COMPOUND ANGLES

Definitions and Formulae :

1. The algebraic sum of two or more angles is called a compound a compound angle. i.e. A + B, A – B, A + B + C, A + B – C, A – B + C, B + C – A, ……. etc. are called compound angles.

2. If A and B are any two angles then

   i) Sin(A + B) = sin A cos B + cos A sin B.

   ii) Sin(A – B) = sin A cos B – cos A sin B

   iii) cos(A + B) = cos A cos B – sin A sin B

   iv) cos(A – B) = cos A cos B + sin A sin B.

3. If A, B, A + B, A – B are not odd multiples of $\pi/2$ then

   i) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

   ii) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

4. If A, B, A + B and A – B are not integral multiples of π, then

   i) $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

   ii) $\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$.

5. i) sin(A + B) + sin(A – B) = 2 sin A cos B

   ii) sin(A + B) – sin(A – B) = 2 cos A sin B

   iii) cos(A + B) + cos(A – B) = 2 cos A cos B

   iv) cos(A + B) – cos(A – B) = –2 sin A sin B

   v) cos(A – B) – cos(A + B) = 2sin A sin B.

6. i) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

   $= \cos^2 B - \cos^2 A$

   ii) $\cos(A + B)\cos(A - B) = \cos^2 - \sin^2 B$

   $= \cos^2 B - \sin^2 A$

   iii) $\tan(A + B)\tan(A - B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A \tan^2 B}$

   iv) $\cot(A + B)\cot(A - B) = \frac{\cot^2 A \cot^2 B - 1}{\cot^2 B - \cot^2 A}$

v) $\tan(45^\circ + \theta) = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

$= \cot(45^\circ - \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$

vi) $\tan(45^\circ - \theta) = \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

$= \cot(45^\circ + \theta) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

and $\tan(45^\circ + \theta) \,.\, \tan(45^\circ - \theta) = 1$.

7. i) $\sin(A + B + C)$

$= \sum(\cos A \cos B \cos C) - \sin A \sin B \sin C$

ii) $\cos(A + B + C)$

$= \cos A \cos B \cos C - \sum(\sin A \sin B \sin C)$

iii) $\tan(A + B + C) = \dfrac{\sum(\tan A) - \pi(\tan A)}{1 - \sum(\tan A \tan B)}$

VSAQ'S

Simplify the following

1. cos 100° • cos 40° + sin 100° • sin 40°

Sol. L.H.S. =

$= \cos 100^\circ \cdot \cos 40^\circ + \sin 100^\circ \cdot \sin 40^\circ$

$= \cos(100^\circ - 40^\circ) = \cos 60^\circ = \dfrac{1}{2}$ = R.H.S.

2. $\tan\left(\dfrac{\pi}{4} + \theta\right) \cdot \tan\left(\dfrac{\pi}{4} - \theta\right)$

Sol. $\dfrac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} \cdot \dfrac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$

$= \dfrac{(1 + \tan\theta)}{(1 - \tan\theta)} \cdot \dfrac{(1 - \tan\theta)}{(1 + \tan\theta)} = 1$

3. tan 75° + cot 75°

Sol. $\tan 75^\circ = 2 + \sqrt{3}$

$\cot 75^\circ = 2 - \sqrt{3}$

$\therefore \tan 75^\circ + \cot 75^\circ = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$

4. Express $\frac{(\sqrt{3}\cos 25° + \sin 25°)}{2}$ as a sine of an angle.

Sol. $\frac{(\sqrt{3}\cos 25° + \sin 25°)}{2}$

$= \frac{\sqrt{3}}{2}\cos 25° + \frac{1}{2}\sin 25°$

$= \sin 60° \cos 25° + \cos 60° \sin 25°$

$= \sin(60° + 25°) = \sin 85°$

5. tan θ in terms of tan α, if sin(θ + α) = cos (θ + α).

Sol. sin(θ + α) = cos (θ + α)

$\frac{\sin(\theta + \alpha)}{\cos(\theta + \alpha)} = 1$

$\tan(\theta + \alpha) = 1$

$\frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = 1$

$\tan\theta + \tan\alpha = 1 - \tan\theta\tan\alpha$

$\tan\theta + \tan\theta\tan\alpha = 1 - \tan\alpha$

$\tan\theta[1 + \tan\alpha] = 1 - \tan\alpha$

$\therefore \tan\theta = \frac{1 - \tan\alpha}{1 + \tan\alpha}$

6. If $\tan\theta = \frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$ and θ is in the third quadrant, find θ.

Sol. $\tan\theta = \frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$

$= \frac{\cos 11°\left[1 + \frac{\sin 11°}{\cos 11°}\right]}{\cos 11°\left[1 - \frac{\sin 11°}{\cos 11°}\right]}$

$= \frac{1 + \tan 11°}{1 - \tan 11°}$

$= \frac{\tan 45° + \tan 11°}{1 - \tan 45° \tan 11°}$ $(\because \tan 45° = 1)$

$\tan\theta = \tan(45° + 11°)$

$= \tan 56°$

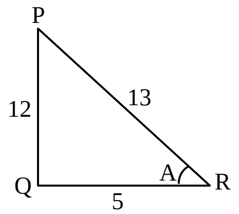
$= \tan(180° + 56°)$

$= \tan 236°$

$\theta = 236°$

7. If 0° < A, B < 90°, such that $\cos A = \frac{5}{13}$ and $\sin B = \frac{4}{5}$, find the value of sin (A – B).

Sol. $\cos A = \frac{5}{13}$ and $\sin B = \frac{4}{5}$

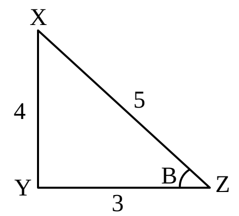


$PQ^2 = PR^2 - QR^2$

$= (13)^2 - 5^2 = 169 - 25 = 144$

$PQ = 12$

$\cos A = \frac{5}{13}, \sin A = \frac{12}{13}$



$YZ^2 = XZ^2 - XY^2$

$= 5^2 - 4^2 = 24 - 16 = 9$

$YZ = 3$

$\sin B = \frac{4}{5}, \cos B = \frac{3}{5}$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$= \frac{12}{13} \times \frac{3}{5} - \frac{5}{13} \times \frac{4}{5} = \frac{36}{65} - \frac{20}{65} = \frac{36-20}{65} = \frac{16}{65}$

8. What is the value of tan 20° + tan 40° + $\sqrt{3}$ tan 20° tan 40°?

Sol. tan 20° + tan 40° + $\sqrt{3}$ tan 20° tan 40°

Consider 20° + 40° = 60°

Tan(20° + 40°) = tan 60°

$\frac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°} = \sqrt{3}$

$\tan 20° + \tan 40° = \sqrt{3}(1 - \tan 20° \tan 40°)$

$\tan 20° + \tan 40° = \sqrt{3} - \sqrt{3} \tan 20° \tan 40°$

$\tan 20° + \tan 40° + \sqrt{3} \tan 20° \tan 40° = \sqrt{3}$

9. Find the value of tan 56° – tan 11° – tan 56° tan 11°.

Sol. We have 56° – 11° = 45°

$$\tan(56°-11°)=\tan 45°$$

$$\frac{\tan 56°-\tan 11°}{1+\tan 56°\tan 11°}=1$$

$$\tan 56°-\tan 11°=1+\tan 56°\tan 11°$$

$$\tan 56°-\tan 11°-\tan 56°\tan 11°=1$$

10. Evaluate $\Sigma\frac{\sin(C-A)}{\cos C\sin A}$ if none of sin A, sin B, sin C is zero.

Sol. $\Sigma\frac{\sin(C-A)}{\sin C\sin A}=\Sigma\frac{\sin C\cos A-\cos C\sin A}{\sin C\sin A}$

$$=\Sigma\frac{\sin C\cos A}{\sin C\sin A}-\frac{\cos C\sin A}{\sin C\sin A}$$

$$=\Sigma\cot A-\cot C$$

$$=\cot A-\cot C+\cot B-\cot A+\cot C-\cot B=0$$

11. tan 72° = tan 18° + 2 tan 54°

Sol. 72° – 18° = 54°

Take tan on both sides

$$\tan(72°-18°)=\tan 54°$$

$$\frac{\tan 72°-\tan 18°}{1+\tan 72°\tan 18°}=\tan 54°$$

$$\frac{\tan 72°-\tan 18°}{1+\tan(90-18)\tan 18°}=\tan 54°$$

$$\frac{\tan 72°-\tan 18°}{1+\tan 18°\tan 18°}=\tan 54°$$

$$\frac{\tan 72°-\tan 18°}{1+1}=\tan 54°$$

$$\frac{\tan 72°-\tan 18°}{2}=\tan 54°$$

$$\tan 72°-\tan 18°=2\tan 54°$$

$$\tan 72°=\tan 18°+2\tan 54°$$

12. sin 750° cos 480° + cos 120° cos 60° = $\frac{-1}{2}$

Sol. sin 750° = sin(2·360° + 30°) = sin 30° = $\frac{1}{2}$

cos 480° = cos(360° + 120°) = cos 120°

= 120° = cos(180° – 60°)

$= -\cos 60° = -\frac{1}{2}$

$\cos 120° = -\frac{1}{2}$

$\cos 60° = \frac{1}{2}$

$\therefore$ L.H.S. =

= sin 750° cos 480° + cos 120° cos 60°

$= \frac{1}{2}\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\frac{1}{2}$

$= -\frac{1}{4} - \frac{1}{4} = \frac{-2}{4} = \frac{-1}{2} = \text{R.H.S.}$

13. $\cos A + \cos\left(\frac{4\pi}{3} - A\right) + \cos\left(\frac{4\pi}{3} + A\right) = 0$

Sol. Consider $\cos\left(\frac{4\pi}{3} - A\right) + \cos\left(\frac{4\pi}{3} + A\right)$

$= \cos(240° + A) + \cos(240° - A)$

$= \cos 240° \cos A - \sin 240° \sin A$

$\quad + \cos 240° \cos A + \sin 240° \sin A$

$= 2\cos 240° \cos A$

$= 2\cos(180° + 60°)\cos A$

$= -2\cos 60° \cos A$

$= -2 \times \frac{1}{2}\cos A = -\cos A$

14. $\cos^2\theta + \cos^2\left(\frac{2\pi}{3} + \theta\right) + \cos^2\left(\frac{2\pi}{3} - \theta\right) = \frac{3}{2}$

Sol. $\cos^2\left(\frac{2\pi}{3} + \theta\right) + \cos^2\left(\frac{2\pi}{3} - \theta\right)$

$$= \cos^2(60° + \theta) + \cos^2(60° - \theta)$$

$$= [\cos 60° \cos\theta - \sin 60° \sin\theta]^2 + [\cos 60° \cos\theta + \sin 60° \sin\theta]^2$$

$$= 2(\cos^2 60° \cos^2\theta + \sin^2 60° \sin^2\theta)[\because (a+b)^2 + (a-b)^2 = 2(a^2 + b^2)]$$

$$= 2\left[\left(\frac{1}{2}\right)^2 \cos^2\theta + \left(\frac{\sqrt{3}}{2}\right)^2 \sin^2\theta\right]$$

$$= 2\left[\frac{1}{4}\cos^2\theta + \frac{3}{4}\sin^2\theta\right]$$

$$= \frac{2}{4}[\cos^2\theta + 3\sin^2\theta]$$

$$= \frac{1}{2}\cos^2\theta + \frac{3}{2}\sin^2\theta$$

$$\therefore \text{L.H.S.} = \cos^2\theta + \frac{1}{2}\cos^2\theta + \frac{3}{2}\sin^2\theta$$

$$= \frac{3}{2}\cos^2\theta + \frac{3}{2}\sin^2\theta$$

$$= \frac{3}{2}\left[\cos^2\theta + \sin^2\theta\right]$$

$$= \frac{3}{2} \quad (\because \cos^2\theta + \sin^2\theta = 1)$$

$$= \text{R.H.S.}$$

15. Evaluate $\sin^2 82\frac{1°}{2} - \sin^2 22\frac{1°}{2}$**.**

Sol. Put $A = \sin^2 82\frac{1°}{2}$ and $B = \sin^2 22\frac{1°}{2}$, then

$$\sin^2 82\frac{1°}{2} - \sin^2 22\frac{1°}{2}$$

$$= \sin^2 A - \sin^2 B$$

$$= \sin(A+B)\sin(A-B)$$

$$= \sin 105° \sin 60°$$

$$= \sin(90° + 15°)\sin 60°$$

$$= \cos 15° \sin 60°$$

$$= \frac{1+\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

16. Prove that $\sin^2 52\frac{1}{2}^0 - \sin^2 22\frac{1}{2}^0 = \frac{\sqrt{3}+1}{4\sqrt{2}}$

17. $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)$

Sol. $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)$

$[\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)]$

$= \sin\left(\frac{\pi}{8}+\frac{A}{2}+\frac{\pi}{8}-\frac{A}{2}\right)\sin\left(\frac{\pi}{8}+\frac{A}{2}-\frac{\pi}{8}+\frac{A}{2}\right)$

$= \sin\left(\frac{2\pi}{8}\right)\sin\left(\frac{2A}{2}\right)$

$= \sin\frac{\pi}{4}\sin A = \sin 45^\circ \sin A = \frac{1}{\sqrt{2}}\sin A$

18. $\cos^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$

Sol. $\cos^2 52\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$

$[\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)]$

$= \cos\left(52\frac{1}{2}^\circ + 22\frac{1}{2}^\circ\right)\cos\left(52\frac{1}{2}^\circ - 22\frac{1}{2}^\circ\right)$

$= \cos 75^\circ \cos 30^\circ$

$= \frac{\sqrt{3}}{2}(\cos 75^\circ)$

$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{3-\sqrt{3}}{4\sqrt{2}}$

19. $\cos^2 112\frac{1}{2}^0 - \sin^2 52\frac{1}{2}^0 = \cos\left(112\frac{1}{2}^0 + 52\frac{1}{2}^0\right)\cos\left(112\frac{1}{2}^0 - 52\frac{1}{2}^0\right)$

$\left(\because \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)\right)$

$= \cos 165^0 . \cos 60^0 = \frac{1}{2}\cos\left(180^0 - 15^0\right) = -\frac{1}{2}\cos 15^0$

$= -\frac{1}{2}\left\{\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$

20. Prove that $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

Solution:

We know that $\tan 3A = \tan(2A + A)$

$$\tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$\tan 3A - \tan 2A \tan A \tan 3 = \tan 2A + \tan A$$

$$\tan 3A - \tan 2 - \tan A = \tan A \tan 2A \tan 3A$$

21. Find the expansion of sin(A + B + C).

Sol. sin(A + B + C)

$= \sin[(A + B) - C]$

$= \sin(A + B)\cos C - \cos(A - B)\sin C$

$= (\sin A \cos B + \cos A \sin B)\cos C - [\cos(A - B)\cos B - \sin A \sin B]\sin c$

$= \sin A \cos B \cos C + \cos A \sin B \cos C - \cos A \cos B \sin C + \sin A \sin B \sin C$

22. Find the expansion of cos (A – B – C).

Sol. cos (A – B – C) = cos[(A – B) – C]

$= \cos(A - B)\cos C + \sin(A - B)\sin C$

$= (\cos A \cos B + \sin A \sin B)\cos C + (\sin A \cos B - \cos A \sin B]\sin C$

$= \cos A \cos B \cos C + \sin A \sin B \cos C + \sin A \cos B \sin C - \cos A \sin B \sin C$

23. For what values of x in the first quadrant $\frac{2\tan x}{1 - \tan^2 x}$ **is positive?**

Sol. $\frac{2\tan x}{1 - \tan^2 x} > 0 \Rightarrow \tan 2x > 0$

$\Rightarrow 0 < 2x < \frac{\pi}{2}$ (since x in the first quadrant)

$\Rightarrow 0 < x < \frac{\pi}{4}$

Therefore, $\frac{2\tan x}{1 - \tan^2 x}$ is positive for $0 < x < \frac{\pi}{4}$

SAQ'S

24. Prove that $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$

Solution:

$$\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$$

$$\left(\sin\frac{\pi}{8}\right)^4+\left\{\sin^2\frac{\pi}{2}-\frac{\pi}{8}\right\}^4+\left\{\sin^2\left(\frac{\pi}{2}+\frac{\pi}{8}\right)\right\}^2+\left\{\sin\left(\pi-\frac{\pi}{8}\right)\right\}^4$$

$$\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}+\sin^4\frac{\pi}{8}$$

$$2\left\{\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}\right\}=2\left\{\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)^2-2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right\}$$

$$=2-4\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$$

$$=2-\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2=2-\sin^2\frac{\pi}{4}=2-\frac{1}{2}=\frac{3}{2}$$

25. Prove that $\cos^4\frac{\pi}{8}+\cos^4 3\frac{\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$

26. Prove that **(i)** $\sin A\sin(60-A)\sin\left(60^0+A\right)=\frac{1}{4}\sin 3A$

(ii) $\cos A\cos(60+A)\cos\left(60^0-A\right)=\frac{1}{4}\cos 3A$

(iii) $\tan A\tan\left(60^0+A\right)\tan\left(60^0-A\right)=\tan 3A$

(iv) $\sin 20^0\sin 40^0\ \sin 60^0\sin 80^0=\frac{3}{16}$

(v) $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{3\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{16}$

27. Prove that $\tan\theta+2\tan 2\theta+4\tan 4\theta+8\cot 8\theta=\cot\theta$

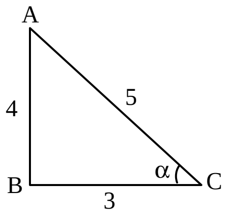
28. If cos $\alpha=\frac{-3}{5}$ **and** $\sin\beta=\frac{7}{25}$**, where** $\frac{\pi}{2}<\alpha<\pi$ **and** $0<\beta<\frac{\pi}{2}$**, then find the values of tan**$(\alpha+\beta)$ **and sin**$(\alpha+\beta)$.

Sol. $\cos\alpha=\frac{-3}{5}$, where $\frac{\pi}{2}<\alpha<\pi$

α in II quadrant

$\sin\beta=\frac{7}{25}$, where $0<\beta<\frac{\pi}{2}$

β in I Quadrant

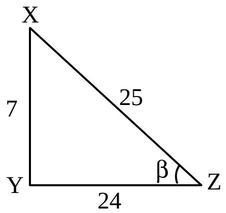


$$AB^2 = AC^2 - BC^2$$

$$= 5^2 - 3^2 = 25 - 9 = 16$$

$$\therefore AB = 4$$

$$\cos\alpha = -\frac{3}{5}, \sin\alpha = \frac{4}{5}, \tan\alpha = -\frac{4}{3}$$



$$YZ^2 = XZ^2 - XY^2$$

$$= 25^2 - 7^2$$

$$= 625 - 49 = 576$$

$$YZ = 24$$

$$\sin\beta = \frac{7}{25}, \cos\beta = \frac{24}{25}, \tan\beta = \frac{7}{24}$$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{-4}{3} + \frac{7}{24}}{1 + \frac{4}{3} \times \frac{7}{24}} = \frac{\frac{-32+7}{4}}{1 + \frac{7}{18}}$$

$$= \frac{\frac{-25}{24}}{\frac{18+7}{18}} = \frac{-25}{24} \times \frac{18}{25} = -\frac{3}{4}$$

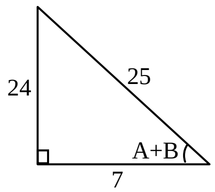
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{4}{5} \times \frac{24}{25} + \frac{-3}{5} \times \frac{7}{25}$$

$$= \frac{96}{125} - \frac{21}{125} = \frac{75}{125} = \frac{3}{5}$$

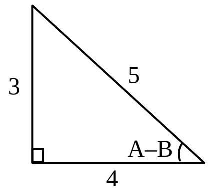
29. If $0 < A < B < \frac{\pi}{4}$ and $\sin(A+B) = \frac{24}{25}$ and $\cos(A-B) = \frac{4}{5}$, then find the value of tan 2A.

Sol. $\sin(A+B) = \frac{24}{25}$



$\tan(A+B) = \frac{24}{7}$

$\cos(A-B) = \frac{4}{5}$



$\tan(A-B) = \frac{3}{4}$

Now 2A = (A + B) + (A – B)

$\tan 2A = \tan\left[(A+B)+(A-B)\right]$

$$= \frac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)}$$

$$= \frac{\frac{24}{7}+\frac{3}{4}}{1-\frac{24}{7}\times\frac{3}{4}} = \frac{96+21}{28-72} = \frac{-117}{44}$$

30. If $A+B, A$ are acute angles such that $\sin\left(A+B\right) = \frac{24}{25}$ and $\tan A = \frac{3}{4}$ find the value of $\cos B$

Solution:

$\sin\left(A+B\right) = \frac{24}{25}$ $\quad\quad \therefore \cos\left(A+B\right) = \frac{7}{25}$

$\tan A = \frac{3}{4}$ $\quad\quad \therefore \sin A = \frac{3}{5}$ $\quad$ & $\quad$ $\cos A = \frac{4}{5}$

$\cos B = \cos\left(A+B-A\right) = \cos\left(A+B\right)\cos A + \sin\left(A+B\right)\sin A$

$$\frac{7}{25}\times\frac{4}{5}+\frac{24}{25}\times\frac{3}{5} = \frac{100}{125} = \frac{4}{5}$$

31. If tan α – tan β = m and cot α – cot β = n, then prove that $\cot(\alpha-\beta)=\frac{1}{m}-\frac{1}{n}$**.**

Sol. We have tan α – tan β = m

$$\frac{1}{\cot\alpha}-\frac{1}{\cot\beta}=m$$

$$\frac{\cot\beta-\cot\alpha}{\cot\alpha\cot\beta}=m$$

$$\therefore \frac{\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}=\frac{1}{m} \quad ...(1)$$

$$\cot\alpha-\cot\beta=n$$

$$-(\cot\beta-\cot\alpha)=n$$

$$\cot\beta-\cot\alpha=-n$$

$$\frac{1}{\cot\beta-\cot\alpha}=-\frac{1}{n} \quad ...(2)$$

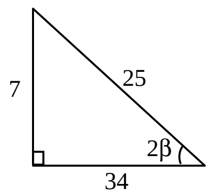
$$\text{L.H.S.}=\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}$$

$$=\frac{\cot\alpha\cot\beta}{\cot\beta-\cot\alpha}+\frac{1}{\cot\beta-\cot\alpha}$$

$$=\frac{1}{m}-\frac{1}{n}\ (\because \text{from}(1)\ \&\ (2))=\text{R.H.S.}$$

32. If tan(α – β) = $\frac{7}{24}$ **and** $\tan\alpha=\frac{4}{3}$**, where α and β are in the firs quadrant prove that α + β = π/2.**

Sol. tan(α – β) = $\frac{7}{24}$ and $\tan\alpha=\frac{4}{3}$



$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$\Rightarrow \frac{\frac{4}{3}-\tan\beta}{1+\frac{4}{3}\tan\beta}=\frac{7}{24}$$

$$\Rightarrow \frac{\frac{4-3\tan\beta}{3}}{\frac{3+4\tan\beta}{3}}=\frac{7}{24}$$

$$\Rightarrow \frac{4-3\tan\beta}{3+4\tan\beta}=\frac{7}{24}$$

$$\Rightarrow 24[4-3\tan\beta]=7(3+4\tan\beta)$$

$$\Rightarrow 96-72\tan\beta=21+28\tan\beta$$

$$\Rightarrow 96-21=28\tan\beta+72\tan\beta$$

$$\Rightarrow 100\tan\beta=75$$

$$\therefore \tan\beta=\frac{75}{100}=\frac{3}{4}$$

$$\therefore \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$=\frac{\frac{4}{3}+\frac{3}{4}}{1-\frac{4}{3}\cdot\frac{3}{4}}=\alpha$$

$$\tan(\alpha+\beta)=\tan\frac{\pi}{2}$$

$$\therefore \alpha+\beta=\frac{\pi}{2}$$

33. If $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{a+b}{a-b}$, then prove that $a\tan\beta=b\tan\alpha$.

Sol. Given that $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{a+b}{a-b}$

By using componendo and dividendo, we get

$$\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)}=\frac{a+b+a-b}{a+b-a+b}=\frac{2a}{2b}=\frac{a}{b}$$

$$\Rightarrow \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)}=\frac{a}{b}$$

$$\Rightarrow \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta-\sin\alpha\cos\beta+\cos\alpha\sin\beta}=\frac{a}{b}$$

$$\Rightarrow \frac{2\sin\alpha\cos\beta}{2\cos\alpha\sin\beta} = \frac{a}{b}$$

$$\Rightarrow \tan\alpha\cot\beta = \frac{a}{b}$$

$$\Rightarrow b\tan\alpha = a\tan\beta$$

hence, $a\tan\beta = b\tan\alpha$

34. **If** $A - B = \frac{3\pi}{4}$**, then show that**

$(1 - \tan A)(1 + \tan B) = 2.$

Sol. $A - B = \frac{3\pi}{4}$

$A - B = 135°$

$\tan(A - B) = \tan 135°$

$= \tan(90° + 45)° = -\cot 45° = -1$

$$\therefore \frac{\tan A - \tan B}{1 + \tan A \tan B} = -1$$

$\tan A - \tan B = -(1 + \tan A \tan B)$

$\tan A - \tan B = -1 - \tan A \tan B$

$\tan A - \tan B + \tan A \tan B = -1$

$\tan B - \tan A - \tan A \tan B = 1$ ...(1)

$\text{L.H.S.} = (1 - \tan A)(1 + \tan B)$

$= 1 + (\tan B - \tan A - \tan A \tan B)$

$= 1 + 1$ ($\because$ from(1))

$= 2 = \text{R.H.S.}$

35. If $A + B + C = \frac{\pi}{2}$ **and if none of A, B, C is an odd multiple of π/2, then prove that cot A + cot B + cot C = cotA cotB cotC.**

Sol. $A + B + C = \frac{\pi}{2}$

$$A + B = \frac{\pi}{2} - C$$

$$\cot(A + B) = \cot\left(\frac{\pi}{2} - C\right)$$

$$\frac{\cot A \cot B - 1}{\cot B + \cot A} = \tan C$$

$$\frac{\cot A \cot B - 1}{\cot B + \cot A} = \frac{1}{\cot C}$$

$\cot C[\cot A \cot B - 1] = \cot B + \cot A$

$\cot A \cot B \cot C - \cot C \cot A + \cot B$

$\cot A \cot B \cot C = \cot A + \cot B + \cot C$

$\therefore \cot A + \cot B + \cot C = \cot A \cot B \cot C$

36. If $A+B+C=\frac{\pi}{2}$ and if none of A, B, C is an odd multiple of π/2, then prove that,

tan A tan B + tan B tan C + tan C tan A = 1.

Sol. $A+B+C=\frac{\pi}{2}$

$$A+B=\frac{\pi}{2}-C$$

$$\tan(A+B)=\tan\left(\frac{\pi}{2}-C\right)$$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=\cot C$$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{1}{\tan C}$$

$$\Rightarrow \tan C[\tan A+\tan B]=1-\tan A\tan B$$

$$\Rightarrow \tan C\tan A+\tan C\tan B=1-\tan A\tan B$$

$$\Rightarrow \tan A\tan B+\tan B\tan C+\tan C\tan A=1$$

37. $\Sigma\frac{\cos(B+C)}{\cos B\cos C}=2$.

Sol. L.H.S. = $\Sigma\frac{\cos(B+C)}{\cos B\cos C}$

$$=\Sigma\frac{\cos B\cos C-\sin B\sin C}{\cos B\cos C}$$

$$=\Sigma\frac{\cos B\cos C}{\cos B\cos C}-\frac{\sin B\sin C}{\cos B\cos C}$$

$$=\Sigma(1-\tan B\tan C)$$

$$=1-\tan B\tan C+1-\tan C\tan A+\quad 1-\tan A\tan B$$

$$=3-(\tan A\tan B+\tan B\tan C+\tan C\tan A)$$

$$=3-1\ (\because \text{from(b)})$$

$$=2=\text{R.H.S.}$$

38. Prove that $\sin^2\alpha+\cos^2(\alpha+\beta)+2\sin\alpha\sin\beta\cos(\alpha+\beta)$ is independent of α.

Sol. Given expression,

$\sin^2\alpha+\cos^2(\alpha+\beta)+2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$$=\sin^2\alpha+1-\sin^2(\alpha+\beta)+2\sin\alpha\sin\beta\cos(\alpha+\beta)$$

$$=1+[\sin^2\alpha-\sin^2(\alpha+\beta)]+2\sin\alpha\sin\beta\cos(\alpha+\beta)$$

$$=1+\sin(\alpha+\alpha+\beta)\sin(\alpha-\alpha-\beta)+2\sin\alpha\sin\beta\cos(\alpha+\beta)$$

$$=1+\sin(2\alpha+\beta)\sin(-\beta)+2\sin\alpha\sin\beta\cos(\alpha+\beta)$$
$$=1-\sin(2\alpha+\beta)\sin\beta+[2\sin\alpha\cos(\alpha+\beta)]\sin\beta$$
$$=1-\sin(2\alpha+\beta)\sin\alpha+[\sin(\alpha+\alpha+\beta)+\sin(\alpha-\alpha-\beta)]\sin\beta$$
$$=1-\sin(2\alpha+\beta)\sin\alpha+[\sin(2\alpha+\beta)-\sin\beta]\sin\beta$$
$$=1-\sin(2\alpha+\beta)\sin\alpha+\sin(2\alpha+\beta)\sin\beta-\sin^2\beta$$
$$=1-\sin^2\beta=\cos^2\beta$$

Thus the given expression is independent of α.

39. Prove that $\cot\frac{\pi}{16}\cdot\cot\frac{2\pi}{16}\cdot\cot\frac{3\pi}{16}...\cot\frac{7\pi}{16}=1$**.**

Sol. $\cot\frac{\pi}{16}\cdot\cot\frac{2\pi}{16}\cdot\cot\frac{3\pi}{16}...\cot\frac{7\pi}{16}$

$$=\left(\cot\frac{\pi}{16}\cdot\cot\frac{7\pi}{16}\right)\left(\cot\frac{2\pi}{16}\cdot\cot\frac{6\pi}{16}\right)\left(\cot\frac{3\pi}{16}\cdot\cot\frac{5\pi}{16}\right)\cdot\cot\frac{4\pi}{16}$$
$$=\left[\cot\frac{\pi}{16}\cdot\cot\left(\frac{\pi}{2}-\frac{\pi}{16}\right)\right]\left[\cot\frac{2\pi}{16}\cdot\cot\left(\frac{\pi}{2}-\frac{2\pi}{16}\right)\right]\left[\cot\frac{3\pi}{16}\cdot\cot\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)\right]\cdot\cot\frac{\pi}{4}$$
$$=\left(\cot\frac{\pi}{16}\cdot\tan\frac{\pi}{16}\right)\left(\cot\frac{2\pi}{16}\cdot\tan\frac{2\pi}{16}\right)\left(\cot\frac{3\pi}{16}\cdot\tan\frac{3\pi}{16}\right)\cdot 1$$
$$=1\times1\times1\times1=1$$

40. Prove that tan 70° – tan 20° = 2 tan 50°.

Sol. tan 50° = tan(70° – 20°)

$$=\frac{\tan 70°-\tan 20°}{1+\tan 70°\tan 20°}$$
$$\Rightarrow \tan 70°-\tan 20°$$
$$=\tan 50°(1+\tan 70°\cdot\tan 20°)$$
$$=\tan 50°(1+\tan 70°\cdot\tan(90°-70°)]$$
$$=\tan 50°[1+\tan 70°\cdot\cot 70°]$$
$$=\tan 50°[1+1]$$
$$=2\tan 50°$$
$$\therefore \tan 70°-\tan 20°=2\tan 50°$$

41. If A + B = 45°, then prove that

i) (1 + tan A) (1 + tan B) = 2

ii) (cot A – 1)(cot B – 1) = 2.

Sol. i) A + B = 45°

$\Rightarrow \tan(A+B) = \tan 45° = 1$

$\Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \tan A + \tan B + \tan A \tan B = 1 ...(1)$

Now, $(1+\tan A)(1+\tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2$

(from(1))

ii) A + B = 45° $\Rightarrow$ cot(A + B) = cot 45° = 1

$\Rightarrow \dfrac{\cot A \cot B - 1}{\cot B + \cot A} = 1$

$\Rightarrow \cot A \cot B - 1 = \cot A + \cot B$

$\Rightarrow \cot A \cot B - \cot A - \cot B = 1 ...(2)$

Now, $(\cot A - 1)(\cot B - 1) = \cot A \cot B - \cot A - \cot B + 1 = 2$

(from(2))

42. If A, B, C are the angles of a triangle and if none of them is equal to π/2, then prove that i) tan A + tan B + tan C = tan A tan B tan C

ii) cotA cotB + cotB cotC + cot C cot A =1

Sol. i) Given A + B + C = π

$\Rightarrow A + B = \pi - C$

$\Rightarrow \tan(A+B) = \tan(\pi - C)$

$\Rightarrow \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\Rightarrow \tan A + \tan B = -\tan C(1 - \tan A \tan B)$

$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

ii) Replacing tan A by $\dfrac{1}{\cot A}$ etc., in (i) above, we get

$\dfrac{1}{\cot A} + \dfrac{1}{\cot B} + \dfrac{1}{\cot C} = \dfrac{1}{\cot A \cot B \cot C}$

$\Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

LAQ'S

43. Let ABC be a triangle such that
$\cot A + \cot B + \cot C = \sqrt{3}$. Then prove that ABC is an equilateral triangle.

Sol. Given that A + B + C = 180°

We get $\Sigma \cot A \cot B = 1$

Now, $\Sigma(\cot A - \cot B)^2 = \Sigma\cot^2 A + \cot^2 B - 2 \cot A \cot B$

$= 2\cot^2 A + 2\cot^2 B + 2\cot^2 C - 2\cot A\cot B - 2\cot B\cot C - 2\cot C\cot A$

(on expanding)

$= 2\{(\cot A + \cot B + \cot C)^2 - 2(\cot A\cot B) - 2\cot B\cot C - 2\cot C\cot A\}$

$-2(\cot A\cot B + \cot B\cot C + \cot C\cot A)$

$= 2(\cot A + \cot B + \cot C)^2 - 6(\cot A\cot B + \cot B\cot C + \cot C\cot A)$

$= 2\cdot 3 - 6 = 0$

$\Rightarrow \cot A = \cot B = \cot C$

$\Rightarrow \cot A = \cot B = \cot C = \dfrac{\sqrt{3}}{3} = \dfrac{1}{\sqrt{3}}$

(since $\cot A + \cot B + \cot C = \sqrt{3}$)

$\Rightarrow A + B + C = 60°$

(Since each angle lies in the interval [0,180°]