Ratio and Proportion 6

STUDY NOTES

- If a and b are two non-zero quantities of the same kind and in the same units, then the fraction $\frac{a}{b}$ is called the ratio between a and b, written as $a : b$.
- A ratio is purely a number. It has no units attached to it.

  In the ratio $a : b$, we call a, the *first term* or *antecedent* and b, the *second term* or *consequent*.
- The value of a ratio remains unchanged, if both of its terms are multiplied or divided by the same non-zero number.
- When a and b have no common factor, other than 1, we say that $a : b$ is in its *lowest terms*. Generally a ratio is expressed in its lowest terms.
- *Comparison of Ratios*

  (i) $(a : b) > (c : d) \Leftrightarrow \frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc.$ (ii) $(a : b) = (c : d) \Leftrightarrow \frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc.$

  (iii) $(a : b) < (c : d) \Leftrightarrow \frac{a}{b} < \frac{c}{d} \Leftrightarrow ad < bc.$
- An equality of two ratios is called a *proportion*.
- Four (non-zero) quantities a, b, c, d are said to be in proportion if $a : b = c : d$, i.e., if $\frac{a}{b} = \frac{c}{d}\cdot$ We write it as $a : b :: c : d$.
- If $a : b :: c : d$, then :

  (i) a, b, c, d are known as *first, second, third* and *fourth terms* respectively.

  (ii) d is called the *fourth proportional* to a, b, c.

  (iii) a and d are called *extremes*, while b and c are *means*.
- If $a : b :: c : d$, then the product of extremes is equal to the product of means

  i.e., $a : b :: c : d \Leftrightarrow \frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc.$
- If $a : b :: b : c :: c : d$ ....., or $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ ...., then the quantities a, b, c, d ... are said to be in *continued proportion*.
- Let a, b, c be in continued proportion.

  Then, $\frac{a}{b} = \frac{b}{c}$ or $b^2 = ac$ or $b = \sqrt{ac}$

  Here, b is called the *mean proportion* or *geometric mean* between a and c.

  Here c is called the third proportional to a and b.
- If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each of the ratios is equal to $\frac{a+c+e}{b+d+f}$. Also, each of the ratios is equal to $\frac{la+mc+ne}{lb+md+nf}$.
- If $\frac{a}{b} = \frac{c}{d}$, then each ratio is equal to $\frac{a-c}{b-d}$.
- *Invertendo* : $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{b}{a} = \frac{d}{c}$
- *Alternendo* : $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$
- *Componendo* : $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$
- *Dividendo* : $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$
- *Componendo and Dividendo* : $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$
- *Convertendo* : $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{a-b} = \frac{c}{c-d}$

QUESTION BANK

A. Multiple Choice Questions [1 Mark]

Choose the correct option:

1. If x^2, 4 and 9 are in continued proportion, then the value of x is :

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) $\frac{16}{9}$

2. If $a : b = 5 : 3$, then $(5a + 8b) : (6a - 7b)$ is equal to :

(a) 40 : 9 (b) 9 : 49 (c) 49 : 9 (d) 25 : 9

3. The fourth proportional to 7, 13 and 35 is :

(a) 65 (b) 62 (c) 52 (d) 50

4. The third proportional to 9 and 15 is :

(a) 10 (b) 15 (c) 18 (d) 25

5. If $\frac{7m+2n}{7m-2n} = \frac{5}{3}$, then $m : n$ is :

(a) 7 : 8 (b) 2 : 7 (c) 8 : 7 (d) 1 : 8

6. The mean proportion between 28 and 63 is :

(a) 42 (b) 45 (c) 36 (d) 32

7. The fourth proportional to 3, 12, 15 is :

(a) 40 (b) 45 (c) 60 (d) 62

8. If $x : y = 2 : 3$, then $\frac{3x+2y}{2x+5y}$ is :

(a) $\frac{12}{19}$ (b) $\frac{19}{12}$ (c) $\frac{12}{13}$ (d) $\frac{19}{21}$

9. The mean proportion between $x - y$ and $x^3 - x^2y$ is :

(a) $x(x + y)$ (b) $x^2(x - y)$ (c) $x^2(x + y)$ (d) $x(x - y)$

10. If $a : b = 2 : 3$ and $b : c = 4 : 5$, then $a : c$ is :

(a) 12 : 15 (b) 15 : 7 (c) 15 : 8 (d) 8 : 15

11. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d}$ is equal to :

(a) $\frac{a}{b}$ (b) $\frac{c}{d}$ (c) both (a) and (b) (d) neither (a) nor (b)

12. The fourth proportional to 3, 6 and 4.5 is :

(a) 10 (b) 9 (c) 8.5 (d) 6

13. The third proportional to $x - y$ and $x^2 - y^2$ is :

(a) $(x + y)(x^2 - y^2)$ (b) $(x - y)(x^2 + y^2)$

(c) $(x^2 - y^2)(x^2 + y^2)$ (d) $(x + y)(x - y)$

14. If $\frac{a}{b} : \frac{c}{d}$, then each ratio is equal to :

(a) $a + b : c + d$ (b) $a + c : b + d$ (c) $a + d : b + c$ (d) $a - b : c - d$

15. The mean proportion between a^2b and $\frac{1}{b}$ is :

(a) a (b) a^2 (c) ab (d) $\sqrt{ab}$

16. If $2x = 3y$ and $4y = 5z$, then $\frac{8x}{z}$ is equal to :

(a) 5 (b) 15 (c) 10 (d) 8

17. Two numbers are in the ratio 1 : 4. If the mean proportion between them is 28 and third proportional to them is 224, then the smaller number is :

(a) 12 (b) 14 (c) 16 (d) 21

18. If three quantities a, b, c are in continued proportion, then the mean proportion between :

(a) a and c is b (b) b and c is a (c) c and a is b (d) all the above are true

19. x, y, z are in continued proportion, then $\frac{x}{z}$ is equal to :

(a) $\frac{y^2}{z^2}$ (b) $\frac{y^2}{x^2}$ (c) x^2y^2 (d) $\frac{x}{y^2}$

20. If $\frac{4a + 9b}{4c + 9d} = \frac{4a - 9b}{4c - 9d}$, then $a : b =$

(a) $c : d$ (b) $d : c$ (c) $c : d + c$ (d) $d : c - d$

Answers

1. (b) **2.** (c) **3.** (a) **4.** (d) **5.** (c) **6.** (a) **7.** (c) **8.** (a) **9.** (d) **10.** (d)
11. (c) **12.** (b) **13.** (a) **14.** (b) **15.** (a) **16.** (b) **17.** (b) **18.** (a) **19.** (a) **20.** (a)

B. Short Answer Type Questions [3 Marks]

1. If $(3a + 2b) : (5a + 3b) = 18 : 29$, find $a : b$.

Sol. $\frac{3a + 2b}{5a + 3b} = \frac{18}{29} \Rightarrow 87a + 58b = 90a + 54b$

$\Rightarrow 3a = 4b \Rightarrow \frac{a}{b} = \frac{4}{3}$

i.e., $a : b = 4 : 3$

2. What least number must be added to each of the numbers 2, 5, 18 and 33, so that the resulting numbers are proportional?

Sol. Let x be added such that $2 + x, 5 + x, 18 + x, 33 + x$. are proportional.

$\therefore$ Product of means = product of extremes

$\Rightarrow (5 + x)(18 + x) = (2 + x)(33 + x)$

$\Rightarrow 90 + 5x + 18x + x^2 = 66 + 33x + 2x + x^2$

$\Rightarrow 12x = 24$

$\Rightarrow x = 2.$

3. If b is the mean proportion between a and c, show that $b(a + c)$ is the mean proportion between $(a^2 + b^2)$ and $(b^2 + c^2)$.

Sol. $\because b^2 = ac$ [Given]

$\therefore$ Mean proportion between $(a^2 + b^2)$ and $(b^2 + c^2) = \sqrt{(a^2+b^2)(b^2+c^2)} = \sqrt{(a^2+ac)(ac+c^2)}$

$= \sqrt{a(a+c)\,c(a+c)} = (a+c)\sqrt{ac} = (a+c)\sqrt{b^2} = b(a + c)$ **Proved.**

4. If $a : b = c : d$, then prove that $(a + b) : (c + d) = \sqrt{a^2+b^2} : \sqrt{c^2+d^2}$.

Sol. Let $\frac{a}{b} = \frac{c}{d} = k$

$\Rightarrow a = bk$ and $c = dk$

L.H.S. $= \frac{a + b}{c + d} = \frac{bk + b}{dk + d} = \frac{b}{d}$.

R.H.S. $= \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \frac{\sqrt{b^2k^2+b^2}}{\sqrt{d^2k^2+d^2}} = \sqrt{\frac{b^2}{d^2}} = \frac{b}{d}$.

Hence, L.H.S. = R.H.S. **Proved.**

5. If $\frac{x}{b - c} = \frac{y}{c - a} = \frac{z}{a - b}$, then prove that $ax + by + cz = 0$.

Sol. Let $\frac{x}{b - c} = \frac{y}{c - a} = \frac{z}{a - b} = k$

$\therefore x = k(b - c), y = k(c - a)$ and $z = k(a - b)$

L.H.S. $= ax + by + cz = ak(b - c) + bk(c - a) + ck(a - b)$

$= abk - ack + bck - abk + ack - bck = 0 =$ R.H.S. **Proved.**

6. If a, b, c, d are in continued proportion, prove that $(b + c)(b + d) = (c + a)(c + d)$.

Sol. $\because a, b, c, d$ are in continued proportion.

$\therefore\ \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$ [Let]

$\Rightarrow a = bk,\ b = ck$ and $c = dk$

$\because\ b = ck = dk.\ k = dk^2$

and $a = bk = dk^2.k = dk^3$

L.H.S. $= (b + c)\ (b + d) = (dk^2 + dk)\ (dk^2 + d) = kd^2\ (k + 1)\ (k^2 + 1)$

R.H.S. $= (c + a)\ (c + d)$

$= (dk + dk^3)\ (dk + d)$

$= dk\ (1 + k^2)\ d(k + 1) = kd^2\ (k + 1)\ (k^2 + 1)$

Hence, L.H.S. = R.H.S. **Proved**.

7. If $\frac{a}{b} = \frac{c}{d}$, then, show that $\frac{3a - 5b}{3a + 5b} = \frac{3c - 5d}{3c + 5d}$.

Sol. Given, $\frac{a}{b} = \frac{c}{d}$

or $\frac{3a}{5b} = \frac{3c}{5d}$ [Multiplying by $\frac{3}{5}$ on both sides]

Using componendo and dividendo, we get

$$\frac{3a + 5b}{3a - 5b} = \frac{3c + 5d}{3c - 5d}$$

$$\Rightarrow \frac{3a - 5b}{3a + 5b} = \frac{3c - 5d}{3c + 5d}.$$ **Proved**.

C. Long Answer Type Questions [4 Marks]

1. If b is the mean proportion between a and c, prove that $\frac{a^2 + b^2 + c^2}{a^{-2} + b^{-2} + c^{-2}} = b^4$.

Sol. $\because$ b is the mean proportion between a and c, then $b^2 = ac$.

$$\text{L.H.S.} = \frac{a^2+b^2+c^2}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} = \frac{a^2+b^2+c^2}{\frac{b^2c^2+a^2c^2+a^2b^2}{a^2b^2c^2}} = \frac{a^2b^2c^2(a^2+b^2+c^2)}{b^2c^2+a^2c^2+a^2b^2} = \frac{b^2\times b^4(a^2+b^2+c^2)}{b^2c^2+b^4+a^2b^2}$$

$$= \frac{b^4\times b^2(a^2+b^2+c^2)}{b^2(c^2+b^2+a^2)} = b^4 = \text{R.H.S.}$$ **Proved**.

2. If a, b, c are in continued proportion, prove that $\frac{2a^2 + 7b^2 - 5ab}{2b^2 + 7c^2 - 5bc} = \frac{a}{c}$

Sol. $\because$ a, b, c are in continued proportion.

$\therefore\ \frac{a}{b} = \frac{b}{c} \Rightarrow b^2 = ac$

$$\text{L.H.S.} = \frac{2a^2 + 7b^2 - 5ab}{2b^2 + 7c^2 - 5bc} = \frac{2a^2 + 7ac - 5ab}{2ac + 7c^2 - 5bc} = \frac{a(2a + 7c - 5b)}{c(2a + 7c - 5b)}$$

$= \frac{a}{c}$ = R.H.S. **Proved**.

3. If a, b, c, d are in continued proportion, prove that

$$\sqrt{ab}+\sqrt{bc}-\sqrt{cd} = \sqrt{(a+b-c)(b+c-d)}.$$

Sol. Since, a, b, c, d are in continued proportion.

$\therefore\ \frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$ [Let]

i.e., $a = bk,\ b = ck,\ c = dk$.

or, $a = dk^3$, $b = dk^2$, $c = dk$.

L.H.S. $= \sqrt{ab}+\sqrt{bc}-\sqrt{cd} = \sqrt{dk^3.dk^2} + \sqrt{dk^2.dk} - \sqrt{dk.d}$

$= \sqrt{d^2k^5} + \sqrt{d^2k^3} - \sqrt{d^2k} = dk^2\sqrt{k} + dk\sqrt{k} - d\sqrt{k} = d\sqrt{k}\ (k^2 + k - 1)$

R.H.S. $= \sqrt{(a+b-c)(b+c-d)} = \sqrt{(dk^3+dk^2-dk)(dk^2+dk-d)}$

$= \sqrt{(dk(k^2+k-1).d(k^2+k-1)} = d\ (k^2 + k - 1).\ \sqrt{k}$.

Hence, L.H.S. = R.H.S. **Proved.**

4. If $a : b = c : d$, then show that $(a^2 + c^2 + ac) : (a^2 + c^2 - ac) = (b^2 + d^2 + bd) : (b^2 + d^2 - bd)$

Sol. Let $\frac{a}{b} = \frac{c}{d} = k$, then, $a = bk$ and $c = dk$.

L.H.S. $(a^2 + c^2 + ac) : (a^2 + c^2 - ac) = \frac{a^2 + c^2 + ac}{a^2 + c^2 - ac} = \frac{b^2k^2 + d^2k^2 + bk.dk}{b^2k^2 + d^2k^2 - bk.dk} = \frac{k^2(b^2 + d^2 + bd)}{k^2(b^2 + d^2 - bd)} = \frac{b^2 + d^2 + bd}{b^2 + d^2 - bd}$

$= (b^2 + d^2 + bd) : (b^2 + d^2 - bd)$ = R.H.S. **Proved**.

5. If $ax = by = cz$, then prove that : $\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}$

Sol. Let $ax = by = cz = k$, then, $x = \frac{k}{a}$, $y = \frac{k}{b}$ and $z = \frac{k}{c}$

L.H.S. $= \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy}$

$= \frac{k^2}{a^2.\frac{k}{b}.\frac{k}{c}} + \frac{k^2}{b^2.\frac{k}{c}.\frac{k}{a}} + \frac{k^2}{c^2.\frac{k}{a}.\frac{k}{b}}$

$= \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}$ = R.H.S. **Proved**.

6. If $x = \frac{\sqrt{a+3b}+\sqrt{a-3b}}{\sqrt{a+3b}-\sqrt{a-3b}}$, prove that $3bx^2 - 2ax + 3b = 0$

Sol. $x = \frac{\sqrt{a+3b}+\sqrt{a-3b}}{\sqrt{a+3b}-\sqrt{a-3b}}$

Using componendo and dividendo, we have

$\frac{x + 1}{x - 1} = \frac{\sqrt{a+3b}}{\sqrt{a-3b}}$

Squaring on both sides, we have

$\frac{x^2 + 2x + 1}{x^2 - 2x + 1} = \frac{a + 3b}{a - 3b}$

Again using componendo and dividendo, we have

$\frac{x^2 + 1}{2x} = \frac{a}{3b} \Rightarrow 3bx^2 + 3b = 2ax$

$\Rightarrow 3bx^2 - 2ax + 3b = 0$. **Proved**.

7. If $p = \frac{4xy}{x + y}$, prove that $\frac{p + 2x}{p - 2x} + \frac{p + 2y}{p - 2y} = 2$.

Sol. $p = \frac{4xy}{x + y} \Rightarrow \frac{p}{2x} = \frac{2y}{x + y}$ and $\frac{p}{2y} = \frac{2y}{x + y}$

$\Rightarrow \dfrac{p}{2x} = \dfrac{2y}{x + y}$

$\Rightarrow \dfrac{p + 2x}{p - 2x} = \dfrac{x + 3y}{y - x}$ [Using componendo and dividendo]

And $\dfrac{p}{2y} = \dfrac{2y}{x + y}$

$\Rightarrow \dfrac{p + 2y}{p - 2y} = \dfrac{3x + y}{x - y}$ [using componendo and dividendo]

So, $\dfrac{p + 2x}{x - 2x} + \dfrac{p + 2y}{p - 2y} = \dfrac{-x - 3y + 3x + y}{x - y} = \dfrac{2(x - y)}{x - y} = 2$ **Proved.**

8. If $\dfrac{\sqrt{x+15}+\sqrt{x-6}}{\sqrt{x+15}-\sqrt{x-6}} = \dfrac{7}{3}$, find the value of x.

Sol. $\dfrac{\sqrt{x+15}+\sqrt{x-6}}{\sqrt{x+15}-\sqrt{x-6}} = \dfrac{7}{3}$

Using componendo and dividendo, we have

$\dfrac{\sqrt{x+15}}{\sqrt{x-6}} = \dfrac{7+3}{7-3} = \dfrac{10}{4}$

Squaring on both sides, we get

$\dfrac{x + 15}{x-6} = \dfrac{100}{16} \Rightarrow 16x + 240 = 100x - 600$

$\Rightarrow 84x = 840 \quad \Rightarrow x = 10.$

9. Using the properties of proportion, solve for x:

$\dfrac{\sqrt{6x}+\sqrt{3x+7}}{\sqrt{6x}-\sqrt{3x+7}} = 11.$

Sol. $\dfrac{\sqrt{6x}+\sqrt{3x+7}}{\sqrt{6x}-\sqrt{3x+7}} = 11$

Using componendo and dividendo, we have,

$\dfrac{\sqrt{6x}}{\sqrt{3x+7}} = \dfrac{11+1}{11-1} = \dfrac{12}{10}.$

Squaring on both sides, we get

$\dfrac{6x}{3x + 7} = \dfrac{144}{100} \Rightarrow 600x = 432x + 1008 \Rightarrow 168x = 1008 \Rightarrow x = 6.$