8 Permutations and Combinations

KEY FACTS

I. Fundamental Principle of Counting

(*a*) Multiplication: Let us understand this basic principle with the help of the following examples:

Ex. 1. Suppose you have 3 shirts and 4 pairs of pants. In how many possible ways can you dress up by wearing a shirt and a pair of pants?

In the above case, you can wear any of the 3 shirts and after wearing one of these shirts any of these pairs of pants with it. If we label the shirts as S_1, S_2, S_3 and the pants as P_1, P_2, P_3 and P_4, then the different ways of dressing up can be as under:

$S_1 P_1$	$S_2 P_1$	$S_3 P_1$
$S_1 P_2$	$S_2 P_2$	$S_3 P_2$
$S_1 P_3$	$S_2 P_3$	$S_3 P_3$
$S_1 P_4$	$S_2 P_4$	$S_3 P_4$

Total number of ways = 12 = **3 × 4.**

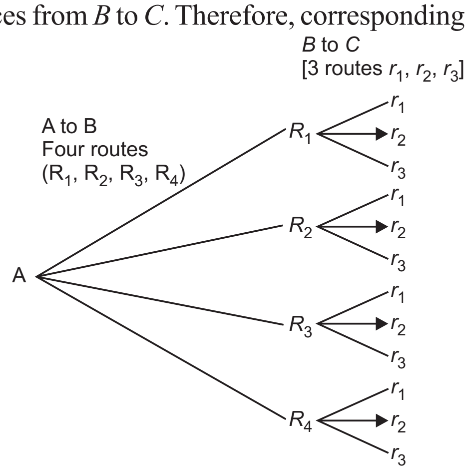
In this illustration, we **multiply** the number of ways in which you can wear a shirt and the number of ways in which you can wear a pair of pants.

Ex. 2. There are 4 different routes between cities *A* and *B*, and 3 different routes between cities *B* and *C*. How many different routes are there from city *A* to city *C* by way of city *B*?

Obviously, one can go from city *A* to city *B* by any of the 4 routes, *i.e.*, in 4 ways. After having gone to *B* by any of the different 4 routes, one can go to city *C* by any of the three routes.

Thus, corresponding to one route taken, from *A* to *B*, he has 3 choices from *B* to *C*. Therefore, corresponding to 4 routes, there are 12 choices in all.

Therefore, he can go from *A* to *C* via *B*, in 4 × 3 = 12 ways as depicted in the given tree diagram.



The possible routes taken are :

R_1r_1	R_2r_1	R_3r_1	R_4r_1
R_1r_2	R_2r_2	R_3r_2	R_4r_2
R_1r_3	R_2r_3	R_3r_3	R_4r_3

Total number of ways =12.

Thus, the examples discussed above illustrate the use of a general principle, called the product rule or the fundamental principle of counting, which is stated below.

If one operation can be performed in m ways, and if corresponding to each of the m ways of performing this operation, there are n ways of performing a second operation, then the number of ways of performing the two operations together is m × n. (This AND That).

Suppose that the first operation is performed in any one of the m ways, the second operation can then be performed in n ways and with the particular first operation, we can associate any one of the n ways of performing the second operation. This means that if the first operation could have been performed only in this one way, there would have been $1 \times n$, *i.e.*, n ways of performing both the operations. But it is given that the first operation can be performed in m ways and there are n ways of performing the second operation for every one way of performing the first operation. Therefore, there are $\boldsymbol{m \times n}$ ways of performing both the operations.

Generalisation. The above principle can be extended to the case in which the different operations can be performed in $m, n, p,$. .....ways. In this case, the number of ways of performing all the operations together would be $\boldsymbol{m \times n \times p}$**.......**

Ex. 3. There are 10 buses running between two towns X and Y. In how many ways can a man go from X to Y and return by a different bus?

Sol. The man can go from X to Y in 10 ways and as he is not to return by the same bus that he took while going, corresponding to each of the 10 ways of going, there are 9 ways of returning. Hence the total number of ways in which he can go to Y and be back is $10 \times 9 = 90$.

Ex. 4. How many different numbers of three digits can be formed with the digits 1, 2, 3, 4, 5, no digit is being repeated?

Sol. The unit's place can be filled with either of these 5 digits and so the unit's place can be filled in 5 ways. The ten's place can be filled in 4 ways corresponding to each way of filling up the unit's place, for we can have any digit here except the one used in the unit's place. Similarly, the hundredth's place can be filled in 3 ways as here we have any of the remaining three digits. Therefore, there are $5 \times 4 \times 3 = 60$ ways of forming a number of three digits with the five given digits.

(*b*) **Addition:** If there are two jobs which can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $m + n$ ways.

For example: Suppose you have 3 full-sleeve and 4 half-sleeve shirts. Since you have the choice of wearing any of these shirts, you can wear one shirt in $3 + 4 = 7$ ways. If in addition, you have 5 T-shirts, then you can wear one of them in $3 + 4 + 5 = 12$ ways.

The above example illustrates one way of counting, which we may call the sum rule and applies when **one event** has to happen out of given disjoint events.

In this case, we use the word '***or***' between various jobs and the meaning of '***or***' is ***addition.***

II. Permutations

Def. : ***Each of the different arrangements which can be made by taking some or all of a number of things at a time is called a permutation.***

Notation: The number of permutations of n things taken r at a time is denoted by $\boldsymbol{^nP_r}$ or $\boldsymbol{P(n, r)}$**.** The letter P is an abbreviation of the word 'permutation'.

Thus $^6P_4$ denotes the number of permutations or arrangements of 6 things taken 4 at a time.

The value of ^nP_r

To find the number of permutations of n different things, taken r at a time or to determine ^nP_r.

The number of permutations of n things taken r at a time will be the same as the number of ways in which r blank places can be filled up with n given things.

As the first place can be filled in by any one of the n things so there are n ways of filling up the first place.

After having filled in the first place by any one of the n things, there are $(n - 1)$ things left. Hence the second place can be filled in $(n - 1)$ ways. Now, as for every one way of filling up the first place, there are $(n - 1)$ ways of filling up the second place, so the first two places can be filled in $n(n - 1)$ ways.

After having filled in the first two places in any one of the above ways, there are $(n-2)$ things left and so the third place can be filled in $(n-2)$ ways. Now for every one way of filling up the first two places, there are $(n-2)$ ways of filling up the third place and so the first three places can be filled up in $n(n-1)(n-2)$ ways.

It may be observed that

(*a*) At every stage the number of factors is equal to the number of places filled up.

(*b*) Every factor is by one less than its preceding factor.

Position of the object	1st	2nd	...	$(r-1)$th	rth
Number of ways	n	$n-1$	...	$n-(r-2)$	$n-(r-1)$

Hence the number of ways of filling up all the r places, *i.e.*, the number of permutations of n different things taken r at a time is $n(n-1)(n-2)......r$ factors

$$= n(n-1)(n-2)\,...(n-\overline{r-1})$$

Hence $\boxed{{}^nP_r = n(n-1)(n-2)\,...\,(n-r+1)}$

Thus, ${}^7P_2 = 7 \times 6$; ${}^{10}P_4 = 10 \times 9 \times 8 \times 7$; ${}^{20}P_3 = 20 \times 19 \times 18$.

Cor. : The number of permutations of n things taken all at a time is

$${}^nP_n = n(n-1)(n-2)......3.\,2.1. \qquad \text{[Putting } n \text{ for } r\text{]}$$

Ex. 1. In how many ways can 5 persons occupy 3 vacant seats?

Sol. Total number of ways $= {}^5P_3 = 5 \times 4 \times 3 = 60$.

Ex. 2. If ${}^{12}P_r = 1320$, find r.

Sol. ${}^{12}P_r = 12 \times 11 \times ......$ to r factors $\Rightarrow 1320 = 12 \times 11 \times 10 \quad \therefore r = 3$.

III. Factorial Notation

The product of n natural numbers from 1 to n is denoted by n ! or $\underline{|n}$ and is read as factorial n.

Thus, $n!$ or $\underline{|n} = 1.\,2.\,3\,......\,(n-1)\,.\,n$

$4! = 1 \times 2 \times 3 \times 4 = 24$; $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$

$(n-1)! = 1 \times 2 \times 3\,......\,(n-1)$.

It is easily seen that $8! = 8 \times (7!)$.

taken out
$n! = n(n-1)!$
One less

IV. Values of ${}^nP_r$ in terms of Factorial Notation

$${}^nP_r = n(n-1)(n-2)\,...(n-r+1) = \frac{n(n-1)...(n-r+1)}{\underline{|(n-r)}}\,.\,\underline{|(n-r)}$$

$$= \frac{n(n-1)(n-2)...(n-r+1)\,.\,(n-r)(n-r-1)...3.\,2.1}{\underline{|(n-r)}}$$

$$= \frac{n(n-1)(n-2)\,...\,3.\,2.1}{\underline{|(n-r)}} = \frac{\underline{|n}}{\underline{|(n-r)}}$$

$$\therefore \qquad \boxed{{}^nP_r = \frac{\underline{|n}}{\underline{|(n-r)}}}$$

Thus, ${}^{18}P_5 = \dfrac{\underline{|18}}{\underline{|13}}$

Cor. 1. Putting $r = 0$, ${}^nP_0 = \dfrac{\underline{|n}}{\underline{|n}} = 1$

Cor. 2. Value of 0 !.

Putting $r = n$, ${}^nP_n = \dfrac{n!}{(n-n)} = \dfrac{n!}{0!}$; But ${}^nP_n = n!$ $\therefore n! = \dfrac{n!}{0!}$ $\boxed{\therefore 0! = 1}$

> **Note.** In fact, 0 ! is meaningless but in order to avoid contradiction in the results, we suppose that 0 ! = 1.

Ex. Find the value of n if ${}^nP_{13} : {}^{n+1}P_{12} = \dfrac{3}{4}$.

Sol. Here, ${}^nP_{13} = \dfrac{n!}{(n-13)!}$ and ${}^{n+1}P_{12} = \dfrac{(n+1)!}{(n+1-12)!} = \dfrac{(n+1)!}{(n-11)!}$ $\left[\text{Using } {}^nP_r = \dfrac{n!}{(n-r)!}\right]$

$${}^nP_{13} : {}^{n+1}P_{12} = \frac{n!}{(n-13)!} \times \frac{(n-11)!}{(n+1)!} = \frac{3}{4} \Rightarrow \frac{n!}{(n-13)!} \times \frac{(n-11)\,(n-12)\,.\,(n-13)!}{(n+1)\,.\,n!} = \frac{3}{4}$$

$$\Rightarrow \frac{(n-11)(n-12)}{(n+1)} = \frac{3}{4} \Rightarrow 4n^2 - 95n + 525 = 0 \text{ or } (n-15)\,(4n-35) = 0$$

$\therefore$ $n = 15$ (Rejecting the fractional value of n).

V. Restricted Permutations

Type I.

Ex. 1. In how many of the permutations of 10 things taken 4 at a time will

(*i*) one thing always occur, (*ii*) never occur?

Sol. (*i*) Keeping aside the particular thing which will always occur, the number of permutations of 9 things taken 3 at a time is ${}^9P_3$. Now this particular thing can take up any one of the four places and so can be arranged in 4 ways. Hence the total number of permutations = ${}^9P_3 \times 4 = 9 \times 8 \times 7 \times 4 = 2016$.

(*ii*) Leaving aside the particular thing which has never to occur, the number of permutations of 9 things taken 4 at a time is ${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$.

Ex. 2. In how many of the permutations of *n* things taken *r* at a time will 5 things

(*i*) always occur, (*ii*) never occur?

Sol. (*i*) Keeping aside the 5 things, the number of permutations of $(r-5)$ things taken out of $(n-5)$ things is ${}^{n-5}P_{r-5}$. Now these 5 things can be arranged in r places in ${}^rP_5$ ways. Hence, the total number of permutations is ${}^rP_5 \times {}^{n-5}P_{r-5}$.

(*ii*) Total number of permutations = ${}^{n-5}P_r = \dfrac{(n-5)!}{(n-r-5)!}$

Type II. *When certain things are not to occur together.*

Case I. When the number of things not occurring together is two.

> **Procedure**
> 1. *Find the total number of permutations –when no restriction is imposed on the manner of arrangement.*
> 2. *Then find the number of permutations when the two things occur together.*
> 3. *The difference of the two results gives the number of permutations in which the two things do not occur together.*

Ex. 3. Prove that the number of ways in which *n* books can be placed on a shelf when two particular books are never together is $(n-2) \times (n-1)!$.

Sol. Regarding the two particular books as one book, there are $(n-1)$ books now which can be arranged in ${}^{n-1}P_{n-1}$, *i.e.*, $(n-1)!$ ways. Now, these two books can be arranged amongst themselves in 2 ! ways. Hence the total number of permutations in which these two books are placed together is 2 ! . $(n-1)$!. The number of permutations of n books without any restriction is n !.

Therefore, the number of arrangements in which these two books never occur together

$= n\,! - 2\,!\,.(n-1)\,! = n\,.\,(n-1)\,! - 2\,.\,(n-1)\,! = (n-2)\,.\,(n-1)\,!$

Case II. When the number of things not occurring together is more than two.

Ex. 4. In how many ways can 6 boys and 4 girls be arranged in a straight line so that no two girls are ever together?

Sol. The seating arrangement may be done as desired in two operations.

(*i*) First we fix the positions of 6 boys. Their positions are indicated by $B_1, B_2, \ldots, B_6$.

$\times\, B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \,\times$

This can be done in 6 ! ways.

(*ii*) Now if the positions of girls are fixed at places (including those at the two ends) shown by the crosses, the four girls will never come together. In any one of these arrangements there are 7 places for 4 girls and so the girls can sit in ${}^7P_4$ ways.

Hence the required number of ways of seating 6 boys and 4 girls under the given condition

$= {}^7P_4 \times 6! = 7 \times 6 \times 5 \times 4 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$ **604800.**

Type III. *Formation of numbers with digits.*

Ex. 5. Suppose the six digits 1, 2, 4, 5, 6, 7 are given to us and we have to find the total number of numbers with no repetition of digits which can be formed under different conditions.

Sol. 1. There is no restriction. The number of 6-digit numbers.

$= {}^6P_6 = 6\,! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

2. Numbers in which a particular digit occupies a particular place.

Suppose we have to form numbers in which 5 always occurs in the ten's place. In this case, the ten's place is fixed and the remaining five places can be filled in by the remaining 5 digits in ${}^5P_5$, *i.e.*, 5 ! = 120 ways.

The number of numbers in which 5 occurs in the ten's place = 120.

3. Numbers divisible by a particular number. Suppose we have to form numbers which may be divisible by 2. These numbers will have 2 or 4 or 6 in the unit's place. Thus the unit's place can be filled in 3 ways. After having filled up the unit's place in any one of the above ways, the remaining five places can be filled in ${}^5P_5 = 5\,! = 120$ ways.

∴ The total number of numbers divisible by 2 = 120 × 3 = 360.

4. Numbers having particular digits in the beginning and the end. Suppose we have to form numbers which begin with 1 and end with 5. Here, the first and the last places are fixed and the remaining four places can be filled in 4 !, *i.e.*, 24 ways by the remaining four digits.

Therefore, the total number of numbers beginning with 1 and ending with 5 = 24.

Note. If the numbers could have 1 or 5 in the beginning or the end, the number would have been 2 ! . 4 !, *i.e.*, 48.

5. Numbers which are smaller than or greater than a particular number. Suppose we have to form numbers which are greater than 4,00,000. In these numbers, there will be 4 or a digit greater than 4, *i.e.* 5, 6 or 7 in the lac's place. Thus this place can be filled in 4 ways. The remaining 5 places can then be filled in 5 ! = 120 ways.

∴ The total number of numbers = 4 × 120 = 480.

Type IV. *Word Building*

The following cases may arise:

1. No letter may be repeated.

2. Some letters may be repeated.

3. There may be a particular letter in the beginning or the end.

4. Some letters may occur together.

Vowels occur together   Consonants occur together

Illustration. Suppose the word 'PENCIL' is given to us and we have to form words with the letters of this word.

Sol. 1. There is no restriction on the arrangement of the letters.

The six different letters can be arranged in $^6P_6 = 6\,! = 720$ ways.

Hence the total number of words formed = **720.**

2. All words begin with a particular letter.

Suppose all words begin with E. The remaining 5 places can be filled with remaining 5 letters in 5 ! = **120 ways.**

3. All words begin and end with particular letters.

Suppose all words begin with L, and end with P. The remaining 4 places can then be filled in 4 ! ways.

$\therefore$ The total number of words formed = 4 ! = **24.**

> **Note.** If the words were to begin or end with E or L, these two positions could have been filled in $^2P_2 = 2$ ways. Hence the number of words in this case would have been = 2 × 24 = 48.

4. *N* is always next to *E*.

P	*EN*	*C*	*I*	*L*

Since N is always next to E, therefore 'EN' is considered to be one letter.

$\therefore$ Required no. of permutations = 5! = 5 × 4 × 3 × 2 × 1 = **120.**

5. Vowels occur together: The vowels are E and I. Regarding them as one letter, the 5 letters can be arranged in 5 ! = 120 ways. These two vowels can be arranged amongst themselves in 2 ! = 2 ways.

$\therefore$ The total number of words = 2 × 120 = **240.**

6. Consonants occur together: Regarding these consonants as one letter the three letters E, I, ($PNCL$) can be arranged in 3 !, *i.e.*, 6 ways. The letters $PNCL$ can be arranged among themselves in 4 ! = 24 ways.

$\therefore$ The number of words in which consonants occur together = 6 × 24 = **144.**

7. Vowels occupy even places.

×	×	×	×	×	×
1	2	3	4	5	6

There are 6 letters and 3 even places. E can be placed in any one of the three even places in $^3P_1$, *i.e.*, 3 ways. Having placed E in any one of these places, I can be placed in any one of the remaining 2 places in $^2P_1$, *i.e.*, 2 ways. Thus, the vowels can occupy even places in 2 × 3 = 6 ways. After the vowels have been placed the remaining 4 letters can take up their positions in $^4P_4$, *i.e.*, 24 ways.

$\therefore$ The total number of words = 6 × 24 = **144.**

VI. Permutations of Alike Things

The number of permutations of n things taken all at a time where p of the things are alike and of one kind, q others are alike and of another kind, r others are alike and of another kind, and so on is

$$\boxed{x = \frac{n!}{p!q!r!..}}$$

Illustration: In how many ways can the letters of the word 'INDIA' be arranged ?

Sol. The word contains 5 letters of which 2 are 'I's.

The number of words possible = $\frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \mathbf{60.}$

VII. Permutations of Repeated Things

The number of permutations of n different things taken r at a time, when each thing may occur any number of times is n^r.

Suppose r places are to be filled with n things. The first place can be filled in n ways and when this has been filled up in any one of these ways, the second place can also be filled in n ways for the thing occupying the first place may occupy the second place also. Thus the first two places can be filled in $n \times n = n^2$ ways. Similarly the third place can also be filled in n ways.

Arguing in the same manner, we conclude that the r places can be filled in $n \times n \times n.....r$ times, *i.e.* n^r ways.

Illustration: In how many ways can 3 prizes be distributed among 4 boys, when

(*i*) no boy gets more than one prize;

(*ii*) a boy may get any number of prizes;

(*iii*) no boy gets all the prizes.

Sol. (*i*) The first prize can be given to any of the four boys. Then, the second prize can be given to any of the three boys. Lastly, the third prize can be given to any one of the remaining 2 boys.

$\therefore$ The number of ways in which all the 3 prizes can be given = $4 \times 3 \times 2$ = **24.**

(*ii*) In this case, each of the three prizes can be given in 4 ways since a boy can receive any number of prizes.

$\therefore$ The number of ways in which all the prizes can be given = $4 \times 4 \times 4 = 4^3$ = **64.**

(*iii*) Since anyone of the 4 boys can get all the prizes, therefore, the number of ways in which a boy gets all the 3 prizes = 4.

$\therefore$ Number of ways in which a boy does not get all the prizes = 64 – 4 = **60.**

VIII. Circular Permutations

Method I. If we have to arrange the four letters *A, B, C, D,* two of the arrangements would be *ABCD*, *DABC* which are two distinct arrangements. Now, if these arrangements are written along the circumference of a circle (Fig. 1 next page), the two arrangements are one and the same. Thus, we conclude that circular permutations are different only when the relative order of the objects is changed otherwise they are the same. Thus the arrangements in Fig. 2 are different.

As the number of circular permutations depends on the relative positions of objects, we *fix the position of one object and then arrange the remaining* $(n-1)$ *objects in all possible ways. This can be done in* $(n-1)!$ *ways.*

Illustration: 20 persons were invited for a party. In how many ways can they and the host be seated around a circular table? In how many of these ways will two particular persons be seated on either side of the host? (*IIT*)

Sol. There is 1 host and 20 guests, they are to be seated around a circular table.

(*i*) Let us fix the seat of one person, say the host, the 20 guests will be seated around the circular table in 20 ! ways, [or, $(n-1)! = (21-1)! = 20!$]

(*ii*) The two particular persons can be seated on either side of the host in 2 ways and for each way of their taking seats, the remaining 18 persons can be seated around the circular table in 18 ! ways

Hence the number of ways of seating two particular persons on either side of the host = $2 \times 18!$. [or, $2! \times (19-1)!$, considering the host and two particular persons as one entity.]

IX. Clockwise and Counter-Clockwise Permutations

We have two types of circular permutations:

(*i*) Those in which counter-clockwise and clockwise are distinguishable. Thus while seating 4 persons *A, B, C, D* around a table, the following permutations are considered different. (Fig. 1)

(*ii*) Those in which counter-clockwise and anti-clockwise are not distinguishable. (Fig. 2)

Thus, (*a*) while forming a garland of roses or jasmine, the following arrangements are not disturbed if we turn the garland over.

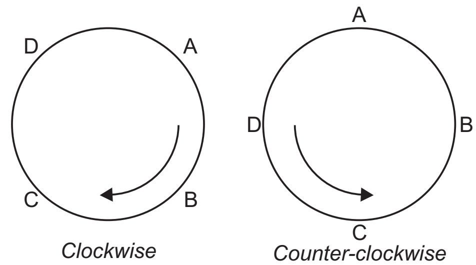


Fig. 1

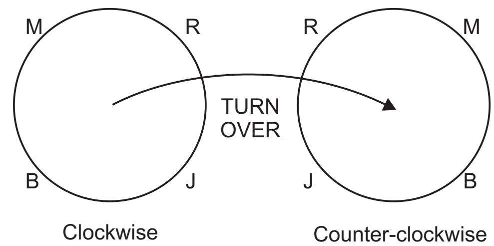


Fig. 2

(*b*) The distinction between clockwise and anti-clockwise is ignored when a number of people have to be seated around a table so as not to have the same neighbours.

Points to remember (Circular permutations)

We can **summarise** the above discussion as under:

1. Number of circular arrangements (permutations) of n different things = $(n - 1)!$
2. Number of circular arrangements of n different things when clockwise and anti-clockwise arrangements are not different = $\frac{1}{2}(n-1)!$
3. Number of circular permutations of n different things taken r at a time when **clockwise and anti-clockwise arrangements are taken as different** is $\frac{^nP_r}{r}$.
4. Number of circular permutations of n different things, taken r at a time, when clockwise and anti-clockwise arrangements are not different = $\frac{^nP_r}{2r}$.

Ex. Find the number of ways in which (*i*) *n* different beads, (*ii*) 10 different beads can be arranged to form a necklace.

Sol. Fixing the position of one bead, the remaining beads can be arranged in $(n - 1)$! ways. As there is no distriction between the clockwise and anti-clockwise arrangements, the total number of ways in which 10 different beads can be arranged = $\frac{(10-1)!}{2} = \frac{1}{2}(9\,!)$

X. Combinations

Def.: ***Each of the different groups or selections which can be made by taking some or all of a number of things at a time (irrespective of the order) is called a combination.***

By the number of combinations of n things taken r at a time is meant the number of groups of r things which can be formed from the n things. The same is denoted by the symbol $^n\boldsymbol{C_r}$ or $\boldsymbol{C(n, r)}$ or $\boldsymbol{\binom{n}{r}}$.

Value of ^nC_r

Each combination consists of r different things which can be arranged among themselves in r ! ways. Therefore, the number of arrangements for all the ^nC_r combinations is $^nC_r \times r$!. This is equal to the permutations of n different things taken r at a time.

$\therefore \qquad {}^nC_r \times r! = {}^nP_r$

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n(n-1)(n-2)...(n-r+1)}{1.2.3...r}$$

$${}^nP_r = \frac{n!}{(n-r)} \qquad \boxed{\therefore {}^nC_r = \frac{n!}{r!(n-r)!}}$$

Some Important Properties of Combinations

1. ${}^nC_r = \dfrac{{}^nP_r}{r!}$
2. ${}^nC_0 = {}^nC_n = 1$
3. ${}^nC_r = {}^nC_{n-r} \quad (0 \le r \le n)$
4. ${}^nC_x = {}^nC_y \Rightarrow x + y = n$ or $x = y$ $(x, y \in W)$
5. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$
6. If n is even, the greatest value of ${}^nC_r = {}^nC_m$ where $m = \dfrac{n}{2}$
7. If n is odd, the greatest value of ${}^nC_r = {}^nC_m$, where, $m = \dfrac{n-1}{2}$ or $m = \dfrac{n+1}{2}$
8. ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_r = 2^n$
9. ${}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$
10. ${}^nC_0 + {}^nC_2 + {}^nC_4 + ... = 2^{2n}$

The proof of Formula 5 above is given below:

$$\boxed{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \ (1 \le r \le n) \quad (\textit{Pascal's Rule})}$$

$${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= n!\left\{\frac{(n-r+1)+r}{r!.(n-r+1)!}\right\} = \frac{(n+1)n!}{r!.(n-r+1)} = \frac{(n+1)!}{r!.(n-r+1)!} = {}^{n+1}C_r.$$

Another form. $C(n, r) + C(n, r-1) = C(n+1, r)$,

i.e., $C(n, r-1) = C(n+1, r) - C(n, r)$

Ex. Find the values of $^6C_3$ and $^{30}C_{28}$.

Sol. (*i*) ${}^6C_3 = \dfrac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20.$

(*ii*) ${}^{30}C_{28} = {}^{30}C_{30-28} = {}^{30}C_2 = \dfrac{30 \times 29}{1 \times 2} = 435.$

XI. Miscellaneous Types

Type I. *Total number of combinations.*

To find the total number of combinations of n dissimilar things taking any number of them at a time.

Case I. When all things are different.

Each thing may be disposed of in two ways. It may either be included or rejected.

$\therefore$ The total number of ways of disposing of all the things = $2 \times 2 \times 2 \times$......$n$ times = 2^n

But this includes the case in which all the things are rejected.

Hence the total number of ways in which *one* or *more* things are taken = $2^n - 1$.

Cor. $2^n - 1$ is also the total number of the combinations of n things taken 1, 2, 3,...... or n at a time. Hence, ${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$.

Ex. There are 5 questions in a question paper. In how many ways can a boy solve one or more questions?

Sol. The boy can dispose of each question in two ways. He may either solve it or leave it. Thus the number of ways of disposing of all the questions = 2^5.

But this includes the case in which he has left all the questions unsolved.

Hence the total number of ways of solving the paper = $2^5 - 1 = 31$.

Case II. When all things are not different.

Suppose, out of $(p + q + r + ...)$ *things, p are alike of one kind, q are alike of a second kind, r alike of a third kind, and the rest different.*

Out of p things we may take 0,1, 2, 3...... or p. Hence they may be disposed of in $(p + 1)$ ways.

Similarly, q alike things may be disposed of in $(q + 1)$ and r alike things in $(r + 1)$ ways. The t different things may be disposed of in 2^t ways.

This includes that case in which all are rejected.

∴ The total number of selections = $(p + 1)(q + 1)(r + 1)2^t - 1$.

Ex. Prove that from the letters of the sentence, 'Daddy did a deadly deed', one or more letters can be selected in 1919 ways.

Sol. In the given sentence, there are 9 *d's* ; 3 *a's* ; 3 *e's* ; 2 *y's* ; 1 *i*; and 1 *l*.

∴ The total number of selections = $(9 + 1)(3 + 1)(3 + 1)(2 + 1)(1 + 1)(1 + 1) - 1$

$= 10 \times 4 \times 4 \times 3 \times 2 \times 2 - 1 = 1919$.

Type II. *Division into groups.*

To find the number of ways in which $p + q$ things can be divided into two groups containing p and q things respectively.

Every time when a set of p things is taken, a second set of q things is left behind. Hence the required number of ways = the number of combinations of $(p + q)$ things taken p at a time

$$= {}^{p+q}C_p = \frac{(p+q)!}{p!q!}$$

Cor. 1. Generalisation. The number of ways in which $p + q + r$ things can be divided into three groups containing p, q and r things respectively

$$= {}^{p+q+r}C_p \times {}^{q+r}C_q \times {}^rC_r = \frac{(p+q+r)!}{p!(q+r)!} \times \frac{(q+r)!}{q!r!} \times 1 = \frac{(p+q+r)!}{p!q!r!}$$

Similarly the result can be extended to the case of dividing a given number of things into more than three groups.

Cor. 2. The number of ways in which $3p$ things can be divided equally into three *distinct* groups is $\frac{(3p)!}{(p!)^3}$. $(q = p, r = p)$

Cor. 3. The number of ways in which $3p$ things can be divided into three *identical* groups is $\frac{(3p)!}{3!(p!)^3}$.

Ex. 1. In how many ways can 15 things be divided into 3 groups containing 8, 4 and 3 things respectively?

Sol. The number of ways = $\frac{(15)!}{8!4!3!} = 225225$.

Ex. 2. In how many ways can 18 different books be divided equally among 3 students?

Sol. The required number of ways = $\frac{(18)!}{(6!)^3}$

SOLVED EXAMPLES

Ex. 1. Each section in first year of plus two course has exactly 30 students. If there are 3 sections, in how many ways can a set of 3 student representatives be selected from each section?

Sol. 1st representative can be selected from first section in 30 ways.

2nd representative can be selected from second section in 30 ways.

3rd representative can be selected from third section in 30 ways.

$\therefore$ Required number of ways = 30 × 30 × 30 = **27000.**

Ex. 2. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?

Sol. Any number between 100 and 1000 is of 3 digits. The unit's place can be filled by 2 or 9 in 2 ways. Similarly ten's place can be filled in 2 ways.

The hundred's place can also be filled in 2 ways.

$\therefore$ Required no. of numbers = 2 × 2 × 2 = **8.**

Ex. 3. How many odd numbers less than 1000 can be formed using the digits 0, 2, 5, 7 repetition of digits are allowed?

Sol. Since the required numbers are less than 1000 therefore, they are 1-digit, 2-digit or 3-digit numbers.

One-digit numbers. Only two odd one-digit numbers are possible, namely, **5** and **7.**

Two-digit numbers. For two-digit odd numbers the unit place can be filled up by 5 or 7 *i.e.*, in two ways and ten's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

$\therefore$ No. of possible 2-digit odd numbers = 2 × 3 = **6.**

Three-digit numbers. For three-digit odd numbers, the unit place can be filled up by 5 or 7 in 2 ways. The ten's place can be filled up by any one of the digits 0, 2, 5, 7 in 4 ways. The hundred's place can be filled up by 2, 5 or 7 (not 0) in 3 ways.

$\therefore$ No. of possible 3-digit numbers = 2 × 4 × 3 = 24

Hence total number of odd numbers = 2 + 6 + 24 = **32.**

Ex. 4. If $\frac{(2n)!}{3!(2n-3)!} : \frac{n!}{2!(n-2)!} = 44:3$, find n.

Sol. Given, $\frac{(2n)!}{3!(2n-3)!} : \frac{n!}{2!(n-2)!} = 44:3$

$$\Rightarrow \frac{(2n)(2n-1)(2n-2)(2n-3)!}{3\times 2!.(2n-3)!} \times \frac{2!(n-2)!}{n(n-1)(n-2)!} = \frac{44}{3}$$

$$\Rightarrow \frac{(2n)(2n-1)(2n-2)}{3} \times \frac{1}{n(n-1)} = \frac{44}{3}$$

$$\Rightarrow \frac{4(n)(2n-1)(n-1)}{3} \times \frac{1}{n(n-1)} = \frac{44}{3}$$

$\Rightarrow 4(2n-1) = 44 \quad \Rightarrow \quad 8n - 4 = 44 \quad \Rightarrow \quad 8n = 48 \quad \Rightarrow \quad n = \mathbf{6}$

Ex. 5. How many numbers greater than 5000 can be formed with the digits 3, 5, 7, 8, 9 no digit being repeated?

Sol. Obviously with the given 5-digits, numbers greater than 5000 are either 4-digit numbers having 5 or 7 or 8 or 9 at the thousand's place or 5-digit numbers.

Since the thousand's place in the required 4-digit numbers cannot take 3 as a value, we have 4 options for thousand's place, the remaining 4 out of 5 (1 earning the one used up at thousand's place) for hundred's place, 3 for tens' place and 2 for one's place.

(**Note:** Repetition of digits is not allowed)

Four-Digit Numbers

Thousands Place 5, 7, 8 or 9	**Hundreds Place** Any of the 4 remaining digits	**Tens Place** Any of the remaining 3 digits	**Ones Place** Any of the remaining 2 digits
4 ways	4 ways	3 ways	2 ways

$\therefore$ Number of 4-digit numbers greater than 5000

That can be formed with given digits = $4 \times 4 \times 3 \times 2 = 96$

For the 5-digit numbers, the various places can be filled up as shown:

TTh	Th	H	T	O
↓	↓	↓	↓	↓
5 ways	4 ways	3 ways	2 ways	1 way

$\therefore$ Number of 5-digit numbers = $5 \times 4 \times 3 \times 2 \times 1 = 120$.

$\therefore$ Total required numbers = $96 + 120 =$ **216.**

Ex. 6. How many 6-digit numbers can be formed from the digits 1, 2, 4, 5, 6, 7 (no digit being repeated) which are divisible by 5?

Sol. A number is divisible by 5 only if its unit's digit is 5 or 0.

So we fix the unit's digit as 5 and fill up the remaining 5 places with the remaining 5 digits in the following way:

L	TTh	Th	H	T	O
↓	↓	↓	↓	↓	↓
5	4	3	2	1	1 (5 is fixed)

Note : Repetition of digits is not allowed

$\therefore$ Required number of numbers divisible by 5 = $5 \times 4 \times 3 \times 2 \times 1 =$ **120.**

Ex. 7. How many numbers can be formed by using any number of the digits 3, 1, 0, 5, 7, 2, 9, no digit being repeated in any number?

Sol. The number of single digit numbers is ${}^7P_1$.

The permutations of 7 digits taken 2 at a time are ${}^7P_2$. But ${}^6P_1$ of these have zero in the ten's place and so reduce to one digit numbers.

Hence the number of two-digit numbers is ${}^7P_2 - {}^6P_1$

Similarly the number of the three-digit numbers is ${}^7P_3 - {}^6P_2$ and so on.

$\therefore$ The total number required

$$= {}^7P_1 + ({}^7P_2 - {}^6P_1) + ({}^7P_3 - {}^6P_2) + ({}^7P_4 - {}^6P_3) + ({}^7P_5 - {}^6P_4) + ({}^7P_6 - {}^6P_5) + ({}^7P_7 - {}^6P_6) = \mathbf{11743.}$$

Ex. 8. How many different numbers can be formed with the digits 1, 3, 5, 7, 9, when taken all at a time, and what is their sum?

Sol. The total number of numbers = 5 ! = 120. Suppose we have 9 in the unit's place. We will have 4 ! = 24 such numbers. The number of numbers in which we have 1, 3, 5 or 7 in the unit's place is also 4 ! = 24 in each case.

Hence the sum of the digits in the unit's place in all the 120 numbers

= 24 (1 + 3 + 5 + 7 + 9) = 600.

The number of numbers when we have any one of the given digits in ten's place is also 41=24 in each case.

Hence the sum of the digits in the ten's place

= 24 (1 + 3 + 5 + 7 + 9) tens = 600 tens = 600 × 10.

Proceeding similarly, the required sum

= 600 units + 600 tens + 600 hundreds + 600 thousands + 600 ten thousands

= 600(1 + 10 + 100 + 1000 + 10000) = 600 × 11111 = **6666600.**

Ex. 9. Find the number of words that can be formed by taking all the letters of the word "COMBINE", such that the vowels occupy odd places. ***(WBJEE 2010)***

Sol. There are 7 letters in the word *COMBINE* of which 4 are consonants and 3 are vowels (*O*, *I*, *E*).

There are 4 odd places in a 7 letter word, so the number of ways 3 vowels can be arranged in 4 places = ${}^4P_3$

After arranging the 3 vowels, there are 4 places left (one at odd position and 3 at even positions). So the 4 consonants can be arranged in these 4 places in 4! ways.

∴ Required number of words = ${}^4P_3 \times 4! = 4 \times 4 \times 3 \times 2 \times 1 =$ **96.**

Ex. 10. How many ways are there to arrange the letter in the word GARDEN with the vowels in alphabetical order? **(*AIEEE 2004*)**

Sol. The word *GARDEN* contains 6 letters — 4 consonants (*G*, *R*, *D*, *N*) and 2 vowels *A*, *E*. The 4 consonants can be arranged in 6 places in ${}^6P_4$ ways. In each of these arrangements two places will remain blank in which the first place will be filled by *A* and the place following it by *E* as vowels have to be in alphabetical order.

This can always be done in only 1 way, *i.e.*, *E* following *A*

∴ Required number of ways = ${}^6P_4 \times 1 = \frac{6!}{(6-4)!} = \frac{6!}{2!} \times 1 = 6 \times 5 \times 4 \times 3 =$ **360.**

Ex. 11. How many signals can be made by hoisting 2 blue, 2 red and 5 yellow flags on a pole at the same time?

Sol. The number of signals = $\frac{9!}{2!2!5!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 5 \times 4 \times 3 \times 2 \times 1} =$ **756.**

Ex. 12. A coin is tossed 6 times. In how many different ways can we obtain 4 heads and 2 tails?

Sol. Whether we toss a coin 6 times or toss 6 coins at a time, the number of arrangements will be the same.

∴ The number of arrangements of 4 heads and 2 tails out of 6 is $\frac{6!}{4!2!} = 15$.

Ex. 13. How many numbers can be formed with digits 1, 2, 3, 4, 3, 2, 1, so that odd digits always occupy the odd places? **(*IIT*)**

Sol. The odd digits having two 1's alike and two 3's alike can be arranged in four odd places in $\frac{4!}{2!2!} = 6$ ways.

The three even digits having two 2's alike can be arranged in the three even places in $\frac{3!}{2!} = 3$ ways.

∴ The number of numbers = 6 × 3 = **18.**

Ex. 14. There are 3 copies each of 4 different books. Find the number of ways of arranging them on a shelf. **(*IIT*)**

Sol. Total number of books = 3 × 4 = 12

Each of the 4 different titles has 3 copies each

∴ Required number of ways of arranging them on a shelf = $\frac{12!}{3!3!3!3!} = \frac{12!}{(3!)^4} =$ **369600.**

Ex. 15. Find the number of arrangements of the letters of the word 'BANANA' in which the two *N's* do not appear adjacently. **(*IIT Prel. 2002*)**

Sol. Considering the two *N's* as one letter, the number of letters to be arranged = 5.

Therefore, the number of arrangements = $\frac{5!}{3!} = 20$ ($\because$ *A* repeated 3 times)

Total number of arrangements if there were no restriction imposed = $\frac{6!}{3!\,2!} = 60.$

(*A* repeated 3 times and *N* repeated 2 times)

$\therefore$ Required number of arrangements = 60 – 20 = **40.**

Ex. 16. How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

Sol. A million is a 7-digit number. So any number greater than 1 million will contain all the seven digits. Since the digit 2 occurs twice and digit 3 occurs thrice and the rest are different, therefore, number of possible numbers which can be formed with the given seven digits = $\frac{7!}{(2!)\,(3!)} = 420.$

These possible numbers include those which have 0 at the millions place. Keeping 0 fixed at the millions place, the remaining 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are different can be arranged in $= \frac{6!}{(2!)(3!)} = 60.$ ways.

$\therefore$ Number of numbers greater than 1 million made from the given digits = 420 – 60 = **360.**

Ex. 17. In how many ways can the letters of the word 'ARRANGE' be arranged such that the two *r's* do not occur together?

Sol. There are two *a's*, two *r's* in the word 'arrange', therefore the number of arrangements = $\frac{7!}{2!\,2!} = 1260.$ ...(1)

The number of arrangements in which the two *r's* occur together = $\frac{6!}{2!} = 360.$ ...(2)

$\therefore$ The number of arrangements in which two *r's* do not occur together = (1) – (2) = 1260 – 360 = **900.**

Ex. 18. If the letters of the word 'AGAIN' be arranged in a dictionary, what is the fiftieth word?

Sol. In a dictionary, the words are arranged in an alphabetical order.

(*i*) Starting with *A*, the remaining 4 letters *G*, *A*, *I*, *N* can be arranged in 4! = 24 ways. These are the first 24 words.

(*ii*) Then, starting with *G*, the remaining letters *A*, *A*, *I*, *N* can be arranged in $\frac{4!}{2!} = 12$ ways. Thus, there are 12 words starting will *G*.

(*iii*) Now, the words will start with *I*. Starting with *I*, the remaining letters *A*, *G*, *A*, *N* can be arranged in $\frac{4!}{2!} = 12$ ways. So, there are 12 words, which start with *I*.

(*iv*) Thus, so far, we have constructed 24 + 12 + 12, *i.e.*, 48 words. The 49th word will start with *N* and is *NAAGI*. Hence, the 50th word is ***NAAIG.***

Ex. 19. The letters of the word 'RANDOM' are written in all possible ways and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.

Sol. The words in a dictionary are written in an alphabetical order, which, here is *ADMNOR*.

(*i*) Starting with *A*, the remaining letters *D*, *M*, *N*, *O*, *R* can be arranged in 5! = 120 ways. So, there are 120 words starting with *A*.

(*ii*) Similarly, number of words starting with *M* = 120, starting with *N* = 120, and starting with *O* =120.

(*iii*) Now, the number of words starting with *R* is also 120. Out of these words, one word is *RANDOM*. First, we find the words starting with *RAD* and *RAM*.

Number of starting with *RAD* = 3! = 6

Number of starting with *RAM* = 3! = 6

(*iv*) Thus, so far, 120 + 120 + 120 + 120 + 120 + 6 + 6 = 612 words have been constructed.

(*v*) Now, the words starting with *RAN* will appear. Their number is also = 3! = 6. One of these words is the word *RANDOM* itself.

The first word beginning with *RAN* is *RANDMO*. It is 613th word and so the next word is *RANDOM*. *RANDOM* is the 614th word.

Hence, rank of *RANDOM* = **614.**

Ex. 20. How many numbers of 3-digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated?

Sol. Since repetition is allowed, each of the 3 places in a 3-digit number can be filled in 5 ways.

H	T	O
5 ways	**5** ways	**5** ways

∴ Required number of 3-digit numbers = $5 \times 5 \times 5 = 5^3 = 125$

Ex. 21. How many numbers each containing four digits can be formed, when a digit may be repeated any number of times?

Sol. There are in all 10 digits, including zero. As the first digit of the number cannot be zero, so it can be chosen in 9 ways. Again, as a digit may occur any number of times in a number, the second, third and fourth digits of the numbers can be any one of the ten digits and so each of the remaining three places can be filled in 10 ways. Hence the total number of 4-digit numbers = $9 \times 10^3 = 9000$.

Verification. All the 4-digit numbers will be between 1000 and 9999 and so their number is 9999 – 999 = 9000.

Ex. 22. Eight different letters of an alphabet are given. Words of 4 letters from these are formed. Find the number of such words with at least one letter repeated. **(*EAMCET 2006*)**

Sol. If any letter can be used any number of times, then the number of words of 4 Letters with 8 different letters is $8 \times 8 \times 8 \times 8 = 8^4 = 4096$

Number of words of 4 letters with at least one letter repetition not allowed = ${}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680$

∴ Number of 4 letter words with at least one letter repeated is $8^4 - {}^8P_4 = 4096 - 1680 =$ **2416.**

Ex. 23. (*a*) In how many ways can a party of 4 boys and 4 girls be seated at a circular table so that no 2 boys are adjacent

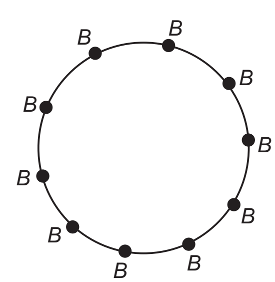
(*b*) In how many ways can 10 boys and 5 girls sit around a circular table, so that no two girls sit together?

Sol. (*a*) Let the girls first take up their seats. They can sit in 3 ! ways. When they have been seated, then there remain 4 places for the boys each between two girls. Therefore the boys can sit in 4 ! ways. Therefore there are 3 ! × 4 !, *i.e.*, 144 ways of seating the party.

(*b*) Let *B*, denote the position of the boys around the table. 10 boys can be seated around the table in 9 ! ways.

There are 10 spaces between the boys, which can be occupied by 5 girls in ${}^{10}P_5$ ways. Hence,

Total number of ways = $9! \times {}^{10}P_5 = \dfrac{\mathbf{9!\,10!}}{\mathbf{5!}}$.



Ex. 24. A round table conference is to be held between delegates of 20 countries. In how many ways can they be seated if two particular delegates are
(*i*) always together, (*ii*) never together?

Sol. (*i*) Let D_1 and D_2 be the two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. 19 delegates can be seated round a circular table in (19 – 1)! = 18 ! ways.
But two particular delegates can seat themselves in 2! ($D_1\,D_2$ or $D_2\,D_1$) ways.
Hence, the total number of ways = 18! × 2! = 2 (18!)

(*ii*) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of ways of seating 20 persons *i.e.*, (20 – 1)! = 19! ways.
Hence the total number of ways in this case = 19! – 2 (18!) = 19 (18!) – 2 (18!) = **17 (18!).**

Ex. 25. Find the number of ways in which 10 different flowers can be strung to form a garland so that 4 particular flowers are never separated.

Sol. Consider 4 particular flowers as one flower. Then, we have 7 flowers which can be strung to form a garland in (7 – 1)! = 6! ways. But 4 particular flowers can be arranged in 4! ways.

Hence, the required number of ways = $\frac{1}{2}(6! \times 4!) = \mathbf{8640.}$

Ex. 26. In how many ways can 7 persons sit around a table so that all shall not have the same neighbours in any two arrangements.

Sol. 7 persons can sit around a table in 6 ! ways but as each person will have the same neighbours in clockwise and anti-clockwise arrangements, the required number = $\frac{1}{2}$ 6! = **360.**

Ex. 27. If $^nC_{r-1} = 36$, $^nC_r = 84$ and $^nC_{r+1} = 126$, then find the value of *r*. ***(DCE 2000, MPPET 2009, IIT, Pb CET)***

Sol. $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{84}{36} \Rightarrow \dfrac{n!}{r!(n-r)!} \times \dfrac{(r-1)!(n-r+1)!}{n!} = \dfrac{84}{36} \Rightarrow \dfrac{n-r+1}{r} = \dfrac{7}{3}$ ...(*i*)

Also, $\dfrac{^nC_{r+1}}{^nC_{r-1}} = \dfrac{126}{84} \Rightarrow \dfrac{n!}{(r=1)!(n-r-1)} \times \dfrac{r!(n-r)!}{n!} = \dfrac{3}{2} \Rightarrow \dfrac{n-r}{r+1} = \dfrac{3}{2}$ ...(*ii*)

From (*ii*) $2n - 2r = 3r + 3 \Rightarrow n = \dfrac{5r+3}{2!}$

Substituting in (*i*), we get $\dfrac{\frac{5r+3}{2} - r + 1}{r} = \dfrac{7}{3} \Rightarrow \boldsymbol{r = 3.}$

Ex. 28. In how many ways can 4 persons be selected from amongst 9 persons? How many times will a particular person be always selected?

Sol. The number of ways in which 4 persons can be selected from amongst 9 persons

$= {}^9C_4 = \dfrac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} = 126.$

The number of ways in which a particular person is always to be selected

$= {}^8C_3 = \dfrac{8 \times 7 \times 6}{1 \times 2 \times 3} = \mathbf{56.}$

Ex. 29. Find the number of diagonals that can be drawn by joining the angular points of a
(*i*) heptagon
(*ii*) a polygon of 20 sides. (***J&K CET 2009***)

Sol. (*i*) A heptagon has seven angular points and seven sides.

The join of two angular points is either a side or a diagonal.

The number of lines joining the angular points = $^7C_2 = \dfrac{7 \times 6}{1 \times 2} = 21.$

But the number of sides = 7 $\therefore$ The number of diagonals = 21 – 7 = **14.**

(*ii*) Number of diagonals in a polygon of n sides = $^nC_2 - n$. Here, $n = 20$

$\therefore$ Required number of diagonals = $^{20}C_2 - 20 = \dfrac{20 \times 19}{2 \times 1} - 20 = \mathbf{170.}$

Ex. 30. A committee of 4 is to be selected from amongst 5 boys and 6 girls. In how many ways can this be done so as to include (*i*) exactly one girl, (*ii*) at least one girl?

Sol. (*i*) In this case we have to select one girl out of 6 and 3 boys out of 5.

The number of ways of selecting 3 boys = $^5C_3 = {}^5C_2 = 10$.

The number of ways of selecting one girl = $^6C_1 = 6$.

$\therefore$ The required committee can be formed in 6 × 10 = 60 ways.

(*ii*) The committee can be formed with

(*a*) one boy and three girls, or (*b*) 2 boys and 2 girls,

or (*c*) 3 boys and one girl, or (*d*) 4 girls alone.

The committee can be formed in (*a*) $^5C_1 \times {}^6C_3$ ways ;

The committee can be formed in (*b*) $^5C_2 \times {}^6C_2$ ways ;

The committee can be formed in (*c*) $^5C_3 \times {}^6C_1$ ways ;

The committee can be formed in (*d*) $^6C_4$ ways.

Hence the required number of ways of forming the committee

$= {}^5C_1 \times {}^6C_3 + {}^5C_2 \times {}^6C_2 + {}^5C_3 \times {}^6C_1 + {}^6C_4 = 100 + 150 + 60 + 15 =$ **325 ways.**

(*ii*) **Method II.** Required ways = (Committees of 4 out of 11 without any restriction) – (Committees in which no girl is included) = $^{11}C_4 - {}^5C_4$ = **325.**

Ex. 31. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five questions. Find the number of choices available to him. (***AIEEE 2003***)

Sol. Two cases are possible:

(*i*) Selecting 4 out of first five questions and 6 out of remaining 8 questions

$\therefore$ Number of choices in this case = $^5C_4 \times {}^8C_6 = {}^5C_1 \times {}^8C_2 = \dfrac{5 \times 8 \times 7}{1 \times 2} = 140$

(*ii*) Selecting 5 out of first five questions and 5 out of remaining 8 questions.

$\Rightarrow$ Number of choices = $^5C_5 \times {}^8C_5 = 1 \times {}^8C_3 = \dfrac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56.$

$\therefore$ Total number of choices = 140 + 56 = **196.**

Ex. 32. How many different words, each containing 2 vowels and 3 consonants, can be formed with 5 vowels and 17 consonants?

Sol. Two vowels can be selected in $^5C_2$ ways.

Three consonants can be selected in $^{17}C_3$ ways.

$\therefore$ 2 vowels and 3 consonants can be selected in $^5C_2 \times {}^{17}C_3$ ways.

Now, each group of 2 vowels and 3 consonants can be arranged in 5 ! ways.

$\therefore$ The total number of words = $^5C_2 \times {}^{17}C_3 \times 5\,! =$ **816000.**

PRACTICE SHEET

1. How many words (with or without meaning) can be formed from three distinct letters of the English alphabet?

(*a*) 17576 (*b*) 15600
(*c*) 14400 (*d*) None of these

2. Two persons enter a railway compartment where there are 6 vacant seats. In how many different ways can they seat themselves?

(*a*) 36 (*b*) 11 (*c*) 30 (*d*) 12

3. How many integers of four digits can be formed with the digits 0, 1, 3, 5, 6 (assuming no repetitions)?

(*a*) 96 (*b*) 120 (*c*) 15 (*d*) 625

4. There are 12 true-false questions in an examination. How many sequences of answers are possible ?

(*a*) 12! (*b*) 120 (*c*) 2^{12} (*d*) $2^{12} - 2$

5. How many seven-digit phone numbers are possible if 0 and 1 cannot be used as the first digit and the first three digits cannot be 555, 411 or 936?

(*a*) 412560 (*b*) 7970000 (*c*) 797000 (*d*) 362880

6. There are 5 roads leading to a town from a village. The number of different ways in which a villager can go to the town and return back is:

(*a*) 5 (*b*) 10 (*c*) 25 (*d*) 20

(*AMU 2002*)

7. Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

(*a*) 120 (*b*) 20 (*c*) 320 (*d*) 240

8. How many numbers are there between 100 and 1000 such that at least one of their digits is 7?

(*a*) 900 (*b*) 648 (*c*) 729 (*d*) 252

9. How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8?

(*a*) 625 (*b*) 125 (*c*) 25 (*d*) 20

10. How many three-digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if repetition of digits is not allowed?

(*a*) 72 (*b*) 108 (*c*) 40 (*d*) 60

11. The number of all 3-digit numbers in each of which the sum of the digits is even is

(*a*) 450 (*b*) 375 (*c*) 365 (*d*) 250

12. Find the total number of ways in which 20 balls can be put in 5 boxes so that the first box contains just one ball.

(*a*) 20^5 (*b*) 5^{20} (*c*) 20×4^{19} (*d*) 4^{20}

13. Find the number of ways in which five large books, four medium sized books and three small books can be placed on a shelf so that all books of the same size together

(*a*) $5 \times 4 \times 3$ (*b*) $5! \times 4! \times 3!$
(*c*) $3 \times 5! \times 4! \times 3!$ (*d*) $3! \times 5! \times 4! \times 3!$

(*VITEE 2006*)

14. If $P(5, r) = 2P(6, r-1)$, find r.

(*a*) 2 (*b*) 4
(*c*) 3 (*d*) Cannot be determined

15. If $^{2n+1}P_n : {}^{2n-1}P_n = 3 : 5$, then n equals

(*a*) 2 (*b*) 3 (*c*) 6 (*d*) 4

(*Kerala PET 2009*)

16. 6 different letters of an alphabet are given. Words with four letters are formed from these given letters. Determine the number of words which have at least one letter repeated.

(*a*) 1296 (*b*) 996 (*c*) 936 (*d*) 360

17. How many eight-distinct letter words can be formed with the letters of the word "COURTESY" beginning with *C* and ending with *Y*?

(*a*) 576 (*b*) 640 (*c*) 336 (*d*) 720

18. How many different words can be formed from with the letters of the word "RAINBOW" so that the vowels occupy odd places.

(*a*) 676 (*b*) 336 (*c*) 576 (*d*) 144

19. How many words can be formed from the letters of the word "DAUGHTER" so that the vowels always come together?

(*a*) 2880 (*b*) 4320 (*c*) 3600 (*d*) 3200

20. How many words can be formed from the letters of the word "SUNDAY" so that the vowels never come together?

(*a*) 480 (*b*) 360 (*c*) 400 (*d*) 720

21. There are 5 boys and 3 girls. In how many ways can they stand in a row so that no two girls are together?

(*a*) 36000 (*b*) 25600 (*c*) 14400 (*d*) 15600

22. Find how many arrangements can be made with the letters of the word "MATHEMATICS" in which the vowels occur together?

(*a*) 10080 (*b*) 120960 (*c*) 14400 (*d*) 36500

23. All the words that can be formed using the letters A, H, L, U, R are written as in a dictionary (no alphabet is repeated). Then the rank of the word RAHUL is

(*a*) 71 (*b*) 72 (*c*) 73 (*d*) 74

(*Kerala PET 2008*)

24. The letters of the word 'COCHIN' are permuted and all the permutations are arranged in alphabetical order as in English dictionary. The number of words that appear before the word COCHIN is

(*a*) 48 (*b*) 96 (*c*) 192 (*d*) 360

(*IIT 2007*)

25. Four friends have 7 shirts, 6 pants and 8 ties. In how many ways can they wear them?

(*a*) 336000 (*b*) 33600 (*c*) 124800 (*d*) 508032000

26. If all the L's occur together and also all I's occur together, when the letters of the word 'HALLUCINATION' are permuted, then the number of such arrangements of letters is:

(*a*) $\frac{7!}{2!2!2!2!}$ (*b*) $\frac{11!}{2!2!}$

(*c*) $\frac{13!}{2!2!2!2!}$ (*d*) $\frac{11!}{2!2!2!2!}$

27. In how many ways can the letters of the word "AFLATOON" be arranged if the consonants and vowels must occupy alternate places?

(*a*) 176 (*b*) 144 (*c*) 136 (*d*) 288

28. How many different numbers greater than 60000 can be formed with the digits 0, 2, 2, 6, 8?

(*a*) 144 (*b*) 48 (*c*) 24 (*d*) 288

29. There are 5 red, 4 white and 3 blue marbles in a bag. They are drawn one by one and arranged in a row. Assuming that all the 12 marbles are drawn, determine the number of different arrangements.

(*a*) 20772 (*b*) 27270 (*c*) 22707 (*d*) 27720

30. How many natural numbers not exceeding 4321 can be formed with the digits 1, 2, 3, 4 if repetition is allowed?

(*a*) 123 (*b*) 113 (*c*) 222 (*d*) 313

31. In how many ways can 8 people sit around a circular Table?

(*a*) 5040 (*b*) 40320 (*c*) 20160 (*d*) 2520

32. A committee of 11 members sit at a round table. In how many ways can they be seated if the "President" and the "Secretary" choose to sit together.

(*a*) $\frac{10!}{2!}$ (*b*) $\frac{9!}{2!}$ (*c*) $9! \times 2!$ (*d*) $\frac{11!}{2!}$

33. In how many ways can 7 men and 7 women sit on a round table such that no two women sit together?

(*a*) $2(7!)$ (*b*) $(6!)^2$ (*c*) $7! \times 6!$ (*d*) $(7!)^2$

(*BITSAT 2007*)

34. In how many ways can 6 gentlemen and 3 ladies be seated round a table so that every gentleman may have a lady by his side?

(*a*) 360 (*b*) 1440 (*c*) 720 (*d*) 1260

35. There are 6 numbered chairs placed around a circular table. 3 boys and 3 girls want to sit on them such that neither of two boys nor two girls sit adjacent to each other. How many such arrangements are possible?

(*a*) 44 (*b*) 72 (*c*) 48 (*d*) 12

36. Find the number of ways in which 10 different flowers can be strung to form a garland so that three particular flowers are always together.

(*a*) $\frac{9!\times 3!}{2}$ (*b*) $\frac{7!\times 3!}{2}$ (*c*) $\frac{8!\times 3!}{2}$ (*d*) $7! \times 2!$

37. If $C(2n, 3) : C(n, 2) = 12 : 1$ find *n*.

(*a*) 2 (*b*) 5 (*c*) 4 (*d*) 3

38. If ${}^nC_r : {}^nC_{r+1} = 1 : 2$ and ${}^nC_{r+1} : {}^nC_{r+2} = 2 : 3$ determine the values of *n* and *r*.

(*a*) $n = 10, r = 6$ (*b*) $n = 12, r = 6$

(*c*) $n = 10, r = 4$ (*d*) $n = 14, r = 4$

39. In how many ways can a committee of five persons be formed out of 8 members when a particular member is taken every time?

(*a*) 35 (*b*) 42 (*c*) 50 (*d*) 56

40. In how many ways can a team of 11 players be selected from 14 players when two of them play as goalkeepers only?

(*a*) 112 (*b*) 132 (*c*) 91 (*d*) 182

41. 15 men in a room shake hands with each other, then the total number of handshakes is

(*a*) 90 (*b*) 105 (*c*) 110 (*d*) 115

(*Rajasthan PET 2002*)

42. A man has 6 friends. Number of different ways, he can invite 2 or more for dinner is

(*a*) 28 (*b*) 57 (*c*) 71 (*d*) 96

(*Orissa JEE 2011*)

43. There are 5 professors and 6 students out of whom a committee of 2 professors and 3 students is to be formed such that a particular student is excluded.

(*a*) 100 (*b*) 150 (*c*) 200 (*d*) 125

44. Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included?

(*a*) 15 (*b*) 25 (*c*) 20 (*d*) 10

45. A committee of 5 persons is to be formed out of 6 gents and 4 ladies. In how many ways can this be done if at most two ladies are included?

(*a*) 168 (*b*) 156 (*c*) 186 (*d*) 165

46. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for at least one candidate, then the number of ways he can vote is

(*a*) 385 (*b*) 1110 (*c*) 5040 (*d*) 6210

(*AIEEE 2006*)

47. In a chess tournament, where participants were to play one game with one another, two players fell ill, having played 6 games each without playing among themselves. If the total number of games played was 117, then the number of participants at the beginning was:

(*a*) 15 (*b*) 16 (*c*) 17 (*d*) 18

(*AMU 2005*)

48. 7 relatives of a man comprise 4 ladies and 3 gentlemen; his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite in a dinner party, 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?

(*a*) 265 (*b*) 375 (*c*) 395 (*d*) 485

(*IIT 1985*)

49. A person invites 15 guests for dinner and wishes to arrange them at two round tables that can accommodate 8 persons and 7 persons respectively. In how many ways can he arrange the guests?

(*a*) $7! \times 6!$ (*b*) $2 \times 8! \times 7!$

(*c*) $\frac{15!}{56}$ (*d*) $15C_2 \times \frac{13!}{56}$

50. How many diagonals can be drawn in a polygon of 15 sides?

(*a*) 16 (*b*) 60 (*c*) 80 (*d*) 90

(*J&K CET 2009*)

51. There are 18 points in a plane such that no three of them are in the same line except five points which are collinear. The number of triangles formed by these points is:

(*a*) 805 (*b*) 806 (*c*) 813 (*d*) 816

(*Rajasthan PET 2007*)

52. In how many ways can a mixed doubles game be arranged from amongst 8 married couples if no husband and wife play in the same game?

(*a*) 840 (*b*) 240

(*c*) 480 (*d*) None of these

53. In how many way can a committee of 4 women and 5 men be chosen from 9 women and 7 men, if Mr. A refuse to serve on the committee if Ms. B is a member?

(*a*) 1608 (*b*) 1860 (*c*) 1680 (*d*) 1806

54. If m parallel lines in a plane are intersected by a family of n parallel lines, find the number of parallelograms formed?

(*a*) m^n (*b*) $(m + 1)(n + 1)$

(*c*) $\frac{(m-n)}{n!}$ (*d*) $\frac{mn(n-1)(m-1)}{4}$

55. In how many ways can a pack of 52 cards be divided equally among four players in order?

(*a*) $(52!)^4$ (*b*) $4 \times (13!)$

(*c*) $\frac{52!}{(13!)^4}$ (*d*) None of these

56. In how many ways can a pack of 52 cards be divided into 4 sets, three of them having 16 cards each and the fourth just 4 cards?

(*a*) $16! \times 52!$ (*b*) $\frac{52!}{(16!)^3}$

(*c*) $\frac{52!}{(3!)^{16}}$ (*d*) $\frac{52!}{(16!)^3 \times (3!)}$

57. 7 persons enter an elevator on the ground floor of a 11 storey hotel. Any one of them can leave the elevator at any of the 10 floors. The number of ways in which the 7 persons can leave the elevator, if each one of them can leave it at any of the ten floors is

(*a*) $10!$ (*b*) 10^7 (*c*) $7!$ (*d*) 7^{10}

58. Eighteen quests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three on the other side. Determine the number of ways in which the seating arrangement can be made?

(*a*) ${}^{18}C_4 \times {}^{14}C_3 \times 9! \times 2$ (*b*) ${}^{11}C_5 \times {}^{6}C_6 \times 9! \times 2!$

(*c*) ${}^{11}C_5 \times {}^{6}C_6 \times 9! \times 9!$ (*d*) ${}^{18}C_4 \times {}^{14}C_3 \times 9! \times 2!$

59. A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour.

(*a*) 452 (*b*) 524 (*c*) 425 (*d*) 254

60. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five. In how many different ways can we place all the balls so that no box remains empty?

(*a*) 6 (*b*) 36 (*c*) 90 (*d*) 150

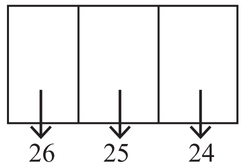
(*IIT 1981, 2012*)

ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*a*)	**4.** (*c*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*d*)	**9.** (*c*)	**10.** (*d*)
11. (*a*)	**12.** (*c*)	**13.** (*d*)	**14.** (*c*)	**15.** (*d*)	**16.** (*c*)	**17.** (*d*)	**18.** (*c*)	**19.** (*b*)	**20.** (a)
21. (*c*)	**22.** (*b*)	**23.** (*c*)	**24.** (*b*)	**25.** (*d*)	**26.** (*b*)	**27.** (*d*)	**28.** (*c*)	**29.** (*d*)	**30.** (*d*)
31. (*a*)	**32.** (*c*)	**33.** (*c*)	**34.** (*b*)	**35.** (*b*)	**36.** (*b*)	**37.** (*b*)	**38.** (*d*)	**39.** (*a*)	**40.** (*b*)
41. (*b*)	**42.** (*b*)	**43.** (*a*)	**44.** (*b*)	**45.** (*c*)	**46.** (*a*)	**47.** (*c*)	**48.** (*d*)	**49.** (*c*)	**50.** (*d*)
51. (*b*)	**52.** (*a*)	**53.** (*d*)	**54.** (*d*)	**55.** (*c*)	**56.** (*d*)	**57.** (*b*)	**58.** (*c*)	**59.** (*c*)	**60.** (*d*)

HINTS AND SOLUTIONS

1. There are 26 letters in the English alphabet, so for words with three distinct letters, the first place has 26 choices of letters, the next has remaining 25 choices and the third place has 24 choices.



$\therefore$ Total number of words = 26 × 25 × 24

= **15600**

2. The first person can sit on any of the six vacant seats.

$\therefore$ Number of ways to seat first person = 6

Now the second person can sit on any of the remaining five seats.

So number of ways to seat second person = 5

$\therefore$ Total number of ways in which both can be seated

= 6 × 5 = **30.**

3. The thousands' place cannot be occupied by 0, as the number will then become three digit, so there are 4 choices for thousands' place. As repetition is not allowed, so leaving the digit occupying thousands' place, there are again 4 choices for hundreds' place (including 0)

Th	H	T	U
4	4	3	2

(Excluding 0)

Similarly for Tens' place there 3 choices, for units' place 2 choices.

$\therefore$ Total number of 4-digit numbers without repetition of digits = 4 × 4 × 3 × 2 = **96.**

4. For each of the 12 questions, there are 2 ways of answering **'true'** or **'false'.** Hence, total number of sequences of answers possible

$= \underbrace{2 \times 2 \times 2 \times .... \times 2}_{12 \text{ times}} = \mathbf{2^{12}}.$

5. For the first digit there are 8 choices (out of 10 digits) as 0 and 1 cannot be used. Since repetition can be done, the 2nd digit and the 3rd digit have 10 choices each. So, the first three digits can be filled in (8 × 10 × 10 – 3) ways (We need to exclude the numbers 555, 411 and 936 also from first three digits)

The last four digits of the telephone number can be filled in (10 × 10 × 10 × 10) ways.

$\therefore$ Total number of seven digit phone numbers

= (8 × 10 × 10 – 3) × 10 × 10 × 10 × 10 = **7970000.**

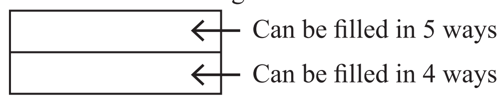
6. Number of ways of going from the village to the town = Number of roads between them = 5

Since all the 5 roads can be used for going back from town to the village, so number of ways of returning = 5.

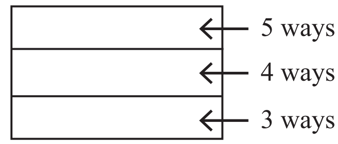
$\therefore$ Total number of ways of going to the town and coming back to the village = 5 × 5 = **25.**

7. At least 2 flags in order means that a signal may consist of either 2 flags, 3 flags, 4 flags in succession one below the other.

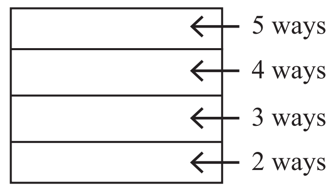
Total number of signals = Number of 2 flag signals



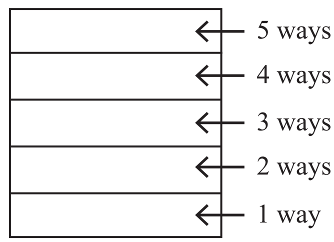
\+ Number of 3 flag signals



\+ Number of 4 flag signals



\+ Number of 5 flag signals



= 5 × 4 + 5 × 4 × 3 + 5 × 4 × 3 × 2 + 5 × 4 × 3 × 2 × 1

= 20 + 60 + 120 + 120 = **320.**

8. A number between 100 and 1000 has 3-digits

$\therefore$ Total number of 3-digit numbers having at least one of their digits as 7 = (Total number of 3-digit numbers) – (Total number of 3-digit numbers in which 7 does not appear at all)

(*a*) **Total number of 3-digit numbers:**

Clearly repetition of digits is allowed.

The hundred's place can be filled in 9 ways, *i.e.*, using any of the digits 1 to 9 excluding zero.

The ten's and the unit's place can be filled in 10 ways using all the digits from 0 – 9 in each place.

$\therefore$ Total number of 3-digit numbers = 9 × 10 × 10 = 900

(*b*) **Total number of 3-digit numbers in which 7 does not appear at all:**

Here, the digits to be used are 0, 1, 2, 3, 4, 5, 6, 8, 9 *i.e.*, 9 in number.

Now, the hundred's place can be filled in 8 ways (excluding 0), the tens' and ones' place can be filled in 9 ways each.

$\therefore$ Total number of 3-digit numbers in which 7 does not appear, = 8 × 9 × 9 = 648

$\therefore$ From (*a*) and (*b*), we have, **total number of 3-digit numbers having at least one of their digits as 7 = 900 – 648 = 252.**

9. A number between 4000 and 5000 will have a 4 in the thousand's place and since the number has to be divisible by 5, it will have 5 at unit's place. The hundreds' and tens' place each can be filled with any of the five digits. So,

Th	H	T	O
1	5	5	1

Total number of required numbers = 1 × 5 × 5 × 1 = **25.**

10. For a number to be odd, we should have 1 or 3 or 5 at the unit's place. So there are 3 ways of filling the unit's place.

As repetition of digits is not allowed, the ten's place can be filled in 5 ways with any of the remaining 5-digts and the hundred's place can be filled in 4 ways by the remaining 4-digits. So,

Required number of three-digit odd numbers = 3 × 5 × 4 = **60.**

11. Out of the 10 digits 0, 1, 2, 3, ..., 9, five digits, *i.e.*, 0, 2, 4, 6, 8 are even and five digits, *i.e.*, 1, 3, 5, 7 and 9 are odd.

The sum of the three digits D_1, D_2, D_3 of the number $D_1 D_2 D_3$ will be even if :

(*i*) All the three digits are even:

$\therefore$ Number of such numbers = 4 × 5 × 5 = 100

(Here the position D_1 cannot be occupied by 0)

(*ii*) One of the digits is even and the rest two are odd :

(*a*) D_1 is even, D_2 is odd, D_3 is odd

$\therefore$ Number of such numbers = 4 × 5 × 5 = 100

(Again since $D_1 \neq 0$, so only 4 choices for D_1)

(*b*) D_1 is odd, D_2 is even, D_3 is odd

$\therefore$ Number of such numbers = 5 × 5 × 5 = 125

(*c*) D_1 is odd, D_2 is odd, D_3 is even

$\therefore$ Number of such numbers = 5 × 5 × 5 = 125

$\therefore$ Total number of required numbers

= 100 + 100 + 125 + 125 = **450.**

12. One ball can be put in first box in 20 ways because we can put any one of the twenty balls in the first box. Now, remaining 19 balls are to be put into remaining 4 boxes. This can be done in 4^{19} ways, because there are 4 choices for each ball. Hence, the required number of ways = $\mathbf{20 \times 4^{19}}$.

13. Let us consider all the books of same size as one entity. Now there are three different entities which have to be arranged on the shelf and this can be done is 3! ways.

Also the five large books can be arranged amongst themselves in 5! ways, four medium books can be arranged amongst themselves in 4! ways and the three small books can be arranged amongst themselves in 3! ways. So,

Required number of ways of arranging the books so that all same sized books are together = **3! × 5! × 4! × 3!**

14. $P(5, r) = 2.\ P(6, r-1)$

$$\Rightarrow \quad {}^5P_r = 2 \times {}^6P_{r-1}$$

$$\Rightarrow \quad \frac{\lfloor 5}{\lfloor 5-r} = 2 \times \frac{\lfloor 6}{\lfloor 6-(r-1)}$$

$$\Rightarrow \quad \frac{\lfloor 5}{\lfloor 5-r} = \frac{2 \times 6 \times \lfloor 5}{\lfloor 7-r}$$

$$\Rightarrow \quad \frac{\lfloor 5}{\lfloor 5-r} = \frac{12 \times \lfloor 5}{(7-r)(6-r)\lfloor 5-r}$$

$$\Rightarrow \quad 1 = \frac{12}{(7-r)(6-r)} \quad \Rightarrow \quad (7-r)(6-r) = 12$$

$$\Rightarrow \quad (7-r)(6-r) = 4 \times 3$$

$$\Rightarrow \quad 7-r = 4 \text{ and } 6-r = 3 \quad \Rightarrow \quad \mathbf{r = 3.}$$

15. ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$

$$\Rightarrow \quad \frac{(2n+1)!}{(2n+1-(n-1))!} : \frac{(2n-1)!}{(2n-1-n)!} = 3:5$$

$$\Rightarrow \quad \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)(2n)\cancel{(2n-1)!}}{(n+2)(n+1)(n)\cancel{(n-1)}} \times \frac{\cancel{(n-1)}!}{\cancel{(2n-1)}!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{4n^2 + 2n}{n(n^2 + 3n + 2)} = \frac{3}{5}$$

$$\Rightarrow \quad (4n^2 + 2n)\,5 = 3 \times (n^3 + 3n^2 + 2n)$$

$$\Rightarrow \quad 20n^2 + 10n = 3n^3 + 9n^2 + 6n$$

$$\Rightarrow \quad 3n^3 - 11n^2 - 4n = 0$$

$$\Rightarrow \quad n(3n^2 - 11n - 4) = 0$$

$$\Rightarrow \quad n(3n^2 - 12n + n - 4) = 0$$

$$\Rightarrow \quad n(3n(n-4) + 1(n-4)) = 0$$

$$\Rightarrow \quad n = 0 \text{ or } 4 \text{ or } -\frac{1}{3}$$

Ignoring $n = 0, -\frac{1}{3}$ as these are in admissible values, we have $\mathbf{n = 4.}$

16. The number of 4-letter words which can be formed from 6 letters when one or more of the letters is repeated = 6 × 6 × 6 × 6 = 1296

Number of 4-letter words which can be formed from the given 6 letters when none of the letters is repeated

= Number of arrangements of 6 letters taken 4 at a time

$$= {}^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 \times 5 \times 4 \times 3 = 360$$

$\therefore$ Number of 4 letter words which have at least one of their letters repeated = 1296 – 360 = **936.**

17. The first place will always be filled by C and the last place will always be filled with Y. The remaining six places can be filled by the remaining 6 letters in ${}^6P_6$ ways.

∴ Total number of words beginning with C and ending with $Y = 1 \times 1 \times {}^6P_6 = 6! =$ **720.**

18. There are 7 letters in the word RAINBOW out of which 4 are consonants and 3 are vowels.

$\underline{1}$ 2 $\underline{3}$ 4 $\underline{5}$ 6 $\underline{7}$

In order that the vowels may occupy odd places, we first of all arrange any 3 consonants in even places in ${}^4P_3$ ways and then the odd places can be filled by 3 vowels and the remaining 1 consonant in ${}^4P_4$ ways. So,

Required number of words $= {}^4P_3 \times {}^4P_4 = 24 \times 24 =$ **576.**

19. There are eight letters in the word "DAUGHTER" including three vowels (A, U, E) and 5 consonants (D, G, H, T, R)

If the vowels are to be together, we consider them as one letter, so the 6 letters now (5 consonants and 1 vowels entity) can be arranged in ${}^6P_6 = 6!$ ways. Also corresponding to each of these arrangements, the 3 vowels can be arranged amongst themselves in 3 ! ways.

∴ Required number of words = 6 ! × 3 !

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 =$ **4320.**

20. Total number of words in which vowels are never together

= Total number of words that can be formed with the letters of the word SUNDAY – Number of words in which vowels are always together

There are 6 letters in the word SUNDAY. So,

Total number of words formed by using all 6 letters $= {}^6P_6 = 6! = 720$.

Considering the vowels U and A as one unit, we have 5 letters that can be arranged in ${}^5P_5$ ways and also corresponding to each of these arrangement of 5 letters the vowels can be arranged amongst themselves in 2 ! ways.

So, number of words in which vowels are always together = 5 ! × 2 ! = **240.**

Hence, total number of words in which vowels are never together = 720 – 240 = **480.**

21. First of all we arrange the 5 boys in 5 ! ways.

Then we arrange the 3 girls in the remaining 6 places between the 5 boys and on the extreme in ${}^6P_3$ ways.

× B × B × B × B × B ×

∴ Required number of ways $= 5! \times {}^6P_3$

$= 120 \times 6 \times 5 \times 4 =$ **144000.**

22. There are 11 letters in the word "MATHEMATICS" out of which 4 are vowels and the rest 7 are consonants.

Let the four vowels be written together. $\boxed{\text{A A E I}}$ M, T, H, M, T, C, S

Consider the four vowels as one as unit, then these 8 letters (7 consonants and the vowel unit) can be permuted in $\frac{8!}{2!2!} = 10080$ ways. (There are two pairs of same letters AA and MM)

Corresponding to each of these permutations, the 4 vowels can be arranged among themselves in $\frac{4!}{2!} = 12$ ways.

∴ Required number of word in which vowels occur together

$= \frac{8!}{2!2!} \times \frac{4!}{2!} = 10080 \times 12 = \mathbf{120960}$.

23. The words coming before RAHUL in the dictionary will have A, H, L or R as their first letters.

I. When A is the first letter, the rest of the 4 letters H, L, R, U can fill the next 4 places in 4 !

∴ Number of words beginning with $A = 24$

II. When H is the first letter, the rest of the 4 letters A, L, R, U can fill the next 4 places in 4!

Number of words beginning with H = 24

Similarly, the number of words beginning with L = 24

Now among the words having R as their first letter, there is only one word which comes before RAHUL and that is RAHLU

Thus there are (24 × 3) + 1 = 73 words before RAHUL.

24. To determine the rank of the word COCHIN when the letters of the word 'COCHIN' are permuted and all permutations arranged in alphabetical order, we see that all words begin with C. So we have the following cases when permuted words are in alphabetical.

I. First two letters are CC

Then the number of ways of arranging the remaining 4 letters 0, H, I, N in remaining 4 places 4 !

∴ Number of word having CC has first two letters = 24

II. First two letters are CH (next to CC in alphabetical order)

Here also, as in case I,

Number of words with CH as first two letters = 4 ! = 24

Similarly, the number of words with CI and CN as their first two letters will be 24 each.

The first word with CO as first two letters is COCHIN

∴ 24 + 24 + 24 + 24 = **96** words appear before COCHIN.

25. 7 shirts can be worn by 4 friends in ${}^7P_4$ ways.

Similarly, 6 pants and 8 ties can be worn by 4 friends in ${}^6P_4$ and ${}^8P_4$ ways respectively.

∴ Total number of ways in which 7 shirts, 6 pants and 8 ties can be worn by 4 friends $= {}^7P_4 \times {}^6P_4 \times {}^8P_4$

$= \frac{7!}{3!} \times \frac{6!}{2!} \times \frac{8!}{4!}$

$= (7 \times 6 \times 5 \times 4) \times (6 \times 5 \times 4 \times 3) \times (8 \times 7 \times 6 \times 5)$

$= 840 \times 360 \times 1680 =$ **508032000.**

26. In the word 'HALLUCINATION' there are total 13 letters out of which 7 are consonants and 6 are vowels. Also there are 2L's, 2N's, 2A's and 2I's.

If L's occur together and I's occur together, then we consider them as one unit each. So, the arrangement can be written as:

$\boxed{L, L}, \boxed{I, I}$, H, A, A, U, C, N, N, T, O

Now these 11 letters (9 letters and 2 units of L and I) can be arranged in $\frac{11!}{2!2!}$ ways. (∵ There are 2A's and 2N's)

27. There are 8 letters in the word AFLATOON out of which 4 are vowels, *i.e.*, A, A, O, O and 4 are consonants, *i.e.*, F, L, T, N.

There are 2 ways in which the 4 consonants and 4 vowels occupy alternate places:

V C V C V C V C or CV CV CV CV

where V–vowel, C–consonant.

The four vowels can be arranged in 4 places in $\frac{4!}{2!2!}$ ways $= 6$ ways

The four consonants can be arranged in their 4 places in 4! ways = 24 ways.

Also, there are 2 cases, so,

∴ Required number of permutations $= 6 \times 24 \times 2 =$ **288.**

28. Numbers greater than 60000 will have either 6 or 8 in the TTh place and will consist of 5-digits.

If the digit 6 occupies the TTh place, the remaining 4 places can be occupied in $\frac{4!}{2!}$ ways. (∵ There are two 2's)

Number of numbers beginning with 6 $= \frac{4!}{2!} = 12$

Similarly, number of numbers beginning with 8 $= \frac{4!}{2!} = 12$

∴ Number of different numbers greater than 60000 formed with digits 0, 2, 2, 6, 8

$= 12 + 12 =$ **24.**

29. There are $5 + 4 + 3 = 12$ marbles of which 5 are red (alike), 4 are white (alike) and 3 are blue (alike).

∴ Required number of arrangements $= \frac{12!}{5! \times 4! \times 3!}$

$= \frac{12 \times 11 \times 10 \times 9 \times 8^{4} \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}$

$=$ **27720.**

30. As there are 4 digits 1, 2, 3, 4 and repetition of digits is allowed.

Total number of 1-digit numbers = 4

Total number of 2-digit numbers $= 4 \times 4 = 16$

Total number of 3-digit numbers $= 4 \times 4 \times 4 = 64$

Number of 4-digit numbers beginning with $1 = 4 \times 4 \times 4 = 64$ (∵ The first place is occupied by 1)

Number of 4-digit numbers beginning with $2 = 4 \times 4 \times 4 = 64$

Number of 4-digit numbers beginning with $3 = 4 \times 4 \times 4 = 64$

Number of 4-digit numbers beginning with $41 = 4 \times 4 = 16$

Number of 4-digit numbers beginning with $42 = 4 \times 4 = 16$

Number of 4-digit numbers beginning with $431 = 4$

Number of 4-digit numbers beginning with $432 = 1$ (4321 only)

∴ Total number of 4-digit numbers $= 64 + 64 + 64 + 16 + 16 + 4 + 1 = 229$

∴ Total number of natural numbers not exceeding 4321 $= 4 + 16 + 64 + 229 =$ **313.**

31. As is known in case of circular permutations, we keep one place fixed, so 8 people can sit around a circular table in $(8 - 1)!$ ways = 7! ways = 5040 ways.

32. Considering the president and secretary as one member, we have now 10 members in all. These 10 members can be seated round the circular table in $(10 - 1)! = 9!$ ways.

Also the president and secretary can seat themselves in 2! ways **PS or SP**

∴ Required number of ways of seating = **9! × 2!.**

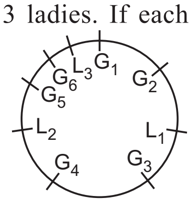
33. 7 men can seat themselves in a round table in $(7 - 1)! = 6!$ ways.

Now there are 7 places vacant between these 7 men.

∴ 7 women can seat themselves in these 7 places in 7 ! ways.

∴ Total number of required arrangement where no two women sit together = **6! × 7!.**

34. There are 9 people, 6 gentlemen and 3 ladies. If each gentleman has to have a lady by his side, the seating arrangement can be done as shown below:



This can be done in 5! (Gentlemen) × 3! (Ladies) = 720 ways.

The arrangements can also be made in opposite direction. So,

Total number of required arrangements $= 2 \times 720 =$ **1440.**

35. Since the chairs are numbered, so they are distinguishable. Therefore 3 boys can be arranged on 3 alternate chairs in 3! ways. 3 girls can be arrenged in 3! ways

Also, the girls can be seated before the boys.

Total number of required ways $= 3! \times 3! + 3! \times 3!$

$= 2 \times (3!)^2$

36. Consider the three particular flowers as one flower. Then we have $(10 - 3) + 1 = 8$ flowers which can be strung in the garland.

Thus the garland can be formed in $(8 - 1)!$, *i.e.*, 7! ways

But the 3 particular flowers can be arranged amongst themselves in 3! ways.

∴ Required number of ways $= \frac{1}{2}(7! \times 3!)$

37. Given ${}^{2n}C_3 : {}^nC_2 = 12 : 1$

$$\Rightarrow \frac{{}^{2n}C_3}{{}^nC_2} = \frac{12}{1} \Rightarrow \frac{\dfrac{(2n)!}{(2n-3)!\,3!}}{\dfrac{n!}{(n-2)!\,2!}} = \frac{12}{1}$$

$$\Rightarrow \frac{(2n)!}{n!} \times \frac{(n-2)!\,2!}{(2n-3)!\,3!} = \frac{12}{1}$$

$$\Rightarrow \frac{(2n)(2n-1)\,(2n-2)\,(2n-3)!}{n(n-1)\,(n-2)!} \times \frac{(n-2)! \times 2!}{(2n-3)! \times 3 \times 2!} = \frac{12}{1}$$

$$\Rightarrow \frac{(2n)\,(2n-1)\,(2n-2)}{3n\,(n-1)} = \frac{12}{1}$$

$$\Rightarrow \frac{4(\cancel{n})\,(2n-1)\,(\cancel{n-1})}{3\cancel{n}\,(\cancel{n-1})} = \frac{12}{1}$$

$\Rightarrow 8n - 4 = 36 \quad \Rightarrow \quad 8n = 40 \quad \Rightarrow$ $\boldsymbol{n = 5.}$

38. Given, ${}^nC_r : {}^nC_{r+1} = 1 : 2$ and ${}^nC_{r+1} : {}^nC_{r+2} = 2 : 3$

$$\Rightarrow \frac{n!}{r!(n-r)!} : \frac{n!}{(r+1)!\,(n-r-1)!} = 1:2 \text{ and}$$

$$\frac{n!}{(r+1)!\,(n-r-1)!} : \frac{n!}{(r+2)!\,(n-r-2)!} = 2:3$$

$$\Rightarrow \frac{n!}{r!\,(n-r)\,(n-r-1)!} \times \frac{(r+1)r!\,(n-r-1)!}{n!} = \frac{1}{2}$$

$$\text{and } \frac{n!}{(r+1)!\,(n-r-1)\,(n-r-2)!} \times \frac{(r+2)\,(r+1)!\,(n-r-2)!}{n!} = \frac{2}{3}$$

$$\Rightarrow \frac{(r+1)}{n-r} = \frac{1}{2} \text{ and } \frac{r+2}{n-r-1} = \frac{2}{3}$$

$\Rightarrow 2r + 2 = n - r$ and $3r + 6 = 2n - 2r - 2$

$\Rightarrow n - 3r = 2$ and $2n - 5r = 8$

Solving the two simultaneous equations, we get
$\boldsymbol{n = 14, r = 4.}$

39. When a particular member is taken every time, then we need to choose the remaining 4 members from 7 members in ${}^7C_4$ ways.

$$\therefore \text{Required number of ways} = {}^7C_4 = \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5}{3 \times 2} = \mathbf{35.}$$

40. As each team of 11 players has one goalkeeper and 10 team members, and out of 14 players there are 2 goalkeepers and 12 team members.

So the number of ways in which a team of 11 can be selected

$$= {}^{12}C_{10} \times {}^2C_1 = \frac{12!}{10! \times 2!} \times 2$$

$$= \frac{12 \times 11}{2} \times 2 = \mathbf{132.}$$

41. Since every person in the room shakes hand with every other person.

So, total number of handshakes = Number of ways of selecting 2 men out of 15 men

$$= {}^{15}C_2 = \frac{15!}{13! \times 2!} = \frac{15 \times 14}{2} = \mathbf{105.}$$

42. He can write 2 or more friends out of 6 friends in the given number of ways:

Invite (*i*) 2 friends out of 6 friends
or (*ii*) 3 friends out of 6 friends
or (*iii*) 4 friends out of 6 friends
or (*iv*) 5 friends out of 6 friends
or (*v*) 6 friends out of 6 friends.

$\therefore$ Total number of ways of inviting 2 or more friends

$= {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$

$$= \frac{6!}{4!\,2!} + \frac{6!}{3!\,3!} + \frac{6!}{4!\,2!} + \frac{6!}{5!\,1!} + \frac{6!}{6!}$$

$$= \frac{6 \times 5}{2} + \frac{6 \times 5 \times 4}{3 \times 2} + \frac{6 \times 5}{2} + 6 + 1$$

$= 15 + 20 + 15 + 6 + 1 =$ **57.**

43. If a particular student is excluded, then the committee has to be chosen as: 2 professors from 5 professors and 3 students from 5 students (as one student is excluded)

$\therefore$ Total number of ways of forming the committee

$= {}^5C_2 \times {}^5C_3$

$$= \frac{5!}{2!\,3!} \times \frac{5!}{3!\,2!}$$

$$= \frac{5 \times 4}{2} \times \frac{5 \times 4}{2} = \mathbf{100.}$$

44. If at least one woman has to be included then the committee can be formed as:

- 1 woman and 2 men
- 2 women and 1 man

$\therefore$ Number of ways of forming the committee

$= {}^2C_1 \times {}^5C_2 + {}^2C_2 \times {}^5C_1$

$= 2 \times 10 + 1 \times 5 = 20 + 5 =$ **25.**

45. A committee of 5 persons, including at most two ladies can be constituted in the following ways :

I. Selecting 5 gents only out of 6 gents.

II. Selecting 4 gents out of 6 gents and 1 lady out of 4 ladies.

III. Selecting 3 gents out of 6 gents and 2 ladies out of 4 ladies.

$\therefore$ Total number of ways of forming the committee

$= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2$

$$= 6 + \frac{6 \times 5}{2} \times 4 + \frac{6 \times 5 \times 4}{3 \times 2} \times \frac{4 \times 3}{2}$$

$= 6 + 60 + 120 =$ **186.**

46. As there are 4 candidates to be elected and the voter votes for at least one candidate, the voter can vote for 1 or 2 or 3 or 4 candidates out of 10 candidates.

$\therefore$ Total number of ways in which the voter can vote

$= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^4C_4$

$= 10 + \frac{10\times 9}{2\times 1} + \frac{10\times 9\times 8}{3\times 2\times 1} + \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}$

$= 10 + 45 + 120 + 210 =$ **385.**

47. Let the number of participants in the beginning be ***n*.**

Total number of games played by the 2 players who fell ill $= 6 + 6 = 12$

Number of remaining players is $(n - 2)$

Number of games played by $(n - 2)$ players with each other $= {}^{n-2}C_2$

$\therefore$ Total number of games played $= {}^{n-2}C_2 + 12$

Given ${}^{n-2}C_2 + 12 = 117$

$\Rightarrow {}^{n-2}C_2 = 105 \quad \Rightarrow \frac{n-2!}{(n-4)!\,2!} = 105$

$\Rightarrow \frac{(n-2)(n-3)}{2} = 105 \Rightarrow n^2 - 5n - 204 = 0$

$\Rightarrow (n - 17)(n + 12)\ 0 \Rightarrow n = 17$ (Neglecting –ve value)

48. The 6 people (3 ladies and 3 gentlemen) to be invited can be selected in the given ways:

(*i*) 3 ladies from man's relatives and 3 gentlemen from wife's relatives, *i.e.*,

No. of ways $= {}^4C_3 \times {}^3C_0 \times {}^3C_0 \times {}^4C_3 = 16$

(*ii*) 2 ladies and 1 gentleman from man's relatives and 1 lady and 2 gentlemen from wife's relatives, *i.e.*,

No. of ways $= {}^4C_2 \times {}^3C_1 \times {}^3C_1 \times {}^4C_2$

$= \frac{4\times 3}{2} \times 3 \times 3 \times \frac{4\times 3}{2} = 324$

(*iii*) 1 lady and 2 gentlemen from man's relatives and 2 ladies and 1 gentleman from wife's relatives, *i.e.*,

No. of ways $= {}^4C_1 \times {}^3C_2 \times {}^3C_2 \times {}^4C_1 = 4 \times 3 \times 4 \times 3$
$= 144$

(*iv*) 3 gentlemen from man's relatives and 3 ladies from wife's relatives, *i.e.*,

No. of ways $= {}^4C_0 \times {}^3C_3 \times {}^3C_3 \times {}^4C_0 = 1 \times 1 = 1$

$\therefore$ Total number of ways of selecting the relatives for dinner party $= 16 + 324 + 144 + 1 =$ **485.**

49. First the 15 guests have to be divided into two groups of 8 persons and 7 persons.

$\therefore$ Number of ways of dividing the guests $= \frac{15!}{8!\,7!}$

Now 8 persons can be seated in one round table in $(8 - 1)! = 7!$ ways

Also 7 persons can be seated in another round table in $(7 - 1)! = 6!$ ways

$\therefore$ Number of ways of arranging the guests $= \frac{15!}{8!\,\cancel{7!}} \times \cancel{7!} \times 6!$

$= \frac{15!}{8\times 7} = \mathbf{\frac{15!}{56}}$.

50. Number of diagonals for a n-sided closed polygon

$= {}^nC_2 - n = \frac{n!}{(n-2)!\,2!} - n$

$= \frac{n(n-1)(n-2)!}{(n-2)!\times 2} - n = \frac{n(n-1) - 2n}{2}$

$= \frac{n^2 - n - 2n}{2} = \frac{n - 3n}{2} = \frac{n(n-3)}{2}$

$\therefore$ Number of diagonals for a 15–sided polygon

$= \frac{15\times(15-3)}{2} = \frac{15\times 12}{2} =$ **90.**

51. Number of triangles that can be drawn with all 18 points in the plane $= {}^{18}C_3$

But 5 points are collinear

$\therefore$ Required number of triangles $= {}^{18}C_3 - {}^5C_3$

$= \frac{18!}{15!\,3!} - \frac{5!}{2!\,3!} = \frac{18\times 17\times 16}{3\times 2} - \frac{5\times 4}{2}$

$= 816 - 10 =$ **806.**

52. We can choose 2 men out of 8 men in ${}^8C_2$ ways. Since no husband and wife are to play in the same game, two women out of the remaining 6 women can be chosen in ${}^6C_2$ ways.

If M_1, M_2, W_1, W_2 are respectively the two men and two women chosen for the teams then the teams can be formed as $(M_1, W_1)\ (M_2, W_2)$ or $(M_1, W_2)\ (M_2, W_1)$, *i.e.*, in 2 ways. Hence, total number of ways of arranging the game $= {}^8C_2 \times {}^6C_2 \times 2$

$= \frac{8\times 7}{2} \times \frac{6\times 5}{2} \times 2 =$ **840.**

53. The required committee can be chosen in the following ways:

I. When Ms. B is a member and Mr. A refuses to serve.

[Selection has to be made of 3 women from 8 women as Ms B is already there and 5 men from 6 men as Mr A is excluded]

II. When Ms B is not a member and Mr. A can serve.

(Selection is to made of 4 women from 8 women as Ms. B is excluded and 5 men from 7 men)

$\therefore$ Total number of ways in which the committee can be formed.

$= {}^8C_3 \times {}^6C_5 + {}^8C_4 \times {}^7C_5$

$= \frac{8\times 7\times 6}{3\times 2} \times 6 + \frac{8\times 7\times 6\times 5}{4\times 3\times 2} \times \frac{7\times 6}{2}$

$= 56 \times 6 + 70 \times 21 = 336 + 1470 =$ **1806.**

54. A parallelogram is formed by choosing two straight lines from the set of ***m*** parallel lines and choosing two straight lines from the set of ***n*** parallel lines.

$\therefore$ Number of parallelograms formed $= {}^mC_2 \times {}^nC_2$

$$= \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn\,(m-1)\,(n-1)}{4}$$

55. This means that first player gets 13 cards, then second player gets 13 cards, then third player gets 13 cards and then fourth gets 13 cards.

$\therefore$ First player gets 13 cards in ${}^{52}C_{13}$ ways

Second player gets 13 cards in ${}^{39}C_{13}$ ways

Third player gets 13 cards in ${}^{26}C_{13}$ ways

Fourth player gets 13 cards in ${}^{13}C_{13}$ ways

$\therefore$ Total number of ways $= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$

$$= \frac{52!}{39!13!} \times \frac{39!}{26!13!} \times \frac{26!}{13!13!} \times 1 = \frac{52!}{(13!)^4}$$

Alternatively, 52 cards can be divided equally among 4 players in $\frac{52!}{(13!)^4 \times 4!}$ ways.

But as here order is important, hence required number of ways

$$= \frac{52! \times 4!}{(13!)^4 \times 4!} = \frac{52!}{(13!)^4}$$

56. First player can get 16 cards in ${}^{52}C_{16}$ ways

Second player can get 16 cards in ${}^{36}C_{16}$ ways

Third player can get 16 cards in ${}^{20}C_{16}$ ways

Fourth player can get 4 cards in ${}^{4}C_{4}$ ways

But the first three sets can be interchanged in 3! ways

$\therefore$ Required number of ways

$$= {}^{52}C_{16} \times {}^{36}C_{16} \times {}^{20}C_{16} \times {}^{4}C_{4} \times \frac{1}{3!}$$

$$= \frac{52!}{(16!)^3 \times 3!}$$

57. The first person can leave the elevator in any of the 10 floors

$\therefore$ Number of ways 1st person can leave the elevator = 10

Similarly, for each of the remaining 6 persons, the number of ways each one can leave the elevator = 10

$\therefore$ All the 7 persons can leave the elevator in

$$= \underbrace{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}_{7 \text{ terms}} = \mathbf{(10)^7} \text{ ways.}$$

58. As four particular guests want to sit on a particular side say (S_1) and three others on the other side, say, (S_2). So we are left with 11 guests out of which we choose 5 for side S_1 from remaining 11 in ${}^{11}C_5$ ways and 6 from remaining 6 for side S_2 in ${}^{6}C_6$ ways.

Also the 9 persons on each side can be arranged amongst themselves in 9! ways.

$\therefore$ Required number of ways of making the seating arrangement $= \mathbf{{}^{11}C_5 \times {}^{6}C_6 \times 9! \times 9!}$

59. 6 balls consisting of at least two balls of each colour from 5 red and 6 white balls can be made in the following ways:

(*a*) Selecting 2 red balls out of 5 red balls and 4 white balls out of 6, *i.e.*,

$$\text{Number of ways} = {}^5C_2 \times {}^6C_4 = \frac{5 \times 4}{2} \times \frac{6 \times 5}{2}$$

$$= 10 \times 15 = 150$$

(*b*) Selecting 3 red balls out of 5 red balls and 3 white balls out of 6, *i.e.*,

$$\text{Number of ways} = {}^5C_3 \times {}^6C_3 = \frac{5 \times 4}{2} \times \frac{6 \times 5 \times 4}{3 \times 2}$$

$$= 10 \times 20 = 200$$

(*c*) Selecting 4 red balls out of 5 red balls and 2 white balls out of 6, *i.e.*,

$$\text{Number of ways} = {}^5C_3 \times {}^6C_2 = 5 \times \frac{6 \times 5}{2} = 5 \times 15 = 75$$

$\therefore$ Total number of ways of selecting at least two balls of each colour = 150 + 200 + 75 = **425.**

60. As no box has to be empty, each box should have at least 1 ball.

Let the boxes be labelled as B_1, B_2 and B_3. Then the distribution of the balls can be shown as:

B_1	B_2	B_3
1	1	3
1	2	2
1	3	1
2	1	2
2	2	1
3	1	1

$\therefore$ Number of ways of placing the balls in different boxes so that no box remains empty

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 + {}^5C_1 \times {}^4C_2 \times {}^2C_2 + {}^5C_1 \times {}^4C_3 \times {}^1C_1 + {}^5C_2 \times {}^3C_1 \times {}^2C_2 + {}^5C_2 \times {}^3C_2 \times {}^1C_1 + {}^5C_3 \times {}^2C_1 \times {}^1C_1$$

$$= 5 \times 4 \times 1 + 5 \times \frac{4 \times 3}{2} \times 1 + 5 \times 4 \times 1 + \frac{5 \times 4}{2} \times 3 \times 1 + \frac{5 \times 4}{2} \times 3 \times 1 + \frac{5 \times 4}{2} \times 2 \times 1$$

= 20 + 30 + 20 + 30 + 30 + 20 = **150.**

SELF ASSESSMENT SHEET

1. How many integers greater than 999 but not greater than 4000 can be formed with the digits 0, 1, 2, 3 and 4 if repetition of digits is allowed?

(*a*) 499 (*b*) 500 (*c*) 375 (*d*) 376

(***CAT 2008***)

2. Consider the word RADAR. Whichever way you read it, from left to right or right to left, you get the same word. Such a word is known as a **palindrome.** Find the maximum possible number of 5–letter palindromes (meaningful or non-meaningful)

(*a*) 676 (*b*) 78 (*c*) 17576 (*d*) 130

3. How many arrangements can be formed out of the letters of the word EXAMINATION so that vowels always occupy odd places?

(*a*) 72000 (*b*) 86400 (*c*) 10800 (*d*) 64000

(***SNAP 2007***)

4. A number lock consists of 3 rings each marked with 10 different numbers. In how many cases the lock cannot be opened?

(*a*) 3^{10} (*b*) 10^3 (*c*) 30 (*d*) 999

(***SNAP 2008***)

5. A six digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 if repetition of digits is not allowed. The total number of ways this can be done is:

(*a*) 120 (*b*) 240 (*c*) 600 (*d*) 720

(***Kerala PET***)

6. How many different nine-digit numbers can be formed from the number 223355888 by rearranging its digits so that odd digits occupy even positions?

(*a*) 16 (*b*) 36 (*c*) 60 (*d*) 96

(***IIT 2000***)

7. In how many ways can the eight directors, the vice-chairman and chairman of a firm be seated at a round table, if the chairman has to sit between the vice-chirman and the director?

(*a*) 9! × 2 (*b*) 2 × 8! (*c*) 2 × 7! (*d*) None of these

(***CAT***)

8. A total of 28 handshakes was exchanged at the conclusion of a party. Assuming that each participants was equally polite toward all the others, the number of people present was

(*a*) 14 (*b*) 28 (*c*) 56 (*d*) 8

(***FMS 2011***)

9. Out of 6 ruling and 5 opposition party members, 4 are to be selected for a delegation. In how many ways can it be done so as to include at least one opposition member?

(*a*) 300 (*b*) 315 (*c*) 415 (*d*) 410

(***JMET 2011***)

10. A team of 8 players is to be chosen from a group of 12 players. Out of the eight players one is to be elected as captain and another as vice-captain. In how many ways can this be done?

(*a*) 27720 (*b*) 13860 (*c*) 6930 (*d*) 495

(***NDA/NA 2010***)

11. There are 10 points on a line and 11 points on another line, which are parallel to each other. How many triangles can be drawn taking the vertices on any of the line?

(*a*) 1050 (*b*) 2550 (*c*) 150 (*d*) 1045

(***CAT***)

12. In how many ways is it possible to choose a white square and a black square on a chess board so that the squares must not lie in the same row or column ?

(*a*) 56 (*b*) 896 (*c*) 60 (*d*) 768

(***CAT 2002***)

13. In a chess competition involving some boys and girls of a school, every student has to play exactly one game with every other student. It was found that in 45 games both the players were girls and in 190 games both were boys. The number of games in which one player was a boy and the other was a girl is

(*a*) 200 (*b*) 216 (*c*) 235 (*d*) 256

(***CAT 2005***)

14. The value of $\sum_{r=1}^{n} \frac{{}^nP_r}{r!}$ is

(*a*) 2^n (*b*) $2^n - 1$ (*c*) 2^{n-1} (*d*) $2^n + 1$

(***IIFT 2007***)

15. In how many ways can four letters of the word 'SERIES' be arranged?

(*a*) 24 (*b*) 42 (*c*) 84 (*d*) 102

(***IIFT 2010***)

ANSWERS

1. (*d*)	**2.** (*c*)	**3.** (*c*)	**4.** (*d*)	**5.** (*c*)	**6.** (*c*)	**7.** (*b*)	**8.** (*d*)	**9.** (*b*)	**10.** (*a*)
11. (*d*)	**12.** (*d*)	**13.** (*a*)	**14.** (*b*)	**15.** (*d*)					

HINTS AND SOLUTIONS

1. The smallest number in the required series is 1000 and the greatest is 4000 (The only number to start with 4)

$\therefore$ The thousands' place can be filled with any of the 3-digits 1, 2 and 3.

The next three places (hundred's, ten's and unit's place) can take any of the five values 0 or 1 or 2 or 3 or 4.

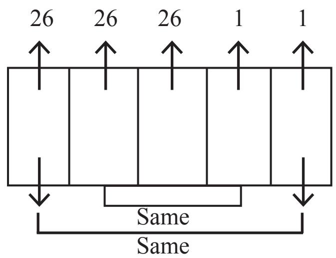
$\therefore$ Number of numbers from 100 to 3999

= 3 × 5 × 5 × 5 = 375

As 4000 is also included, so required number = **376.**

2. The first letter can be chosen in 26 ways as there are 26 letters in the English alphabet.

Having chosen the first letter, we have 26 choices for the second letter also.



Similarly for the third letter also, we have 26 choices.

The fourth letter has to be same as the second and the fifth letter has to be same as the first letter so 1 choice for each.

$\therefore$ Maximum possible number of 5-letter palindromes

$= 26 \times 26 \times 26 \times 1 \times 1 =$ **17576.**

3. There are 11 letters in the word EXAMINATION of which there are 2A's, 2I's.

Also there are 5 consonants and 6 vowels.

The 6 vowels can be arranged in 6 odd places in $\frac{6!}{2!\,2!}$ ways (2A's, 2I's)

The 5 consonants can be arranged in 5 even possible $= \frac{5!}{2!}$ ways (2N's)

$\therefore$ Total number of arrangements possible $= \frac{6!}{2! \times 2!} \times \frac{5!}{2!}$

$= \frac{6 \times 5 \times 4 \times 3}{2} \times 5 \times 4 \times 3 =$ **10800.**

4. The first ring can be marked with any of the 10 numbers, *i.e.*, Number of ways of marking the first ring = 10

Similarly the second and third ring can also be marked with any of the 10 numbers.

$\therefore$ Number of number combinations that can appear on the 3 rings $= 10 \times 10 \times 10 = 1000$

The lock can be opened with a single combination only.

$\therefore$ Number of cases in which the lock cannot be opened $= 1000 - 1 =$ **999.**

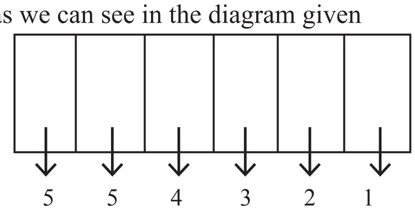
5. As repetition of digits is not allowed and the six digit number has to be formed using all the 6-digits 0, 1, 2, 3, 4, 5 implies that all the six digits will be used in making the number.

Sum of the 6-digits $= 0 + 1 + 2 + 3 + 4 + 5 = 15$

Hence any 6-digit number formed with these 6 digits will be divisible by 3.

The first place from the left can be filled with any of the non zero numbers.

Now, as we can see in the diagram given



The second place with remaining 5 numbers (including 0), the third with remaining 4, the fourth with remaining 3 and so on.

$\therefore$ Number of ways in which all the six places can be filled

= Number of ways of forming the 6-digit number divisible by 3

$= 5 \times 5 \times 4 \times 3 \times 2 \times 1 =$ **600.**

6. There are 9-digits in the given number of which there are 4 odd-digits (Two 3's, Two 5's) and 5 even digits (Two 2's, Three 8's).

In case of a 9-digit number there are 5 odd positions and 4 even positions

The 4 odd digts can occupy the 4 even positions in $\frac{4!}{2!\,2!}$ ways (Two 3's, Two 5's)

— X — X — X — X — $\left(\begin{array}{l} \text{X Odd} \\ \text{— Even} \end{array}\right)$

Also the 5 even digits can occupy the 5 odd positions in $\frac{5!}{3!\,2!}$ ways (Three 8's, Two 2's)

$\therefore$ Required number of 9-digit numbers $= \frac{4!}{2!\,2!} \times \frac{5!}{3!\,2!}$

$= 6 \times 10 =$ **60.**

7. Let the vice-chairman and chairman form 1 unit along with 8 directors. So now we have 9 units to arrange in a circle which can be done in $(9 - 1)!$ ways = 8! ways.

D C VC D

or

D VC C D

Also the chairman and vice chairman can be arranged amongst themselves in 2 different ways.

$\therefore$ Required number of ways = **2 × 8!.**

8. Let the number of people present be n. Then if n people shake hands with one another, number of handshakes $= {}^nC_2$.

$\Rightarrow \; {}^nC_2 = 28$

$\Rightarrow \; \frac{n!}{(n-2)!\,2!} = 28 \Rightarrow \frac{n(n-1)}{2} = 28$

$\Rightarrow \; n(n-1) = 56 \quad \Rightarrow n(n-1) = 8 \times 7 \quad \Rightarrow n =$ **8.**

9. 4 members of the delegation can be selected in the following ways:

I. 1 opposition member and 3 Ruling party members, *i.e.*,

Number of ways of this selection $= {}^5C_1 \times {}^6C_3$

II. 2 opposition members and 2 ruling party members, *i.e.*,

Number of ways of this selection $= {}^5C_2 \times {}^6C_2$

III. 3 opposition members and 1 ruling party member, *i.e.*,

Number of ways of this selection $= {}^5C_3 \times {}^6C_1$

IV. 4 opposition members, *i.e.*,

Number of ways of this selection $= {}^5C_4$.

$\therefore$ Total number of ways for required selection

$= {}^5C_1 \times {}^6C_3 + {}^5C_2 \times {}^6C_2 + {}^5C_3 \times {}^6C_1 + {}^5C_4$

$= 5\times\frac{6\times5\times4}{3\times2}+\frac{5\times4}{2}\times\frac{6\times5}{2}+\frac{5\times4}{2}\times6+5$

= 100 + 150 + 60 + 5 = **315.**

10. Number of ways to choose 8 players from 12 players

$= {}^{12}C_8 = \frac{12!}{8!\,4!} = 495$

Number of ways to choose a captain and a vice captain
$= {}^8C_1 \times {}^7C_1$
= 8 × 7 = 56

∴ Required number of ways = 495 × 56 = **27720.**

11. For a triangle, two points on one line and one on the other has to be chosen.

No. of ways = 2 points from 10 points and 1 point from 11 points **or** 1 point from 10 points and 2 points from 11 points
$= {}^{10}C_2 \times {}^{11}C_1 + {}^{10}C_1 \times {}^{11}C_2$

$= \frac{10\times9}{2}\times11+10\times\frac{11\times10}{2}$ = 495 + 550 = **1045.**

12. There are 32 black and 32 white squares on a chess board. Then number of ways of choosing one white and one black square on the chose board
$= {}^{32}C_1 \times {}^{32}C_1 = 32 \times 32 = 1024$

There are 8 rows and 8 columns on a chess board each containing 4 white squares and 4 black squares.

∴ Number of ways to choose a white and a black square from the same row $= {}^4C_1 \times {}^4C_1 \times 8 = 128$

Similarly, number of ways to choose a white and a black square from the same column $= {}^4C_1 \times {}^4C_1 \times 8 = 128$

∴ Number of ways of choosing a white square and a black square on the chess board so that squares do not lie in the same row or column
= 1024 – (128 + 128) = 1024 – 256 = **768.**

13. Let the number of **girls** be ***x*** and let the number of **boys** be ***y***.

Then, as in 45 games 2 girls played against each other,
${}^xC_2 = 45$ ...(*i*)
Similarly, ${}^yC_2 = 190$ ...(*ii*)

${}^xC_2 = 45 \Rightarrow \frac{x!}{(x-2)!\,2!} = 45 \Rightarrow x(x-1) = 90$

$\Rightarrow x(x-1) = 10 \times 9 \Rightarrow x =$ **10**

${}^yC_2 = 190 \Rightarrow \frac{y(y-1)}{2} = 190 \Rightarrow y(y-1) = 380$

$\Rightarrow y(y-1) = 20 \times 19 \Rightarrow y =$ **20.**

∴ There are 10 girls and 20 boys.

Hence, the total number of games in which one player is a girl and other player is a boy = 10 × 20 = **200.**

14. $\sum_{r=1}^{n}\frac{{}^nP_r}{r!} = \sum_{r=1}^{n}\frac{n!}{(n-r)!\,r!} = \sum_{r=1}^{n}{}^nC_r$

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_{n-1} + {}^nC_n$

We know that,
${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n$

$\therefore \sum_{r=1}^{n}{}^nC_r = 2^n - {}^nC_0 = \mathbf{2^n - 1.}$

15. The given word SERIES has 2 S and 2 E and the rest are distinct. The number of ways of arranging the 4 letters of word are as follows:

I. 4 letters are distinct, *i.e.*, S, E, R, I

∴ Number of ways of arranging these distinct 4 letters = 4! = 24

II. 2 letters are same and 2 are distinct, *i.e.*
SSRI, SSRE, SSIE, EERI, EERS, EEIS

∴ Number of ways of arranging letters in this way

$= \frac{4!}{2!}\times 6 = 72$

III. Two are same of one kind and two are same of other kind

∴ Number of ways of above arrangement $= \frac{4!}{2!\,2!} = 6$

∴ Total number of ways = 24 + 72 + 6 = **102.**