{5}{3a^2}=1$$

$$a^2=\frac{32}{3}$$

$$b^2=\frac{32}{5}$$

$\therefore$ Required equation of ellipse $3x^2+5y^2-32=0$

Sol 11: (D) Semi minor axis b = 2

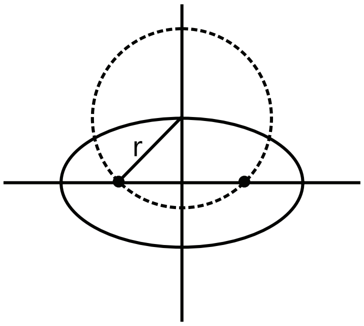
Semi major axis a = 4

Equation of ellipse $=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\Rightarrow\frac{x^2}{16}+\frac{y^2}{4}=1$$

$$\Rightarrow x^2+4y^2=16.$$

Sol 12: (A)



$$a=4,b=3,e=\sqrt{1-\frac{9}{16}}\quad\Rightarrow\frac{\sqrt{7}}{4}$$

Focii is $(\pm\,ae,\,0)\qquad\Rightarrow\left(\pm\sqrt{7},0\right)$

$$r=\sqrt{(ae)^2+b^2}$$

$$\sqrt{7+9}=4$$

Now equation of circle is $(x-0)^2+(y-3)^2=16$

$$x^2+y^2-6y-7=0$$

Sol 13: (A)

Here ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2=6,b^2=2$

Now, equation of any variable tangent is

$y=mx\pm\sqrt{a^2m^2+b^2}$ ....(i)

where m is slope of the tangent

So, equation of perpendicular line drawn from centre to tangent is $y=\frac{-x}{m}$ ....(ii)

Eliminating m, we get

$$\left(x^2+y^2\right)^2=a^2x^2+b^2y^2$$

$$\Rightarrow\boxed{\left(x^2+y^2\right)^2=6x^2+2y^2}$$

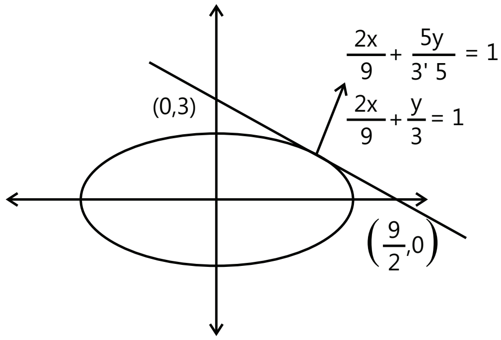
Sol 14: (D) $\frac{x^2}{9}+\frac{y^2}{5}=1$

$a=3,\ b=\sqrt{5}\qquad\left(ae,\frac{b^2}{a}\right)$

$\frac{b^2}{a}=\frac{5}{3},\qquad\left(2,\frac{5}{3}\right)$

$$e=\sqrt{1-\frac{5}{9}}=\frac{2}{3}$$

Area = $4\left(\frac{1}{2}\times\frac{9}{2}\times 3\right)=27$



JEE Advanced/Boards

Exercise 1

Sol 1: Let X = x – 1

And Y = y – 2

$\therefore$ Centre = (0, 0)

Focus (F_1) (5, 0)

F_2 = (–5, 0) and point on ellipse

= (3, 4)

$F_1P + F_2P = 2a$

$= \sqrt{80}+\sqrt{20} = 2a$

$3\sqrt{20} = 2a$

$\therefore \boxed{a = 3\sqrt{5}}$

ae = 5

$\therefore a^2 - b^2 = 25$

$\therefore b^2 = 20$

$b = 2\sqrt{5}$

$\therefore$ Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{9}+\frac{y^2}{4}=5$

$\Rightarrow \frac{(x-1)^2}{9}+\frac{(y-2)^2}{4} = 5$

$4x^2 + 9y^2 - 8x - 36y - 175 = 0$

Sol 2: Let T = (h, k)

$\therefore$ AP is

$xh + yk = a^2$

It passes through (a, 0)

$\therefore$ h = a

$\therefore$ T = (a, k)

TB is

$y = \frac{(k-0)}{2a} \times (x + a)$

$\therefore 2ay = kx + ka \Rightarrow kx - 2ay + ka = 0$

and $ax + ky = a^2$

Let point of intersection be (x_1, y_1)

$y_1 = \frac{2a^2k}{k^2+2a^2}$ and $x_1 = \frac{2a^3-ak^2}{2a^2-k^2}$

$x_1 = \frac{a(2a^2-k^2)}{2a^2+k^2}$

$(x_1)^2 + 2(y_1)^2 = a^2$

$\because \left(\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{\left(\frac{a}{\sqrt{2}}\right)^2}=a^2\right)$

$\therefore$ Eccentricity is = $\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}$

Sol 3: Equation of auxiliary circle is

$x^2 + y^2 = a^2$ .... (i)

Equation of tangent at P(α) is

$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1$ ...(ii)

Equation of pair of lines OA, OB is obtained by making homogenous equation of i w. r. t. (ii)

$\therefore x^2 + y^2 = a^2\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)^2$

$\therefore (1-\cos^2\alpha)x^2 - \frac{2xya\sin\alpha\cos\alpha}{b}$

$+ y^2\left(1-\frac{a^2}{b^2}\sin^2\alpha\right)=0$

But $\angle AOB = 90°$

$\therefore$ coeff of x^2 + coeff of y^2 = 0

$\therefore 1 - \cos^2\alpha + 1 - \frac{a^2}{b^2}\sin^2\alpha = 0$

$1 = \frac{a^2-b^2}{b^2}\sin^2\alpha$

$1 = \frac{a^2e^2}{a^2(1-e^2)}\sin^2\alpha$

$\Rightarrow e^2 = \frac{1}{(1+\sin^2\alpha)}$ or $e = (1 + \sin^2\alpha)^{-1/2}$

Sol 4: $(-3, 1) = (a\cos a_1, b\sin a_1)$

$(a_1 - \alpha) = (a\cos a_2, b\sin a_2)$

$\therefore \frac{9}{a^2} + \frac{1}{b^2} = 1$ & $\frac{4}{a^2} + \frac{4}{b^2} = 1$

$\therefore \frac{32}{a^2} = 3$

$\therefore a^2 = \frac{32}{3}$ & $b^2 = \frac{32}{5}$

$\therefore$ Equation of ellipse is $\frac{x^2}{\frac{32}{3}} + \frac{y^2}{\frac{32}{5}} = 1$

$\therefore 3x^2 + 5y^2 = 32$

Sol 5: Let α and β form a chord which interests the major axis at (c, 0)

$\therefore$ Equation of chord is

$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\frac{(\alpha+\beta)}{2} = \cos\left(\frac{\alpha-\beta}{2}\right)$

$\frac{c}{a}\cos\frac{(\alpha+\beta)}{2} = \cos\left(\frac{\alpha-\beta}{2}\right)$

$\Rightarrow \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{a}{c}$

$\Rightarrow \frac{\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{a+c}{a-c}$

$\Rightarrow \frac{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{-2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} = \frac{a+c}{a-c}$

$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{c-a}{c+a}$

Similarly let γ and δ intersect major axis at (– c, 0)

$\therefore \tan\frac{\gamma}{2}\tan\frac{\delta}{2} = \frac{-c-a}{a-c}$

$\tan\frac{\gamma}{2}\tan\frac{\delta}{2}\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{c-a}{c+a}\times\frac{-(c+a)}{-(c-a)} = 1$

Sol 6: (a) m of line $= -\frac{3}{4}$

$\therefore$ Slope of line $\perp$ to given line $= \frac{4}{3}$

Equation of ellipse is $\frac{x^2}{(3)^2} + \frac{y^2}{5} = 0$

$\therefore$ Equation of tangent is $y = \frac{4}{3}x \pm \sqrt{9\times\left(\frac{4}{3}\right)^2 + 5}$

$y = \frac{4x}{3} \pm \sqrt{21}$

$\therefore 3y = 4x \pm 3\sqrt{21}$

(b) Equation of normal to the ellipse is $ax\sec\theta - by\,\mathrm{cosec}\theta = a^2 - b^2$

or $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

the normal given is $\frac{ax}{3} + \frac{by}{4} = c$

$\therefore \frac{3}{\cos\theta} = \frac{4}{-\sin\theta} = \frac{a^2-b^2}{c}$

$\therefore \frac{(3c)^2}{(a^2-b^2)^2} + \frac{(4c)^2}{(a^2-b^2)^2} = 1$

$\therefore 5c = a^2 - b^2$ or $5x = a^2e^2$

Sol 7: Equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the ends of latus rectum are

$L_1\left(ae, \frac{b^2}{9}\right)$ & $L_2\left(ae, -\frac{b^2}{9}\right)$

For double contact the centre of circle should lie on normal of L_1 & L_2. By symmetry y-coordinates of centre = 0

Equation of normal at L_1 is

$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$

For centre y = 0

$\therefore$ x centre $= \frac{\ell a^2}{a}\left(1 - \frac{b^2}{a^2}\right) = ae^3$

$\therefore$ Equation of circle is

$(x - ae^3)^2 + y^2 = (ae - ae^3)^2 + \left(\frac{b^2}{9}\right)^2$

$(x - ae^3)^2 + y^2 = (ae - ae^3)^2 + \left(\frac{b^2}{9}\right)^2$

$x^2 - 2ae^3 + a^2e^6 + y^2 = (ae - ae^3)^2 + a^2(1 - e^2)^2$

$\therefore x^2 - 2ae^3 + y^2 = a^2(1 - e^2 - e^4)$

Sol 8: Since lines have equal intercept on axis

$\therefore$ Slope = – 1

$\therefore$ Equation is $y = -x \pm \sqrt{a^2m^2 + b^2}$

$y = -x \pm \sqrt{25}$

or $x + y \pm 5 = 0$ are the equation of tangents.

Sol 9: $P = \left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right) = (a\cos\theta, b\sin\theta)$

$\therefore$ Equation of tangent is

$$\frac{x\cos\theta}{4} + \frac{y\sin\theta}{\frac{16}{\sqrt{11}}} = 1$$

It is also tangent to circle

$x^2 + y^2 - 2x - 15 = 0$

$c = (1, 0)$ and $r = 4$

$\therefore$ Distances from center = radius

$$\therefore \frac{\frac{\cos\theta}{4} - 1}{\sqrt{\frac{\cos^2\theta}{16} + \frac{11\sin^2\theta}{256}}} = 4$$

$$\Rightarrow \frac{(\cos\theta - 4)^2}{16} = \cos^2\theta + \frac{11\sin^2\theta}{16}$$

$(\cos\theta - 4)^2 = 16\cos^2\theta + 11\sin^2\theta$

$\cos^2\theta - 8\cos\theta + 16 = 11 + 5\cos^2\theta$

$4\cos^2\theta + 8\cos\theta - 5 = 0$

$4\cos^2\theta + 10\cos\theta - 2\cos\theta - 5 = 0$

$\therefore \cos\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$

Sol 10: $y = mx \pm \sqrt{a^2m^2 + b^2}$

$$y = -\frac{4}{3}x \pm \sqrt{18 \times \left(\frac{4}{3}\right)^2 + 32}$$

$y = -\frac{4}{3}x \pm 8$

$\therefore A = \pm \frac{3}{4} \times 8 = \pm 6$ & $B = \pm 8$

$\therefore$ Area of A $= \left|\frac{1}{2}A \times B\right| = \frac{1}{2} \times 6 \times 8 = 24$

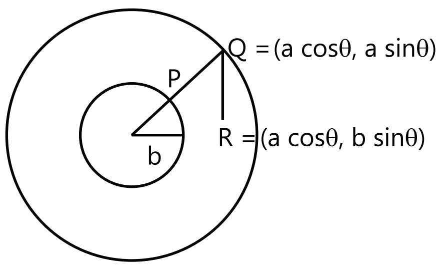
Sol 11: $P = (b\cos\theta, b\sin\theta)$

$Q = (a\cos\theta, a\sin\theta)$

$R = (a\cos\theta, b\sin\theta)$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is locus of R which is

An ellipse since focus lie on inner circle



$\therefore b = ae$

$\Rightarrow b^2 = a^2\left(1 - \frac{b^2}{a^2}\right)$

$\therefore \frac{b^2}{a^2} = \frac{1}{2} \Rightarrow \frac{b}{a} = \frac{1}{\sqrt{2}}$

And $e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$

Sol 12: Let B = (a, 0) C = (–a, 0) & A = (a, 0)

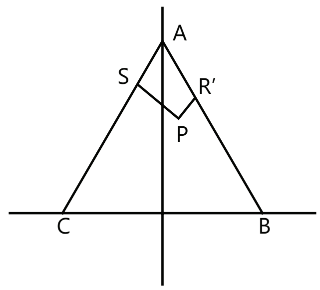
Equation of line AB is

$x + y = a$ and that of AC is $y = x + a$

Let P= (x, y) distance from BC = y

And area of PRAS = PR x PS

$= \frac{(x + y - a)}{\sqrt{2}} \times \frac{(x - y + a)}{\sqrt{2}}$



according to Q.

$$y^2 = \pm \frac{1}{2}\frac{(x^2-(y-a)^2)}{2}$$

$\therefore \pm 4y^2 = x^2 - y^2 - 2ay - a^2$

When it is $+4y^2$ it forms a hyperbola.

When it is $-4y^2$ it forms an ellipse

$\therefore x^2 + 3y^2 - 2ay - a^2 = 0$

$$x^2 + 3\left(y-\frac{1}{3}\right)^2 = \frac{4a^2}{3}$$

$$\therefore \frac{x^2}{\frac{4a^2}{3}} + \frac{\left(y-\frac{1}{3}\right)^2}{\frac{4a^2}{9}} = 1$$

$$\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow e = \sqrt{\frac{2}{3}}$$

Sol 13: Let equation of tangent be

$y = mx \pm \sqrt{16m^2+9}$ is passes through (2, 3)

$\therefore (3 - 2m) = \pm \sqrt{16m^2+9}$

$\Rightarrow 4m^2 - 12m + 9 = 16m^2 + 9$

$\Rightarrow 12m^2 + 12m = 0$

$m = 0$ or $m = -1$

Rechecking we get when $m = 0$ $C > 0$ & when $m = -1$ $C > 0$

$\therefore$ Equation of tangent is

$y = 3$ and $y = -x + 5$ or $x + y = 5$.

Sol 14: Let $y = mx + \frac{1}{m}$ be tangent to parabola $y^2 = 4x$.

It will touch ellipse $\frac{x^2}{4^2} + \frac{y^2}{(\sqrt{6})^2} = 1$ if

$$\frac{1}{m^2} = 16m^2 + 6$$

$\Rightarrow 16m^4 + 6m^2 - 1 = 0$

$\Rightarrow (8m^2 - 1)(2m^2 + 1) = 0$

$$m = \pm \frac{1}{2\sqrt{2}}$$

we know that a tangent at slope m touches parabola at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

so coordinate of point of contact are $A(8, 4\sqrt{2})$ and $B(8, -4\sqrt{2})$ we also know that tangent of slope m touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at

$$\left(\frac{\mp a^2m}{\sqrt{a^2m^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

$$\therefore C = \left(-2, \frac{3}{\sqrt{2}}\right) \ \& \ D\left(-2, \frac{-3}{\sqrt{2}}\right)$$

AB || CD $\therefore$ Quadrilateral is trapezium

$$\text{Area} = \frac{1}{2} \times h\,(AB + CD)$$

$$= \frac{1}{2} \times 10 \times \left(8\sqrt{2} + \frac{6}{\sqrt{2}}\right) = 55\sqrt{2} \text{ sq. units}$$

Sol 15: Equation of normal at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{by}{mn\theta} = a^2 - b^2$$

$$G = \left(\frac{\cos\theta}{a}(a^2-b^2), 0\right)$$

$$\text{And } g = \left(0, \frac{-\sin\theta}{b}(a^2-b^2)\right)$$

$\therefore a^2CG^2 + b^2(Cg)^2$

$$= a^2 \times \frac{\cos^2\theta}{a^2}(a^2-b^2)^2 + b^2\frac{\sin^2\theta}{b^2}(a^2-b^2)^2$$

$= (a^2 - b^2)^2$

$$CG = a\cos\theta\left(1 - \frac{b^2}{a^2}\right)$$

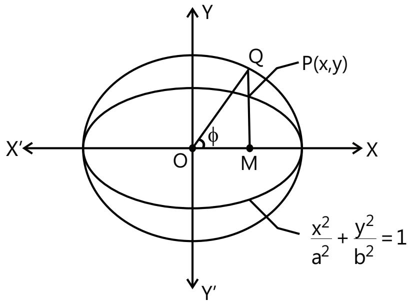
$= a\cos q e^2 = e^2 \times$ abscissa of P

Sol 16: $\therefore (ae + r\cos\theta, r\sin\theta)$ lies on ellipse

$$\therefore \frac{(ae+r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$\therefore \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)r^2 + \frac{2e\cos\theta}{a}r + e^2 - 1 = 0$$

To find the ℓ chord we have to find $(r_1 - r_2)$ as r_1 +ve and $r_2 < 0$



$\therefore (r_1 - r_2)^2$

$$= \frac{4e^2\cos^2\theta}{a^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)^2} - \frac{4(e^2-1)}{\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)}$$

$$= \frac{4e^2a^2b^4\cos^2\theta}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}$$

$$- \frac{4a^2b^2(b^2\cos^2\theta + a^2\sin^2\theta)(e^2-1)}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}$$

$$= \frac{4\left(1-\frac{b^2}{a^2}\right)a^2b^4\cos^2\theta + 4b^4(b^2\cos^2\theta + a^2\sin^2\theta)}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}$$

$$= \frac{4a^2b^4}{(b^2\cos^2\theta + a^2\sin^2\theta)^2}$$

$$\therefore \quad (r_1 - r_2) = \frac{2ab^2}{(b^2\cos^2\theta + a^2\sin^2\theta)}$$

Sol 17: The tangent at P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

where p = (acosθ, bsinθ)

$\therefore\ T = \left(\frac{a}{\cos\theta}, 0\right)$ & N = (acosθ, 0)

equation of circle with TN as diameter is

$\left(x - \frac{a}{\cos\theta}\right)(x - a\cos\theta) + y^2 = 0$

$\Rightarrow x^2 - a\left(\frac{1}{\cos\theta} + \cos\theta\right)x + y^2 + a^2 = 0$

Equation of auxiliary circle is $x^2 + y^2 - a^2 = 0$

$2gg_1 + 2ff_1 = 2\,\frac{a}{2}\left(\frac{1}{\cos\theta} + \cos\theta\right)\times 0 + 2\times 0\times 0 = 0$

$c_1 + c_2 = a^2 - a^2 = 0$

The two circle as orthogonal

Sol 18: Let P = (acosα), bsinα) and Q = (a cosβ, bsinβ)

Since tangents at P & Q are ⊥ is

$\therefore\ \frac{-b}{a\tan} \times \frac{-b}{a\tan\beta} = -1$

$\therefore\ \tan a \tan\beta = -\frac{b^2}{a^2}$

the point of intersection to tangents is

$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

Find the point of interaction of normal from their equations.

You can easily show that

slop ON = slope OT

$\therefore$ N lies on $\frac{y}{y_1} = \frac{x}{x_1}$

Sol 19: Let (h, k) be the point the chord of contact is

$\frac{xh}{a^2} + \frac{ky}{b^2} - 1 = 0$

It touches circle $x^2 + y^2 = c^2$

$\therefore\ \frac{|-1|}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} = c$

$\therefore\ 1 = c^2\left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right)$

on $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$ is locus of P.

Sol 20: Equation of tangent to ellipse

$\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$

$y = mx \pm \sqrt{(a^2+b^2)m^2 + b^2}$

It is also tangent to the ellipse,

$\frac{x^2}{a^2} + \frac{y^2}{a^2+b^2} = 1$

$\therefore c^2 = a^2m^2 + (a^2 + b^2)$

$\therefore (a^2 + b^2)m^2 + b^2 = a^2m^2 + (a^2 + b^2)$

$\therefore\ m^2 = \frac{a^2}{b^2}\quad m = \pm\frac{a}{b}$

$\therefore$ Tangents are

by = ± ax ± $\sqrt{a^4 + a^2b^2 + b^4}$

Sol 21: P = (acosθ, bsinθ) &

Q = (acosθ, asinθ)

tangent at P is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$\therefore\ T = \left(\frac{a}{\cos\theta}, 0\right)$$

$$\text{Slope QT} = \frac{a\sin\theta}{a\cos\theta - \frac{a}{\cos\theta}} = -\frac{1}{\tan\theta}$$

$\therefore$ QT ⊥ OQ $\therefore$ QT is tangent to the auxiliary circle

Sol 22: Normal at P(θ) is

ax secθ – bycosecθ = $a^2 - b^2$

It passes through Q(acos2θ, bsin2θ)

$$\therefore\ \frac{a^2\times\cos^2\theta}{\cos\theta} - \frac{b^2\times 2\sin\theta\cos\theta}{\cos\theta\sin\theta} = a^2 - b^2$$

$\therefore\ a^2(2\cos^2\theta - 1) - 2b^2\cos^2\theta = (a^2 - b^2)\cos\theta.$

$\therefore\ 18\cos^2\theta - 9\cos\theta - 14 = 0$

$18\cos^2\theta - 21\cos\theta - 12\cos\theta - 14 = 0$

$\therefore 3\cos\theta(6\cos\theta - 7) + 2(6\cos\theta - 7) = 0$

$\therefore\ \cos\theta = -\frac{2}{3}$

Sol 23: Let equation of tangent to ellipse be

y = mx + $\sqrt{a^2m^2 + b^2}$

Now it touches circle

$\therefore\ c^2 = r^2(m^2 + 1)$

$\therefore\ a^2m^2 + b^2 = r^2m^2 + 1$

$$\boxed{m^2 = \frac{r^2 - b^2}{a^2 - r^2}}$$

Equation of PQ is y = m(x – ae)

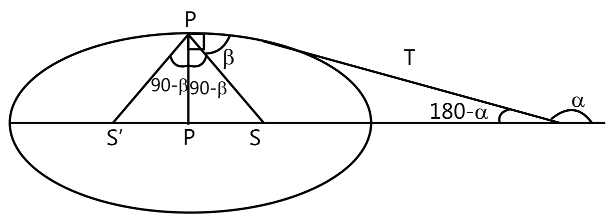
or mx – y – mae = 0

Length of ⊥ from centre to PQ (LP)

$$= \frac{+mae}{\sqrt{m^2+1}} = \frac{+\sqrt{\frac{r^2-b^2}{a^2-r^2}}\times ae}{\sqrt{\frac{a^2-b^2}{a^2-r^2}}}$$

$$= +\sqrt{\frac{r^2-b^2}{a^2-b^2}}\times a\times\frac{1}{a}\sqrt{a^2-b^2} = \sqrt{r^2-b^2}$$

$$PQ = 2\sqrt{r^2 - LP^2} = 2\sqrt{b^2} = 2b$$



Sol 24:

∠SPN = 90 – b

∠S′PN = ∠SPN. As normal bisects angle between S′P and SP

∠SPS′ = 180 – 2β

∠PSS′ = 180 – α + b

∠PS′S = α + β – 180

Applying sine rule on DSPS′

$$\frac{\sin\angle PS'S}{PS} = \frac{\sin\angle PSS}{PS'} = \frac{\sin\angle S'PS}{SS'}$$

$$\therefore\ \frac{\sin(\alpha+\beta-180^\circ)}{PS}$$

$$= \frac{\sin(180^\circ - (\alpha-\beta))}{PS'} = \frac{\sin(180^\circ - 2\beta)}{2ae}$$

PS + PS′ = 2a

$$\Rightarrow\ \frac{\sin(\alpha-\beta) - \sin(\alpha+\beta)}{2a} = \frac{\sin 2\beta}{2ae}$$

$$\therefore\ e = \left|\frac{2\sin\beta\cos\beta}{2\sin\beta\cos\alpha}\right| = \left|\frac{\cos\beta}{\cos\alpha}\right|$$

Sol 25: Let P = (acosθ, bsinθ)

normal is $\frac{5x}{\cos\theta} - \frac{4y}{\sin\theta} = 9$

It passes through (ae, 0) = (3, 0)

when sinθ ≠ 0

$\frac{15}{\cos\theta}$ = 9 $\therefore$ cosθ = $\frac{5}{3}$ × not possible

When sinθ = 0 equation of normal is y = 0 which passes through (3, 0)

$\therefore$ Radius = a – ae = 5 – 3 = 2

Sol 26: PF = QF and PC = RC

$\therefore$ Ae – 1 + b – 1 = $\sqrt{a^2e^2+b^2}$

$(ae + b - 2)^2 = (a^2e^2 + b^2)$

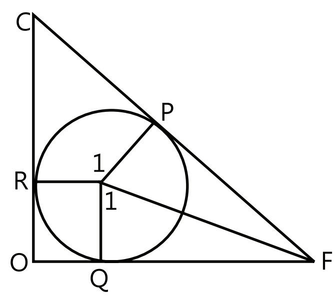
$\therefore$ 4 – 4ae – 4b + 2aeb = 0

2 = 2ae + 2b – aeb

ae = 6

$\therefore$ 2 = 12 + 2b – 6b

$\therefore$ b = $\frac{5}{2}$



$a^2l^2 = a^2 - b^2$

36 = $a^2 - \frac{25}{4}$

$a^2 = \frac{16a}{4}$ $\therefore$ a = $\frac{13}{2}$

AB. CD = 4ab = 65

Exercise 2

Single Correct Choice Type

Sol 1: (A) Tangent to ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is

$\frac{x}{4}\cos\theta + \frac{y}{3}\sin\theta = 1$

Let P be point of intersection of x-axis & Q be Let Q be the point on tangent and y-axis

$\therefore$ P = $\left(\frac{4}{\cos\theta}, 0\right)$ & Q = $\left(0, \frac{3}{\sin\theta}\right)$

Let M = (x, y)

x = $\frac{2}{\cos\theta}$ & $y = \frac{3}{2\sin\theta}$

$\therefore \frac{4}{x^2}+\frac{9}{4y^2}=1$

$\therefore 16y^2 + 9x^2 = 4x^2y^2$

Sol 2: (C) P = (acosθ, bsinθ)

Q = (acosθ, asinθ)

normal at P is

$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}$ = $a^2 - b^2$

$\Rightarrow \frac{4x}{\cos\theta}-\frac{3y}{\sin\theta}=7$

Equation of CQ is y = tanθx

$\therefore$ $\frac{x}{\cos\theta}$ = 7 ⇒ x = 7cosθ & y = 7sinθ s

$\therefore$ R = (7cosθ, 7sinθ)

$\therefore$ ℓ(CR) = 7

Sol 3: (A) Lines that C = (1, 2)

Patting through P = (4, 6)

$\because$ Centre and focus have same y coordie

$\therefore$ This will be ellipse where major axis is horizontal so.

Eqn. will be $\frac{(x-n)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Centre is (h, k)

The left focus (h – c, k)

Right focus (h + c, k)

Where $c^2 = a^2 - b^2$

On putting values

$$\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1 \qquad \text{(i)}$$

Patting through (9, 6)

$\Rightarrow \frac{3^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{9}{a^2}+\frac{16}{b^2}=1$ (ii)

The given frees (6, 2) must be right focuses

$\Rightarrow (h+c,k)=(6,2)$

$\Rightarrow h+c=6 \quad k=2$

$\Rightarrow \quad c=5 \quad (\because h=1)$

Now, $C^2=a^2-b^2 \Rightarrow a^2-b^2=25$ (iii)

Solving (ii) and (iii)

Substituting in (i)

$\frac{(x-1)^2}{45}+\frac{(y-2)^2}{20}=1$

Sol 4: (A) Let the lines be x-axis & y-axis

Let P = (h, 0) and Q = (0, k)

$h^2 + k^2 = (a + b)^2$

$$(x, y) = \left(\frac{bh}{(a+b)}, \frac{ak}{(a+b)}\right)$$

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

∴ Equation of P is an ellipse

Sol 5: (A) Now sum of distance of points from two foci = constant = 2a for an ellipse

∴ Necessary length of string = 2a = 6

a = 3 and b = 2

Distance between pins = 2ae

$$= 6\sqrt{1-\frac{b^2}{a^2}} = \frac{6\times\sqrt{5}}{3} = 2\sqrt{5}$$

Sol 6: (B) It is a known property that

$SF_1 \cdot SF_2 = b^2 = 3$

Sol 7: (B) Ellipse 1 is

$$\frac{x^2}{\left(\sqrt{\frac{3}{a}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{3}{b}}\right)^2} = 1$$

And E_2 is

$$\frac{x^2}{\left(\sqrt{\frac{3}{a+b}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{3}{a-b}}\right)^2} = 1$$

are of ellipse = pab

∴ $pa_1b_1 = pa_2b_2$

$$\Rightarrow \frac{3}{\sqrt{ab}} = \frac{3}{\sqrt{a^2-b^2}}$$

$\Rightarrow a^2 - b^2 = ab$

$$\Rightarrow \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) - 1 = 0$$

$$\therefore \frac{a}{b} = \frac{1+\sqrt{5}}{2}$$

Sol 8: (B) Let P = (h, k)

Foot of perpendicular from focus to any tangent of the ellipse lies on its auxiliary circle.

∴ Midpoint of P & S lies on auxiliary circle

$$\therefore M = \left(\frac{h \pm ae}{2}, \frac{k}{2}\right)$$

$$\Rightarrow \left(\frac{h \pm ae}{2}\right)^2 + \left(\frac{k}{2}\right)^2 = a^2.$$

$$\Rightarrow \left(\frac{x \pm ae}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = a^2.$$

Sol 9: (C) Equation of normal is

$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$

$$Q = \left(\frac{a^2-b^2}{a}\right)\cos\theta$$

$$R = -\left(\frac{a^2-b^2}{b}\right)\sin\theta$$

$$M = (x, y) = \left(\frac{a^2-b^2}{2a}\times\cos\theta, -\frac{(a^2-b^2)}{2b}\sin\theta\right)$$

∴ Locus of M is

$$(ax)^2 + (by)^2 = \frac{(a^2-b^2)^2}{4}$$

$$\therefore \frac{x^2}{\left(\frac{a^2-b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{a^2-b^2}{2b}\right)^2} = 0$$

coeff of y is > coeff of x.

$$e' = 1 - \frac{\left(\frac{a^2-b^2}{2a}\right)^2}{\left(\frac{a^2-b^2}{2b}\right)^2} = 1 - \frac{b^2}{a^2} = 0$$

Sol 10: (C) Equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and of a circle is

$x^2 + y^2 = a^2 - b^2$

$$\therefore \frac{a^2-b^2-y^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow (a^2 - b^2)y^2 = a^2b^2 - a^2b^2 + b^4$

$\therefore y^2 = \dfrac{b^4}{a^2 - b^2}$

$y = \dfrac{b^2}{\sqrt{a^2 - b^2}}$

$2a = 17 \therefore\ a = \dfrac{17}{2}$

And aexy = 30

$\therefore \sqrt{a^2 - b^2} \times \dfrac{b^2}{\sqrt{a^2 - b^2}} = 30$

$\therefore$ Distance between foci

$= 2ae = 2\sqrt{a^2 - b^2} = 2\sqrt{\dfrac{289}{4} - 30}$

$= 2 \times \dfrac{13}{2} = 13$

Sol 11: (A) Equation of tangent is

$\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} - 1 = 0$

$\Rightarrow \dfrac{x}{\sqrt{2}a} + \dfrac{y}{\sqrt{2}b} - 1 = 0$

$\therefore P' = \dfrac{1}{\sqrt{\dfrac{1}{2a^2} + \dfrac{1}{2b^2}}}$

Equation of normal is $\sqrt{2}ax - \sqrt{2}by = a^2 - b^2$

$\therefore P^2 = \dfrac{a^2 - b^2}{\sqrt{2(a^2 + b^2)}}$

$\therefore$ Area of rectangle = P_1P_2

$= \dfrac{\sqrt{2}ab}{\sqrt{a^2 + b^2}} + \dfrac{(a^2 - b^2)}{\sqrt{2(a^2 + b^2)}} = \dfrac{(a^2 - b^2)ab}{(a^2 + b^2)}$

Sol 12: (A) If for an ellipse S & S′ are focus, then

$\tan\left(\dfrac{PSS'}{2}\right) \times \tan\left(\dfrac{PS'S}{2}\right) = \dfrac{1 - e}{1 + e}$

$\therefore$ Centre of ellipse = (5, 0)

$\dfrac{1 - e}{1 + e} = \dfrac{1}{4}$

$\therefore 5e = 3e = \dfrac{3}{5}$

$2ae = 6$

$\therefore a = 5$

$\therefore 1 - \dfrac{b^2}{a^2} = \dfrac{a}{25}$

$\therefore b = 4$

$\therefore$ Equation of ellipse can be $\dfrac{(x-5)^2}{25} + \dfrac{y^2}{16} = 1$

Multiple Correct Choice Type

Sol 13: (A, C, D) (A) ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

Equation of director circle is $x^2 + y^2 = a^2 + b^2$

$\Rightarrow x^2 + y^2 = 14$

(B) Sum of focal distances = 2b(when b > 0)

and 2a when a > 0

$\therefore S = 2 \times 6 = 12$

(C) Free (a known property of parabola)

(D) Line passes through focus.

$\dfrac{2at_2 - 0}{at_2^2 - a} = \dfrac{2at_1}{at_1^2 - a}$

(slope of PF = QF)

$\therefore t_2(t_1^2 - 1) = t_1(t_2^2 - 1)$

$t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$

$\therefore t_1 = t_2$ or $t_1t_2 = -1$

But points are distinct

$\therefore t_1t_2 = -1$

Sol 14: (C, D) Equation of tangent is

$y = mx \pm \sqrt{\dfrac{5}{3}m^2 + \dfrac{5}{2}}$

It passes through (1, 2)

$(m - 2)^2 = \dfrac{5}{3}m^2 + \dfrac{5}{2}$

$m^2 - 4m + 4 = \dfrac{5}{3}m^2 + \dfrac{5}{2}$

$\dfrac{2}{3}m^2 + 4m - \dfrac{3}{2} = 0$

$4m^2 + 24m - 9 = 0$

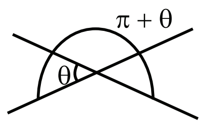
angle between tangents

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2}\right|$

$= \left|\frac{\sqrt{6^2 + \frac{9}{4}\times 4}}{1-\frac{9}{4}}\right| = \left|\frac{3\sqrt{5}}{5}\times 4\right| = \frac{12}{\sqrt{5}}$

$\therefore \theta = \tan^{-1}\frac{12}{\sqrt{5}}$

other angle $= \pi - \tan^{-1}\frac{12}{\sqrt{5}} = \pi - \cot^{-1}\left(\frac{\sqrt{5}}{12}\right)$



If angle between lines is E_0

$\therefore$ $\pi + \theta$ can also be considered angle between lines,

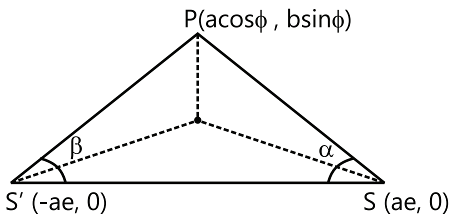
$\therefore$ D is also correct

Sol 15: (A, B, C) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(A) and (B) are true

By sine rule in DPSS' then

$\frac{SP}{\sin(\beta)} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin(\pi-(\alpha+\beta))}$



$\frac{SP}{\sin\beta} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin(\alpha+\beta)}$

$\frac{SP+S'P}{\sin\alpha+\sin\beta} = \frac{SS'}{\sin(\alpha+\beta)}$

$\therefore \frac{2a}{\sin\alpha + m\beta} = \frac{2ae}{\sin(\alpha+\beta)}$

$\therefore \frac{1}{e} = \frac{\sin\beta+\sin\alpha}{\sin(\alpha+\beta)}$

$\frac{1}{e} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}$

$\frac{1}{e} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$

$\frac{1-e}{1+e} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)}$

$\therefore \tan\frac{\alpha}{2}\ \tan\frac{\beta}{2} = \frac{1-e}{1+e} = \frac{(1-e)^2}{1-e^2}$

$= \frac{1+1-\frac{b^2}{a^2}-2\sqrt{1-\frac{b^2}{a^2}}}{\frac{b^2}{a^2}} = \frac{2a^2-b^2-2a\sqrt{a^2-b^2}}{b^2}$

Sol 16: (A, C) Equation of tangent to parabola is

$y = mx + \frac{1}{m}$

for ellipse $c^2 = a^2m^2 + b^2$

$\therefore \frac{1}{m^2} = 8m^2 + 2$

$8m^4 + 2m^2 - 1 = 0$

$8m^4 + 4m^2 - 2m^2 - 1 = 0$

$\therefore m^2 = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$

$\therefore$ Equation of tangents

are $y = \frac{1}{2}x + 2$ or $x - 2y + 4 = 0$

and $y = -\frac{1}{2}x - 2$ or $2y + x + 4 = 0$

Sol 17: (A, B, C, D) $a^2\cos^2\theta + b^2\sin^2\theta = 4$

$6\cos^2\theta + 2\sin^2\theta = 4$

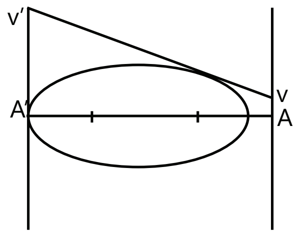
$4\cos^2\theta = 2$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\therefore$ Eccentric angle is $\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

Sol 18: (A, C, D) Equation of tangent is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$



A =(a, 0)

$$\therefore\ V=\left(a,\frac{b(1-\cos\theta)}{\sin\theta}\right)$$

A′ = (–a, 0)

$$\therefore\ V'=\left(-a,\frac{b(1+\cos\theta)}{\sin\theta}\right)$$

$$AV\times A'V=\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}$$

We know that V′V subtend a right angle at each of the foci

∴ VV′ SS′ lie on a circle with VV′ as diameter.

Previous Years' Questions

Sol 1: (A) Given, $y=mx-b\sqrt{1+m^2}$ touches both the circles, so distance from centre = radius of both the circles.

$$\frac{\left|ma-0-b\sqrt{1+m^2}\right|}{\sqrt{m^2+1}}=b$$

and $\dfrac{\left|-b\sqrt{1+m^2}\right|}{\sqrt{m^2+1}}=b$

$$\Rightarrow\left|ma-b\sqrt{1+m^2}\right|=\left|-b\sqrt{1+m^2}\right|$$

$$\Rightarrow m^2a^2-2abm\sqrt{1+m^2}+b^2=b^2(1+m^2)$$

$$\Rightarrow ma-2b\sqrt{1+m^2}=0$$

$$\Rightarrow m^2a^2=4b^2(1+m^2)$$

$$\Rightarrow m=\frac{2b}{\sqrt{a^2-4b^2}}$$

Sol 2: **(B)** Given tangent is drawn at $(3\sqrt{3}\cos\theta,\sin\theta)$ to $\frac{x^2}{27}+\frac{y^2}{1}=1$.

∴ Equation of tangent is $\frac{x\cos\theta}{3\sqrt{3}}+\frac{y\sin\theta}{1}=1$

Thus, sum of intercepts

$$=\left(\frac{3\sqrt{3}}{\cos\theta}+\frac{1}{\sin\theta}\right)=f(\theta)\ \text{(say)}$$

$$\Rightarrow f'(\theta)=\frac{3\sqrt{3}\sin^3\theta-\cos^3\theta}{\sin^2\theta\cos^2\theta}\ \text{Put } f'(\theta)=0$$

$$\Rightarrow\ \sin^3\theta=\frac{1}{3^{3/2}}\cos^3\theta$$

$$\Rightarrow\ \tan\theta=\frac{1}{\sqrt{3}}\text{ , i.e., }\theta=\frac{\pi}{6}$$

and at $\theta=\frac{\pi}{6}$, $f''(0)>0$

∴ Hence, tangent is minimum at $\theta=\frac{\pi}{6}$.

Sol 3: (C) There are two common tangents to the circle $x^2+y^2=1$ and the hyperbola $x^2-y^2=1$. These are x = 1 and x = –1. But x = 1 is nearer to the point P(1/2, 1).

Therefore, directrix of the required ellipse is x = 1.

$$\therefore\ e=\frac{\sqrt{3}}{2}$$

$$\therefore\ x=\pm4\times\frac{\sqrt{3}}{2}=\pm2\sqrt{3}\ (\because x=\pm ae)\qquad\text{...(ii)}$$

On solving equation (i) and (ii), we get $\frac{4}{49}\times12+y^2=1$

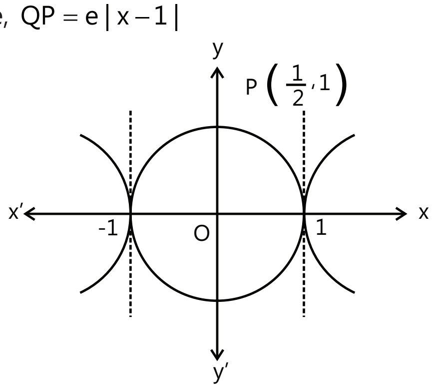
$$\Rightarrow y^2=1-\frac{48}{49}=\frac{1}{49}\ \Rightarrow y=\pm\frac{1}{7}$$

∴ Required points $\left(\pm2\sqrt{3},\pm\frac{1}{7}\right)$.

Sol 4: Now, If Q(x, y) is any point on the ellipse, then its distance from the focus is

$$QP=\sqrt{(x-1/2)^2+(y-1)^2}$$

and its distance from the directrix is $|x-1|$ by definition of ellipse, $QP=e|x-1|$



$$\Rightarrow \sqrt{\left(x-\frac{1}{2}\right)^2+(y-1)^2}=\frac{1}{2}|x-1|$$

$$\Rightarrow \left(x-\frac{1}{2}\right)^2+(y-1)^2=\frac{1}{4}(x-1)^2$$

$$\Rightarrow x^2-x+\frac{1}{4}+y^2-2y+1=\frac{1}{4}(x^2-2x+1)$$

$$\Rightarrow 4x^2-4x+1+4y^2-8y+4=x^2-2x+1$$

$$\Rightarrow 3x^2-2x+4y^2-8y+4=0$$

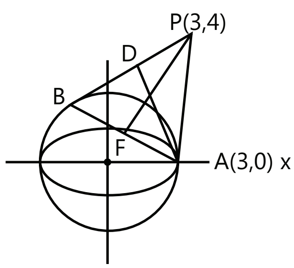
$$\Rightarrow 3\left[\left(x-\frac{1}{3}\right)^2-\frac{1}{9}\right]+4(y-1)^2=0$$

$$\Rightarrow 3\left(x-\frac{1}{3}\right)^2+4(y-1)^2=\frac{1}{3}$$

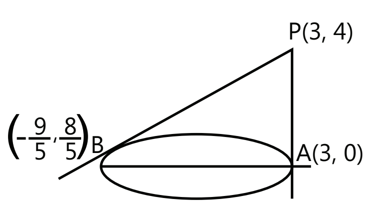
$$\Rightarrow \frac{\left(x-\frac{1}{3}\right)^2}{1/9}+\frac{(y-1)^2}{1/12}=1$$

Comprehenesion Type

Sol 5: (D) Figure is self explanatory.



Sol 6: (C) Equation of AB is



$$y-0=\frac{\frac{8}{5}}{-\frac{9}{5}-3}(x-3)$$

$$=\frac{8}{-24}(x-3)$$

$$\Rightarrow y=-\frac{1}{3}(x-3)$$

$\Rightarrow x + 3y = 3$ ...(i)

Equation of the straight line perpendicular to AB through P is $3x-y=5$.

Equation of PA is $x-3=0$.

The equation of straight line perpendicular to PA through $B\left(-\frac{9}{5},\frac{8}{5}\right)$ is $y=\frac{8}{5}$.

Hence, the orthocenter is $\left(\frac{11}{5},\frac{8}{5}\right)$.

Sol 7: (A) Equation of AB is $y-0=-\frac{1}{3}(x-3)$

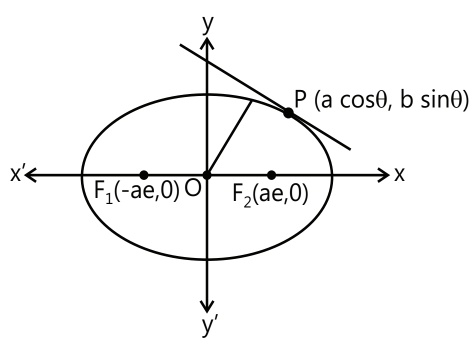
$x+3y-3=0\ \ |x+3y-3|^2=10[(x-3)^2+(y-4)^2]$

(Look at coefficient of x^2 and y^2 in the answers).

Sol 8: Let the coordinates of point P be $(a\cos\theta, b\sin\theta)$. Then equation fo tangent at P is

$$\frac{x}{a}\cos\theta+\frac{y}{b\sin\theta}=1 \quad \text{...(i)}$$

We have , d = length of perpendicular from O to the tangent at P.



$$d=\frac{|0+0-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

$$\Rightarrow \frac{1}{d}=\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}$$

$$\Rightarrow \frac{1}{d^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$$

We have, to prove $(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$.

Now, RHS = $4a^2\left(1-\frac{b^2}{d^2}\right)$

$$=4a^2-\frac{4a^2b^2}{d^2}$$

$= 4a^2 - 4a^2b^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)$

$= 4a^2 - 4b^2\cos^2\theta - 4a^2\sin^2\theta$

$= 4a^2(1-\sin^2\theta) - 4b^2\cos^2\theta$

$= 4a^2\cos^2\theta - 4b^2\cos^2\theta$

$= 4\cos^2\theta(a^2 - b^2)$

$= 4\cos^2\theta a^2e^2\left(\because e = \sqrt{a-(b/a)^2}\right)$

Again, $PF_1 = e\left|a\cos\theta + a/e\right|$

$= a\left|e\cos\theta + 1\right|$

$= a(e\cos\theta + 1)$ $(\because -1 \le \cos\theta \le 1$ and $0 < e < 1)$

Similarly, $PF_2 = a(1 - e\cos\theta)$

Therefore, LHS = $(PF_1 - PF_2)^2$

$= [a(e\cos\theta + 1) - a(1 - e\cos\theta)]^2$

$= (ae\cos\theta + a - a - ae\cos\theta)^2$

$= (2ae\cos\theta)^2 = 4a^2e^2\cos^2\theta$

Hence, LHS = RHS.

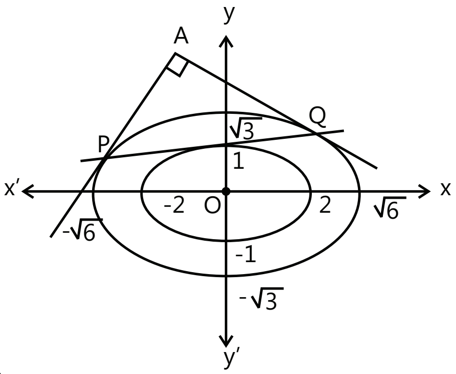
Sol 9: Given, $x^2 + 4y^2 = 4$

or $\frac{x^2}{4} + \frac{y^2}{1} = 1$ ...(i)

Equation of any tangent to the ellipse on (i) can be written as

$\frac{x}{2}\cos\theta + y\sin\theta = 1$ ...(ii)

Equation of second ellipse is



$x^2 + 2y^2 = 6$

$\Rightarrow \frac{x^2}{6} + \frac{y^2}{3} = 1$ ...(iii)

Suppose the tangents of P and Q meets in A(h,k). Equation of the chord of contact of the tangents through A(h,k) is

$\frac{hx}{6} + \frac{ky}{3} = 1$ ...(iv)

But Eqs. (iv) and (ii) represent the same straight line, so comparing Eqs. (iv) and (ii), we get

$\frac{h/6}{\cos\theta/2} = \frac{k/3}{\sin\theta} = \frac{1}{1} \Rightarrow h = 3\cos\theta$ and $k = 3\sin\theta$

Therefore, coordinates of A are $(3\cos\theta, 3\sin\theta)$.

Now, the joint equation of the tangents at A is given by $T^2 = SS_1$

i.e, $\left(\frac{hx}{6} + \frac{ky}{3} - 1\right)^2$

$= \left(\frac{x^2}{6} + \frac{y^2}{3} - 1\right)\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right)$ ...(v)

In equation (v).

Coefficient of $x^2 = \frac{h^2}{36} - \frac{1}{6}\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right)$

$= \frac{h^2}{36} - \frac{h^2}{36} - \frac{k^2}{18} + \frac{1}{6} = \frac{1}{6} - \frac{k^2}{18}$

And coefficient of $y^2 = \frac{k^2}{9} - \frac{1}{3}\left(\frac{h^2}{6} + \frac{k^2}{3} - 1\right)$

$= \frac{k^2}{9} - \frac{h^2}{18} - \frac{k^2}{9} + \frac{1}{3} = -\frac{h^2}{18} + \frac{1}{3}$

Again, coefficient of x^2 + coefficient of y^2

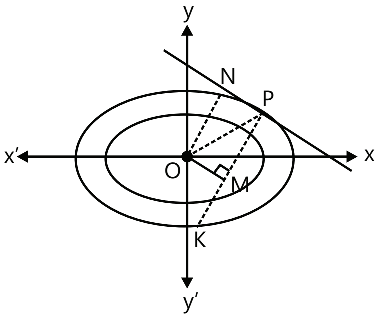
$= -\frac{1}{18}(h^2 + k^2) + \frac{1}{6} + \frac{1}{3}$

$= -\frac{1}{18}(9\cos^2\theta + 9\sin^2\theta) + \frac{1}{2}$

$= -\frac{9}{18} + \frac{1}{2} = 0$

Which shows that two lines represent by equation (v) are at right angles to each other.

Sol 10: Let the coordinates of P be $(a\cos\theta, b\sin\theta)$. Equations of tangents at P is



$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Again, equation of normal at point P is

$$ax\sec\theta-by\,\text{cosec}\,\theta=a^2-b^2$$

Let M be foot of perpendicular from O to PK, the normal at P.

Area of $\Delta OPN=\frac{1}{2}$ (Area of rectangle OMPN)

$=\frac{1}{2}$ ON. OM

Now,

$$ON=\frac{1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

($\perp$ from O, to line NP)

$$\text{and } OM=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\text{cosec}^2\theta}}$$

$$=\frac{(a^2-b^2).\cos\theta.\sin\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

Thus area of

$$\Delta OPN=\frac{ab(a^2-b^2).\cos\theta.\sin\theta}{2(a^2\sin^2\theta+b^2\cos^2\theta)}$$

$$=\frac{ab(a^2-b^2)\tan\theta}{2(a^2\tan^2\theta+b^2)}$$

Let $f(\theta)=\frac{\tan\theta}{a^2\tan^2\theta+b^2}$ $(0<\theta<\pi/2)$

$$f'(\theta)=\frac{\sec^2\theta(a^2\tan^2\theta+b^2)-\tan\theta(2a^2\tan\theta\sec^2\theta)}{(a^2\tan^2\theta+b^2)^2}$$

$$=\frac{\sec^2\theta(a^2\tan^2\theta+b^2-2a^2\tan^2\theta)}{(a^2\tan^2\theta+b^2)^2}$$

$$=\frac{\sec^2\theta(a\tan\theta+b)(b-a\tan\theta)}{(a^2\tan^2\theta+b^2)^2}$$

For maximum or minimum, we put

$f'(\theta)=0\Rightarrow b-a\tan\theta=0$

$[\sec^2\theta\neq 0.\ a\tan\theta+b\neq 0,\ 0<\theta<\pi/2]$

$\Rightarrow\ \tan\theta=b/a$

$$\text{Also, } f'(\theta)\begin{cases}>0, & \text{if } 0<\theta<\tan^{-1}(b/a)\\ <0, & \text{if} \tan^{-1}(b/a)<\theta<\pi/2\end{cases}$$

Therefore, $f(\theta)$ has maximum, when

$$\theta=\tan^{-1}\left(\frac{b}{a}\right)\Rightarrow\tan\theta=\frac{b}{a}$$

$$\text{Again}\sin\theta=\frac{b}{\sqrt{a^2+b^2}}-\cos\theta=\frac{a}{\sqrt{a^2+b^2}}$$

By using symmetry, we get the required points

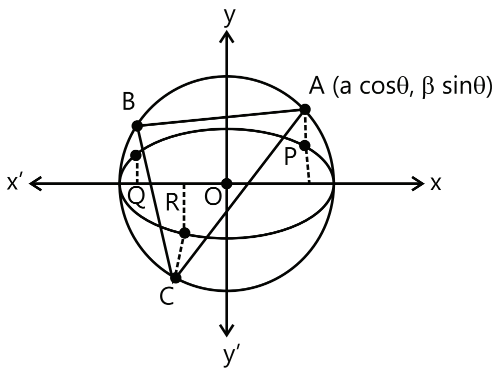
$$\left(\frac{\pm a^2}{\sqrt{a^2+b^2}},\frac{\pm b^2}{\sqrt{a^2+b^2}}\right).$$

Sol 11: Let the coordinates of $A\equiv(a\cos\theta, b\sin\theta)$, so that the coordinates of

$B=\{a\cos(\theta+2\pi/3),\ a\sin(\theta+2\pi/3)\}$

and $C=\{a\cos(\theta+4\pi/3),\ a\sin(\theta+4\pi/3)\}$

According to the given condition, coordinates of P are $(a\cos\theta, b\sin\theta)$ and that of Q are $\{a\cos(\theta+2\pi/3), b\sin(\theta+2\pi/3)\}$ and that of R are $\{a\cos(\theta+4\pi/3), b\sin(\theta+4\pi/3)\}$.



[It is given that P, Q, R are on the same side of x-axis as A, B and C].

Equation of the normal to the ellipse at P is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \text{ or } ax\sin\theta-by\cos\theta$$

$$=\frac{1}{2}(a^2-b^2)\sin2\theta \qquad ......(i)$$

Equation of normal to the ellipse at Q is

$$ax\sin\left(\theta+\frac{2\pi}{3}\right)-by\cos\left(\theta+\frac{2\pi}{3}\right)=$$

$$\frac{1}{2}(a^2-b^2)\sin\left(2\theta+\frac{4\pi}{3}\right) \qquad ...(ii)$$

Equation of normal to the ellipse at R is

$$ax\sin\left(\theta+\frac{4\pi}{3}\right)-by\cos\left(\theta+\frac{4\pi}{3}\right)=$$

$$\frac{1}{2}(a^2-b^2)\sin\left(2\theta+\frac{8\pi}{3}\right) \qquad ...(iii)$$

$$\text{But} \sin\left(\theta+\frac{4\pi}{3}\right)=\sin\left(2\pi+\theta-\frac{2\pi}{3}\right)=\sin\left(\theta-\frac{2\pi}{3}\right)$$

$$\text{and } \cos\left(\theta+\frac{4\pi}{3}\right)=\cos\left(2\pi+\theta-\frac{2\pi}{3}\right)=\cos\left(\theta-\frac{2\pi}{3}\right)$$

$$\text{and } \sin\left(2\theta+\frac{8\pi}{3}\right)=\sin\left(4\pi+2\theta-\frac{4\pi}{3}\right)=\sin\left(2\theta-\frac{4\pi}{3}\right)$$

Now, eq. (iii) can be written as

$$ax\sin(\theta-2\pi/3)-by\cos(\theta-2\pi/3)=$$

$$\frac{1}{2}(a^2-b^2)\sin(2\theta-4\pi/3) \qquad ...(iv)$$

For the liens (i), (ii) and (iv) to be concurrent, we must have the determinant

$$\Delta_1=\begin{vmatrix} a\sin\theta & -b\cos\theta & \frac{1}{2}(a^2-b^2)\sin2\theta \\ a\sin\left(\theta+\frac{2\pi}{3}\right) & -b\cos\left(\theta+\frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin\left(2\theta+\frac{4\pi}{3}\right) \\ a\sin\left(\theta-\frac{2\pi}{3}\right) & -b\cos\left(\theta-\frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}=0$$

Thus, line (i), (ii) and (iv) are concurrent.

Sol 12: Any point on the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be P $(a\cos\theta, b\sin\theta)$.

The equation of tangent at point P is given by;

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

The equation of line perpendicular to tangent is,

$$\frac{x\sin\theta}{b}-\frac{y\cos\theta}{a}=\lambda$$

Since, it passes through the focus (ae, 0), then

$$\frac{ae\sin\theta}{b}-0=\lambda$$

$$\Rightarrow \lambda=\frac{ae\sin\theta}{b}$$

$$\therefore \text{ Equation is } \frac{x\sin\theta}{b}-\frac{y\cos\theta}{a}=\frac{ae\sin\theta}{b} \qquad ...(i)$$

Equation of line joining centre and point of contact $P(a\cos\theta, b\sin\theta)$ is

$$y=\frac{b}{a}(\tan\theta)x \qquad ...(ii)$$

Point of intersection Q of Eqs. (i) and (ii) has x coordinate $\frac{a}{e}$.

Hence, Q lies on the corresponding directrix $x=\frac{a}{e}$.

Sol 13: (B, C) $\frac{x^2}{4}+\frac{y^2}{1}=1$

$$b^2=a^2\left(1-e^2\right)$$

$$\Rightarrow e=\frac{\sqrt{3}}{2}$$

$$\Rightarrow P\left(\sqrt{3},-\frac{1}{2}\right) \text{ and } Q\left(-\sqrt{3},-\frac{1}{2}\right)$$

(given y_1 and y_2 less than 0).

Co-ordinates of mid-point of PQ are

$R\equiv\left(0,-\frac{1}{2}\right)$.

$PQ=2\sqrt{3}$ = length of latus rectum.

$\Rightarrow$ two parabola are possible whose vertices are

$\left(0,-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$ and $\left(0,\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$.

Hence the equations of the parabolas are

$$x^2-2\sqrt{3}y=3+\sqrt{3}$$

And $x^2+2\sqrt{3}y=3-\sqrt{3}$.

Sol 14: (A, B) Ellipse and hyperbola will be confocal

$$\Rightarrow(\pm ae,0)\equiv(\pm 1,0)$$

$\Rightarrow\left(\pm a\times\frac{1}{\sqrt{2}},0\right)\equiv(\pm 1,0)$

$\Rightarrow a=\sqrt{2}$ and $e=\frac{1}{\sqrt{2}}$

$\Rightarrow b^2=a^2\left(1-e^2\right)\Rightarrow b^2=1$

$\therefore$ Equation of ellipse $\frac{x^2}{2}+\frac{y^2}{1}=1$

Sol 15: (D) Equation of line AM is x + 3y −3 = 0

Perpendicular distance of line from origin $=\frac{3}{\sqrt{10}}$

Length of AM $=2\sqrt{9-\frac{9}{10}}=2\times\frac{9}{\sqrt{10}}$

$\Rightarrow$ Area $=\frac{1}{2}\times 2\times\frac{9}{\sqrt{10}}\times\frac{3}{\sqrt{10}}=\frac{27}{10}$ sq. units

Sol 16: $A\to p; B\to s, t; C\to r; D\to q, s$

(p) $\frac{1}{k^2}=4\left(1+\frac{h^2}{k^2}\right)$

$\Rightarrow 1=4\left(k^2+h^2\right)$

$\therefore h^2+k^2=\left(\frac{1}{2}\right)^2$ which is a circle.

(q) If $\left|z-z_1\right|-\left|z-z_2\right|=k$ where $k<\left|z_1-z_2\right|$ the locus is a hyperbola.

(r) Let t = $\tan\alpha$

$\Rightarrow x=\sqrt{3}\cos 2\alpha$ and $\sin 2\alpha=y$

or $\cos 2\alpha=\frac{x}{\sqrt{3}}$ and $\sin 2\alpha=y$

$\therefore \frac{x^2}{3}+y^2=\sin^2 2\alpha+\cos^2 2\alpha=1$ which is an ellipse.

(s) If eccentricity is $[1,\infty)$, then the conic can be a parabola (if e = 1) and a hyperbola if $e\in(1,\infty)$.

(t) Let $z=x+iy;\ x,y\in R$

$\Rightarrow (x+1)^2-y^2=x^2+y^2+1$

$\Rightarrow y^2=x$; which is a parabola.

Sol 17: (B) Let equation of tangent to ellipse

$\frac{\sec\theta}{3}x-\frac{\tan\theta}{2}y=1$

$2\sec\theta x-3\tan\theta y=6$

It is also tangent to circle $x^2+y^2-8x=0$

$\Rightarrow\frac{\left|8\sec\theta-6\right|}{\sqrt{4\sec^2\theta+9\tan^2\theta}}=4$

$\left(8\sec\theta-6\right)^2=16\left(13\sec^2\theta-9\right)$

$\Rightarrow 12\sec^2\theta+8\sec\theta-15=0$

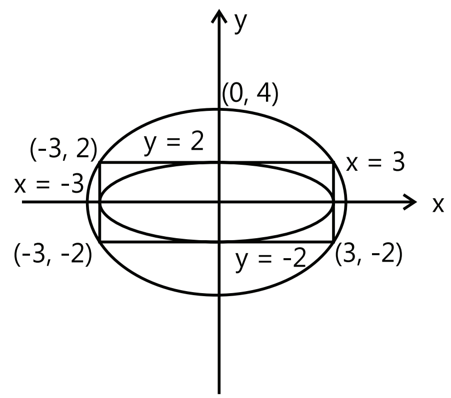
$\Rightarrow \sec\theta=\frac{5}{6}$ and $-\frac{3}{2}$ but $\sec\neq\frac{5}{6}$

$\Rightarrow \sec\theta=-\frac{3}{2}$ and $\Rightarrow \tan\theta=\frac{\sqrt{5}}{2}$

$\therefore$ Slope is positive

Equation of tangent $=2x-\sqrt{5}\,y+4=0$

Sol 18: (C)



Equation of ellipse is

$(y+2)(y-2)+\lambda(x+3)(x-3)=0$

It passes through (0, 4) $\Rightarrow\lambda=\frac{4}{3}$

Equation of ellipse is $\frac{x^2}{12}+\frac{y^2}{16}=1$

$e=\frac{1}{2}$.

Sol 19: (9)

$\frac{x^2}{4}+\frac{y^2}{3}=1$

$y=\frac{\sqrt{3}}{2}\sqrt{4-h^2}$ at x = h

Let $R\left(x_1,0\right)$

PQ is chord of contact, so $\frac{xx_1}{4}=1 \Rightarrow x=\frac{4}{x_1}$

which is equation of PQ, x = h

So $\frac{4}{x_1}=h \Rightarrow x_1=\frac{4}{h}$

$\Delta(h)$ = area of $\Delta PQR=\frac{1}{2}\times PQ\times RT$

$=\frac{1}{2}\times\frac{2\sqrt{3}}{2}\sqrt{4-h^2}\times(x_1-h)=\frac{\sqrt{3}}{2h}\left(4-h^2\right)^{3/2}$

$\Delta'(h)=\frac{-\sqrt{3}\left(4+2h^2\right)}{2h^2}\sqrt{4-h^2}$

which is always decreasing.

So Δ_1 = maximum of $\Delta(h)=\frac{45\sqrt{5}}{8}$ at $h=\frac{1}{2}$

Δ_2 = minimum of $\Delta(h)=\frac{9}{2}$ at h = 1

So $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=\frac{8}{\sqrt{5}}\times\frac{45\sqrt{5}}{8}-8.\frac{9}{2}=45-36=9$

Sol 20: (D)

The equation of P_1 is $y^2-8x=0$ and P_2 is $y^2+16x=0$

Tangent to $y^2-8x=0$ passes through (-4, 0)

$\Rightarrow 0=m_1(-4)+\frac{2}{m_1}\Rightarrow\frac{1}{m_1^2}=2$

Also tangent to $y^2+16x=0$ passes through (2, 0)

$\Rightarrow 0=m_2\times 2-\frac{4}{m_2}\Rightarrow m_2^2=2$

$\Rightarrow\frac{1}{m_1^2}+m_2^2=4$

Sol 21: (A, B, D)

Tangent at P, $xx_1-yy_1=1$ intersects x axis at $M\left(\frac{1}{x_1},0\right)$

Slope of normal $=-\frac{y_1}{x_1}=\frac{y_1-0}{x_1-x_2}$

$\Rightarrow x_2=2x_1\Rightarrow N\equiv(2x_1,0)$

For centroid $\ell=\frac{3x_1+\frac{1}{x_1}}{3}, m=\frac{y_1}{3}$

$\frac{d\ell}{dx_1}=1-\frac{1}{3x_1^2}$

$\frac{dm}{dy_1}=\frac{1}{3},\frac{dm}{dx_1}=\frac{1}{3}\frac{dy_1}{dx_1}=\frac{x_1}{3\sqrt{x_1^2-1}}$

Sol 22: (C)

Equation of tangent at M is $\frac{x\times 3}{2\times 9}+\frac{y\sqrt{6}}{8}=1$

Put y = 0 as intersection will be on x-axis.

$\therefore R\equiv(6,0)$

Equation of normal at M is

$\sqrt{\frac{3}{2}}X+y=2\sqrt{\frac{3}{2}}+\left(\sqrt{\frac{3}{2}}\right)^3$

Put y = 0, $x=2+\frac{3}{2}=\frac{7}{2}$

$\therefore Q\equiv\left(\frac{7}{2},0\right)$

$\therefore$ Area $(\Delta MQR)=\frac{1}{2}\times\left(6-\frac{7}{2}\right)\times\sqrt{6}=\frac{5}{4}\sqrt{6}$ sq. units.

Area of quadrilateral

$(MF_1NF_2)=2\times$ Area (ΔF_1F_2M)

$=2\times\frac{1}{2}\times 2\times\sqrt{6}=2\sqrt{6}$

$\therefore$ Required Ratio $=\frac{5/4}{2}=\frac{5}{8}$