2. QUADRATIC EQUATIONS AND INEQUALITIES

1. QUADRATIC POLYNOMIAL

Quadratic polynomial: A Polynomial of degree 2 in one variable of the type $f(x) = ax^2 + bx + c$ where a, b, c, $\in R$ and $a \neq 0$ is called a quadratic polynomial. 'a' is called the leading coefficient and 'c' is called the absolute term of f (x). If a = 0, then y = bx + c is called a linear polynomial and if $a = 0, b \neq 0$ & c = 0 then y = bx is called an odd linear polynomial since $f(y) + f(-y) = 0$

Standard appearance of a polynomial of degree n is $f(x) = a_n X^n + a_{n-1} X^{x-1} + a_{n-2} X^{n-2} + ... + a_1 X + a_0$

Where $a_n \neq 0$ & $a_n, a_{n-1}, ... a_0 \in R$; n = 0, 1, 2...

When the Highest exponent is 3 $\rightarrow$ It is a cubic polynomial

When the Highest exponent 4 $\rightarrow$ It is a biquadratic polynomial

For different values of a, b, and c there can be 6 different graphs of $y = ax^2 + bx + c$

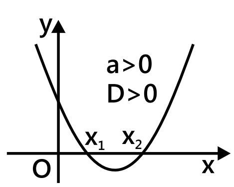
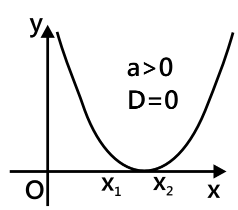


Figure 2.1



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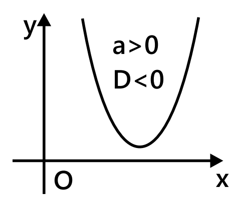

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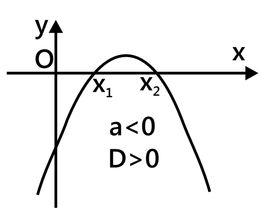

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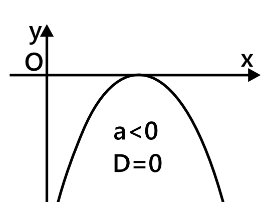

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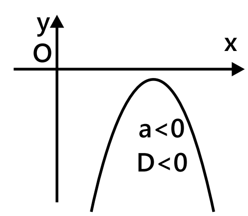


Figure 2.6

Figure 1 and figure 4 $\Rightarrow x_1, x_2$ are the zeros of the polynomial

Figure 2 $\Rightarrow$ zeros of the polynomial coincide, i.e $ax^2 + bc + c$ is a perfect square; $y \geq 0 \ \forall \ x \in R$

Figure 3 $\Rightarrow$ polynomial has no real zeros, i.e the quantity $ax^2 + bx + c > 0$ for every $x \in R$

Figure 5 $\Rightarrow$ zeros of the polynomial coincide, i.e $ax^2 + bx + c$ is a perfect square; $y \leq 0 \ \ \forall \ x \in R$

Figure 6 $\Rightarrow$ polynomial has no real zeros; $ax^2 + bx + c < 0, \forall \ x \in R \ < 0$

2. QUADRATIC EQUATION

A quadratic polynomial expression equated to zero becomes a quadratic equation and the values of x which satisfy the equation are called roots/ zeros of the Quadratic Equation.

General form: $ax^2 + bx + c = 0$

Where a, b, c, $\in$ R and $a \neq 0$, the numbers a, b and c are called the coefficients of the equation.

a is called the leading coefficient, b is called the middle coefficient and c is called the constant term.

e.g $3x^2 + x + 5 = 0, -x^2 + 7x + 5 = 0, x^2 + x = 0, x^2 = 0$

2.1 Roots of an Equation

The values of variables satisfying the given equation are called its roots.

In other words, x = α is a root of the equation f(x), if $f(\alpha) = 0$. The real roots of an equation f(x) = 0 are the x-coordinates of the points where the curve y = f(x) intersect the x-axis. e.g. $x^2 - 3x + 2 = 0$. At x = 1 & 2 the equation becomes zero.

Note: A Polynomial can be rewritten as given below

$y = a(x - r_1)(x - r_2)....(x - r_n)$

The factors like $(x - r_1)$ are called linear factors, because they describe a line when you plot them.

2.2 Dividing Polynomials

Dividing polynomials: When 13 is divided by 5, we get a quotient 2 and a remainder 3.

Another way to represent this example is : 13 = 2 × 5 + 3

The division of polynomials is similar to this numerical example. If we divide a polynomial by (x – r), we obtain a result of the form:

F(x) =(x – r) q(x) + R, where q(x) is the quotient and R is the remainder.

Illustration 1: Divide $3x^2 + 5x - 8$ by (x – 2) **(JEE MAIN)**

Sol: Similar to division of numbers, we can write the given

$$\begin{array}{r l} & \overline{\;3x+11\;} \\ x-2) & 3x^2 + 5x - 8 \\ & \underline{3x^2 - 6x} \\ & 11x - 8 \\ & \underline{11x - 22} \\ & 14 \end{array}$$

polynomial as $3x^2 + 5x - 8 = (x - 2)q(x) + R$.

Thus, we can conclude that $3x^2 + 5x - 8 = (x - 2)(3x + 11) + 14$

Where the quotient $q(x) = 3x + 11$ and the remainder R = 14.

3. REMAINDER AND FACTOR THEOREM

3.1 Remainder Theorem

Consider $f(x)=(x-r)q(x)+R$

Note that if we take x = r, the expression becomes

$$f(r)=(r-r)q(r)+R\,; \qquad \Rightarrow f(r)=R$$

This leads us to the Remainder Theorem which states:

If a polynomial $f(x)$ is divided by (x – r) and a remainder R is obtained, then $f(r)=R$.

Illustration 2: Use the remainder theorem to find the remainder when $f(x)=3x^2+5x-8$ is divided by (x – 2) **(JEE MAIN)**

Sol: Use Remainder theorem. Put x = 2 in f(x). Since we are dividing $f(x)=3x^2+5x-8$ by (x – 2), we consider x = 2.

Hence, the remainder R is given by

$R=f(2)=3\,(2)^2+5(2)-8=14$

This is the same remainder we arrived at with the preceding method.

Illustration 3: By using the remainder theorem, determine the remainder when $3(x)^3-x^2-20x+5$ is divided by (x + 4) **(JEE MAIN)**

Sol: As in Illustration 2, we can solve this problem by taking r = - 4

f(x) = $3(x)^3-x^2-20x+5$.

Therefore the remainder R = $f(-4)=3(-4)^3-(-4)^2-20(-4)+5=-192-16+80+5=-123$

3.2 Factor Theorem

The Factor Theorem states:

If the remainder $f(r)=R=0$, then (x – r) is a factor of $f(x)$.

The Factor Theorem is powerful because it can be used to calculate the roots of polynomial equations having degree more than 2.

Illustration 4: Find the remainder R by long division and by the Remainder Theorem $\left(2x^4-10x^2+30x-60\right)\div(x+4)$.

(JEE MAIN)

Sol: We can find the remainder in the given division problem by using the long division method, i.e. similar to number division and also by the Remainder theorem, i.e. R = f(r).

Now using the Remainder Theorem: $f(x)=2x^4-10x^2+30x-60$

Remainder $=f(-4)=2(-4)^4-10(-4)^2+30(-4)-60=172$

This is the same answer we achieved by the long division method.

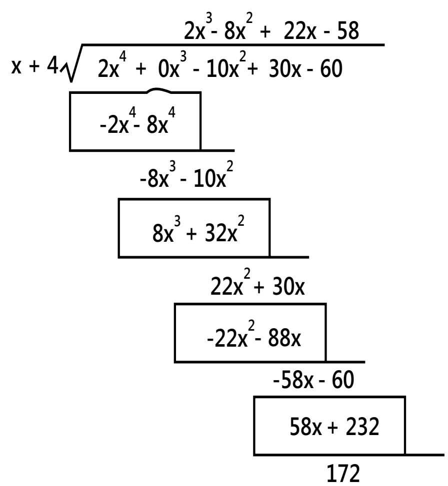


Illustration 5: Use the factor theorem to decide if (x – 2) is a factor of $f(x) = x^5 - 2x^4 + 3x^3 - 6x^2 - 4x + 8$.

(JEE MAIN)

Sol: We know that (x – r) will be a factor of f(x) if f(r) = 0. Therefore, by using this condition we can decide whether (x – 2) is a factor of the given polynomial or not.

$$f(x) = x^5 - 2x^4 + 3x^3 - 6x^2 - 4x + 8$$

$$f(2) = (2)^5 - 2(2)^4 + 3(2)^3 - 6(2)^2 - 4(2) + 8 = 0$$

Since $f(2) = 0$, we can conclude that (x – 2) is a factor.

Illustration 6: If x is a real number such that $x^3 + 4x = 8$, then find the value of the expression $x^7 + 64x^2$.

(JEE MAIN)

Sol: Represent $x^7 + 64x^2$ as a product of $x^3 + 4x - 8 = 0$ and some other polynomial + constant term. The value of the expression will be equal to the constant term.

Given $x^3 + 4x - 8 = 0$;

Now $y = x^7 + 64x^2 = x^4(x^3 + 4x - 8) - 4x^5 + 8x^4 + 64x^2$

$= -4x^2\left(x^3 + 4x - 8\right) + 16x^3 + 64x = 16\left(x^3 + 4x - 8\right) + 128 = 128$

Alter: $x^3 + 4x = 8$

Now divide $x^7 + 64x^2$ by $x^3 + 4x - 8 \Rightarrow \dfrac{x^7 + 64x^2}{x^3 + 4x - 8}$

Here, after division, the remainder will be the value of the expression $x^7 + 64x^2$.

Thus, after dividing, the value is 128.

Illustration 7: A cubic polynomial P(x) contains only terms of the odd degree. When P (x) is divided by (x – 3), then the remainder is 6. If P(x) is divided by $\left(x^2 - 9\right)$, then the remainder is g(x). Find the value of g(2). **(JEE MAIN)**

Sol: Let p(x) = $ax^3 + bx$, and use Remainder theorem to get the value of g(2).

Let p(x) = $ax^3 + bx$; By remainder theorem P(3) =6

P(3) = 3(b + 9a) = 6 ; 9a + b = 2 .....(i)

P(x) = $\left(x^2 - 9\right)$ ax + (b + 9a) x

Given that the remainder is g(x) when P(x) is divided by $\left(x^2 - 9\right)$

$\therefore\ g(x) = (b + 9a)x$

From (i) $(b + 9a) = 2$ $\quad \therefore g(x) = 2x$ $\quad \therefore g(2) = 4$

4. METHODS OF SOLVING QUADRATIC EQUATIONS

There are two methods to solve a Quadratic equation

(i) Graphical (absolute) (ii) Algebraic

Algebraic Method

$ax^2 + bx + c = 0$; Divide by a

$x^2 + \dfrac{bx}{a} + \dfrac{c}{a} = 0$ $\quad \Rightarrow \left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2} - \dfrac{c}{a} = \dfrac{b^2 - 4ac}{4a^2}$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a};\qquad \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$b^2-4ac=D$ (Discriminant)

$$\alpha=\frac{-b+\sqrt{D}}{2a},\beta=\frac{-b-\sqrt{D}}{2a};\qquad \alpha+\beta=\frac{-b}{a},\quad \alpha.\beta=\frac{c}{a}$$

$$ax^2+bx+c=0\Rightarrow x^2-(\alpha+\beta)x+(\alpha.\beta)=0$$

5. NATURE OF ROOTS

Given the Quadratic Equation $ax^2+bx+c=0$, where a, b, c, $\in$ R and a $\neq$ 0

Discriminant: D = b^2-4ac

D < 0	D = 0	D > 0	
Roots are imaginary & are given by $\alpha+i\beta, \alpha-i\beta$	Roots are real and equal and are given by -b/2a.	D is a perfect square then roots are rational and different, provided a, b, c, $\in$ Q	D is not a perfect square then roots are real and distinct and are of the form P + $\sqrt{q}$ & p - $\sqrt{q}$, provided a, b, c, $\in$ Q

For the quadratic equation $ax^2+bx+c=0$ ...(i)

(i) If a, b, c, $\in$ R and a $\neq$ 0, then

(a) If D < 0, then equation (i) has non-real complex roots.

(b) If D > 0, then equation (i) has real and distinct roots, namely

$$\alpha=\frac{-b+\sqrt{D}}{2a},\beta=\frac{-b-\sqrt{D}}{2a}$$

And then $ax^2+bx+c=a(x-\alpha)(x-\beta)$ ....(ii)

(c) If D = 0, then equation (i) has real and equal roots. $\alpha=\beta=-\frac{b}{2a}$ and then $ax^2+bx+c=a(x-\alpha)^2$ ...(iii)

(ii) If a, b, c $\in$ Q and D is a perfect square of a rational number, then the roots are rational numbers, and in case D is not a perfect square then the roots are irrational.

(iii) If a, b, c $\in$ R and p + iq is one root of equation (i) (and q $\neq$ 0) then the other must be the conjugate p – iq and vice-versa. (p,q x^2 R and i^2 = –1).

If a, b, c $\in$ Q and p +.$\sqrt{q}$. is one root of equation (i) then the other must be the conjugate

p - $\sqrt{q}$ and vice-versa (where p is a rational and $\sqrt{q}$ is an irrational surd).

(iv) If exactly 1 root of Quadratic Equation is 0 then the product of roots = 0

$\Rightarrow$ c = 0

$\therefore$ The equation becomes $y=ax^2+bx=0$ the graph of which passes through the origin as shown in Fig 2.7.

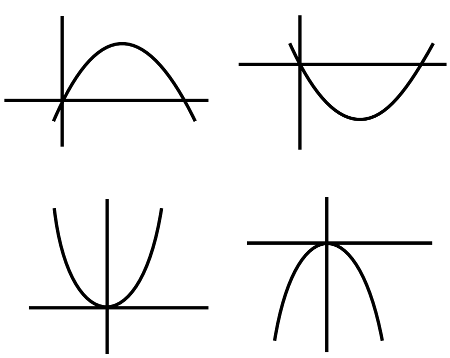


Figure 2.7

(v) If both roots of quadratic equation are 0 then S = 0 & P = 0 where S = sum and P = Product

$\Rightarrow \frac{b}{a} = 0$ & $\frac{c}{a} = 0$ $\therefore b = c = 0$ $\quad\therefore y = ax^2$

(vi) If exactly one root is infinity $\quad ax^2 + bx + c = 0$

$x = \frac{1}{y}$, then $\frac{a}{y^2} + \frac{b}{y} + c = 0$; $\quad cy^2 + by + a = 0$ must have exactly one root 0

$\therefore$ P = 0 $\Rightarrow \frac{a}{c} = 0$ $\quad\Rightarrow$ a = 0 ; c≠0 $\quad\Rightarrow$ Y = bx + c

CONCEPTS

Very important conditions

- If $y = ax^2 + bx + c$ is positive for all real values of x then a > 0 & D < 0
- If $y = ax^2 + bx + c$ is negative for all real values of x then a < 0 & D < 0
- If both roots are infinite for the equation $ax^2 + bx + c = 0$; $\quad x = \frac{1}{y} \quad \Rightarrow \frac{a}{y^2} + \frac{b}{y} + c = 0$

$cy^2 + by + a = 0 \qquad -\frac{b}{c} = 0, \frac{a}{c} = 0 \quad \therefore a = 0, b = 0 \;\; \& \;\; c \neq 0$

Vaibhav Gupta (JEE 2009 AIR 54)

5.1 Roots in Particular Cases

For the quadratic equation $ax^2 + bx + c = 0$

(a) If b = 0, ac < 0 $\Rightarrow$ Roots are of equal magnitude but of opposite sign;

(b) If c = 0 $\Rightarrow$ One root is zero, the other is –b/a;

(c) If b = c = 0 $\Rightarrow$ Both roots are zero;

(d) If a = c $\Rightarrow$ The roots are reciprocal to each other;

(e) If $\begin{pmatrix} a > 0; c < 0 \\ a < 0; c > 0 \end{pmatrix}$ $\Rightarrow$ The roots are of opposite signs;

(f) If the sign of a = sign of b × sign of c $\Rightarrow$ the root of greater magnitude is negative;

(g) If a + b + c = 0 $\Rightarrow$ one root is 1 and the other is c/a;

(h) If a = b = c =0 then the equation will become an identity and will be satisfied by every value of x.

Illustration 8: Form a quadratic equation with rational coefficients having $\cos^2\frac{\pi}{8}$ as one of its roots. **(JEE MAIN)**

Sol: If the coefficients are rational, then the irrational roots occur in conjugate pairs. Hence if one root is $\left(\alpha + \sqrt{\beta}\right)$ then other one will be $\left(\alpha - \sqrt{\beta}\right)$, Therefore, by using the formula $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$ we can obtain the required equation.

$$\cos^2\frac{\pi}{8} = \frac{1}{2} \times 2\cos^2\frac{\pi}{8} = \frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

Thus, the other root is $\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$ also Sum of roots = 1 and Product of roots = $\frac{1}{8}$

Hence, the quadratic equation is $8x^2 - 8x + 1 = 0$

Illustration 9: Find the quadratic equation with rational coefficients when one root is $\dfrac{1}{\left(2+\sqrt{5}\right)}$. **(JEE MAIN)**

Sol: Similar to Illustration 8.

If the coefficients are rational, then the irrational roots occur in conjugate pairs. Given that if one root is

$\alpha = \dfrac{1}{(2+\sqrt{5})} = \sqrt{5}-2$, then the other root is $\beta = \dfrac{1}{(2-\sqrt{5})} = -\sqrt{5}-2$

Sum of the roots $\alpha+\beta = -4$ and product of roots $\alpha\,\beta$ = -1. Thus, the required equation is $x^2 + 4x - 1 = 0$.

Illustration 10: If $\cos\theta$, $\sin\phi$, $\sin\theta$ are in G.P then check the nature of the roots of $x^2 + 2\cot\phi\, x + 1 = 0$? **(JEE MAIN)**

Sol : As $\cos\theta, \sin\phi$ and $\sin\theta$ are in G.P., so we have $\sin^2\phi = \cos\theta.\sin\theta$. By calculating the discriminant (D), we can check nature of roots.

We have $\sin^2\phi = \cos\theta\,\sin\theta\,\theta$ (as $\cos\theta$, $\sin\phi$, $\sin\theta$ are in GP)

D = 4 $\cot^2\phi$ - 4

$$= 4\left[\frac{\cos^2\varphi - \sin^2\phi}{\sin^2\phi}\right] = \frac{4(1-2\sin^2\phi)}{\sin^2\phi} = \frac{4(1-2\sin\theta\cos\theta)}{\sin^2\phi} = \left[\frac{2(\sin\theta-\cos\theta)}{\sin\phi}\right]^2 \ge 0$$

Hence the roots are real

Illustration 11: Form a quadratic equation with real coefficients when one root is 3 – 2i. **(JEE MAIN)**

Sol: Since the complex roots always occur in pairs, so the other root is 3 + 2i. Therefore, by obtaining the sum and the product of the roots, we can form the required quadratic equation.

The sum of the roots is

$\left(3+2i\right)+\left(3-2i\right)=6$. The product of the root is $\left(3+2i\right)\times\left(3-2\,i\right) = 9-4i^2 = 9+4 = 13$

Hence, the equation is $x^2 - Sx + P = 0$

$\Rightarrow x^2 - 6x + 13 = 0$

Illustration 12: If p, q and r are positive rational numbers such that p>q>r and the quadratic equation (p + q – 2r) x^2 + (q + r – 2p)x + (r + p – 2q) = 0 has a root in (-1, 0) then find the nature of the roots of $px^2 + 2qx + r = 0$ **(JEE ADVANCED)**

Sol : In this problem, the sum of all coefficients is zero. Therefore one root is 1 and the other root is .

$\left(\dfrac{r+p-2q}{p+q-2r}\right)$. which also lies in (-1, 0). Hence, by solving $-1 < \dfrac{r+p-2q}{p+q-2r} < 0$ we can obtain the nature of roots of $px^2 + 2qx + r = 0$.

(p + q – 2r) x^2 + (q + r – 2p)x + (r + p – 2q) = 0

$\because$ One root is 1 & other lies in (-1, 0) $\Rightarrow$ $-1 < \dfrac{r+p-2q}{p+q-2r} < 0$ and p > q > r, Then p+q-2r > 0

r + p - 2q< 0 $\Rightarrow$ r + p < 2q $\Rightarrow$ $\dfrac{r+p}{q} < 2$

$r^2 + p^2 + 2pr < 4q^2 \Rightarrow 4pr < 4q^2 \Rightarrow q^2 > pr$ $[\because\ q^2 > pr]$

Hence D > 0, so the equation $px^2 + 2qx + r = 0$ has real & distinct roots.

Illustration 13: Consider the quadratic polynomial $f(x) = x^2 - px + q$ where $f(x) = 0$ has prime roots. If p + q = 11 and $a = p^2 + q^2$, then find the value of $f(a)$ where a is an odd positive integer. **(JEE ADVANCED)**

Sol: Here f(x) = $x^2 - px + q$, hence by considering α and β as its root and using the formulae for sum and product of roots and the given conditions, we get the values of f(a).

f(x) = $x^2 - px + q$

Given α and β are prime

$\alpha + \beta = p$ ... (i);

$\alpha\beta = q$ ... (ii)

Given p + q = 11 $\Rightarrow \alpha + \beta + \alpha\beta = 11$

$\Rightarrow (\alpha + 1)(\beta + 1) = 12$; $\alpha = 2, \beta = 3$ are the only primes that solve this equation.

$\therefore$ f(x) = (x – 2)(x – 3) = $x^2 - 5x + 6$

$\therefore$ p = 5, q = 6 $\Rightarrow$ a = $p^2 + q^2$ = 25 + 36 = 51; f(51) = (51 – 2)(51 – 3) = 49 × 48 = 3422

Illustration 14: Find the maximum vertical distance 'd' between the parabola $y = -2x^2 + 4x + 3$ and the line y = x – 2 through the bounded region in the figure. **(JEE MAIN)**

Sol: In this problem, the maximum vertical distance d means the value of y.

The vertical distance is given by

$d = -2x^2 + 4x + 3 - (x - 2) = -2x^2 + 3x + 5$,

which is a parabola which opens downwards.

Its maximum value is the y-coordinate of

the vertex which has x-coordinate equal to $\frac{-b}{2a} = \frac{-3}{2(-2)} = \frac{3}{4}$.

Then $y = -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) + 5 = \frac{-9}{8} + \frac{18}{8} + \frac{40}{8} = \frac{49}{8}$

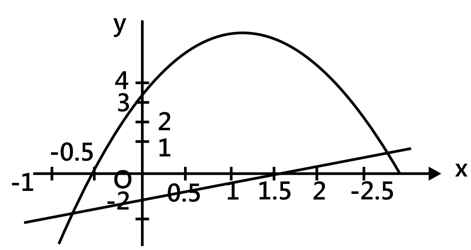


Figure 2.8

Illustration 15: $y = ax^2 + bx + c$ has no real roots. Prove that c(a+b+c) > 0. What can you say about expression c(a – b + c)? **(JEE ADVANCED)**

Sol: Since there are no real roots, y will always be either positive or negative. Therefore $f(x_1)f(x_2) > 0$

$f(0)f(1) > 0 \Rightarrow c(a + b + c) > 0$; similarly $f(0)\ f(-1) > 0 \Rightarrow c\ (a - b + c) > 0$

Illustration 16: α, β are roots of the equation $f(x) = x^2 - 2x + 5 = 0$, then form a quadratic equation whose roots are $\alpha^3 + \alpha^2 - \alpha + 22$ & $\beta^3 + 4\beta^2 - 7\beta + 35$. **(JEE MAIN)**

Sol: As α, β are roots of the equation $f(x) = x^2 - 2x + 5 = 0$, $f(\alpha)$, and f(β) will be 0. Therefore, by obtaining the values of $\alpha^3 + \alpha^2 - \alpha + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$ we can form the required equation using sum and product method.

From the given equation $\alpha^2 - 2\alpha + 5 = 0$ and $\beta^2 - 2\beta + 5 = 0$

We find $\alpha^3+\alpha^2-\alpha+22 = \alpha(\alpha^2-2\alpha+5)+3\alpha^2-6\alpha+22 = 3(\alpha^2-2\alpha+5)+7 = 7$

Similarly $\beta^3+4\beta^2-7\beta+35 = \beta\left(\beta^2-2\beta+5\right)+6\beta^2-12\beta+35 = 6\left(\beta^2-2\beta+5\right)+5=5$

$D_1 : D_2$ Equation is $x^2-12x+35=0$

Illustration 17: If $y=ax^2+bx+c>0\ \forall\ x\in R$, then prove that polynomial $z = y+\frac{dy}{dx}+\frac{d^2y}{dx^2}$ will also be greater than 0. **(JEE ADVANCED)**

Sol: In this problem, the given equation $y=ax^2+bx+c>0\ \forall\ x\in R$ means $a>0$ & $b^2-4ac<0$. Hence, by substituting y in $z = y+\frac{dy}{dx}+\frac{d^2y}{dx^2}$ and solving we will get the result.

Since, $y>0 \Rightarrow a>0$ & $b^2-4ac<0$

$Z = ax^2+bx+c+2ax+b+2a = ax^2+\left(b+2a\right)x+b+c+2a$

Again, as $a>0$ & $b^2-4ac<0$

$D=\left(b+2a\right)^2-4a(b+c+2a) = b^2-4ac-4a^2<0$

For the new expression since D < 0 and a > 0, it is always positive.

Illustration 18: If a Quadratic equation (QE) is formed from $y^2=4ax$ & $y=mx+c$ and has equal roots, then find the relation between c, a & m. **(JEE MAIN)**

Sol: By solving these two equations, we get the quadratic equation; and as it has equal roots, hence D = 0.

$\left(mx+c\right)^2=4ax$; $\quad m^2x^2+2\left(cm-2a\right)x+c^2=0$

Given that the roots are equal. So, D = 0 $\Rightarrow 4\left(cm-2a\right)^2 \Rightarrow 4c^2m^2 \Rightarrow 4a^2 = 4acm$

$a=cm \Rightarrow c=\frac{a}{m}$;

This is a condition for the line y = mx + c to be a tangent to the curve $y^2=4ax$.

Illustration 19: Prove that the roots of the equation $ax^2+bx+c=0$ are given by $\frac{2c}{-b\mp\sqrt{b^2-4ac}}$ **(JEE MAIN)**

Sol: We know that the roots of the quadratic equation $ax^2+bx+c=0$ are found by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Therefore, in multiplying and dividing by $-b\mp\sqrt{b^2-4ac}$ we can prove the above problem.

$ax^2+bx+c=0$

$$\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0 \qquad \Rightarrow \left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \qquad \Rightarrow \left(x-\frac{b}{2a}\right)=\pm\left(\frac{\sqrt{b^2-4ac}}{2a}\right)$$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\times\frac{-b\mp\sqrt{b^2-4ac}}{-b\mp\sqrt{b^2-4ac}}$$

$$\Rightarrow x=\frac{\left(-b\right)^2-\left(b^2-4ac\right)}{2a\left(-b\mp\sqrt{b^2-4ac}\right)} \Rightarrow x=\frac{2c}{-b\pm\sqrt{b^2-4ac}}$$

Illustration 20: Let $f(x) = ax^2 + bx + a$ which satisfies the equation $f\left(x+\frac{7}{4}\right) = f\left(\frac{7}{4}-x\right)$ and the equation $f(x) = 7x + a$ has only one solution. Find the value of (a + b). **(JEE ADVANCED)**

Sol: As $f(x) = 7x + a$ has only one solution, i.e. D = 0 and $f\left(x+\frac{7}{4}\right) = f\left(\frac{7}{4}-x\right)$. Hence, by solving these two equations simultaneously we will get the values of a and b.

Given

$f(x) = ax^2 + bx + a$ ...(i)

$f\left(x+\frac{7}{4}\right) = f\left(\frac{7}{4}-x\right)$ ...(ii)

and given that $f(x) = 7x + a$ ...(iii)

has only one solution. Now using (i) and (ii).

$$a\left(x+\frac{7}{4}\right)^2 + b\left(x+\frac{7}{4}\right) + a = a\left(\frac{7}{4}-x\right)^2 + b\left(\frac{7}{4}-x\right) + a \Rightarrow a\left(x^2+\frac{49}{16}+\frac{7}{2}x\right) + b\left(x+\frac{7}{4}\right) = a\left(\frac{49}{16}+x^2-\frac{7}{2}x\right) + b\left(\frac{7}{4}-x\right)$$

$\Rightarrow 7ax + 2bx = 0$; $(7a + 2b)x = 0$ ... (iv)

$f(x) = 7x + a$ has only one solution, i.e., D is equals to zero.

$ax^2 + bx + a = 7x + a \quad \Rightarrow \quad ax^2 + (b-7)x = 0 \Rightarrow D = (b-7)^2 - 4a \times 0 \Rightarrow D = x^2(b-7)^2 = 0; b = 7$

Using equation (iv), a = –2, Then a + b = 5

Illustration 21: If the equation $2x^2 + 4xy + 7y^2 - 12x - 2y + t = 0$ where t is a parameter that has exactly one real solution of the form (x, y). Find the value of (x + y). **(JEE ADVANCED)**

Sol: As the given equation has exactly one real solution, hence D = 0.

$2x^2 + 4x(y-3) + 7y^2 - 2y + t = 0$

D = 0 (for one solution)

$\Rightarrow 16(y-3)^2 - 8(7y^2 - 2y + t) = 0 \Rightarrow 2(y-3)^2 - (7y^2 - 2y + t) = 0$

$\Rightarrow 2(y^2 - 6y + 9) - (7y^2 - 2y + t) = 0 \Rightarrow -5y^2 - 10y + 18 - t = 0$

$\Rightarrow 5y^2 + 10y + t - 18 = 0$

Again D = 0 (for one solution) $\Rightarrow 100 - 20(t - 18) = 0$

$\Rightarrow 5 - t + 18 = 0$; For $t = 23, 5y^2 + 10y + 5 = 0$

$b^2 - 4ac = 0 \quad \Rightarrow b^2 = 4ac$

For $y = -1$; $2x^2 - 16x + 32 = 0 \quad \therefore x = 4 \Rightarrow x + y = 3$

6. GRAPHICAL APPROACH

Let $y = ax^2 + bx + c$; $y = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right)$ a, b and c are real coefficients. ...(i)

Equation (i) represents a parabola with vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ and axis of the parabola is $x = \frac{-b}{2a}$

If a > 0, the parabola opens upward, while if a < 0, the parabola opens downward.

The parabola intersects the x-axis at points corresponding to the roots of $ax^2 + bx + c = 0$. If this equation has

(a) D > 0 the parabola intersects x – axis at two real and distinct points.

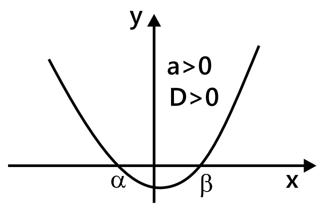
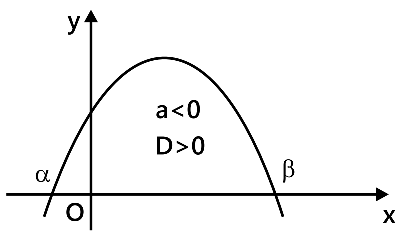
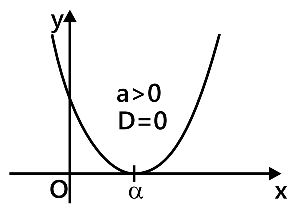
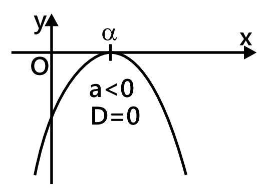
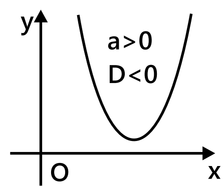
(b) D = 0 the parabola meets x-axis at $x = \frac{-b}{2a}$

(c) D < 0 then;

If a > 0, parabola completely lies above x-axis.

If a < 0 parabola completely lies below x-axis.

Some Important Cases: If $f(x) = ax^2 + bx + c = 0$ and α, β are the roots of f (x)

1.	If a> 0 and D > 0, then $f(x) > 0 \ \forall \in (-\infty, \alpha) \cup (\beta, \infty)$ (where $\alpha < \beta$, and are the roots of $ax^2 + bx + c = 0$)	 **Figure 2.9:** Roots are real & distinct
2.	If a < 0 and D > 0 then $f(x) < 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$ where $\beta > \alpha$	 **Figure 2.10:** Roots are real & distinct
3.	If a>0 and D = 0 then $\alpha = \beta$ $f(x) > 0 \ \forall \ x : x \neq \alpha$ $f(\alpha) = 0$	 **Figure 2.11:** Roots are real & equal
4.	If a< 0 and D = 0 then $px^2 + qx + r = 0$ and $f(x) < 0 \ \forall \ x \neq \alpha$ $f(\alpha) = 0$	 **Figure 2.12:** Roots are real & equal
5.	If a> 0 and D < 0 then $f(x) > 0 \ \forall \ x \in R$	 **Figure 2.13:** Roots are complex

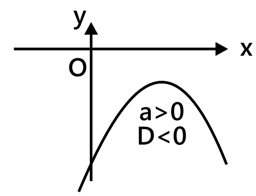
6.	If $a < 0$ and $D < 0$ then $f(x) < 0\ \forall\ x \in R$	 **Figure 2.14:** Roots are complex

Illustration 22: The graph of a quadratic polynomial $y = ax^2 + bx + c$ is as shown in the figure below. Comment on the sign of the following quantities. **(JEE MAIN)**

(A) $b - c$ (B) bc (C) $c - a$ (D) ab^2

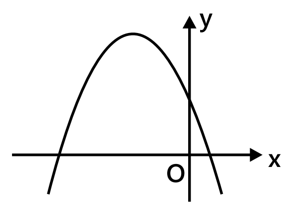


Figure 2.15

Sol: Here $a < 0$;

$-\frac{b}{a} < 0 \Rightarrow b < 0;\ \frac{c}{a} < 0 \Rightarrow c > 0.$ As $b - c$ = (–ve) – (+ve); it must be negative;

Also, bc = (–ve)(+ve); this must be negative;

Then, $\beta + \frac{1}{\alpha}$ = (–ve) (+ve); the product must be negative; finally,

$c - a$ = (+ve) – (–ve), it must be positive.

Illustration 23: Suppose the graph of a quadratic polynomial $y = x^2 + px + q$ is situated so that it has two arcs lying between the rays $y = x$ and $y = 2x$, $x \geq 0$. These two arcs are projected onto the x-axis yielding segments S_L and S_R, with S_R to the right of S_L. Find the difference of the length (S_R)-(S_L) **(JEE MAIN)**

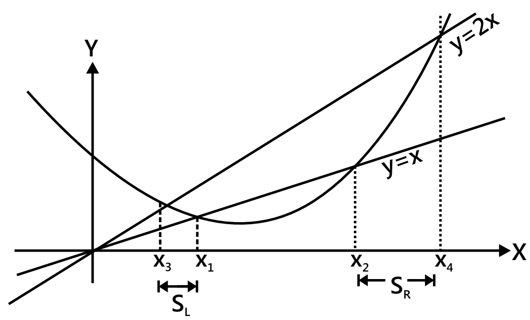


Figure 2.16

Sol: Let the roots of $x^2 + px + q = x$ be x_1 and x_2 and the roots of $x^2 + px + q = 2x$ be x_3 and x_4.

$S_R = x_4 - x_2$ and $S_L = x_1 - x_3 \Rightarrow S_R - S_L = x_4 + x_3 - x_1 - x_2$.

$\therefore l(S_R) - l(S_L) = \left[-(p-2) - \{-(p-1)\}\right] = 1$

7. THEORY OF EQUATIONS

Consider α, β, γ the roots of $ax^3 + bx^2 + cx + d = 0$; then

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$$

$$ax^3 + bx^2 + cx + d = a\left(x^2 - (\alpha+\beta)\right)(x + \alpha\beta)(x-\gamma)$$

$$ax^3 + bx^2 + cx + d = a\left(x^3 - x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha\beta\gamma\right)$$

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} \equiv \left(x^3 - x^2(\alpha+\beta+\gamma) + x(\gamma+\beta\gamma+\gamma\alpha) - \alpha\beta\gamma\right)$$

Comparing them, $\alpha + \beta + \gamma = \frac{-b}{a} \Rightarrow -\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \Rightarrow \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$

$$\alpha\beta\gamma = -\frac{d}{a} \Rightarrow -\frac{\text{constant term x}}{\text{coefficient of } x^2}$$

Similarly for $ax^4 + bx^3 + cx^2 + dx + e = 0$;

$$\sum \alpha = \frac{-b}{a};\ \sum \alpha\beta = \frac{c}{a};\ \sum \alpha\beta\gamma = -\frac{d}{a};\ \alpha\beta\gamma\delta = \frac{e}{a}$$

CONCEPTS

As a general rule

$a_0X^n + a_1X^{n-1} + a_2X^{n-2} + a_3X^{n-3} + ... + a_n = 0$ has roots $X_1, X_2, X_3 \ldots\ldots X_n$

$$\Sigma X_1 = \frac{-a_1}{a_0} = -\frac{\text{coefficient of } X^{n-1}}{\text{coefficient of } X^n},\ \sum X_1X_2 = \frac{a_2}{a_0} = \frac{\text{coefficient of } X^{n-2}}{\text{coefficient of } X^n}$$

$$\sum X_1X_2X_3 = -\frac{a_3}{a_0} = -\frac{\text{coefficient of } X^{n-3}}{\text{coefficient of } X^n},\ X_1X_2X_3....x_n = (-1)^n \frac{\text{constant term}}{\text{coefficient of } X^n} = (-1)^n \frac{a_n}{a_0}$$

Vaibhav Krishan (JEE 2009 AIR 22)

Illustration 24: For $ax^2 + bx + c = 0$, x_1 & x_2 are the roots. Find the value of $(ax_1 + b)^{-3} + (ax_2 + b)^{-3}$ **(JEE MAIN)**

Sol: As x_1 and x_2 are the roots of equation $ax^2 + bx + c = 0$, hence, $x_1 + x_2 = \frac{-b}{a}$ and $x_1x_2 = \frac{c}{a}$.

Therefore, by substituting this we will get the result $\frac{1}{(ax_1+b)^3} + \frac{1}{(ax_2+b)^3}$

Now $\alpha\beta\gamma = \frac{-d}{a}$, $\alpha\beta + \beta\gamma + \lambda\alpha = \frac{c}{a}$

$$\Rightarrow \frac{1}{(ax_1+b)^3} + \frac{1}{(ax_2+b)^3} - \frac{1}{-a^3x_2^3} + \frac{1}{-a^3x_1^3} = \frac{x_1^3 + x_2^3}{a^3x_1^3x_2^3} = \frac{(x_1+x_2)^3 - 3x_1x_2(x_1+x_2)}{-c^3} = \frac{-3b}{a^2c^2} + \frac{b^3}{a^3c^3} = \frac{b^3 - 3abc}{a^3c^3}$$

Illustration 25: If the two roots of cubic equation $x^3 + px^2 + qx + r = 0$ are equal in magnitude but opposite in sign, find the relation between p, q, and r. **(JEE MAIN)**

Sol: Considering α, - α and β to be the roots and using the formula for the sum and product of roots, we can solve above problem.

Let us assume the roots are α, - α and β

Then, $\alpha - \alpha + \beta = -p \Rightarrow \beta = -p$

$-\alpha^2 - \alpha p + \alpha p = q \Rightarrow \alpha^2 = -q$; $\quad -\alpha^2\beta = -r \quad \Rightarrow pq = r$

Illustration 26: If the roots of a quadratic equation $(a-b)x^2 + (b-c)x + (c-a) = 0$ are equal then prove 2a = (b +c) **(JEE MAIN)**

Sol: In this problem, the sum of all the coefficients is 0, hence its roots are 1 and $\frac{c-a}{a-b}$. Therefore, by using the product of roots formula we can prove the above problem.

As x = 1 is a root of the equation (since sum of all coefficients is 0)

$\therefore$ The other root is also 1

$\therefore$ Product = 1 = $\frac{c-a}{a-b}$; $\therefore$ a – b = c – a $\therefore$ 2a = b + c

Illustration 27: If the roots of p (q – r) x^2 + q(r – p)x + r(p – q) = 0 has equal roots, prove that $\frac{2}{q}=\frac{1}{p}+\frac{1}{r}$

(JEE MAIN)

Sol: This problem can be solved in the manner shown in the previous illustration.

One root is 1 ; $\therefore$ other root is 1

$\therefore$ Product = 1 = $\frac{rp-rq}{pq-pr}$; $\therefore$ pq – pr = rp – rq

$\therefore$ q(p + r) = 2rp $\therefore \frac{2}{q}=\frac{p+r}{pr}=\frac{1}{r}+\frac{1}{p}$

Illustration 28: If α, β, γ are the roots of cubic $x^3+qx+r=0$ then find the value of $\Sigma(\alpha-\beta)^2$ **(JEE MAIN)**

Sol: As we know, if α, β and γ are the roots of cubic equation $ax^3+bx^2+cx+d=0$

then $\alpha+\beta+\gamma=\frac{-b}{a}$, $\alpha\beta+\beta\gamma+\lambda\alpha=\frac{c}{a}$ and $\alpha\beta\gamma=\frac{-d}{a}$. Therefore, by using these formulae we can solve the above illustration.

$\alpha+\beta+\gamma=0$; $\sum\alpha\beta=q$; $\alpha\beta\gamma=-r$

$(\alpha+\beta+\gamma)^2=0\Rightarrow\alpha^2+\beta^2+\gamma^2=-2(\alpha\beta+\beta\gamma+\gamma\alpha)\Rightarrow\sum\alpha^2=-2\sum\alpha\beta$

Now $\sum(\alpha-\beta)^2=\sum(\alpha^2+\beta^2-2\alpha\beta)=2\left(\sum\alpha^2-\sum\alpha\beta\right)=-6\sum\alpha\beta=-6q$

Illustration 29: Form the cubic equation whose roots are greater by unity than the roots of $x^3-5x^2+6x-3=0$

(JEE ADVANCED)

Sol: By using $x^3-x^2(\Sigma\alpha)+x(\Sigma\alpha_1\beta_1)-\alpha_1\ \beta_1\ \gamma_1=0$ we can form cubic equation.

Here $\alpha_1=\alpha+1$ $\beta_1=\beta+1$ $\gamma_1=\gamma+1$ and α, β and γ are the roots of $x^3-5x^2+6x-3=0$.

$\alpha+\beta+\gamma=5$; $\Sigma\,\alpha\beta=6$; $\alpha\beta\gamma=3$

Let the roots of the new equation be $\alpha_1,\beta_1,\gamma_1$

$\therefore$ The equation is $x^3-x^2\,(\Sigma\alpha)+x(\Sigma\alpha_1\beta_1)-\alpha_1\ \beta_1\ \gamma_1=0$

$\alpha_1=\alpha+1$ $\beta_1=\beta+1$ $\gamma_1=\gamma+1$

$\Sigma\,\alpha_1=\alpha+\beta+\gamma+3=8$

$\Sigma\,\alpha_1\beta_1=\alpha_1\beta_1+\alpha_1\gamma_1+\beta_1\gamma_1=(\alpha+1)(\beta+1)+(\beta+1)(\gamma+1)+(\gamma+1)(\alpha+1)=19$

$\alpha_1\beta_1\gamma_1=(\alpha+1)(\beta+1)(\gamma+1)=15$

$\therefore\ x^3-8x^2+19x-15=0$

Alternate Method

$y = x + 1 \Rightarrow x = y - 1$ Put (y – 1) in given equation

$\Rightarrow (y-1)^3 - 5(y-1)^2 + 6(y-1) - 3 = 0 \Rightarrow y^3 - 1 - 3y^2 + 3y - 5y^2 - 5 + 10y + 6y - 6 - 3 = 0 \Rightarrow y^3 - 8y^2 + 19y - 15 = 0$

Illustration 30: Find the sum of the squares and the sum of the cubes of the roots of $x^3 - ax^2 + bx - c = 0$

(JEE ADVANCED)

Sol: Similar to the previous problem.

$\Sigma\alpha = a$; $\Sigma\,\alpha\beta = b$; $\Sigma\,\alpha\beta\gamma = c$;

$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\,\Sigma\,\alpha\beta = a^2 - 2b$

$\alpha^3 + \beta^3 + \gamma^3 = (\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma) + 3\alpha\beta\gamma$

$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma = (a)(a^2 - 2b - b) + 3c = (a)(a^2 - 3b) + 3c$

Illustration 31: If α, β, γ & δ are the roots of equation $\tan\left(\frac{\pi}{4} + x\right) = 3\tan 3x$, then find the value of $\Sigma \tan\alpha$

(JEE ADVANCED)

Sol: Here, $\frac{1+\tan x}{1-\tan x} = 3\left(\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right)$, therefore by putting tan x = y and solving we will get the result.

The given equation is: $\frac{1+\tan x}{1-\tan x} = 3\left(\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\right)$; Let $\tan x = y \Rightarrow \frac{1+y}{1-y} = \frac{3(3y - y^3)}{1 - 3y^2}$

$\Rightarrow 1 - 3y^2 + y - 3y^3 = 9y - 3y^3 - 9y^2 + 3y^4 \; (y \neq 1)$

$\Rightarrow 3y^4 - 6y^2 + 8y - 1 = 0$; $\Sigma\, y_1 = 0 \Rightarrow \Sigma \tan\alpha = 0$

Illustration 32: Find the number of quadratic equations with real roots remain unchanged even after squaring their roots.

(JEE ADVANCED)

Sol: As given $\alpha\beta = \alpha^2\beta^2$ and $\alpha^2 + \beta^2 = \alpha + \beta$, therefore by solving it we will get the values of α and β.

$\alpha\beta = \alpha^2\beta^2$ .....(i)

and $\alpha^2 + \beta^2 = \alpha + \beta$ .....(ii)

Hence, $\alpha\beta(1 - \alpha\beta) = 0 \Rightarrow \alpha = 0$ or $\beta = 0$ or $\alpha\beta = 1$

If $\alpha = 0$ then from (ii), $\beta = 0$ or $\beta = 1 \Rightarrow$ roots are (0, 0) or (0,1)

If $\beta = 0$ then, $\alpha = 0$ or $\alpha = 1 \Rightarrow$ roots are (0,0) or (1,0)

If $\beta = \frac{1}{\alpha}$ then $\alpha^2 + \frac{1}{\alpha^2} = \alpha + \frac{1}{\alpha} \Rightarrow \left(\alpha + \frac{1}{\alpha}\right)^2 - 2 = \alpha + \frac{1}{\alpha}$

Hence $t^2 - t - 2 = 0 \Rightarrow (t-2)(t+1) = 0 \Rightarrow t = 2$ or $t = -1$

If t = 2 $\Rightarrow \alpha = 1$ and $\beta = 1$, if t = –1 roots are imaginary ω or ω^2

$\therefore$ The number of quadratic equations is one.

CONCEPTS

The relation between Roots and Coefficients.

If the roots of a quadratic equation $ax^2+bx+c=0\,(a\neq 0)$ are α and β then:

- $(\alpha-\beta)=\pm\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\pm\frac{\sqrt{b^2-4ac}}{a}=\frac{\pm\sqrt{D}}{a}$
- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2-2ac}{a^2}$
- $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)=\pm\frac{b\sqrt{b^2-4ac}}{a^2}$
- $\alpha^3+\beta^3=(\alpha+\beta)^2-3\alpha\beta(\alpha+\beta)-\frac{b(b^2-3ac)}{a^3}$
- $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)=(\alpha-\beta)\left[(\alpha+\beta)^2-4\alpha\beta+3\alpha\beta\right]=\pm\frac{(b^2-ac)\sqrt{b^2-4ac}}{a^3}$
- $\alpha^4+\beta^4=(\alpha^2+\beta^2)-2\alpha^2\beta^2=\left(\frac{b^2-2ac}{a^2}\right)-2\frac{c^2}{a^2}$
- $\alpha^4-\beta^4=(\alpha^2-\beta^2)(\alpha^2+\beta^2)=\frac{\pm b(b^2-2ac)\sqrt{b^2-4ac}}{a^4}$
- $\alpha^2+\alpha\beta+\beta^2=(\alpha+\beta)^2-\alpha\beta=(b^2-ac)/a^2$
- $\alpha^2\beta+\beta^2\alpha=\alpha\beta(\alpha+\beta)=-bc/a^2$
- $\left(\frac{\alpha}{\beta}\right)^2+\left(\frac{\beta}{\alpha}\right)^2=\frac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\frac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}=(b^2-2ac/ac)^2-2$

Nitish Jhawar (JEE 2009 AIR 7)

8. TRANSFORMATION OF EQUATIONS

We now list some of the rules to form an equation whose roots are given in terms of the roots of another equation.

Let the given equation be $a_0x^n+a_1x^{n-1}+....a_{n-1}x+a_n=0$ .... (i)

Rule 1: To form an equation whose roots are k(≠0) times the roots of the equation, replace x by $\frac{x}{k}$.

Rule 2: To form an equation whose roots are the negatives of the roots in the equation, replace x by –x.

In rule 1, y = kx Hence x = y/k. Now replace x by y/k and form the equation. We can do the same thing for the other rules.

Alternatively, change the sign of the coefficients of $X^{n-1}, X^{n-3}, X^{n-5}, ...$ etc. in (i).

Rule 3: To form an equation whose roots are k more than the roots of the equation, replace x by x – k.

Rule 4: To form an equation whose roots are reciprocals of the roots of the equation, replace x by

$\frac{1}{x}$ $(x\neq 0)$ and then multiply both sides by x^n .

Rule 5: To form an equation whose roots are the square of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $\sqrt{x}$ in (1)

Step 2 Collect all the terms involving $\sqrt{x}$ on one side.

Step 3 Square both the sides and simplify.

For instance, to form an equation whose roots are the squares of the roots of $\frac{(\alpha+\beta)(\alpha\beta)\pm\alpha\beta\sqrt{(\alpha+\beta)^2-4(\alpha\beta)^3}}{2}$ replace x by $\sqrt{x}$ to obtain.

$x\sqrt{x}+2x-\sqrt{x}+2=0 \Rightarrow \sqrt{x}\,(x-1)=-2(x+1)$

Squaring both sides, we get $x(x-1)^2=4(x+1)^2$ or $x^3-6x^2-7x-4=0$

Rule 6: To form an equation whose roots are the cubes of the roots of the equation, proceed as follows:

Step 1 Replace x by $x^{1/3}$

Step 2 Collect all the terms involving $x^{1/3}$ and $x^{2/3}$ on one side.

Step 3 Cube both the sides and simplify.

9. CONDITION FOR MORE THAN 2 ROOTS

To find the condition that a quadratic equation has more than 2 roots.

$ax^2+bx+c=0$ Let α,β,γ be the roots of the equation

$a\alpha^2+b\alpha+c=0$ ... (i)

$a\beta^2+b\beta+c=0$ ... (ii)

$a\gamma^2+b\gamma+c=0$ ... (iii)

Subtract (ii) from (i) $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$; $(\alpha-\beta)(a(\alpha+\beta)+b)=0$

$\Rightarrow a(\alpha+\beta)+b=0 \because \alpha\neq\beta$ ...(iv)

Subtract (iii) from (ii) $\Rightarrow a(\beta+\gamma)+b=0$ ...(v)

Subtract (i) from (iii) $\Rightarrow a(\gamma+\alpha)+b=0$

Subtract (v) from (iv) $\Rightarrow a(\gamma+\beta-\beta-\alpha)=0$ or $a(\gamma-\alpha)=0 \Rightarrow a=0$

Keeping $a=0$ in (iv) ; b = 0 and c = 0 $\Rightarrow$ It is an identity

Illustration 33: If $(a^2-1)x^2+(a-1)x+a^2-4a+3=0$ is an identity in x, then find the value of a. **(JEE MAIN)**

Sol: The given relation is satisfied for all real values of x, so all the coefficients must be zero.

$$\left.\begin{aligned} a^2-1=0 &\Rightarrow a=\pm1 \\ a-1=0 &\Rightarrow a=1 \\ a^2-4a+3=0 &\Rightarrow 1,3 \end{aligned}\right\}\text{Common value a is 1}$$

Illustration 34: If the equation $a(x-1)^2+b(x^2-3x+2)+x-a^2=0$ is satisfied for all $x\in$ R, find all possible ordered pairs (a, b). **(JEE ADVANCED)**

Sol: Similar to illustration 33, we can solve this illustration by taking all coefficients to be equal to zero.

$a(x-1)^2+b(x^2-3x+2)+x-a^2=0$

$$\Rightarrow (a+b)x^2 - (2a+3b-1)x + 2b - a^2 + a = 0$$

Since the equation is satisfied for all α, it becomes an identity

Coeff. of x^2 = 0	Coeff. of x = 0	Constant term = 0
$a + b = 0$	$2a + 3b - 1 = 0$	$2b - a^2 + a = 0$; $2 - a^2 + a = 0$
$a = -b$ .....(i)	using (i) ; $\Rightarrow$ $-2b + 3b = 1$;	$a^2 - a - 2 = 0$
	$\Rightarrow$ $b = 1$	$\Rightarrow$ $(a + 1)(a - 2) = 0$; $a = -1, 2$

But from (i) $a = -b$ $\Rightarrow$ only $a = -1$ is the possible solution. Hence $(a, b) = (-1, 1)$

10. SOLVING INEQUALITIES

10.1 Intervals

Given $E(x) = (x - a)(x - b)(x - c)(x - d) \geq 0$

To find the solution set of the above inequality we have to check the intervals in which E(x) is greater/less than zero.

Intervals

Closed interval | **Open interval** | **Open-closed interval** | **Closed-open interval**

(a) Closed Interval: The set of all values of x, which lies between a & b and is also equal to a & b is known as a closed interval, i.e. if $a \leq x \leq b$ then it is denoted by $x \in [a, b]$.

(b) Open Interval: The set of all values of x, which lies between a & b but equal to a & b is known as an open interval, i.e. if $a < x < b$ then it is denoted by $a \in (a, b)$

(c) Open-Closed Interval: The set of all values of x, which lies between a & b, equal to b, but not equal to a is known as an open-closed interval, i.e. if $a < x \leq b$ then it is denoted by $x \in (a, b]$.

(d) Closed-open Interval: The set of all values of x, which lies between a & b, equal to a but not equal to b is called a closed-open interval, i.e. if $a \leq x < b$, then it is denoted by $x \in [a, b)$.

Note: (i) $x \geq a \Rightarrow [a, \infty)$ (ii) $x > a \Rightarrow (a, \infty)$

(iii) $x \leq a \Rightarrow (-\infty, a)$ (iv) $x < a \Rightarrow (-\infty, a)$

10.2 Some Basic Properties of Intervals

(a) In an inequality, any number can be added or subtracted from both sides of inequality.

(b) Terms can be shifted from one side to the other side of the inequality. The sign of inequality does not change.

(c) If we multiply both sides of the inequality by a non-zero positive number, then the sign of inequality does not change. But if we multiply both sides of the inequality by a non-zero negative number then the sign of the inequality does get changed.

(d) In the inequality, if the sign of an expression is not known then it cannot be cross multiplied. Similarly, without knowing the sign of an expression, division is not possible.

(i) $\frac{x-2}{x-5} > 1 \Rightarrow x - 2 > x - 5$ (Not valid because we don't know the sign of the expression)

(ii) $\frac{x-2}{(x-5)^2} > 1 \Rightarrow (x-2) > (x-5)^2$ (valid because $(x-5)^2$ is always positive)

10.3 Solution of the Inequality

(a) Write all the terms present in the inequality as their linear factors in standard form i.e. $x \pm a$.

(b) If the inequality contains quadratic expressions, $f(x) = ax^2 + bx + c$; then first check the discriminant $(D = b^2 - 4ac)$

(i) If $D > 0$, then the expression can be written as $f(x) = a(x - \alpha)(x - \beta)$. Where α and β are given by $\alpha, \beta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(ii) If $D = 0$, then the expression can be written as $f(x) = a\left(x-\alpha\right)^2$, where $\alpha = \dfrac{-b}{2a}$.

(iii) If $D < 0$ & if

- $a > 0$, then $f(x) > 0 \ \forall X \in R$ and the expression will be cross multiplied and the sign of the inequality will not change.
- $a < 0$, then $f(x) < 0 \ \forall X \in R$ and the expression will be cross multiplied and the sign of the inequality will change.
- If the expression (say 'f') is cancelled from the same side of the inequality, then cancel it and write $f \neq 0$ e.g.,

(i) $\dfrac{(x-2)(x-3)}{(x-2)(x-5)} > 1 \Rightarrow \dfrac{(x-3)}{(x-5)} > 1$ iff $x - 2 \neq 0$

(ii) $\dfrac{(x-5)^2(x-8)}{(x-5)} \geq 0 \Rightarrow (x-5)(x-8) \geq 0$ iff $x - 5 \neq 0$

(iii) Let $f(x) = \dfrac{(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a)^{k_n}}{(x-b_1)^{r_1}(x-b_2)^{r_2}\ldots(x-b)^{r_n}}$

Where $k_1, k_2 \ldots\ldots k_n$ & $r_1, r_2 \ldots\ldots r_n \in N$ and $a_1, a_2, \ldots\ldots a_n$ & $b_1, b_2 \ldots\ldots b_n$ are fixed real numbers. The points where the numerator becomes zero are called zeros or roots of the function and points where the denominator becomes zero are called poles of the function. Find poles and zeros of the function f(x). The corresponding zeros are $a_1, a_2, \ldots\ldots a_n$ and poles are $b_1, b_2 \ldots\ldots b_n$. Mark the poles and zeros on the real numbers line. If there are n poles & n zeros the entire number line is divided into 'n+1' intervals. For f(x), a number line is divided into '2n+1' intervals.

Place a positive sign in the right-most interval and then alternate the sign in the neighboring interval if the pole or zero dividing the two interval has appeared an odd number of times. If the pole or zero dividing the interval has appeared an even number of times then retain the sign in the neighboring interval. The solution of $f(x) > 0$ is the union of all the intervals in which the plus sign is placed, and the solution of $f(x) < 0$ is the union of all the intervals in which minus sign is placed. This method is known as the WAVY CURVE method.

Now we shall discuss the various types of inequalities.

Type I: Inequalities involving non-repeating linear factors $(x - 1)(x - 2) \geq 0$

1st condition $\left.\begin{array}{l}(x-1) > 0 \Rightarrow x > 1 \\ (x-2) > 0 \Rightarrow x > 2\end{array}\right\} x \geq 2$

2nd condition $\left.\begin{array}{l}x-1 < 0 \Rightarrow x < 1 \\ x-2 < 0 \Rightarrow x < 2\end{array}\right\} x \leq 1$

$\therefore x \in (-\infty, 1] \cup [2, \infty)$

Illustration 35: (x – 3) (x+1) $\left(x-\frac{12}{7}\right)$ < 0, find range of x **(JEE MAIN)**

Sol: Comparing all brackets separately with 0, we can find the range of values for x.

$x<-1$ and $\frac{12}{7}<x<3$; $\therefore x\in(-\infty,-1)\cup\left(\frac{12}{7},3\right)$

Type II: Inequalities involving repeating linear factors

$(x-1)^2(x+2)^3(x-3)\le 0$

$\Rightarrow (x+1)^2(x+2)^2(x+2)(x-3)\le 0$

$\Rightarrow$ (x+2)(x – 3) $\le$ 0 ; x $\in$ [-2, 3]

Illustration 36: Find the greatest integer satisfying the equation.

$(x+1)^{101}(x-3)^2(x-5)^{11}(x-4)^{200}(x-2)^{555}<0$ **(JEE MAIN)**

Sol: Comparing all brackets separately with 0, we can find the greatest integer.

The inequality $\{-(x-2)\}^2-(x-2)-2=0$

$\Rightarrow$ (x + 1)(x – 5)(x – 2) < 0

x = -1, 3 , 5, 4, 2

x $\in$ $(-\infty,-1)$ U (2, 3) U (3, 4) U (4, 5) .

Type III: Inequalities expressed in rational form.

Illustration 37: $\frac{(x-1)(x+2)}{(x+3)(x-4)}\ge 0$ **(JEE MAIN)**

Sol: If $\frac{(x+a)(x+b)}{(x+c)(x+d)}\ge 0$ then $(x+c)(x+d)\ne 0$, and $(x+a)(x+b)=0$

Hence, $x\ne -3,4$ & x = 1, -2 ; $x\in(-\infty,-3)$ U [-2, 1] U (4, ∞)

Illustration 38: $\frac{x^2(x+1)}{(x-3)^3}<0$ **(JEE MAIN)**

Sol: Similar to the illustration above.

$\frac{x+1}{x-3}<0$ x ≠ 3, -1, 0 ; x $\in$ (–1, 0) U (0, 3)

Illustration 39: $\frac{x^2-1}{x^2-7x+12}\ge 1$ **(JEE MAIN)**

Sol: First reduce the given inequalities in rational form and then solve it in the manner similar to the illustration above.

$\frac{x^2-1}{(x-4)(x-3)}\ge 1$

$$\therefore \frac{(x+1)(x-1)}{(x-4)(x-3)} \ge 1 \quad \Rightarrow \frac{x^2-1}{x^2-7x+12} - 1 \ge 0$$

$$\therefore \frac{x^2-1-x^2+7x-12}{(x-4)(x-3)} \ge 0 \qquad \therefore \frac{7x-13}{(x-4)(x-3)} \ge 0$$

$$\therefore x \ne 3,4 \; ; \qquad x \in \left[\frac{13}{7}, 3\right) \cup (4, \infty)$$

Type IV: Double inequality

Illustration 40: $1 < \dfrac{3x^2-7x+8}{x^2+1} \le 2$ **(JEE ADVANCED)**

Sol: Here $3x^2-7x+8 > x^2+1$ therefore if D < 0 & if a > 0, then f(x) > 0 and always positive for all real x.

$3x^2-7x+8 > x^2+1 \Rightarrow 2x^2-7x+7 > 0$;

D= b^2-4ac = 49 – 56 = -7

$\therefore D < 0$ & $a > 0$ $\quad\therefore$ always positive for all real x

$3x^2-7x+8 \le 2x^2+2 \Rightarrow x^2-7x+6 \le 0 \Rightarrow (x-1)(x-6) \le 0$

$x \in [1, 6]$; $x \in [1, 6] \cap R$

Type V: Inequalities involving biquadrate expressions

Illustration 41: $(x^2+3x+1)(x^2+3x-3) \ge 5$ **(JEE ADVANCED)**

Sol: Using $x^2+3x=y$,we can solve this problem

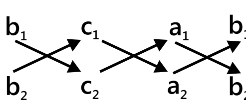
Let $x^2+3x=y$ $\qquad \therefore (y+1)(y-3) \ge 5$

$y^2-2y-8 \ge 0$ $\qquad \therefore$ (y – 4) (y + 2) $\ge$ 0

$\therefore$ (x + 4) (x – 1)(x + 2)(x + 1) $\ge$ 0 $\Rightarrow x \in (-\infty, -4] \cup [-2, -1] \cup [1, \infty)$

11. CONDITION FOR COMMON ROOTS

Consider that two quadratic equations are $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$



(i) One root is common

Let α, be the common root. then α satisfies

$a_1\alpha^2+b_1\alpha+c_1=0$

$a_3\alpha^3+b_2\alpha+c_2=0$

By cross multiplication method, $\dfrac{\alpha^2}{b_1c_2-b_2c_1} = \dfrac{\alpha}{-(a_1c_2-c_1a_2)} = \dfrac{1}{a_1b_2-b_1a_2}$

$$\frac{\alpha^2}{b_1c_2-b_2c_1} = \frac{\alpha}{c_1a_2-a_1c_2} = \frac{1}{a_1b_2-b_1a_2}$$

$$\alpha^2 = \frac{b_1c_2-b_2c_1}{a_1b_2-b_1a_2} \qquad \text{... (i)}$$

$\alpha = \dfrac{c_1a_2 - a_1c_2}{a_1b_2 - b_1a_2}$ ... (ii)

Divide (1)/(2)

$\alpha = \dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$ ... (iii)

equating (i) and (ii) ; $\left(c_1 a_2 - c_2 a_1\right)^2 = \left(a_1 b_2 - b_1 a_2\right)\left(b_1 c_2 - b_2 c_1\right)$ is the condition for a common root.

(ii) If both roots are common, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Illustration 42: Determine the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root. **(JEE MAIN)**

Sol: Consider α to be the common root of the given equations. Then, α must satisfy both the equations. Therefore, by using a multiplication method we can solve this problem.

$3\alpha^2 + 4m\alpha + 2 = 0$; $\quad 2\alpha^2 + 3\alpha - 2 = 0$

Using the cross multiplication method, we have

$(-6 - 4)^2 = (9 - 8m)(-8m - 6)$

$\Rightarrow 50 = (8m - 9)(4m + 3) \Rightarrow 32m^2 - 12m - 77 = 0$

$\Rightarrow 32m^2 - 56m + 44 = 0 \Rightarrow 8m(4m - 7) + 11(4m - 7) = 0$

$\Rightarrow (8m + 11)(4m - 7) = 0 \quad \Rightarrow \quad m = -\dfrac{11}{8}, \dfrac{7}{4}$

Illustration 43: The equation $ax^2 + bx + c$ and $y \ge 0$ have two roots common, Find the value of (a + b). **(JEE ADVANCED)**

Sol: We can reduce $x^3 - 2x^2 + 2x - 1 = 0$ to $(x - 1)\left(x^2 - x + 1\right) = 0$ as the given equations have two common roots, therefore $-\omega$ and $-\omega^2$ are the common roots (as both roots of a quadratic equation are either real or non-real).

We have $x^3 - 2x^2 + 2x - 1 = 0 \Rightarrow (x - 1)\left(x^2 - x + 1\right) = 0$

$\Rightarrow x = 1$ or $x = -\omega, -\omega^2$, where $\omega = \dfrac{-1 + \sqrt{3}i}{2}$

Since $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ have two roots in common, therefore $-\omega$ and $-\omega^2$ are the common roots (as both roots of a quadratic equation are either real or non-real), also $-\omega$ is a root of $ax^2 + bx + a = 0$. Hence.

$a\left(1 + \omega^2\right) - b\omega = 0 \quad \Rightarrow a(-\omega) - b\,\omega = 0$ (as $1 + \omega + \omega^2 = 0$)

$\Rightarrow a + b = 0$

12. MAXIMUM AND MINIMUM VALUE OF A QUADRATIC EQUATION

$y = ax^2 + bx + c$ attains its minimum or maximum value at $x = \dfrac{-b}{2a}$ according to a > 0 or a < 0

MAXIMUM value case

When a < 0 then $y_{max} = \dfrac{-D}{4a}$ i.e. $y \in \left(-\infty, \dfrac{-D}{4a}\right]$

MINIMUM value case

When a > 0 then $y_{min} = \frac{-D}{4a}$ i.e. $y \in \left[\frac{-D}{4a}, \infty\right)$

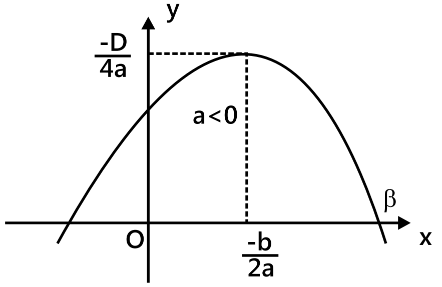


Figure 2.17

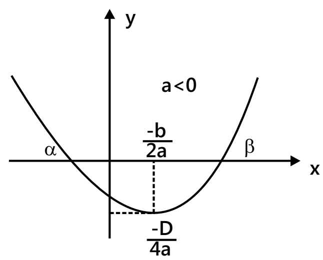


Figure 2.18

CONCEPTS

If α is a repeated root, i.e., the two roots are α , α of the equation f(x) = 0, then α will be a root of the derived equation f′(x) = 0 where $f'(x) = \frac{df}{dx}$

If α is a repeated root common in f(x) = 0 and ϕ (x) = 0, then α is a common root both in f′(x) = 0 and ϕ ′(x) = 0.

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Illustration 44: Find the range of $y = \frac{x}{x^2 - 5x + 9}$ **(JEE ADVANCED)**

Sol: Here as $x \in R$ therefore $D \geq 0$. Hence, by solving these inequalities we can find the required range.

$x = yx^2 - 5yx + 9y$; $yx^2 - 5yx + 9y - x = 0$.

$yx^2 - (5y+1)x + 9y = 0$; $\because x \in R$ $D \geq 0$

$\therefore (5y+1)^2 - 36y^2 \geq 0$; $\therefore 25y^2 + 1 + 10y - 36y^2 \geq 0$

$\therefore -11y^2 + 10y + 1 \geq 0$ $11y^2 - 11y + y - 1 \leq 0$

(11y +1)(y – 1) ≤ 0 ; $y \in \left[\frac{-1}{11}, 1\right]$

Putting the end points in the eq. $1 = \frac{x}{x^2 - 5x + 9}$; $x^2 - 6x + 9 = 0$ $\therefore (x-3)^2 = 0$

If D < 0, then 1 would be open, i.e. excluded; $\frac{-1}{11} = \frac{x}{x^2 - 5x + 9}$

$\Rightarrow -(x^2 - 5x + 9) = 11x$; $\therefore x^2 + 6x + 9 = 0$; $\therefore (x+3)^2 = 0$; $\therefore \frac{-1}{11}$ remains closed

Alternative Solution: $y = \frac{x}{x^2 - 5x + 9} = \frac{1}{\left(x - 5 + \frac{9}{x}\right)}$

Apply the concept of Arithmetic mean ≥ Geometric mean for the values for x and 9/x

We have $\dfrac{\left(x+\dfrac{9}{x}\right)}{2} \geq \sqrt{x * \dfrac{9}{x}}$

Thus $x+\dfrac{9}{x} \geq 6$ for x ≥ 0 and $x+\dfrac{9}{x} \leq 6$ for x < 0

Since the term is in the denominator if we consider its maximum value, we will get the minimum value of y and vice versa.

The maximum value of y will be $\dfrac{1}{6-5} = 1$ and

The minimum value of will be $\dfrac{1}{-6-5} = \dfrac{-1}{11}$.

Thus the range of y is $\left[\dfrac{-1}{11}, 1\right]$

Illustration 45: Find range of $y = \dfrac{x^2+2x-3}{x^2+2x-8}$ **(JEE ADVANCED)**

Sol: Similar to the preceding problem, by taking $b^2 - 4ac \geq 0$ we can solve it.

$x^2 + 2x - 3 = yx^2 + 2xy - 8y$

$(y-1)x^2 + (2y-2)x - (8y-3) = 0$; $b^2 - 4ac \geq 0$

$\therefore (2y-2)^2 + (4)(8y-3)(y-1) \geq 0 \Rightarrow 4y^2 + 4 - 8y + 4(8y^2 - 3y - 8y + 3) \geq 0$

$4y^2 + 4 - 8y - 44y - 32y^2 + 12 \geq 0 \Rightarrow 36y^2 - 52y + 16 \geq 0$

$\therefore 9y^2 - 13y + 4 \geq 0 \Rightarrow (y-1)(9y-4) \geq 0$; $\therefore y \in \left(-\infty, \dfrac{4}{9}\right] \cup (1, \infty)$

To verify if the bracket is open or closed, apply the end points in the equation,

Check for $y = \dfrac{4}{9}$; $\dfrac{4}{9} = \dfrac{x^2+2x-3}{x^2+2x-8}$

$4x^2 + 8x - 32 = 9x^2 + 18x - 27$; $\therefore 5x^2 + 10x + 5 = 0$

$\therefore x^2 + 2x + 1 = 0$; $\therefore (x+1)^2 = 0$

$\therefore x = -1$ $\therefore \dfrac{4}{9}$ is closed

Check for 1, $1 = \dfrac{x^2+2x-3}{x^2+2x-8}$

$\therefore x^2 + 2x - 3 = x^2 + 2x - 8$ Since no value of x can be found, 1 is open

Illustration 46: Find the limits of 'a' such that $y = \dfrac{ax^2 - 7x + 5}{5x^2 - 7x + a}$ is capable of all the values of 'x' being a real quantity. **(JEE ADVANCED)**

Sol: Similar to Illustration 45.

$5yx^2 - 7xy + ay = ax^2 - 7x + 5$

$(5y-a)x^2 - 7x(y-1) + ay - 5 = 0$;

$D = 49(y^2 + 1 - 2y) - 4(ay-5)(5y-a)$; $D \geq 0$

$49y2 - 49 - 98y - 20ay^2 + 100y + 4a^2y - 20\,a \geq 0$

$y^2\,(49 - 20a) + y(2 + 4a^2) + 49 - 20\,a \geq 0$

$(2 + 4a^2)^2 - [2(49 - 20a)]^2 \leq 0$; $\therefore\ (2 + 4a^2 + 2(49 - 20a))(2 + 4a^2 - a^2(49 - 20a))$

$(1 + 2a^2 + 49 - 20a)(1 + 2a^2 - 49 + 20\,a) \leq 0$

$(a^2 - 10a + 25)\,(a^2 + 10\,a - 24) \leq 0$; $\therefore\ (a + 12)(a - 2) \leq 0$

13. LOCATION OF ROOTS

Let $f(x) = ax^2 + bx + c;\ \ a,b,c \in \mathbb{R}$ a is not equal to 0 and $\alpha,\ \beta$ be the roots of $f(x) = 0$

Type I: If both the roots of a quadratic equation $f(x) = 0$ are greater than a specified number, say ‘d’, then

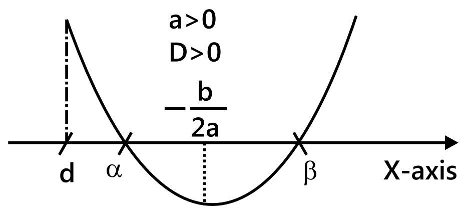


Figure 2.19

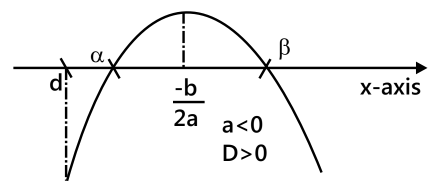


Figure 2.20

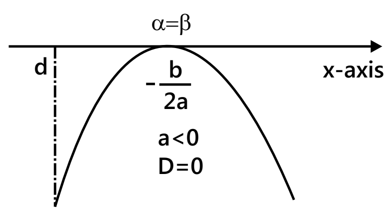


Figure 2.21

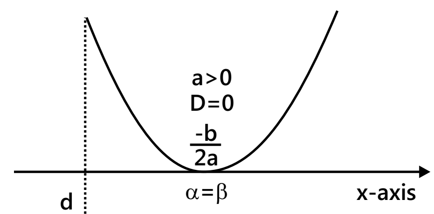


Figure 2.22

(i) $D \geq 0$ (ii) $f(x) > 0$ (iii) $\frac{-b}{2a} > d$

Type II: If both the roots are less than a specified number, say ‘d’, then

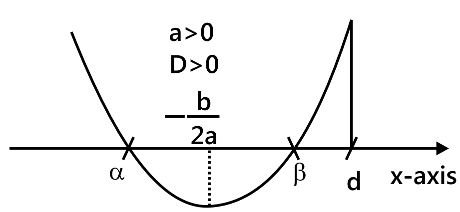


Figure 2.23

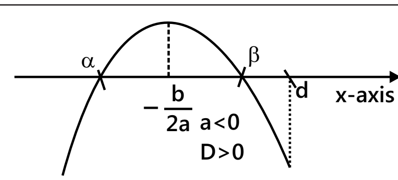


Figure 2.24

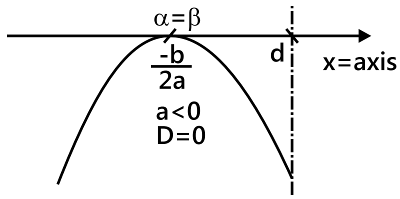


Figure 2.25

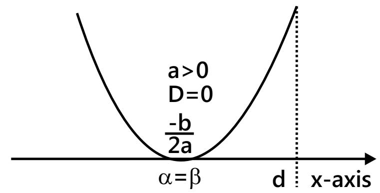


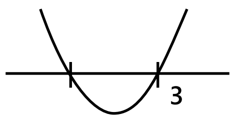
Figure 2.26

(i) $D \geq 0$ (ii) $f(x) > 0$ (iii) $\frac{-b}{2a} < d$

Illustration 47: If both the roots of the quadratic equation $x^2 + x(4-2k) + k^2 - 3k - 1 = 0$ are less than 3, then find the range of values of k. **(JEE MAIN)**

Sol: Here both the roots of the given equation is less than 3, hence, $D \geq 0, \frac{-b}{2a} < 3$ and f (3) > 0.

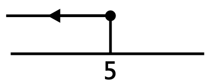
The equation is $f(x) = x^2 + x(4-2k) + k^2 - 3k - 1 = 0$



$D \geq 0$ ...(i)

$\frac{-b}{2a} < 3$ ....(ii)

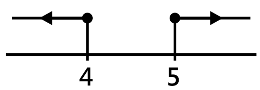
$f(3) > 0$ ....(iii)



(i) $D \geq 0 \Rightarrow (4-2k)^2 - 4(k^2 - 3k - 1) \geq 0$

$\Rightarrow (k2 - 4k + 4) - (k^2 - 3k - 1) \geq 0$

$\Rightarrow -k + 5 \geq 0 \qquad \Rightarrow k - 5 \leq 0$;



$k \in (-\infty, 5]$

(ii) $\frac{-4(4-2k)}{2} < 3$; $k - 2 < 3$; $\quad k < 5$

(iii) $f(3) > 0 \qquad \Rightarrow 9 + 3(4-2k) + k^2 - 3k - 1 > 0$

$\Rightarrow k^2 - 9k + 20 > 0 \qquad \Rightarrow (k-4)(k-5) > 0$

$k \in (-\infty, 4) \cup (5, \infty)$; Combining all values we get $k \in (-\infty, 4)$

Type III: A real number d lies between the roots of $f(x) = 0$ or both the roots lie on either side of a fixed number say 'd' then af(d)<0, and D>0.

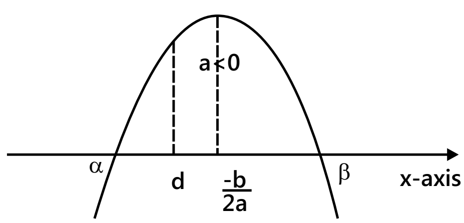


Figure 2.27

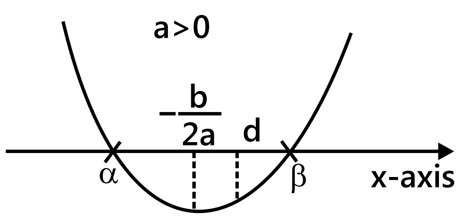


Figure 2.28

Type IV: Exactly one root lies in the interval (d, e) when d < e, then f(d)·f(e)<0

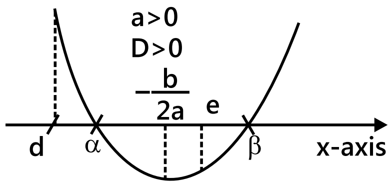


Figure 2.29

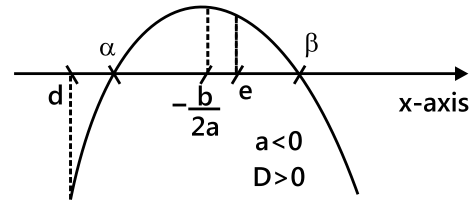


Figure 2.30

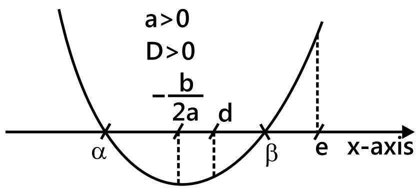


Figure 2.31

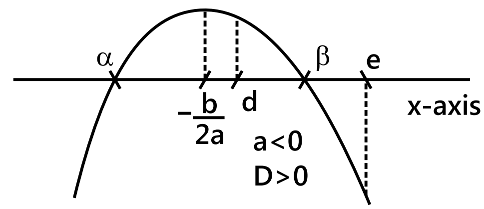


Figure 2.32

Type V: If both the roots of $f(x) = 0$ are confined between real numbers'd' and 'e', where d < e. Then

(i) D ≥ 0, (ii) f(d)f(e) >0, (iii) $d < -\frac{b}{2a} < e$.

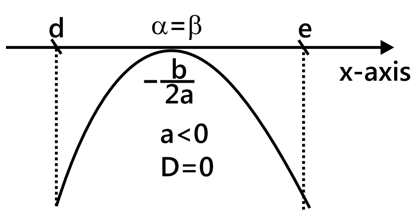


Figure 2.33

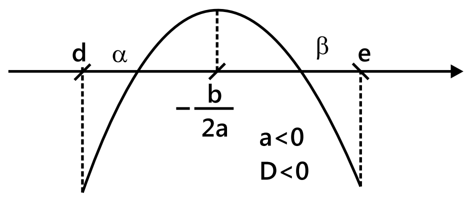


Figure 2.34

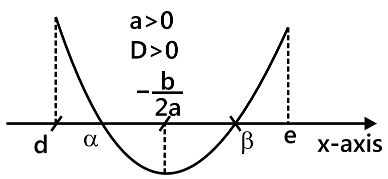


Figure 2.35

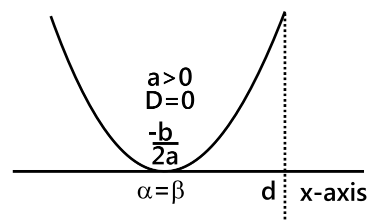


Figure 2.36

Type VI: One root is greater than e and the other root is less than 'd'.

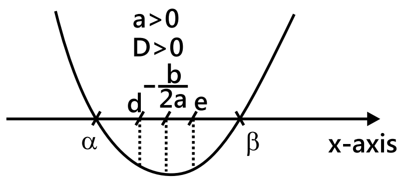


Figure 2.37

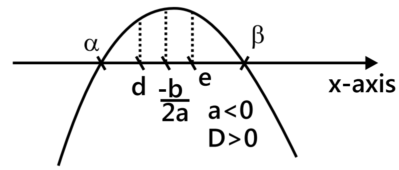


Figure 2.38

14. QUADRATIC EXPRESSION IN TWO VARIABLES

The general quadratic expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be factorized into two linear factors. The corresponding quadratic equation is in two variables

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

or $ax^2 + 2(hx + g)x + by^2 + 2fy + c = 0$ ...(i)

$$\therefore x = \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2+2fy+c)}}{2a} \Rightarrow x = \frac{-(hy+g) \pm \sqrt{h^2y^2+g^2+2ghy-2afy-ac-aby^2}}{a}$$

$$\Rightarrow ax + hy + g = \pm\sqrt{h^2y^2+g^2+2ghy-aby^2-2afy-ac} \quad .....(ii)$$

At this point, the expression (i) can be resolved into two linear factors if

$(h^2-ab)y^2 + 2(gh-af)y + g^2 - ac$ is a perfect square and $h^2 - ab > 0$.

But $(h^2-ab)y^2 + 2(gh-af)y + g^2 - ac$ will be a perfect square if D = 0

$\Rightarrow g^2h^2 + a^2f^2 - 2afgh - h^2g^2 + abg^2 + ach^2 - a^2bc = 0$ and $h^2 - ab > 0$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 - ab > 0$

This is the required condition. The condition that this expression may be resolved into two linear rational factors is

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 - ab > 0$

This expression is called the discriminant of the above quadratic expression.

Illustration 48: If the equation $x^2 + 16y^2 - 3x + 2 = 0$ is satisfied by real values of x & y, then prove that $x \in [1, 2]$, $y \in \left[\frac{-1}{8}, \frac{1}{8}\right]$ **(JEE MAIN)**

Sol: For real values of x and y, $D \geq 0$. Solve this by taking the x term and the y term constant one by one.

$x^2 - 3x + 16y^2 + 2 = 0$; $D \geq 0$ as $x \in R$

$\Rightarrow 9 - 4(16y^2 + 2) \geq 0$; $\Rightarrow 9 - 64y^2 - 8 \geq 0$

$\therefore 64y^2 - 1 \leq 0$

$\Rightarrow (8y - 1)(8y + 1) \leq 0$ $\therefore y \in \left[\frac{-1}{8}, \frac{1}{8}\right]$

To find the range of x, in $16y^2 + x^2 - 3x + 2 = 0$ $D \geq 0$

Hence, $-64(x^2 - 3x + 2) \geq 0$

Solving this, we get $x \in [1, 2]$

Illustration 49: Show that in the equation $x^2 - 3xy + 2y^2 - 2x - 3y - 35 - 0$, for every real value of x there is a real value of y. **(JEE MAIN)**

Sol: By using the formula $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ we will get $x = \frac{3y+2+\sqrt{y^2+24y+144}}{2}$.

Here, the quadratic equation in y is a perfect square.

$$x^2 - x(3y+2) + (2y^2 - 3y - 35) = 0$$

Now, $x = 3y + 2 + 2\sqrt{\text{quadratic in y}}$. As the quadratic equation in y is a perfect square$\left((y+12)^2\right)$.

$\therefore$ The relation between x & y is a linear equation which is a straight line.

$\therefore \forall x \in R$, y is a real value.

Illustration 50: If $(a_1x^2 + b_1x + c_1)y + (a_2x^2 + b_2x + c_2) = 0$ find the condition that x is a rational function of y **(JEE ADVANCED)**

Sol: For x is a rational function of y, its discriminant will be greater than or equal to zero, i.e. $D \geq 0$.

$$x^2 - (a_1y + a_2) + x(b_1y + b_2) + (c_1y + c_2) = 0$$

$$x = \frac{-(b_1y+b_2) \pm \sqrt{(b_1y+b_2)^2 - 4(a_1y+a_2)(c_1y+c_2)}}{2(a_1y+a_2)}$$

For the above relation to exist $(b_1y+b_2)^2 - 4(a_1y+a_2)(c_1y+c_2) \geq 0$

$$\Rightarrow (b_1{}^2 - 4a_1c_1)y^2 + 2(b_1b_2 - 2a_1c_2 - 2a_2c_1)y + (b_2{}^2 - 4a_2c_2) \geq 0$$

$$\Rightarrow b_1{}^2 - 4a_1c_1 > 0 \text{ and } D \leq 0$$

Solving this will result in a relation for which x is a rational function of y.

15. NUMBER OF ROOTS OF A POLYNOMIAL EQUATION

(a) If f(x) is an increasing function in [a, b], then f(x) = 0 will have at most one root in [a, b].

(b) Let f(x) = 0 be a polynomial equation. a, b are two real numbers. Then f(x) =0 will have at least one real root or an odd number of real roots in (a, b) if f(a) and f(b) (a < b) are of opposite signs.

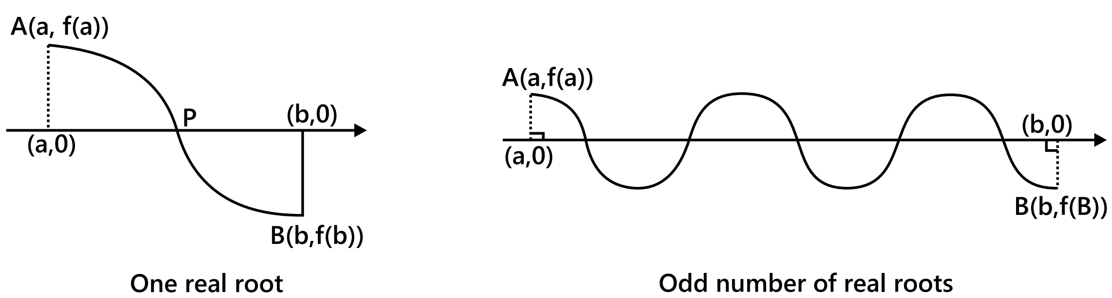
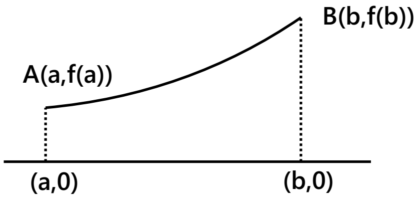
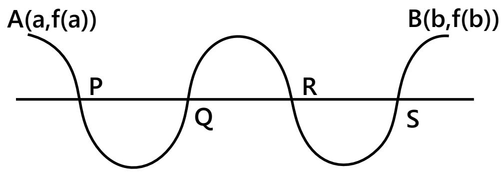


Figure 2.39

But if f(a) and f(b) are of the same sign, then either f(x) = 0 have one real root or an even number of real roots in (a, b)



No real root



Even number of real roots

Figure 2.40

(c) If the equation f(x) = 0 has two real roots a and b, then f′(x) = 0 will have at least one real root lying between a and b (using Rolle's theorem).

CONCEPTS

Descartes' rule of sign for the roots of a polynomial

Rule 1: The maximum number of positive real roots of a polynomial equation

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ is the number of changes of the sign of coefficients from positive to negative and negative to positive. For instance, in the equation $x^3 + 3x^2 + 7x - 11 = 0$ the sign of the coefficients are +++- as there is just one change of sign, the number of positive roots of $x^3 + 3x^2 + 7x - 11 = 0$ is at most 1.

Rule 2 : The maximum number of negative roots of the polynomial equation f(x) = 0 is the number of changes from positive to negative and negative to positive in the sign of the coefficient of the equation f(–x) = 0.

Shivam Agarwal (JEE 2009 AIR 27)

PROBLEM-SOLVING TACTICS

Some hints for solving polynomial equations:

(a) To solve an equation of the form $(x-a)^4 + (x-b)^4 = A$; Put y = $x - \frac{a+b}{2}$

In general to solve an equation of the form $(x-a)^{2n} + (x-b)^{2n} = A$, where $n \in Z^+$, put y = $x - \frac{a+b}{2}$

(b) To solve an equation of the form, $a_0 f(x)^{2n} + a_1 (f(x))^n + a_2 = 0$ we put $(f(x))^n = y$ and solve $a_0y^2 + a_1y + a_2 = 0$ to obtain its roots y_1 and y_2.

Finally, to obtain the solution of (1) we solve, $(f(x))^n = y_1$ and $(f(x))^n = y_2$

(c) An equation of the form $(ax^2 + bx + c_1)(ax^2 + bx + c_2).....(ax^2 + bx + c_n) = A$. Where $c_1, c_2, ...c_n, A \in R$, can be solved by putting $ax^2 + bx = y$.

(d) An equation of the form (x – a)(x – b)(x – c)(x – d) = ⇒ Awhere ab = cd, can be reduced to a product of two quadratic polynomials by putting y = x + $\frac{ab}{2}$.

(e) An equation of the form (x – a) (x – b)(x – c)(x – d) = A where a < b < c < d, b – a = d – c can be solved by a change of variable $y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4} = x - \frac{1}{4}(a+b+c+d)$

(f) A polynomial f(x, y) is said to be symmetric if f(x, y) = f(y, x) $\forall$ x, y. A symmetric polynomial can be represented as a function of x + y and xy.

Solving equations reducible to quadratic

(a) To solve an equation of the type $ax^4 + bx^2 + c = 0$, put $x^2 = y$.

(b) To solve an equation of the type $a(p(x))^2 + bp(x) + c = 0$ (p(x) is an expression of x), put p(x) = y.

(c) To solve an equation of the type $ap(x) + \frac{b}{p(x)} + c = 0$ where p(x) is an expression of x, put p(x)= y

This reduces the equation to $ay^2 + cy + b = 0$

(d) To solve an equation of the form $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$, put $x + \frac{1}{x} = y$

and to solve $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x - \frac{1}{x}\right) + c = 0$, put $x - \frac{1}{x} = y$

(e) To solve a reciprocal equation of the type $ax^4 + bx^3 + cx^2 + bx + a = 0, a \neq 0,$

we divide the equation by $\frac{d^2y}{dx^2}$ to obtain $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$, and then put $x + \frac{1}{x} = y$

(f) To solve an equation of the type (x + a)(x + b)(x + c)(x + d) + k = 0 where a+b =c+d, put x^2 +(a+ b)x = y

(g) To solve an equation of the type $\sqrt{ax+b} = cx + d$ or $\sqrt{ax^2 + bx + c} = dx + e$, square both the sides.

(h) To solve an equation of the type = $\sqrt{ax+b} \pm \sqrt{cx+d} = e$, proceed as follows.

Step 1: Transfer one of the radical to the other side and square both the sides.

Step 2: Keep the expression with radical sign on one side and transfer the remaining expression on the other side

Step 3: Now solve as in 7 above.

FORMULAE SHEET

(a) **A quadratic equation is represented as :** $ax^2 + bx + c = 0,\ a \neq 0$

(b) **Roots of quadratic equation:** $x = \frac{-b \pm \sqrt{D}}{2a}$, where D(discriminant) = $b^2 - 4ac$

(c) **Nature of roots: (i)** D > 0 ⇒ roots are real and distinct (unequal)

(ii) D = 0 ⇒ roots are real and equal (coincident)

(iii) D < 0 ⇒ roots are imaginary and unequal

(d) The roots $(\alpha+i\beta)$, $(\alpha-i\beta)$ and $(\alpha+\sqrt{\beta}),(\alpha-\sqrt{\beta})$ are the conjugate pair of each other.

(e) **Sum and Product of roots :** If α and β are the roots of a quadratic equation, then

(i) $S=\alpha+\beta=\frac{-b}{a}=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$ **(ii)** $P=\alpha\beta=\frac{c}{a}=\frac{\text{constant term}}{\text{Coefficient of } x^2}$

(f) **Equation in the form of roots:** $x^2-(\alpha+\beta)x+(\alpha.\beta)=0$

(g) In equation $ax^2+bx+c=0,\ a\neq 0$ If

(i) b = 0 $\Rightarrow$ roots are of equal magnitude but of opposite sign.

(ii) c = 0 $\Rightarrow$ one root is zero and other is –b/a

(iii) b = c = 0 $\Rightarrow$ both roots are zero.

(iv) a = c $\Rightarrow$ roots are reciprocal to each other.

(v) a > 0, c < 0 or a < 0, c > 0 $\Rightarrow$ roots are of opposite signs.

(vi) a > 0, b > 0, c > 0 or a < 0, b < 0, c < 0 $\Rightarrow$ both roots are –ve.

(vii) a > 0, b < 0, c > 0 or a < 0,b > 0, c < 0 $\Rightarrow$ both roots are +ve.

(h) The equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ have

(i) One common root if $\frac{b_1c_2-b_2c_1}{c_1a_2-c_2a_1}=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

(ii) Both roots common if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(i) In equation $ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right]$

(i) If a>0, the equation has minimum value $=\frac{4ac-b^2}{4a}$ at $x=\frac{-b}{2a}$ and there is no maximum value.

(ii) If a < 0, the equation has maximum value $\frac{4ac-b^2}{4a}$ at $x=\frac{-b}{2a}$ and there is no minimum value.

(j) For cubic equation $ax^3+bx^2+cx+d=0$,

(i) $\alpha+\beta+\gamma=\frac{-b}{a}$

(ii) $\alpha\beta+\beta\gamma+\lambda\alpha=\frac{c}{a}$

(iii) $\alpha\beta\gamma=\frac{-d}{a}$ ... where α,β,γ are its roots.

Solved Examples

JEE Main/Boards

Example 1: For what values of 'm' does the quadratic equation (1 + m) $x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ have equal roots?

Sol: The roots are equal if discriminant (D)= 0.

$4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0 \Rightarrow 4\left(m^2 - 3m\right) = 0$

$\Rightarrow \quad m = 0, 3$

Example 2: When pr = 2(q + s), where p, q, r, s are real numbers, show that at least one of the equations $x^2 + px + q$ and $x^2 + rx + s = 0$ has real roots.

Sol: For at least one of the given

equations to have real roots means one of

their discriminant must be non negative.

The given equations are

$f(\alpha) = 0 + px + q = 0$ ... (i)

$f(\alpha) = 0 + rx + s = 0$ ... (ii)

consider D_1 and D_2 be the discriminants of equations (i) and (ii) respectively,

$D_1 + D_2 = p^2 - 4q + r^2 - 4s$

$= p^2 + r^2 - 4(q + s)$

$= p^2 + r^2 - 2pr$

$= (p - r)^2 \geq 0$ [$\because$ p and r are real]

$\because$ At least one of D_1 and D_2 must be non negative.

Hence, at least one of the given equation has real roots.

Example 3: Find the quadratic equation where one of the roots is $\frac{1}{2+\sqrt{5}}$.

Sol: If one root is $\left(\alpha + \sqrt{\beta}\right)$ then other one will be $\left(\alpha - \sqrt{\beta}\right)$.

given $\alpha = \frac{1}{2+\sqrt{5}}$

Multiplying the numerator and denominator by

$2 - \sqrt{5}$, we get

$= \frac{2-\sqrt{5}}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}$

$= \sqrt{5} - 2$

Then the other root, $x^2 + px + q = x$ will be $-2-\sqrt{5}$,

$\alpha + \beta = -4$ and $\alpha\beta = -1$

Thus, the required quadratic equation is :

$x^2 - \left(\alpha + \beta\right)x + \alpha\beta = 0$ Or , $x^2 + 4x - 1 = 0$

Example 4: The quadratic equations $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ have a common root and the second equation has equal roots, show that $b + q = \frac{ap}{2}$.

Sol: By considering α and β to be the roots of eq. (i) and α to be the common root, we can solve the problem by using the sum and product of roots formulae.

The given quadratic equations are

$x^2 - ax + b = 0$ ... (i)

$x^2 - px + q = 0$ ... (ii)

Consider α and β to be the roots of eq. (i) and α to be the common root.

From (i) $\alpha + \beta = a$, $\alpha = b$

From (ii) $2\alpha = p$, $\alpha^2 = q$

$\therefore$ $b + q = \alpha\beta + \alpha^2 = \alpha\left(\alpha + \beta\right) = \frac{ap}{2}$

Example 5: If α and α^n are the roots of the quadratic equation $ax^2 + bx + c = 0$, then show that $\left(ac^n\right)^{\frac{1}{n+1}} + \left(a^nc\right)^{\frac{1}{n+1}} + b = 0$.

Sol: By using the sum and product of roots formulae we can prove this.

Given that α and α^n are the roots.

$\therefore \alpha.\alpha^n = \frac{c}{a}$

$\Rightarrow \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$

And $\alpha + \alpha^n = \frac{-b}{a}$

$$\Rightarrow \left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{1}{n+1}} = \frac{-b}{a}$$

Or $\left(ca^n\right)^{\frac{1}{n+1}} + \left(c^n a\right)^{\frac{1}{n+1}} + b = 0$.

Example 6: $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ have a non zero common root and a $\neq$ b, then show that the other roots are roots of the equation, $x^2 + cx + ab = 0$, $c \neq 0$.

Sol: By considering α to be the common root of the equations and β, γ to be the other roots of the equations respectively, and then by using the sum and product of roots formulae we can prove this.

Further, $\alpha + \beta = -a$ and $\alpha\beta = bc$;

$\alpha + \gamma = -b$, $\alpha.\gamma = ca$

$2\alpha + \beta + \gamma = -(a+b)$ and $\alpha^2\beta\gamma = abc^2$ ... (i)

$\therefore \ \beta + \gamma = c - 2c = -c$ ... (ii)

and $c^2\beta\gamma = c^2ba$

$\therefore \ \beta\gamma = ab$ ... (iii)

From equation (ii) and (iii),

β and γ are the roots of the equation $x^2 + cx + ab = 0$

Example 7: Solve for x when

$\log_{10}\left(\sqrt{\log_{10} x}\right) = \log_{\left(x^2\right)} x : x > 1$

Sol: By using the formula

$\log_a M^x = x \log_a .M$ and $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$ we can solve this problem.

$$\log_{10}\left(\sqrt{\log_{10} x}\right) = \log_{x^2} x = \frac{\log_{10} x}{\log_{10} x^2} = \frac{1}{2}$$

Let $y = \log_{10} x$; then $\frac{1}{2} = \log_{10} \sqrt{y}$;

$\Rightarrow \frac{1}{2} = \frac{1}{2}\log_{10} y \ \therefore y = 10$ and thus $x = 10^{10}$

Example 8: If α is a root of the equation $4x^2 + 2x - 1 = 0$, then prove that $4\alpha^3 - 3\alpha$ is the other root.

Sol: Consider α, β to be the two roots of the given equation $4x^2 + 2x - 1 = 0$, therefore, by solving this we can get the result.

$\therefore \alpha + \beta = \dfrac{-1}{2}$ and $4\alpha^2 + 2\alpha - 1 = 0$

$$4\alpha^3 - 3\alpha = \left(4\alpha^2 + 2\alpha - 1\right)\left(\alpha - \frac{1}{2}\right) - \left(\alpha + \frac{1}{2}\right) = \beta$$

Hence $4\alpha^3 - 3\alpha$ is the other root.

Example 9: the roots of $\dfrac{1}{x+p} + \dfrac{1}{x+q} = \dfrac{1}{r}$ are equal in magnitude, but opposite in sign, show that p+ q = 2r and the product of the roots $= -\dfrac{p^2+q^2}{2}$

Sol: By considering α and $-\alpha$ as the roots of the given equation and then by using the sum and product of roots formulae we can solve it.

$$\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r} \quad \text{....(i)}$$

$\Rightarrow (x + q + x + p)r = x^2 + (p + q)x + pq$

$\Rightarrow x^2 + (p + q - 2r)x + pq - r(p + q) = 0$

Since, its roots are equal in magnitude but opposite in sign

consider roots are $\alpha, -\alpha$.

$\therefore \alpha - \alpha = p + q - 2r$

$\Rightarrow p + q = 2r$

Product of roots = $pq - r(p + q)$

$$= pq - \frac{(p+q)^2}{2} = -\frac{p^2+q^2}{2}$$

Example 10: If α, β are the roots of $x^2 + px + q = 0$. Prove that $\dfrac{\alpha}{\beta}$ is a root of $qx^2 + \left(2q - p^2\right)x + q = 0$

Sol: For $\dfrac{\alpha}{\beta}$ to be a root of $qx^2 + \left(2q - p^2\right)x + q = 0$ it must satisfy thegiven equation. Hence by using sum and product of roots formula, we can find out the value of $\dfrac{\alpha}{\beta}$.

As $\alpha_1 \beta_1 \gamma_1$ are the roots of $x^2 + px + q = 0$

$\alpha + \beta = -p$ and $\alpha\beta = q$

We need to show that $\dfrac{\alpha}{\beta}$ is a root of

$ax^2 + \left(2q - p^2\right)x + q = 0$

That means

$$q\frac{\alpha^2}{\beta^2} + \left(2q - p^2\right)\frac{\alpha}{\beta} + q = 0$$

i.e., $q\alpha^2 + \left(2q - p^2\right)\alpha\beta + q\beta^2 = 0$

i.e., $q\left(\alpha^2 + 2\alpha\beta + \beta^2\right) - p^2\alpha\beta = 0$

i.e., $q\left(\alpha + \beta\right)^2 - p^2\alpha\beta = 0$

i.e., $p^2q - p^2q = 0$ which is obviously true.

Example 11: Find the value of 'a' for which

$3x^2 + 2\left(a^2 + 1\right)x + a^2 - 3a + 2 = 0$ possesses roots with opposite signs.

Sol: Roots of the given equation are of opposite sign, hence, their product is negative and the discriminant is positive.

$\therefore$ Product of roots is negative

$\therefore \dfrac{a^2 - 3a + 2}{3} < 0$

= (a – 2) (a – 1) < 0 and a $\in$ (1, 2) And D > 0

$4\left(a^2 + 1\right) - 4.3\left(a^3 - 3a + 2\right) > 0$

This equation will always hold true for a ∈ (1, 2)

Example 12: If x is real, find the range of the quadratic expression $\dfrac{x^2 + 14x + 9}{x^2 + 2x + 3}$.

Sol: By considering $\dfrac{x^2 + 14x + 9}{x^2 + 2x + 3} = y$ and as x is real its discriminant must be greater than or equal to zero.

Let $\dfrac{x^2 + 14x + 9}{x^2 + 2x + 3} = y$

$\Rightarrow x^2 + 14x + 9 = x^2y + 2xy + 3y$

$\Rightarrow x^2 \ (1 - y) + 2x(7 - y) + 3\ (3 - y) = 0$

Hence, $D \ge 0$

$4(7 - y)\ \beta - 12(1 - y)(3 - y)\ y_2\ 0$

$-\ 2y\gamma - 2y + 40\ y_3\ 0$

$\Rightarrow y^2 + y - 20 \le 0$

$\Rightarrow$ (y + 5)(y – 4) ≤ 0 $\Rightarrow$ –5 ≤ y ≤ 4

JEE Advanced/Boards

Example 1: Prove that y = $\dfrac{ax^2 + x - 2}{a + x - 2x^2}$

takes all real values for x $\in$ R only if a $\in$ [1, 3]

Sol: Consider y $\in$ R and also that given as x $\in$ R. Hence, the discriminant of y = $\dfrac{ax^2 + x - 2}{a + x - 2x^2}$ must be greater than or equal to zero.

Let y $\in$ R; then,

$y = \dfrac{ax^2 + x - 2}{a + x - 2x^2}$ for some x $\in$ R

(a + 2y) x^2 + (1 – y)x – 2 – ay = 0

$\therefore$ $(1 - y)^2 + 4(a + 2y)(2 + ay) \ge 0$; $\forall y \in R$

Or $\left(8a + 1\right)y^2 + \left(4a^2 + 14\right)y + 8a + 1 \ge 0$

$\forall y \in R$ $\therefore$ 8a + 1 > 0 and

$(4a2 + 14)^2 - 4(8a+1)^2 \le 0$

Or $a > -\dfrac{1}{8}$ and $\left(a^2 - 4a + 3\right)\left(a + 2\right) \le 0$

Or $a > -\dfrac{1}{8}$ and (a – 3)(a – 1) < 0

i.e. a $\in$ [1, 3]

Example 2: Find the value of x if

2x + 5 + |x^2 + 4x +3| = 0

Sol: For 2x + 5 + |x^2 + 4x +3| = 0, 2x + 5 must be less than or equal to zero. And whether x^2 + 4x +3 will be positive or negative depends on the value of x.

$\Rightarrow$ 2x + 5 + |x^2 + 4x + 3| = 0

Case -I When x ≤ –3 or x ≥ -1

x^2 + 4x + 3 + 2x + 5 = 0

(x + 2)(x + 4) = 0 ; $\Rightarrow$ x = –4

Case-II –3 < x < -1

x^2 + 4x + 3 = 2x + 5 ; x^2 + 2x – 2 = 0

$\Rightarrow x = \dfrac{-1 - \sqrt{3}}{2}$

Example 3: Solve the equation $2^{|x+1|} - 2^x = \left|2^x - 1\right| + 1$

Sol: By taking the conditions as $x \ge 0$ and $x \le 0$ we can solve this problem.

$$\left|2^x - 1\right| = \begin{cases} 2^x - 1 & \text{if } x \ge 0 \\ -\left(2^x - 1\right) & \text{if } x < 0 \end{cases}$$

Case-I $x \ge 0$

$2^{|x+1|} - 2^x = 2^x - 1 + 1$

This is true $\forall\ x \geq 0$

Case-II $x < 0$; $2^{|x+1|} - 2^x = 1 - 2^x + 1$

$2^{|x+1|} = 2$; $|x + 1| = 1$; $x = -2$

Example 4: For what values of a are the roots of the equation $(a+1)x^2 - 3ax + 4a = 0 (a \neq -1)$ real and less than 1?

Sol: Here the roots of the given equation have to be real and less than 1, therefore $D \geq 0$; $f(1).(a + 1) > 0$ and the x-coordinate of the vertex < 1 .

Let $f(x) = (a + 1)x^2 - 3ax + 4a$

$D \geq 0$; $f(1).(a + 1) > 0$ and x-coordinate of vertex < 1

$D \geq 0 \Rightarrow -\frac{16}{7} \leq a \leq 0$ ... (i)

$(a + 1)f(1) > 0 \Rightarrow (2a + 1)(a + 1) > 0$

$\Rightarrow a < -1$ or $a > -\frac{1}{2}$ ... (ii)

By (i) & (ii) $a \in \left[-\frac{16}{7}, -1\right) \cup \left(\frac{-1}{2}, 0\right]$ ... (iii)

Since x coordinate of vertex x < 1, we have

Combined with (iii) we get: $a \in \left(\frac{-1}{2}, 0\right]$

Example 5: Find all the values of x satisfying the inequality $\Rightarrow \left(2x - \frac{3}{4}\right) > 2$.

Sol: First, we can reduce the given inequality as $\log_x\left(2x - \frac{3}{4}\right) > \log_x x^2$. Then, by applying each case of $x > 1$ and $\frac{3}{8} < x < 1$ we can solve this problem.

$\log_x\left(2x - \frac{3}{4}\right) > 2$ ($\because x \neq 1$ and $x > \frac{3}{8}$)

$\Rightarrow \log_x\left(2x - \frac{3}{4}\right) > \log_x x^2$ ... (i)

Case I : Let $x > 1$; $2x - \frac{3}{4} > x^2$

Or $4\Rightarrow - 8x + 3 < 0$

Or $4\left(x - \frac{1}{2}\right)\left(x - \frac{3}{2}\right) < 0 \ \therefore x \in \left(1, \frac{3}{2}\right)$

Case II : Let $\frac{3}{8} < x < 1$; $2x - \frac{3}{4} < x^2$

Or $4x^2 - 8x + 3 > 0$

$(2x - 3)(2x - 1) > 0$; $\therefore x \in \left(\frac{3}{8}, \frac{1}{2}\right)$

Example 6: Solve the equation

$\left(2x^2 - 3x + 1\right)\left(2x^2 + 5x + 1\right) = 9x^2$

Sol: This problem is solved by dividing both sides by x^2 and taking $y = 2x + \frac{1}{x}$

$\left(2x^2 - 3x + 1\right)\left(2x^2 + 5x + 1\right) = 9x^2$ ... (i)

Clearly, x = 0 does not satisfy (i), Therefore, we can rewrite equation (i) as

$\left(2x - 3 + \frac{1}{x}\right)\left(2x + 5 + \frac{1}{x}\right) = 9$ ... (ii)

$\therefore (y - 3)(y + 5) = 9$ where $y = 2x + \frac{1}{x}$

Or $y^2 + 2y - 24 = 0$

$\Rightarrow (y + 6)(y - 4) = 0 \Rightarrow y = 4, -6$

When $y = -6$, $2x + \frac{1}{x} = -6$

$\Rightarrow 2x^2 + 6x + 1 = 0$

$\Rightarrow x = \frac{-6 \pm \sqrt{36 - 8}}{4} = \frac{-3 \pm \sqrt{7}}{2}$

When $y = 4$, $2x + \frac{1}{x} = 4$

$\Rightarrow 2x^2 - 4x + 1 =$

$\Rightarrow x = \frac{4 \pm \sqrt{16 - 8}}{4} = \frac{-2 \pm \sqrt{7}}{2}$

Thus, the solutions are $x = \frac{-3 \pm \sqrt{7}}{2}, \frac{-2 \pm \sqrt{2}}{2}$.

Example 7: If α and β are the roots of the equation $ax^2 + bx + c = 0$, then find the equation whose roots are, $\alpha^2 + \beta^2, \frac{1}{\alpha^2} + \frac{1}{\beta^2}$?

Sol: Using the sum and product of roots formulae, we can get the value of α and β and then by using

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

we can arrive at the required equation.

Let S be the sum and P be the product of the roots

$\alpha^2+\beta^2, \dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$

As S = $\left(\alpha^2+\beta^2\right)+\dfrac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \left(\dfrac{b^2-2ac}{a^2}\right)+\left(\dfrac{b^2-2ac}{c^2}\right)$

$= \left(b^2-2ac\right)\left(\dfrac{a^2+c^2}{a^2c^2}\right)$

Now the product of the roots will be

P = $\dfrac{\left(\alpha^2+\beta^2\right)}{\alpha^2\beta^2} = \left(\dfrac{b^2-2ac}{a^2}\right)\times\dfrac{\frac{1}{c^2}}{a^2}$

Hence equation is

$$(acx)^2-\left(b^2-2ac\right)\left(a^2+c^2\right)x+\left(b^2-2ac\right)^2=0$$

Example 8: If a_n are the roots of $a x^2$ + bx + c = 0 and γ, δ the roots of $\ell x^2+mx+n=0$, then find the equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$?

Sol: In the method similar to example 8.

Here S = $\left(\alpha\gamma+\beta\delta\right)+\left(\alpha\delta+\beta\gamma\right)$

$=\alpha\left(\gamma+\delta\right)+\beta\left(\gamma+\delta\right)=\left(\alpha+\beta\right)\left(\gamma+\delta\right)$

$=\left(\dfrac{-b}{a}\right)\left(\dfrac{-m}{\ell}\right)=\dfrac{bm}{a\ell}$ ... (i)

Also $P=\left(\alpha\gamma+\beta\delta\right)\left(\alpha\delta+\beta\gamma\right)$

$=\left(\alpha^2+\beta^2\right)\gamma\delta+\alpha\beta\left(\gamma^2+\delta^2\right)$ ... (ii)

$=b^2 n\ell+m^2ac-4\ acn\ \ell\,/\,a^2\ \ell^2$

Hence, from $x^2 - Sx + P = 0$

$$x^2-\frac{bm}{a\ell}x+\frac{b^2n\ell+m^2ac-4acn\ell}{a^2\ \ell^2}=0$$

Example 9: The expression $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$ must have a common factor and $a \neq 0$, Find the common factor and then the common root.

Sol: Here consider $(x-\alpha)$ to be the common factor then $x = \alpha$ becomes the root of the corresponding equation. Hence, by substituting $x = \alpha$ in both the equations and solving we will get the result.

$\therefore\ \alpha^2-11\alpha+a=0,\ \alpha^2-14\ \alpha+2a=0$

Subtracting $3\alpha-a=0\Rightarrow\alpha=\dfrac{a}{3}$

Hence $\dfrac{a^2}{9}-11\ \dfrac{a}{3}+a=0$, a = 0 or a = 24

Since $a \neq 0$, a = 24

$\therefore$ the common factor of $\begin{cases}x^2-11x+24=0\\ x^2-14x+48=0\end{cases}$

is clearly x – 8 or the common root is x = 8.

Note: A shorter method is in eliminating a from both expressions

$\left.\begin{array}{l}2x^2-22x+2a\\ x^2-14x+2a\end{array}\right\};x^2-8x=0\Rightarrow x(x-8)=0$

$\therefore x \neq 0, \therefore (x-8)$

Example 10: α and β are the roots of

$a x^2$ + bx+ c= 0 and γ, δ be the roots of

$p x^2$ +qx + r = 0; . If $\alpha,\beta,\gamma,\delta$ are

in A.P., then find the ratio of their Discriminants.

Sol: As $\alpha,\beta,\gamma,\delta$ are in A.P., hence, $\beta-\alpha=\delta-\gamma$, by squaring both side and substituting their values we will get the result.

Consider D_1 and D_2 be their discriminants respectively

We have $\alpha+\beta=\dfrac{-b}{a}, \alpha\beta=\dfrac{c}{a}$

and $\gamma+\delta=\dfrac{-q}{p}, \gamma\ \delta=\dfrac{c}{a}$

Since, $\alpha,\beta,\gamma,\delta$ are in A.P.

$\Rightarrow\beta-\alpha=\delta-\gamma;\left(\beta-\alpha\right)^2=\left(\delta-\gamma\right)^2$

$\left(\beta+\alpha\right)^2-4\alpha\beta=\left(\gamma+\delta\right)^2-4\gamma\delta$

$\Rightarrow\dfrac{b^2}{a^2}-\dfrac{4c}{a}=\dfrac{q^2}{p^2}-\dfrac{4r}{p}$

$\Rightarrow\dfrac{b^2-4ac}{a^2}=\dfrac{q^2-4qr}{p^2}$

$\dfrac{D_1}{a^2}=\dfrac{D_2}{p^2}\Rightarrow\dfrac{D_1}{D_2}=\dfrac{a^2}{p^2}$

Example 11: The equation $\dfrac{p}{2x}=\dfrac{a}{x+c}+\dfrac{b}{x-c}$ has two equal roots and $c\neq0$, then find the possible values of p?

Sol: For equal roots discriminant(D) must be zero.

As given $\dfrac{p}{2x}=\dfrac{a}{x+c}+\dfrac{b}{x-c}$

$\Rightarrow \frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$

$\Rightarrow p(x^2 - c^2) = 2(a + b)x^2 - 2c(a - b)x$

$\Rightarrow (2a + 2b - p)x^2 - 2c(a - b)x + pc^2 = 0$

For this equation to have equal roots

$4c^2(a-b)^2 - 4pc^2(2a+2b-p) = 0$

$\Rightarrow (a-b)^2 - 2p(a+b) + p^2 = 0$

$\Rightarrow p^2 - 2p(a+b) = -(a-b)^2$

$\Rightarrow p^2 - 2p(a+b) + (a+b)^2 = (a+b)^2 - (a-b)^2$

$[p - (a + b)]^2 = 4ab$

$\Rightarrow p - (a + b) = \pm 2\sqrt{ab}$

$\Rightarrow p = a + b \pm 2\sqrt{ab} = \left(\sqrt{a} \pm \sqrt{b}\right)^2$

Example 12: Solve $(x + 10)(x - 4)(x - 8)(x + 6) = 660$

Sol: By multiplying $(x + 10)(x - 8)(x - 4)(x + 6)$ we get $(x^2 + 2x - 80)(x^2 + 2x - 24) = 660$.

Therefore by putting $x^2 + 2x = y$ and using $x = \frac{-b \pm \sqrt{D}}{2a}$ we can solve this.

Put $x^2 + 2x = y$ ... (i)

$(y - 80)(y - 24) = 660$

$\Rightarrow y^2 - 104y + 1920 - 660 = 0$

$\Rightarrow y^2 - 104y + 1920 = 0$

$\Rightarrow (y - 90)(y - 14) = 0 \quad \Rightarrow y = 90$ or 14

When $y = 90$ (i) gives $x^2 + 2x - 90 = 0$

$x = \frac{-2 \pm \sqrt{4^2 - 4 \times (-90)}}{2} = -1 \pm \sqrt{94}$

When $y = 14$, (i) gives $x^2 + 2x - 14 = 0$

$x = \frac{-2 \pm \sqrt{4^2 - 4 \times (-14)}}{2} = -1 \pm 3\sqrt{2}$

$\frac{x}{x^2 - 5x + 9}$ The solutions are: $-1 \pm 3\sqrt{2}$ & $-1 \pm \sqrt{94}$

JEE Main/Boards

Exercise 1

Q.1 If the sum of the roots of the equation $px^2+qx+r=0$ be equal to the sum of their squares, show that $2pr = pq + q^2$

Q.2 Show that the roots of the equation $(a+b)^2x^2 - 2(a^2 - b^2)x + (a-b)^2 = 0$ are equal.

Q.3 Find the value of m, for which the equation $5x^2 - 4x + 2 + m(4x^2 - 2x - 1) = 0$ has

(i) equal roots

(ii) product of the roots as 2

(iii) The sum of the roots as 6

Q.4 If one root of the equation $5x^2 + 13x + k = 0$ be reciprocal of the other, find k.

Q.5 If the difference of the roots of $x^2 - px + q = 0$ is unity, then prove that $p^2 - 4q = 1$

Q.6 Determine the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

Q.7 If α and β be the roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$.

Q.8 Solve for $x : \frac{4x}{x^2 + 3} \geq 1$

Q.9 If c, d are the roots of the equation $(x - a)(x - b) - k = 0$ show that a, b are the roots of the equation $(x - c)(x - d) + k = 0$.

Q.10 Find the real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$.

Q.11 Let a, b, c, be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β.

Q.12 If a and b are integers and the roots of equation $x^2 + ax + b = 0$ are rational, show that they will be integers.

Q.13 For what values of m, can the following expression be split as product of two linear factors?

(i) $3x^2 - xy - 2y^2 + mx + y + 1$

(ii) $6x^2 - 7xy - 3y^2 + mx + 17y - 20$

Q.14 Prove that the expression $\frac{x^2 - 2x + 4}{x^2 + 2x + 4}$ lies between $\frac{1}{3}$ and 3 for all real values of x.

Q.15 Find all the values of a for which the roots of the equation (1 + a) x+ – 3ax + 4a = 0 exceed unity.

Q.16 If P(x) = ax^2 + bx + c and Q(x) = $-ax^2$ + bx + c where ac $\neq$ 0, show that the equation P(x). Q(x) = 0 has at least two real roots.

Q.17 If roots of the equation $ax^2 + 2bx + c = 0$ be α and β and those of the equation $Ax^2 + 2Bx + C = 0$ be $\alpha + k$ and $\beta + k$, prove that:

$$\frac{b^2 - ac}{B^2 - AC} = \left(\frac{a}{A}\right)^2$$

Q.18 Solve for x: $\left(15 + 4\sqrt{14}\right)^t + \left(15 - 4\sqrt{14}\right)^t = 30$ where t = $x^2 - 2|x|$.

Q.19 Show that (x – 2)(x - 3) – 8 (x – 1)(x – 3) + 9 (x – 1) (x – 2) = 2 x^2 is an identity.

Q.20 For which values of a does the equation $(1+a)\left(\frac{x^2}{x^2+1}\right)^2 - 3a\left(\frac{x^2}{x^2+1}\right) + 4a = 0$ have real roots?

Q.21 If one root of the equation $(l-m)x^2 + lx + 1 = 0$ be double of the other and if l be real, show that $m \le \frac{9}{8}$.

Q.22 If $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have a common root and $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in A.P show that a_1, b_1, c_1 are in G.P.

Q.23 If the ratio of the roots of the equation ax^2 + bx + c = 0 be equal to that of the roots of the equation $a_1x^2 + 2b_1x + c_1 = 0$, prove that $\left(\frac{b}{b_1}\right)^2 = \frac{ca}{c_1a_1}$

Q.24 Let α be a root of the equation $ax^2 + bx + c = 0$ and β be a root of the equation $-ax^2 + bx + c = 0$. Show that there exists a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ that lies between α and β $(\alpha, \beta \neq 0)$.

Q.25 Let a, b and c be integers with a > 1, and let p be a prime number. Show that if $ax^2 + bx + c$ is equal to p for two distinct integral values of x, then it cannot be equal to "2p" for any integral value of x. (a ≠ p).

Q.26 For a ≤ 0, determine all real roots of the equation: $x^2 - 2a|x - a| - 3a^2 = 0$.

Q.27 Find the values of a for which the inequality $x^2 + ax + a^2 + 6a < 0$ is satisfied for all x $\in$ (1, 2).

Q.28 If the roots of $2x^3 + x^2 - 7 = 0$ are α, β and $f(x) = x^2 + x(4 - 2k) + k^2 - 3k - 1 = 0$,

find the value of $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$.

Q.29 Find all values of k for which the inequality (x – 3k) (x – k – 3)< 0 is satisfied for all x in the interval [1, 3].

Exercise 2

Single Correct Choice Type

Q.1 If $a^2 + b^2 + c^2 = 1$ then ab + bc + ca lies in the interval (a, b, c, $\in$ R)

(A) $\left[\frac{1}{2}, 2\right]$ (B) [–1, 2] (C) $\left[-\frac{1}{2}, 1\right]$ (D) $\left[-1, \frac{1}{2}\right]$

Q.2 If P(x) = ax^2 + bx + c and Q(x) = $-ax^2$ + dx + c, where ac ≠ 0, then P(x). Q(x) = 0 has

(A) Exactly one real root

(B) At least two real roots

(C) Exactly are real roots

(D) All four are real roots

Q.3 If α and β be the roots of the equation $x^2 + 3x + 1 = 0$ then the value of $\left(\frac{\alpha}{1+\beta}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2$ is equal to

(A) 15 (B) 18 (C) 21 (D) None of these

Q.4 Let a > 0, b > 0 & c > 0. Then both the roots of the equation $ax^2 + bx + c = 0$

(A) Are real & negative

(B) Have negative real parts

(C) Are rational numbers

(D) None

Q.5 The equation $x^2 + bx + c = 0$ has distinct roots. If 2 is subtracted from each root, the results are reciprocals of the original roots. The value of $(b^2 + c^2 + bc)$ equals

(A) 7 (B) 9 (C) 10 (D) 11

Q.6 If a, b, c are real numbers satisfying the condition a + b + c = 0 then the roots of the quadratic equation $3ax^2 + 5bx + 7c = 0$ are:

(A) Positive (B) Real & distinct

(C) Negative (D) Imaginary

Q.7 If one solution of the equation $x^3 - 2x^2 + ax + 10 = 0$ is the additive inverse of another, then which one of the following inequalities is true ?

(A) $-40 < a < -30$ (B) $-30 < a < -20$

(C) $-20 < a < -10$ (D) $-10 < a < 0$

Q.8 The sum of the roots of the equation

$(x + 1) = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$ is

(A) 3954 (B) $\log_2 11$

(C) $\log_2 3954$ (D) Indeterminate

Q.9 The quadratic equation $x^2 - 1088x + 295680 = 0$ has two positive integral roots whose greatest common divisor is 16. The least common multiple of the two roots is

(A) 18240 (B) 18480

(C) 18960 (D) 19240

Q.10 If x is real and $4y^2 + 4xy + x + 6 = 0$, then the complete set of values of x for which y is real is

(A) $x \le -2$ or $x \ge 3$ (B) $x \le 2$ or $x \ge 3$

(C) $x \le -3$ or $x \ge 2$ (D) $-3 \le x \le 2$

Q.11 If exactly one root of the quadratic equation $f(x) = 0 - (a + 1)x + 2a = 0$ lies in the interval (0, 3) then the set of values 'a' is given by

(A) $(-\infty, 0) \cup (6, \infty)$ (B) $(-\infty, 0] \cup (6, \infty)$

(C) $(-\infty, 0] \cup [6, \infty)$ (D) (0, 6)

Q.12 If α, β are roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ then the number of integral values of m for which $\alpha, \beta \in (-2, 4)$ is

(A) 0 (B) 1 (C) 2 (D) All of these

Q.13 If x be the real number such that $x^3 + 4x = 8$ then the value of the expression $x^7 + 64x^2$ is

(A) 124 (B) 125 (C) 128 (D) 132

Q.14 If a and b are positive integers and each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then the smallest possible value of (a + b) is

(A) 3 (B) 4 (C) 5 (D) 6

Q.15 Let 'a' be a real number. Number of real roots of the equation $(x^2 + ax + 1)(3x^2 + ax - 3) = 0$ is

(A) At least two (B) At most two

(C) Exactly two (D) All four

Q.16 Let $f(x) = x^2 + ax + b$. If the maximum and the minimum values of f(x) are 3 and 2 respectively for $0 \le x \le 2$, then the possible ordered pair (s) of (a, b) is/are

(A) (–2, 3) (B) $\left(-\frac{3}{2}, 2\right)$ (C) $\left(-\frac{5}{2}, 3\right)$ (D) $\left(-\frac{5}{2}, 2\right)$

Previous Years' Questions

Q.1 The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is .... ***(2009)***

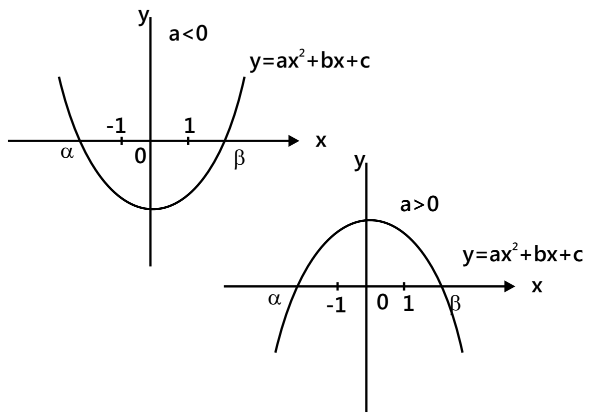
Q.2 Find the set of all x for which

$$\frac{2x}{2x^2 + 5x + 2} > \frac{1}{x + 1}$$ ***(1987)***

Q.3 Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β. ***(2001)***

Q.4 If α, β are the roots of $ax^2 + bx + c = 0$, ($a \neq 0$) and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$, ($A \neq 0$) for some constant δ, then prove that $\dfrac{b^2-4ac}{a^2}=\dfrac{B^2-4AC}{A^2}$ ***(2000)***

Q.5 Let a, b, c, be real. If $ax^2+bx+c=0$ has two roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that $1+\dfrac{c}{a}+\left|\dfrac{b}{a}\right|<0$ ***(1995)***



Assertion Reasoning Type

For the following questions, choose the correct answer from the codes (a). (b), (c) and (d) defined as follows.

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I.

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

Q.6 Let a, b, c, p, q be the real numbers. Suppose $f(k_2)$ are the roots of the equation

$x^2+2px+q=0$ and $\alpha, \dfrac{1}{\beta}$ are the roots of the equation $ax^2+2bx+c=0$, where $\beta^2 \notin \{-1,\ 0,\ 1\}$.

Statement-I : $(p^2-q)(b^2-ac)\geq 0$ and

Statement-II : $b \neq pa$ or $c \neq qa$ ***(2008)***

Q.7 The sum of all real roots of the equation $|x-2|^2+|x-2|-2=0$ is ...... ***(1997)***

Q.8 A value of b for which the equations $x^2+bx-1=0$, $x^2+x+b=0$ have one root in common is ***(2011)***

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

Q.9 Let α, β be the roots of the equation $x^2-px+r=0$ and $\dfrac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2-qx+r=0$. Then, the value of r is ***(2007)***

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

Q.10 If one root is square of the other root of the equation $x^2+px+q=0$, then the relation between p and q is ***(2004)***

(A) $p^3-3(3p-1)q+q^2=0$

(B) $p^3-q(3p+1)+q^2=0$

(C) $p^3+q(3p-1)+q^2=0$

(D) $p^3+q(3p+1)+q^2=0$

Q. 11 For all 'x', $x^2+2ax+(10-3a)>0$, then the interval in which 'a' lies is ***(2004)***

(A) $a < -5$ (B) $-5 < a < 2$

(C) $a > 5$ (D) $2 < a < 5$

Q.12 The set of all real numbers x for which $x^2-|x+2|+a>0$ is ***(2002)***

(A) $(-\infty, -2)\cup(2, \infty)$ (B) $(-\infty, -\sqrt{2})\cup(\sqrt{2}, \infty)$

(C) $(-\infty, -1)\cup(1, \infty)$ (D) $(\sqrt{2}, \infty)$

Q.13 The number of solutions of $\log_4(x-1)=\log_2(x-3)$ is ***(2001)***

(A) 3 (B) 1 (C) 2 (D) 0

Q.14 If α and β $(\alpha<\beta)$ are the roots of the equation $x^2+bx+c=0$ where $c < 0 < b$, then ***(2000)***

(A) $0<\alpha<\beta$ (B) $\alpha<0<\beta<|\alpha|$

(C) $\alpha<\beta<0$ (D) $\alpha<0\,|\alpha|<\beta$

Q.15 The equation $\sqrt{x+1}-\sqrt{x-1}=\sqrt{4x-1}$ has ***(1997)***

(A) No solution (B) One solution

(C) Two solution (D) More than two solution

Q.16 The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. Then the common root is ***(2008)***

(A) 1 (B) 4 (C) 3 (D) 2

Q.17 If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is ***(2009)***

(A) Greater than 4ab (B) Less than 4ab

(C) Greater than – 4ab (D) Less than – 4ab

Q.18 Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for , then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to : ***(2015)***

(A) 6 (B) – 6 (C) 3 (D) –3

Q.19 Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is ***(2014)***

(A) $\frac{\sqrt{34}}{9}$ (B) $\frac{2\sqrt{13}}{9}$

(C) $\frac{\sqrt{61}}{9}$ (D) $\frac{2\sqrt{17}}{9}$

Q.20 If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, a,b,c R, have a common root, then a : b : c is ***(2013)***

(A) 1 : 2 : 3 (B) 3 : 2 : 1

(C) 1 : 3 : 2 (D) 3 : 1 : 2

JEE Advanced/Boards

Exercise 1

Q.1 A quadratic polynomial $f(x) = x^2 + ax + b$ is formed with one of its zeros being $\frac{4+3\sqrt{3}}{2+\sqrt{3}}$ where a and b are integers Also, $g(x) = x^4 + 2x^3 - 10x^2 + 4x - 10$ is a biquadrate polynomial such that $g\left(\frac{4+3\sqrt{3}}{2+\sqrt{3}}\right) = c\sqrt{3} + d$ where c and d are also integers. Find the values of a, b, c and d.

Q.2 Find the range of values of a, such that f(x) $= \frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$ is always negative.

Q.3 Let a, b, be arbitrary real numbers. Find the smallest natural number 'b' for which the equation $x^2 + 2(a+b)x + (a-b+8) = 0$ has unequal real roots for all $a \in R$.

Q.4 When $y^2 + my + 2$ is divided by (y – 1) then the quotient is f(y) and the remainder is R_1. When $y^2 + my + 2$ is divided by (y + 1) then quotient is g(y) and the remainder is R_2. If $R_1 = R_2$, find the value of m.

Q.5 Find the value of m for which the quadratic equations $x^2 - 11x + m = 0$ and $x^2 - 14x + 2m = 0$ may have common root.

Q.6 The quadratic polynomial P(x) $= ax^2 + bx + C$ has two different zeroes including –2. The quadratic polynomial $Q(x) = ax^2 + cx + b$ has two different zeroes including 3. If α and β be the other zeroes of P(x) and Q(x) respectively then find the value of $\frac{\alpha}{\beta}$.

Instructions for Q.7 and Q.8

Let α, β, γ be distinct real numbers such that

$$a\alpha^2 + b\alpha + c = (\sin\theta)\alpha^2 + (\cos\theta)\alpha$$
$$a\beta^2 + b\beta + c = (\sin\theta)\beta^2 + (\cos\theta)\beta$$
$$a\gamma^2 + b\gamma + c = (\sin\theta)\gamma^2 + (\cos\theta)\gamma$$

(where a, b, c, $\in$ R.)

Q.7 $\left(\log_{|x+6|} 2\right).\log_2\left(x^2-x-2\right) \geq 1$

Q.8 If $\vec{V}_1 = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ makes an angle $\pi/3$ with $\vec{V}_2 = \hat{i} + \hat{j} + \sqrt{2}\,\hat{k}$ then find the number of values of $\theta \in [0, 2\pi]$.

Q.9 (a) If α, β are the roots of the quadratic equation $ax^2+bx+c=0$ then which of the following expressions in α, β will denote the symmetric functions of roots.

Give proper reasoning.

(i) $f(\alpha, \beta) = \alpha^2 - \beta$

(ii) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

(iii) $f(\alpha, \beta) = \ln\frac{\alpha}{\beta}$

(iv) $f(\alpha, \beta) = \cos(\alpha - \beta)$

(b) If (a, β) are the roots of the equation $x^2 - px + q = 0$, then find the quadratic equation the roots of which are $\left(\alpha^2-\beta^2\right)\left(\alpha^3-\beta^3\right)$ & $\alpha^3\beta^2 - \alpha^2\beta^3$.

Q.10 Find the product of the real roots of the equation $x^2+18x+30 = 2\sqrt{x^2+18x+45}$

Q.11 Let f(x) = $\frac{\sqrt{x^2+ax+4}}{\sqrt{x^2+bx+4}}$ is defined for all real, then find the number of possible ordered pairs

(a – b) (where a, b, $\in$ I).

Q.12 If the equation $9x^2-12ax+4-a^2=0$ has a unique root in (0, 1) then find the number of integers in the range of a.

Q.13 (a) Find all real numbers x such that.

$$\left(x-\frac{1}{x}\right)^{\frac{1}{2}} + \left(1-\frac{1}{x}\right)^{\frac{1}{2}} = x$$

(b) Find the minimum value of $\frac{\left(x+\frac{1}{x}\right)^6 - x^6 - \frac{1}{x^6} - 2}{\left(x+\frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}}$ for x > 0

Q.14 If the range of m, so that the equations

$\left(x^2+2mx+7m-12\right) = 0$

$\left(4x^2-4mx+5m-6\right) = 0$

have two distinct real roots, is (a, b) then find (a + b).

Q.15 Match the column

Column I	**Column II**
(A) Let α and β be the roots of a quadratic equation $4x^2-(5p+1)x+5p=0$ if $\beta = 1+\alpha$ Then the integral value of p, is	(p) 0
(B) Integers laying in the range of the expression $y = \frac{x^2-3x+4}{x^2+3x+4}$ is (are)	(q) 1
(C) Positive integral values of x satisfying $\frac{x+1}{x-1} \geq \frac{x+5}{x+1}$, is (are)	(r) 2
(D) The value of expression $\frac{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}+\sin\frac{4\pi}{7}}{\sin\frac{8\pi}{7}+\sin\frac{8\pi}{7}\sin\frac{2\pi}{7}}$, is	(s) 3 4

Q.16 Find the product of uncommon real roots of the two polynomials

$P(x) = x^4+2x^3-8x^2-6x+15$ and

$Q(x) = x^3+4x^2-x-10$.

Q.17 Solve the following where $x \in R$.

(a) $(x-1)\left|x^2-4x+3\right| + 2x^2+3x-5 = 0$

(b) $3\left|x^2+4x+2\right| = 5x-4$

(c) $\left|x^3+1\right| + x^2 - x - 2 = 0$

(d) $2^{(x+2)} - \left|2^{x+1}-1\right| = 2^{x+1}+1$

(e) For $a \le 0$, determine all real roots of the equation $x^2 - 2a\,|x-a| - 3a^2 = 0$.

Q.18 (a) Let α, β and γ are the roots of the cubic $x^3 - 3x^2 + 1 = 0$. Find a cubic whose roots are $\frac{\alpha}{\alpha-2}, \frac{\beta}{\beta-2}$ and $\frac{\gamma}{\gamma-2}$.

Hence or otherwise find the value of $(\alpha-2)(\beta-2)(\gamma-2)$.

(b) If α, β, γ are roots of the cubic 2011 $x^3 + 2x^2 + 1 = 0$, then find

(i) $(\alpha\beta)^{-1} + (\beta\gamma)^{-1} + (\gamma\alpha)^{-1}$ (ii) $\alpha^{-2} + \beta^{-2} + \gamma^{-1}$

Q.19 If the range of parameter t in the interval (0, 2π), satisfying

$$\frac{\left(-2x^2 + 5x - 10\right)}{(\sin t)x^2 + 2(1+\sin t)x + 9\sin t + 4}$$

for all real value of x is (a, b), then $(a+b) = k_\pi$.

Find the value of k.

Q.20 Find all numbers p for each of which the least value of the quadratic trinomial $4x^2 - 4px + p^2 - 2p + 2$ on the interval $0 \le x \le 2$ is equal to 3.

Q.21 Let $P(x) = x^2 + bx + c$ where b and c are integers. If P(x) is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$. Find the value of P(1).

Q.22 If α, β are the roots of the equation, $x^2 - 2x - a^2 + 1 = 0$ and γ, δ are the roots of the equation, $x^2 - 2(a+1)x + a(a-1) = 0$ such that $\alpha, \beta \in (\gamma, \delta)$ then find the value of 'a'.

Q.23 Let A denotes the set of values of x for which $\frac{x+2}{x-4} \le 0$ and B denotes the set of values of x for which $x^2 - ax - 4 \le 0$. If B is the subset of A, then find the number of possible integral values of a.

Q.24 The quadratic $ax^2 + bx - c = 0$ has two different roots including the root −2. The equation $ax^2 + cx + b = 0$ has two different roots including the root 3. The absolute value of the product of the four roots of the equation expressed in lowest rational is $\left(\frac{p}{q}\right)$. Find (p+ q).

Q.25 Find the complete set of real values of 'a' for which both roots of the quadratic equation $\left(a^2 - 6a + 5\right)x^2 - \sqrt{a^2 + 2a}\,x + \left(6a - a^2 - 8\right) = 0$ lie on either side of the origin.

Solve the inequality.

Q.26 $\left(\log_2 x\right)^4 \left(\log_{\frac{1}{2}} \frac{x^5}{4}\right)^2 - 20\log_2 x + 148 < 0$

Q.27 $(\log 100\,x)^2 + (\log 10\,x)^2 + \log x \le 14$

Q.28 $\log_{1/2}(x+1) > \log_2(2-x)$

Q.29 $\log_{1/5}\left(2x^2 + 5x + 1\right) < 0$

Exercise 2

Single Correct Choice Type

Q.1 Let r_1, r_2 and r_3 be the solutions of equation $x^3 - 2x^2 + 4x + 5074 = 0$ then the value of $(r_1+2)(r_2+2)(r_3+2)$ is

(A) 5050 (B) 5066 (C) −5050 (D) −5066

Q.2 For every $x \in R$, the polynomial $x^8 - x^5 + x^2 - x + 1$ is

(A) Positive

(B) Never positive

(C) Positive as well as negative

(D) Negative

Q.3 If the equation $a(x-1)^2 + b(x^2 - 3x + 2) + x - a^2 = 0$ is satisfied for all $x \in R$ then the number of ordered pairs of (a, b) can be

(A) 0 (B) 1 (C) 2 (D) Infinite

Q.4 The inequality The inequality $y(-1) \ge -4$, $y(1) \le 0$ and $y(3) \ge 5$ are known to hold for $y = ax^2 + bx + c$ then the least value of 'a' is :

(A) – 1/4 (B) −1/3 (C) 1/4 (D) 1/8

Q.5 If $x = \frac{4\lambda}{1+\lambda^2}$ and $y = \frac{2-2\lambda^2}{1+\lambda^2}$ where

λ is a real parameter, and $x^2 - xy + y^2$ lies between [a, b] then (a + b) is

(A) 8 (B) 10 (C) 13 (D) 25

Multiple Correct Choice Type

Q.6 If the quadratic equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root then the equation containing their other roots is/are:

(A) $x^2 + a(b+c)x - a^2bc = 0$

(B) $x^2 - a(b+c)x + a^2bc = 0$

(C) $a(b+c)x^2 - (b+c)x + abc = 0$

(D) $a(b+c)x^2 + (b+c)x - abc = 0$

Q.7 If one of the roots of the equation $4x^2 - 15x + 4p = 0$ is the square of the other, then the value of p is

(A) 125/64 (B) –27/8 (C) –125/8 (D) 27/8

Q.8 For the quadratic polynomial f(x) = $4x^2 - 8kx + k$, the statements which hold good are

(A) There is only one integral k for which f(x) is non negative $\forall x \in R$

(B) for k < 0 the number zero lies between the zeros of the polynomial.

(C) f(x) = 0 has two distinct solution in (0, 1) for k $\in$ (1/4, 4/7)

(D) Minimum value of y $\forall k \in R$ is k (1+ 12k)

Q.9 The roots of the quadratic equation $x^2 - 30x + b = 0$ are positive and one of them is the square of the other. If the roots are r and s with r > s then

(A) b + r – s = 145 (B) b + r + s = 50
(C) b – r – s = 100 (D) b – r + s = 105

Comprehension Type

Consider the polynomia

$$P(x) = (x - \cos 36°)(x - \cos 84°)(x - \cos 156°)$$

Q.10 The coefficient of x^2 is

(A) 0 (B) 1 (C) $-\frac{1}{2}$ (D) $\frac{\sqrt{5}-1}{2}$

Q.11 The absolute term in P(x) has the value equal to

(A) $\frac{\sqrt{5}-1}{4}$ (B) $\frac{\sqrt{5}-1}{16}$ (C) $\frac{\sqrt{5}+1}{16}$ (D) $\frac{1}{16}$

Assertion Reasoning Type

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I

(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I.

(C) Statement-I is true, statement-II is false.

(D) Statement-I is false, statement-II is true.

Q.12 Consider a cubic function

$f(x) = ax^3 + bx + c$ where a, b, c $\in$ R.

Statement-I: f(x) can not have 3 non - negative real roots.

Statement-II: Sum of roots is equal to zero.

Q.13 Consider two quadratic functions

$f(x) = ax^2 + ax + (a+b)$ and g(x) = $ax^2 + 3ax + 3a + b$, where a and b non-zero real numbers having same sign.

Statement-I: Graphs of the both y = f(x) and y = g(x) either completely lie above x-axis or lie completely below x-axis $\forall x \in R$ .because

Statement-II: If discriminant of f(x), D < 0, then y = f(x) $\forall x \in R$ is of same sign and f(x+1) will also be of same sign as that of f(x) . $\forall x \in R$

Match the Columns

Q.14 It is given that α, β $(\beta \geq \alpha)$ are the roots of the equation if $f(x) = ax^2 + bx + c$. Also a f(t) > 0.

Match the condition given in column I with their corresponding conclusions given in column II.

Column I		**Column II**	
(A)	a > 0 and $b^2 > 4ac$	(p)	$t \neq \alpha$
(B)	a > 0 and $b^2 = 4ac$	(q)	no solution
(C)	a < 0 and $b^2 > 4ac$	(r)	$\alpha < t < \beta$
(D)	a < 0 and $b^2 = 4ac$	(s)	t < α or r > β

Q.15 Match the conditions on column I with the intervals in column II.

Let $f(x) = x^2 - 2px + p^2 - 1$, then

Column I		Column II	
(A)	Both the roots of f(x) = 0 are less than 4, if $p \in R$	(p)	$(-1, \infty)$
(B)	Both the roots of f(x) = 0 are greater than –2 if $p \in R$	(q)	$(-\infty, 3)$
(C)	exactly one root of f(x) = 0 lie in (–2, 4), if $p \in R$	(r)	(0, 2)
(D)	1 lies between the roots of f(x) = 0, if $p \in R$	(s)	(–3, –1) U (3,5)

Q.16

Column I		Column II	
(A)	The minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}}$ for x>0	(p)	2
(B)	The integral values of the parameters c for which the inequality $1+\log_2\left(2x^2+2x+\frac{7}{2}\right) \geq \log_2(cx^2+c)$ has at least one solution is	(q)	4
(C)	Let P(x) = x^2+bx+c, where b and c are integers. If P(x) is a factor of both x^4+6x^2+25 and $3x^4+4x^2+28x^5$, then the value of P(1) equals	(r)	6
		(s)	8

Q.17

Column I		Column II	
(A)	α, β are the roots of the equation K $(x^2-x) + x + 5 = 0$. If K_1 & K_2 are the two values of K for which the roots α, β are Connected by the relation $(\alpha / \beta) + (\beta / \alpha) = 4/5$. The value of $(K_1/K_2)+(K_2/K_1)$ equals.	(p)	146
(B)	If the range of the function f(x) = $\dfrac{x^2+ax+b}{x^2+2x+3}$ is [-5, 4], Then, the value of a^2+b^2 equals to	(q)	254
(C)	Suppose a cubic polynomial f(x) = $x^3+px^2+qx+72$ is divisible by both x^2+ax+b and x^2+bx+a (where a, b, p, q are cubic polynomial and a ≠ b). The sum of the squares of the roots of the cubic polynomial, is	(r)	277
		(s)	298

Previous Years' Questions

Q.1 Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations

$3x - y - z = 0, -3x + z = 0, -3x + 2y + z = 0$. Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is ……… ***(2009)***

Q.2 If $x^2 - 10ax - 11b = 0$ have roots c and d. $x^2 - 10cx - 11d = 0$ have roots a and b, then find a + b + c + d. ***(2006)***

Q.3 If $x^2 + (a-b)x + (1-a-b) = 0$ where a, b, ∈ R, then find the values of a for which equation has real and unequal roots for all values of b. ***(2003)***

Q.4 Let $-1 \leq p < 1$. Show that the equation $4x^3 - 3x - p=0$ has a unique root in the interval [1/2, 1] and identify it, ***(2001)***

Q.5 Let $f(x) = Ax^2 + Bx + C$ where, A, B, C, are real numbers. Prove that if f(x) is an integer whenever x is an integer, then the numbers 2A, A + B and C are all integers. Conversely prove that if the numbers 2A, A + B and C are all integers, then f(x) is an integer whenever x is an integer. ***(1998)***

Q.6 Find the set of all solution of the equation $2^{|y|} - \left|2^{y-1} - 1\right| = 2^{y-1} + 1$ ***(1997)***

Q.7 Solve x in the following equation

$\log_{(2x+3)}\left(6x^2+23x+21\right) = 4 - \log_{(3x+7)}\left(4x^2+12x+9\right)$ ***(1987)***

Passage Based Questions

Read the following passage and answer the questions.

Paragraph 1: If a continuous f defined on the real line R, assumes positive and negative values in R, then the equation f(x) = 0 has a root in R. For example, If it is known that a continuous function f on R is positive at some point and its minimum value is negative.

Then the equation f(x) = 0 has a root in R. Consider $f(x) = ke^x - x$ for all real x where k is real constant. ***(2007)***

Q.8 The line y = x meets $y = ke^x$ for k ≤ 0 at

(A) No point (B) One point

(C) Two point (D) More than two points

Q.9 The positive value of k for which $ke^x - x = 0$ has only one root is

(A) $\frac{1}{e}$ (B) 1 (C) e (D) $\log_e 2$

Q.10 For k > 0, the set of all values of k for which $ke^x - x = 0$ has two distinct root, is

(A) $\left(0, \frac{1}{e}\right)$ (B) $\left(\frac{1}{e}, 1\right)$ (C) $\left(\frac{1}{e}, \infty\right)$ (D) (0, 1)

Q.11 Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of f(x) and let t = |s| . The real numbers s lies in the interval ***(2010)***

(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, -\frac{3}{4}\right)$

(C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

Q.12 The area bounded by the curve y = f(x) and the lines x= 0, y = 0 and x = t, lies in the interval ***(2010)***

(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$

(C) (9, 10) (D) $\left(0, \frac{21}{64}\right)$

Q.13 The function f'(x) is ***(2010)***

(A) Increasing in $\left(-t, -\frac{1}{4}\right)$ and Decreasing $\left(-\frac{1}{4}, -t\right)$

(B) Decreasing in $\left(-t, -\frac{1}{4}\right)$ and Increasing $\left(-\frac{1}{4}, -t\right)$

(C) Increasing in (–t, t)

(D) Decreasing in (–t, t)

Q.14 Let α and β be the roots of $x^2 - 6x - 2 = 0$, with α >β. If $a_n = \alpha^n - \beta^n$ for n ≥ 1, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is. ***(2011)***

(A) 1 (B) 2 (C) 3 (D) 4

Q.15 Let p and q be real numbers such that p ≠ 0, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are non-zero complex numbers satisfying α +β = −p and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is ***(2010)***

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$

(B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$

(D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

Q.16 If a, b, c, are the sides of a triangle ABC such that $x^2 - 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ has real roots, then ***(2006)***

(A) $\lambda < \frac{4}{3}$ (B) $\lambda < \frac{5}{3}$

(C) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$

Q.17 If b > a, then the equation (x – a)(x – b) – 1 = 0 has ***(2000)***

(A) Both roots in (a, b)

(b) Both roots in $(-\infty, a)$

(C) Both roots in $(b + \infty)$

(D) One root in $(-\infty, a)$ and the other in (b, ∞)

Q.18 If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then ***(1999)***

(A) a < 2 (B) 2 ≤ a ≤ 3 (C) 3 < a ≤ 4 (D) a > 4

Q.19 Let f(x) be a quadratic expression which is positive for all real values of x. If g(x) = f(x) + f'(x) + f''(x), then for any real x ***(1990)***

(A) g(x) < 0 (B) g(x) > 0 (C) g(x) = 0 (D) g(x) ≥ 0

Q.20 Let α, β be the roots of the equation $x^2 - px + r = 0$ and be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is ***(2007)***

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$

(C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

Q.21 Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and α, $\frac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta2 \notin \{-1, 0, 1\}$.

Statement-I: $(p^2 - q)(b^2 - ac) \geq 0$ and

Statement-II: $b \neq pa$ or $c \neq qa$ ***(2008)***

(A) Statement-I is True, statement-II is True; statement-II is a correct explanation for statement-I

(B) Statement-I is True, statement-II is True; statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is True, statement-II is False

(D) Statement-I is False, statement-II is True

Q.22 Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is ***(2011)***

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

Q.23 A value of b for which the equations

$x^2 + bx - 1 = 0$

$x^2 + x + b = 0$,

have one root in common is ***(2011)***

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$

(C) $i\sqrt{5}$ (D) $\sqrt{2}$

Questions

JEE Main/Boards

Exercise 1

Q. 12 Q. 15 Q. 18
Q. 20 Q. 22 Q. 24

Exercise 2

Q. 3 Q. 8 Q. 9
Q. 11 Q. 14 Q. 17

Previous Years' Questions

Q. 2 Q. 5 Q. 6
Q. 15

JEE Advanced/Boards

Exercise 1

Q. 7 Q. 10 Q. 13
Q. 17 Q. 19 Q. 22
Q. 25 Q. 30

Exercise 2

Q. 5 Q. 9 Q. 12
Q. 16

Previous Years' Questions

Q. 5 Q. 6 Q. 8
Q. 13

Answer Key

JEE Main/Boards

Exercise 1

Q.3 (i)m = $\frac{-6}{5}$, 1 (ii) m = $\frac{-8}{9}$ (iii) m = $\frac{-13}{11}$

Q.4 K = 5

Q.6 m = $\frac{-11}{8}, \frac{7}{4}$

Q.7 $qx^2 - p(q+1)x + (q+1)^2 = 0$

Q.8 $1 \le x \le 3$

Q.10 $-1 \le x < 1$ or $2 < x \le 4$

Q.11 $\alpha^2 \beta, \alpha \beta^2$

Q.13 (i) $4, \frac{-7}{2}$ (ii) $7, \frac{98}{3}$

Q.15 $a \in \left[\frac{-16}{7}, -1\right]$

Q.18 $x = \pm 1, \pm\left(1+\sqrt{2}\right)$

Q.20 0

Q.26 $-a\left(1+\sqrt{6}\right), a\left(1+\sqrt{2}\right)$

Q.27 $\frac{-7-3\sqrt{5}}{2} \le a \le -4+2\sqrt{3}$

Q.28 –3

Q.29 $k \in \left(0, \frac{1}{3}\right)$

Exercise 2

Single Correct Choice Type

Q.1 C	**Q.2** B	**Q.3** B	**Q.4** B	**Q.5** A	**Q.6** C
Q.7 D	**Q.8** B	**Q.9** B	**Q.10** A	**Q.11** B	**Q.12** D
Q.13 C	**Q.14** D	**Q.15** A	**Q.16** B		

Previous Years' Questions

Q.1 k = 2

Q.2 $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

Q.3 $x = \alpha^2 \beta, \alpha \beta^2$

Q.6 B

Q.7 4

Q.8 B	**Q.9** D	**Q.10** A	**Q.11** B	**Q.12** B	**Q.13** B
Q.14 B	**Q.15** A	**Q.16** D	**Q.17** C	**Q.18** C	**Q.19** B
Q.20 A					

JEE Advanced/Boards

Exercise 1

Q.1 a = 2, b = –11, c = 4, d = –1

Q.2 $a \in \left(-\infty, -\frac{1}{2}\right)$;

Q.3 5

Q.4 0

Q.5 0 or 24

Q.6 11

Q.7 $x < -7, -5 < x \le -2, x \ge 4$

Q.8 3

Q.9 (a) (ii) and (iv); (b) $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

Q.10 20

Q.11 135

Q.12 10

Q.13 (a) $x = \frac{\sqrt{5}+1}{2}$; (b) (a) $y_{min} = 6$

Q.14 6

Q.15 (A) S; (B) Q,R,S,T (C) R, S ; (D) P

Q.16 6

Q.17 (a) x = 1 ; (b) x = 2 or 5 ; (c) x = –1 or 1 (d) $x \geq -1$ or x = –3 ; (e) $x = \left(1-\sqrt{2}\right)a$ or $\left(\sqrt{6}-1\right)a$

Q.18 (a) $3y^3 - 9y^2 - 3y + 1 = 0$; $\left(\alpha-2\right)\left(\alpha-2\right)\left(\gamma-2\right) = 3$; (b) (i) 2 ; (ii) – 4

Q.19 3

Q.20 $a = 1-\sqrt{2}$ or $5+\sqrt{10}$

Q.21 P(1) = 4

Q.22 $a \in \left(-\frac{1}{4}, 1\right)$

Q.23 3

Q.24 115

Q.25 $\left(-\infty,-2\right] \cup \left[0,1\right) \cup \left(2,4\right) \cup \left(5,\infty\right)$

Q.26 $x \in \left(\frac{1}{16}, \frac{1}{8}\right) \cup \left(8, 16\right)$

Q.27 $\frac{1}{\sqrt{10}^9} \leq x \leq 10$

Q.28 $-1 < x < \frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2} < x < 2$

Q.29 $\left(-\infty, -2.5\right) \cup \left(0\ ,\ \infty\right)$

Exercise 2

Single Correct Choice Type

Q.1 C **Q.2** A **Q.3** B **Q.4** D **Q.5** A

Multiple Correct Choice Type

Q.6 B, D **Q.7** C, D **Q.8** A,B,C **Q.9** A, D

Comprehenstion Type

Q.10 A **Q.11** B

Assertion Reasoining Type

Q.12 D **Q.13** A

Match the Columns

Q.14 $A \rightarrow$ p,s; $B \rightarrow$ p, s; $C \rightarrow$ p, s; $D \rightarrow$ p, s

Q.15 $A \rightarrow$ q; $B \rightarrow$ p; $C \rightarrow$ s; $D \rightarrow$ r

Q.16 $A \rightarrow$ r; $B \rightarrow$ p, q, r, s; $C \rightarrow$ q

Q.17 $A \rightarrow$ q; $B \rightarrow$ r; $C \rightarrow$ p

Previous Years' Questions

Q.1 7 **Q.2** 1210 **Q.3** a > l **Q.6** $y \in \left\{-1\right\} \cup \left[1,\infty\right)$ **Q.7** $-\frac{1}{4}$

Q.8 B **Q.9** A **Q.10** A **Q.11** C **Q.12** A **Q.13** B

Q.14 C **Q.15** B **Q.16** A **Q.17** D **Q.18** A **Q.19** B

Q.20 D **Q.21** B **Q.22** B **Q.23** B

Solutions

Exercise 1

Sol 1: Equation $px^2+qx+r=0$.

The sum of roots of a quadratic equation is: $\frac{-q}{p}$.

Let roots be $\frac{c}{a}=1 \Rightarrow r_1+r_2=\frac{-q}{p}$

Given that:- $r_1^2+r_2^2=r_1+r_2 \Rightarrow (r_1+r_2)^2-2r_1r_2=r_1+r_2$

Product of roots is $=\frac{+r}{p}=r_1r_2$

$$\Rightarrow \left(\frac{-q}{p}\right)^2-\frac{2r}{p}=\frac{-q}{p} \Rightarrow \frac{q^2}{p^2}-\frac{2r}{p}=\frac{-q}{p}$$

$$\Rightarrow q^2-2pr=-qp \Rightarrow q^2+pq=2pr$$

Sol 2: Equation

$(a+b)^2x^2-2(a^2-b^2)x+(a-b)^2=0$

For an eq. $ax^2+bx+c=0$, if roots are equal then $b^2 = 4ac$

$\therefore$ for above eq.

$$D = \left[-2\left(a^2-b^2\right)\right]^2-4\left(a+b\right)^2\left(a-b\right)^2$$

$$=4(a^2-b^2)^2-4[(a+b)(a-b)]^2$$

$$=4(a^2-b^2)^2-4(a^2-b^2)^2=0$$

Hence the roots are equal.

Sol 3: Eq. is $5x^2-4x+2+m(4x^2-2x-1)=0$

$$\Rightarrow (5+4m)x^2-(4+2m)x+(2-m)=0$$

(i) If the eq. has equal roots then $b^2-4ac=0$

$$\Rightarrow [-(4+2m)]^2-4(5+4m)(2-m)=0$$

$$\Rightarrow 4m^2+16m+16-4(-4m^2+3m+10)=0$$

$$20m^2+4m-24=0$$

$$5m^2+m-6=0$$

$$\Rightarrow (m-1)(5m+6)=0 \Rightarrow m=1 \text{ or } m=-6/5.$$

(ii) Product of roots is 2

$$\Rightarrow \frac{c}{a}=2 \Rightarrow \frac{(2-m)}{5+4m}=2 \Rightarrow 9m=-8$$

$$\Rightarrow m=\frac{-8}{9}$$

(iii) Sum of roots is 6

$$\Rightarrow \frac{-b}{a}=6 \Rightarrow \frac{(4+2m)}{5+4m}=6 \Rightarrow 22m=-26$$

$$\Rightarrow m=\frac{-13}{11}$$

Sol 4: Eq. is $5x^2+13x+k=0$

One root is reciprocal of other $\Rightarrow r_1=\frac{1}{r_2}$

$$\therefore \frac{c}{a}=1 \Rightarrow k=5$$

Sol 5: Difference of roots is 1

$$\Rightarrow |r_1-r_2|=1 \quad \therefore (r_1-r_2)^2=1$$

$$\Rightarrow (r_1+r_2)^2-4r_1r_2=1$$

$$\Rightarrow \frac{b^2}{a^2}-\frac{4c}{a}=1 \qquad \text{... (i)}$$

eq. is $x^2-px+q=0$

$\therefore$ Putting in eq. (i)

$$\frac{(-p)^2}{1}-4q=1 \Rightarrow p^2-4q=1$$

Sol 6: Equations an $3x^2+4mx+2=0$

and $2x^2+3x-2=0$.

Let the common root be α

$$\Rightarrow 3\alpha^2+4m\alpha+2=0 \qquad \text{... (i)}$$

$$\text{and } 2\alpha^2+3\alpha-2=0 \qquad \text{... (ii)}$$

Solving equation (ii) we get

$$2\alpha^2+4\alpha-\alpha-2=0 \Rightarrow (2\alpha-1)(\alpha+2)=0$$

$$\therefore \alpha=\frac{1}{2} \text{ or } \alpha=-2$$

$\frac{3}{4}+2m+2=0$ (Putting $\alpha=\frac{1}{2}$)

$$m=-\frac{11}{8}$$

and $3\times 4-8m+2=0$ (Putting $\alpha=-2$)

$$\Rightarrow m=\frac{14}{8}=\frac{7}{4}$$

Sol 7: $\alpha \& \beta$ are roots of the equation $x^2 - px + q = 0$

$\Rightarrow \alpha + \beta = p$ and $\alpha\beta = q$

The equations whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$ is

$$\left(x - \left(\alpha + \frac{1}{\beta}\right)\right)\left[x - \left(\beta + \frac{1}{\alpha}\right)\right] = 0$$

$$\Rightarrow x^2 - \left(\alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}\right)x + \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right) = 0$$

$$x^2 - \left((\alpha+\beta) + \frac{(\alpha+\beta)}{\alpha\beta}\right)x + \left(\alpha\beta + 2 + \frac{1}{\alpha\beta}\right) = 0$$

$$\Rightarrow x^2 - \left[p + \frac{p}{q}\right]x + \left[q + \frac{1}{q} + 2\right] = 0$$

$\therefore$ Eq. is $qx^2 - (pq + p)x + (q^2 + 2q + 1) = 0$

Which is $qx^2 - p(q+1)x + (q+1)^2 = 0$

Sol 8: $\dfrac{4x}{x^2 + 3} \geq 1$

Since $x^2 + 3$ is positive, we can directly take it to other side.

$\Rightarrow 4x \geq x^2 + 3$

$\Rightarrow x^2 - 4x + 3 \leq 0$

$\Rightarrow (x-1)(x-3) \leq 0$

The critical points are 1,3

+ − +

1 3

Hence solution is [1,3]

Sol 9: $(x-a)(x-b) - k = 0$ and c and d are the roots of the equation

The equation with root c and d is $(x-c)(x-d) = 0$

$\therefore (x-c)(x-d) = (x-a)(x-b) - k$

$\therefore (x-a)(x-b) = (x-c)(x-d) + k$

$\Rightarrow$ a and b are roots of equation $(x-c)(x-d) + k = 0$

Sol 10: $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$

From the first equation, we can write $(x-1)(x-2) > 0$

+ − +

1 2

$\therefore x \in (-\infty, 1) \cup (2, \infty)$

In the second equation, we have

$x^2 - 4x + x - 4 \leq 0; \Rightarrow (x+1)(x-4) \leq 0$

$\therefore x \in [-1, 4]$

$m - n = 1 - \frac{1}{x}$

$\therefore$ The values of x which satisfies both the equations

$= -[(-\infty, 1) \cup (2, \infty)] \cap [-1, 4] \Rightarrow x \in [-1, 1) \cup (2, 4]$

Sol 11: $ax^2 + bx + c = 0$ (α and β are roots of this eq.)

$\Rightarrow \alpha + \beta = \frac{-b}{a}$ & $\alpha\beta = \frac{c}{a}$

Given eq. $a^3x^2 + abcx + c^3 = 0$

$\Rightarrow$ Let the roots be r & s

$r + s = \frac{-abc}{a^3} = \frac{-b}{a} \times \frac{c}{a}$

$= (\alpha + \beta) \times \alpha\beta = \alpha^2\beta + \alpha\beta^2$

$\Rightarrow rs = \frac{c^3}{a^3} = \left(\alpha^3\beta^3\right)$

$\therefore$ We can see here that $r = \alpha^2\beta$ and $s = \alpha\beta^2$

$\therefore$ The given equation will become

$(x - \alpha^2\beta)(x - \alpha\beta^2) = 0$

Sol 12: a and b are integer

Roots of $x^2 + ax + b = 0$ are rational

Let the roots be α and β putting α in eq.

$\alpha^2 + a\alpha = -b$

$\alpha(\alpha + a) = -b$

a is an integer and b is an integer

$\therefore \alpha$ has to be an integer

Sol 13: An equation $ax^2 + 2hxy + by^2 + 2gx + 2fx + c = 0$

can be factorized into two linear factors

If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ and $h^2 - ab > 0$.

(i) The expression is $3x^2 - xy - 2y^2 + mx + y + 1$

$h = \frac{-1}{2}$, a = 3, b = –2, $g = \frac{m}{2}$, $f = \frac{1}{2}$ and c = 1

And $h^2 - ab > 0$ which is $\frac{1}{4} + 6 > 0$ true.

And $\begin{vmatrix} 3 & -\frac{1}{2} & \frac{m}{2} \\ -\frac{1}{2} & -2 & \frac{1}{2} \\ \frac{m}{2} & \frac{1}{2} & 1 \end{vmatrix} = 0$

$\Rightarrow 3\left(-2-\frac{1}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}-\frac{m}{4}\right)+\frac{m}{2}\left(-\frac{1}{4}+m\right)=0$

$\Rightarrow -\frac{22}{4}-\frac{(m+2)}{8}+\frac{m(4m-1)}{8}=0$

$\Rightarrow -54-m-2+4m^2-m=0 \Rightarrow 4m^2-2m-56=0$,

$\Rightarrow 2m^2-m-28=0, \Rightarrow 2m^2-8m+7m-28=0$

$\Rightarrow (2m+7)(m-4)=0, \Rightarrow m=\frac{-7}{2}, 4$

(ii) $6x^2-7xy-3y^2+mx+17y-20$

$\Rightarrow a=6, h=\frac{-7}{2}, b=-3 \Rightarrow g=\frac{m}{2}, \ f=\frac{17}{2}, c=-20$

And $h^2-ab>0$ which is $\left(\frac{-7}{2}\right)^2+18>0$ True.

and $\begin{vmatrix} 6 & \frac{-7}{2} & \frac{m}{2} \\ \frac{-7}{2} & -3 & \frac{17}{2} \\ \frac{m}{2} & \frac{17}{2} & -20 \end{vmatrix} = 0$

$\Rightarrow 6\left(60-\frac{289}{4}\right)+\frac{7}{2}\left(70-\frac{17m}{4}\right)+\frac{m}{2}\left(\frac{-119}{4}+\frac{3m}{2}\right)=0$

$\Rightarrow 12(-49)+7(280-17m)+m(-119+6m)=0$

$6m^2-238\,m+1272=0$

$\therefore m=7, \frac{98}{3}$ are solutions of this equation of this equation

Sol 14: $\frac{x^2-2x+4}{x^2+2x+4}=y$

$x^2(1-y)-x(2+2y)+4(1-y)=0$

Since x is real $\therefore b^2-4ac \ge 0$

$\Rightarrow (2+2y)^2-16(1-y)^2 \ge 0$

$\Rightarrow 4y^2+8y+4-16y^2+32y-16 \ge 0$

$\Rightarrow 12y^2-40y+12 \le 0, \Rightarrow 3y^2-10y+3 \le 0$

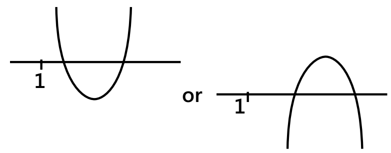
$\Rightarrow (3y-1)(y-3) \le 0$

$\therefore y \in \left[\frac{1}{3}, 3\right]$

Sol 15: $(1+a)x^2-3ax+4a=0$

Let $f(x)=(a+1)x^2-3ax+4a$ and d = 1.

The roots exceed unity

$\Rightarrow$ 

The conditions are

(i) $D \ge 0$

(i) $9a^2-16a(1+a) \ge 0$

$\Rightarrow 9a^2-16a-16a^2 \ge 0, \Rightarrow 7a^2+16a \le 0$

$a(7a+16) \le 0 \quad a \in \left[\frac{-16}{7}, 0\right]$

(ii) *a* f (d) > 0

Note that this *a* is the co-efficient of x^2 and not to be confused with 'a

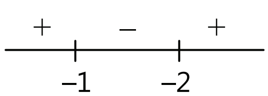
$\Rightarrow (1+a)(1+a-3a+4a) > 0$

$\Rightarrow (1+a)(2a+1) > 0$

$\therefore a \in (-\infty, -1) \cup \left(\frac{-1}{2}, \infty\right)$

(iii) $\frac{-b}{2a} > d \Rightarrow \frac{3a}{2(1+a)} > 1$

$\Rightarrow \frac{3a}{2(1+a)} - 1 > 0 \Rightarrow \frac{a-2}{(a+1)} > 0$



$\therefore \ a \in (-\infty, -1) \cup (2, \infty)$

So taking intersection to all 3 solutions

$a \in \left[\frac{-16}{7}, -1\right]$

Sol 16: $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$

$P(x).Q(x) = 0$

$\Rightarrow (ax^2 + bx + c)(-ax^2 + bx + c) = 0$

$\Rightarrow$ D of P(x) = $b^2 - 4ac$

$\Rightarrow$ D of Q(x) = $b^2 + 4ac$

Clearly both cannot be less than zero at the same time.

Hence the equation has at least 2 real roots

Sol 17: We have $ax^2 + 2bx + c = 0$

$\Rightarrow \alpha + \beta = \frac{-2b}{a}, \alpha\beta = \frac{c}{a}$

For equation $Ax^2 + 2Bx + c = 0$

$\Rightarrow (\alpha + \beta) + 2k = \frac{-2B}{A}$, $\Rightarrow k = \frac{b}{a} - \frac{B}{A}$

Also, $(\alpha + k)(\beta + k) = \frac{C}{A}$, $\Rightarrow k^2 + (\alpha + \beta)k + \alpha\beta = \frac{C}{A}$

$$\Rightarrow \left(\frac{b}{a} - \frac{B}{A}\right)^2 + \left(\frac{-2b}{a}\right)\left(\frac{b}{a} - \frac{B}{A}\right) + \frac{c}{a} = \frac{C}{A}$$

$$\Rightarrow \left(\frac{b}{a} - \frac{B}{A}\right)\left(-\frac{b}{a} - \frac{B}{A}\right) + \frac{c}{a} = \frac{C}{A}$$

$$\Rightarrow \frac{B^2}{A^2} - \frac{b^2}{a^2} + \frac{c}{a} = \frac{C}{A} \Rightarrow \frac{B^2}{A^2} - \frac{C}{A} = \frac{b^2}{a^2} - \frac{c}{a}$$

$$\therefore \frac{B^2 - AC}{A^2} = \frac{b^2 - ac}{a^2} \Rightarrow \frac{b^2 - ac}{B^2 - AC} = \left(\frac{a}{A}\right)^2$$

Sol 18: We have $\left(15 + 4\sqrt{14}\right)^t + \left(15 - 4\sqrt{14}\right)^t = 30$

and $t = x^2 - 2|x|$

Let $\left(15 + 4\sqrt{14}\right)^t = y$

$\Rightarrow y + \frac{1}{y} = 30$

$\Rightarrow y^2 - 30y + 1 = 0 \Rightarrow y = \frac{30 \pm \sqrt{896}}{2}$

$\Rightarrow y = 15 \pm \frac{2}{2}\sqrt{224} \Rightarrow y = 15 \pm 4\sqrt{14}$

$\left(15 + 4\sqrt{14}\right)^t = 15 \pm 4\sqrt{14}$

$\therefore t = 1$ or $t = -1$

When x > 0

$x^2 - 2|x| = x^2 - 2x$

$\Rightarrow x^2 - 2x - 1 = 0$ or $x^2 - 2x + 1 = 0$

$\Rightarrow x = \frac{2 + 2\sqrt{2}}{2}$ or $x = 1$

When $x < 0$

$x^2 - 2|x| = x^2 + 2x$

$\Rightarrow x^2 + 2x - 1 = 0$ or $x^2 + 2x + 1 = 0$

$\Rightarrow x = \frac{-2 - 2\sqrt{2}}{2}$ or $x = -1$

The values of x are $-1, \left(-1 - \sqrt{2}\right), 1, \left(1 + \sqrt{2}\right)$

Sol 19:

$\text{LHS} = (x-2)(x-3) - 8(x-1)(x-3) + 9(x-1)(x-2)$

$= x^2 - 5x + 6 - 8x^2 + 32x - 24 + 9x^2 - 27x + 18 = 2x^2$

Which is always equal to RHS no matter what the value of x

$\therefore$ The equation is an identify

Sol 20: $(1+a)\left(\frac{x^2}{x^2+1}\right)^2 - 3a\left(\frac{x^2}{x^2+1}\right) + 4a = 0$

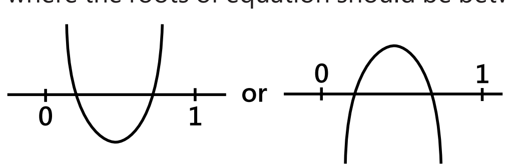
Let $y = \frac{x^2}{x^2+1} \Rightarrow x^2(1-y) - y = 0$

Since x is real, $\Rightarrow 4y(1-y) \geq 0$

$\therefore y \in [0, 1]$

$\therefore$ The given equation becomes $(1+a)y^2 - 3a(y) + 4a = 0$

where the roots of equation should be between $(0 \,\&\, 1)$



These conditions should be satisfied

(i) $D \geq 0$

$\therefore 9a^2 - 16a(a+1) \geq 0 \Rightarrow 7a^2 + 16a \leq 0$

$\therefore a \in \left[\frac{-16}{7}, 0\right]$

$af(d) > 0$ & $af(e) > 0$

(ii) $(1+a)f(0) \geq 0 \Rightarrow (1+a)4a \geq 0$

$\therefore a \in (-\infty, -1] \cup [0, \infty)$

and $(1+a)f(1) > 0 \Rightarrow (1+a)(2a+1) > 0$

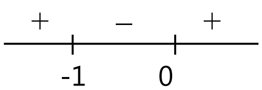
$\therefore a \in (-\infty, -1] \cup [-\frac{1}{2}, \infty)$

(iii) $d \leq \frac{-b}{2a} < e$ as the range is from [0,1]

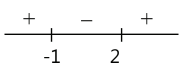
$0 \leq \frac{3a}{2(1+a)} \leq 1$

$\Rightarrow \frac{3a}{2(1+a)} \geq 0$ & $\frac{3a}{2(1+a)} - 1 < 0$

$\Rightarrow \frac{a-2}{2(a+1)} < 0$



$a \in (-\infty, -1) \cup (0, \infty)$

a ∈ 

$a \in (-1, 2)$

$\therefore a \in (0, 2)$

Taking intersection at all 3 possibilities a=0 is the only possible solution.

Sol 21: $(l-m)x^2 + lx + 1 = 0$

Let one root be α: other root = 2α

$\Rightarrow \alpha + 2\alpha = \frac{l}{m-l} \Rightarrow \alpha = \frac{l}{3(m-l)}$

$2\alpha^2 = \frac{1}{l-m}$ (Product of roots)

$\Rightarrow \frac{2l^2}{9(m-l)^2} = \frac{1}{(l-m)}$

$\Rightarrow 2l^2 = 9(l-m)$

$\Rightarrow 2l^2 - (9)l + 9m = 0$

The roots are real $\Rightarrow b^2 - 4ac \geq 0$

$81 - 8 \times 9m \geq 0 \Rightarrow m \leq \frac{9}{8}$

Sol 22: From condition of common root

$(ca_1 - ac_1)^2 = (2bc_1 - 2b_1c)(2ab_1 - 2a_1b)$

$(a_1c_1)^2\left(\frac{c}{c_1} - \frac{a}{a_1}\right)^2 = 4b_1c_1\left(\frac{b}{b_1} - \frac{c}{c_1}\right)a_1b_1\left(\frac{a}{a_1} - \frac{b}{b_1}\right)$ ...(i)

$\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in AP

Let the difference be d.

$\therefore \frac{b}{b_1} - \frac{a}{a_1} = \frac{c}{c_1} - \frac{b}{b_1} = d = \frac{\left(\frac{c}{c_1} - \frac{a}{a_1}\right)}{2}$ ... (ii)

Using (i) and (ii)

$\therefore (a_1c_1)^2 \times 4d^2 = 4a_1c_1b_1^2 \times d^2$

$\therefore b_1^2 = a_1c_1$

a_1, b_1, c_1 are in G.P.

Sol 23: $\frac{\alpha}{\beta} = \frac{\alpha_1}{\beta_1} \Rightarrow \frac{\beta}{\alpha} = \frac{\beta_1}{\alpha_1}$

$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha_1}{\beta_1} + \frac{\beta_1}{\alpha_1} \Rightarrow \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{\alpha_1^2 + \beta_1^2}{\alpha_1\beta_1}$

$\Rightarrow \frac{\alpha^2 + \beta^2}{\alpha\beta} + 2 = \frac{\alpha_1^2 + \beta_1^2}{\alpha_1\beta_1} + 2 \Rightarrow \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{(\alpha_1 + \beta_1)^2}{\alpha_1\beta_1}$

$\Rightarrow \frac{(-b/a)^2}{c/a} = \frac{(-b_1/a_1)^2}{c_1/a_1}$

$\Rightarrow \frac{b^2}{ac} = \frac{b_1^2}{a_1c_1} \Rightarrow \left(\frac{b}{b_1}\right)^2 = \frac{ca}{c_1a_1}$

Sol 24: α is root of equation $ax^2 + bc + c = 0$

$\Rightarrow a\alpha^2 + b\alpha + c = 0$

Similarly $-a\beta^2 + b\beta + c = 0$

Let $f(x) = \frac{ax^2}{2} + bx + c$

$f(\alpha) = \frac{a\alpha^2}{2} + b\alpha + c = \frac{-a\alpha^2}{2}$

$f(\beta) = \frac{a\beta^2}{2} + b\beta + c = \frac{3a\beta^2}{2}$

$\therefore f(\alpha).f(\beta) = \frac{-3a^2}{4}\alpha^2\beta^2 < 0$

$\therefore$ By mean value theorem, there exists a root of f(x) between α and β

Sol 25: Given, $ax^2+bx+c-p=0$ for two distinct $\alpha \& \beta$

$\therefore$ α and β are root of eq^n

$ax^2+bx+(c-p)=0$

$\therefore \alpha+\beta=\frac{-b}{a}$ & $\alpha\beta=\frac{c-p}{a}$

To prove $ax^2+bx+c-2p\neq 0$ for any integral value of x, let us assume these exist integer R satisfying

$ax^2+bx+c-2p=0$

$\Rightarrow ak^2+bk+c-2p=0$

or $\frac{k^2+bk}{a}+\frac{c-p}{a}=\frac{p}{a}$

or $(k-\alpha)(k-\beta)=\frac{p}{a}$ =an integer

Since p is a prime number $\Rightarrow \frac{p}{a}$ is an integer if a=p or a=1 but $a>1 \therefore a=p$

$\Rightarrow (k-\alpha)\ (k-\beta) = 1$

$\therefore$ either $k-\alpha=-1$ and $k-\beta=-1$

$\Rightarrow$ $\alpha=\beta$ (not possible)

$\therefore$ There is contradiction

Sol 26: We are given that $a\le 0$ and

$x^2-2a|x-a|-3a^2=0$

for $x>a$

Equation becomes $x^2-2ax-a^2=0$

$\therefore x=\frac{2a\pm\sqrt{8a^2}}{2}$

$=a\pm a\sqrt{2}$ but since $x>a \& a<0$

$\therefore a\left(1-\sqrt{2}\right)$ is the only solution

For $x<a$

Eqn becomes $x^2+2ax-5a^2=0$

$\Rightarrow x=\frac{-2a\pm\sqrt{24a^2}}{2} = -a\pm a\sqrt{6}$

But since x < a

$\therefore a\left(\sqrt{6}-1\right)$ is the only possible solution.

Sol 27: $x^2+ax+a^2+6a<0$ is satisfied for all $x\in(1, 2)$

$\therefore$ (1 and 2 exists between the roots)

$\therefore$ Bring condition for the given case.

$f(1)\times 1<0$ & $f(2)\times 1<0$

$\therefore a^2+7a+1<0$ & $a^2+8a+4<0$

Solving we get $\frac{-7-3\sqrt{5}}{2}\le a\le -4+2\sqrt{3}$

Sol 28: Given that, $2x^3+x^2-7=0$

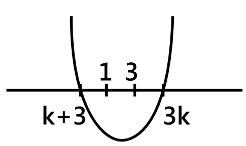
For roots, $\alpha+\beta+\gamma=\frac{-1}{2}$ $\alpha\beta+\beta\alpha+\gamma\alpha=0$ & $\alpha\beta\gamma=\frac{-7}{2}$

For $\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$

$=\left(\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}+1\right)+\left(\frac{\beta}{\alpha}+\frac{\beta}{\gamma}+1\right)+\left(\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}+1\right)-3$

$=\frac{\sum\alpha\beta}{\beta\gamma}+\frac{\sum\alpha\beta}{\alpha\gamma}+\frac{\sum\alpha\beta}{\gamma\beta}-3=0-3=-3$

Sol 29: $(x-3k)(x-(k+3))<0$



$\Rightarrow f(1)<0$ and $f(3)<0$

(using condition to given are)

$(1-3k)(1-(k+3))<0$ and $(3-3k)(-k)<0$

$\therefore k\in\left(-2,\frac{1}{3}\right)$ & $k\in(0,1)$

$\therefore k\in\left(0,\frac{1}{3}\right)$

Exercise 2

Single Correct Choice Type

Sol 1: (C) Given that $a^2+b^2+c^2=1$

We know that

$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$

$\Rightarrow (a+b+c)^2=1+2(ab+bc+ca)$

$\Rightarrow ab + bc + ca = \dfrac{(a+b+c)^2 - 1}{2}$ ...(i)

Also, $2(a^2 + b^2 + c^2) - 2ab - 2bc - 2ac$

$= (a-b)^2 + (b-c)^2 + (c-a)^2$

Now, $(a-b)^2 + (b-c)^2 + (c-a)^2 > 0$

$\therefore ab + bc + ca < a^2 + b^2 + c^2 < 1$

Here, min $(ab + bc + ac) = \dfrac{-1}{2}$

Max $(ab + bc + ac) = 1$

Sol 2: (B) $P(x) = ax^2 + bx + c$

$D(P) = b^2 - 4ac$

If $D(P) < 0 \Rightarrow 4ac > b^2$

If $D(Q) < 0 \Rightarrow 4ac < -d^2 \Rightarrow D(P) > 0$

$\therefore$ At least one of P and Q is real.

$\therefore$ $P(x)$ & $Q(x) = 0$ has atleast 2 real roots

Sol 3: (B) Given that $x^2 + 3x + 1 = 0$

For roots, $\alpha + \beta = -3 \;\; \alpha\beta = 1$

$$\left(\frac{\alpha}{1+\beta}\right)^2 + \left(\frac{\beta}{\alpha+1}\right)^2 = \frac{[\alpha(\alpha+1)]^2 + [\beta(\beta+1)]^2}{(\alpha+1)^2(\beta+1)^2}$$

$$= \frac{\left((\alpha^2+\alpha)+(\beta^2+\beta)\right)^2 - 2\alpha\beta(\alpha+1)(\beta+1)}{(\alpha\beta+\alpha+\beta+1)^2}$$

$$= \frac{\left[(\alpha+\beta)^2 - 2\alpha\beta + (\alpha+\beta)\right]^2 - 2\alpha\beta(\alpha\beta+\alpha+\beta+1)}{(\alpha\beta+\alpha+\beta+1)^2}$$

$$= \frac{[9-2-3]^2 - 2\times 1(-3+2)}{(1-3+1)^2} = 16 + 2 = 18$$

Sol 4: (B) $ax^2 + bx + c = 0$

$a > 0, b > 0$ & $c > 0$

$\Rightarrow \alpha = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If α is real $\Rightarrow \sqrt{b^2 - 4ac} < b$

$\therefore$ α is negative

If $b^2 - 4ac < 0$ then real part of α is always negative

$\therefore$ The roots have negative real parts

Sol 5: (A) $\alpha - 2 = \dfrac{1}{\alpha}$

$\Rightarrow \alpha^2 - 2\alpha - 1 = 0 \Rightarrow b = -2, c = -1$

$\therefore b^2 + c^2 + bc = (-2)^2 + (-1)^2 + (-2)(-1) = 7$

Sol 6: (C) $a + b + c = 0$

D of eq. $= 25b^2 - 84ac = 25(a+c)^2 - 84ac$

$= 25c^2 - 34ac + 25a^2$

$= a^2\left(25\left(\dfrac{c}{a}\right)^2 - 34\left(\dfrac{c}{a}\right) + 25\right)$

D_2 of this eq. < 0

$\therefore$ The eq. is always positive when $a \neq 0$

Sol 7: (D) One root is α $\therefore$ The other root $= -\alpha$

Let third root $= \beta$

$\Rightarrow \alpha - \alpha + \beta = 2 \Rightarrow \beta = 2$

Putting this value in the given equation

$2^3 - 2^3 + 2a + 10 = 0 \Rightarrow a = -5$

$\therefore a \in (-10, 0)$

Sol 8: (B) $x + 1 = \log_2(2^x + 3)^2 - 2\log(1980 - 2^{-x})$

$\Rightarrow 2^{x+1} = \dfrac{(2^x+3)^2}{1980 - 2^{-x}}$

$\Rightarrow 1980 \times 2 \times 2^x - 2 = (2^x + 3)^2$

Let $2^x = t$

$\Rightarrow t^2 + 6t + 11 + 1980 \times 2t = 0$

Now $2^\alpha \times 2^\beta = 11$

$\Rightarrow 2^{\alpha+\beta} = 11 \Rightarrow \alpha + \beta = \log_2 11$

Sol 9: (B) Product of H.C.F. & L.C.M. of two numbers = product of the nos

$\therefore 16 \times LCM = 295680$

$\therefore LCM = \dfrac{295680}{16} = 18480$

Sol 10: (A) Given that $4y^2 + 4xy + x + 6 = 0$

$y = \text{real} \Rightarrow b^2 - 4ac \geq 0$

$\Rightarrow 16x^2 - 16(x+6) \geq 0$

$x^2 - x - 6 \geq 0 \Rightarrow (x-3)(x+2) \geq 0$

$\therefore x \leq -2$ or $x \geq 3$

Sol 11: (B) If exactly one root lies in (0, 3) (as interval is open)

$\Rightarrow f(0)f(3) < 0$

$\therefore 2a(6-a) < 0$

$\Rightarrow a \in (-\infty, 0) \cup (6, \infty)$

Now we check at boundaries

At a = 0 $\Rightarrow x^2 - x = 0$

$\therefore$ Other root = 1 which lies in (0, 3)

$\therefore$ Now at a = 6, $\Rightarrow f(x) = x^2 - 7x + 12 = 0$

$\Rightarrow x = 3, 4$

No root lies in (0, 3)

$\therefore a \in (-\infty, 0] \cup (6, \infty)$

Sol 12: (D) $x^2 - 2mx + m^2 - 1 = 0$

Since both roots lies between (–2, 4)

$\Rightarrow D \geq 0 \;\; af(d) > 0 \;\&\; af(e) > 0$ and $d < \dfrac{-b}{2a} < e$

(i) $\Rightarrow 4m^2 - 4(m^2 - 1) \geq 0$

$4 \geq 0$ ....true

(ii) $1.f(-2) > 0$

$\Rightarrow (4 + 4m + m^2 - 1) > 0$

$\Rightarrow m \in (-\infty, -3) \cup (-1, \infty)$

(iii) $1.f(4) > 0$

$\Rightarrow (16 - 8m + m^2 - 1) > 0$

$\Rightarrow m \in (-\infty, 3) \cup (5, \infty)$

$\Rightarrow -2 < \dfrac{-b}{2a} < 4 \Rightarrow -2 < m < 4$

Combining all the above three conditions, we get

$\therefore m \in (-1, 3)$

$\therefore$ Integral values of m are 0, 1, 2

Sol 13: (C) $(x^3 + 4x)^2 = 82$

$\Rightarrow x^6 + 16x^2 + 8x^4 = 64;$

Mulitiply both sides by x

$\Rightarrow x^7 + 16x^3 + 8x^4 = 64;$

Add $16x^3$ in both sides

$\Rightarrow x^7 + 8x^5 + 32x^3 = 16x^3 + 64;$

$\Rightarrow x^7 + 8x^2(x^3 + 4x) = 16(x^3 + 4x);$

$\Rightarrow x^7 + 8x^2 \times 8 = 16 \times 8;$

$\Rightarrow x^7 + 64x^2 = 128$

Sol 14: (D) Given equations have real roots so,

$a^2 - 8b \geq 0 \Rightarrow a^2 \geq 8b$ and $4b^2 - 4a \geq 0 \therefore \Rightarrow b^2 \geq a$

$\Rightarrow b^4 \geq a^2 \geq 8b$

$\Rightarrow b \geq 2 \;\&\; a \geq 4$

Hence, $(a+b)_{min} = 2 + 4 = 6$

Sol 15: (A) $(x^2 + ax + 1)(3x^2 + ax - 3) = 0$

$D_1 = a^2 - 4$

$D_2 = a^2 + 36$

D_2 is always > 0

$\therefore$ The equation has atleast two real roots.

Sol 16: (B) f(x) = x2 + ax + b

For $X \in [0, 2]$

$f(x)_{max} = 3$ and $f(x)_{min} = 2$

f(0) = b = 2 ... (i)

f(2) = 4 + 2a + b = 3 ... (ii)

By solving (i) and (ii)

$a = -\dfrac{3}{2}; \quad b = 2$

Previous Years' Questions

Sol 1: (i) Given $x^2 - 8kx + 16\left(k^2 - k + 1\right) = 0$

Now, $D = 64\left\{k^2 - \left(k^2 - k + 1\right)\right\} = 64\left(k - 1\right) > 0$

$\therefore k > 1$

(ii) $-\dfrac{b}{2a} > 4 \Rightarrow \dfrac{8k}{2} > 4 \Rightarrow k > 1$

(iii) $f(4) \geq 0$

$16 - 32k + 16\left(k^2 - k + 1\right) \geq 0 \Rightarrow k^2 - 3k + 2 \geq 0$

$\Rightarrow (k-2)(k-1) \geq 0$

$\Rightarrow k \leq 1$ or $k \geq 2$ Hence, k = 2.

Sol 2: Given $\dfrac{2x}{2x^2+5x-2} > \dfrac{1}{x+1}$

$\Rightarrow \dfrac{2x}{(2x+1)(x+2)} - \dfrac{1}{(x+1)} > 0$;

$\left(a_1x^2 + b_1x + c_1\right)y + \left(a_2x^2 + b_2x + c_2\right) = 0$

$x^2\left(a_1y + a_2\right) + x\left(b_1y + b_2\right) + \left(c_1y + c_2\right) = 0$

$\Rightarrow \dfrac{-(3x+2)}{(2x+1)(x+1)(x+2)} > 0$

Using number line rule

$\therefore x \in (-2, -1) \cup \left(-\dfrac{2}{3}, -\dfrac{1}{2}\right)$

Sol 3: Since $ax^2 + bx + c = 0$ in terms of α, β.

$\Rightarrow \alpha + \beta = -b/a$ and $\alpha\beta = c/a$

Now, $a^3x^2 + abcx + c^3 = 0$ ....(i)

On dividing the equation by c^2, we get

$\dfrac{a^3}{c^2}x^2 + \dfrac{abcx}{c^2} + \dfrac{c^3}{c^2} = 0$

$\Rightarrow a\left(\dfrac{ax}{c}\right)^2 + b\left(\dfrac{ax}{c}\right) + c = 0$

$\Rightarrow \dfrac{ax}{c} = \alpha, \beta$ are the roots

$\Rightarrow x = \dfrac{c}{a}\alpha, \dfrac{c}{a}\beta$ are the roots

$\Rightarrow$ x = $\alpha\beta\alpha, \alpha\beta\beta$ are the roots

$\Rightarrow x = \alpha^2\beta, \alpha\beta^2$ are the roots

Alternate solution

Divide the Eq. (i) by a^3, we get $x^2 + \dfrac{b}{a}.\dfrac{c}{a}\ x + \left(\dfrac{c}{a}\right)^3 = 0$

$\Rightarrow x^2 - (\alpha+\beta).(\alpha\beta)x + (\alpha\beta)^3 = 0$

$\Rightarrow x^2 - \alpha^2\beta x - \alpha\beta^2 x + (\alpha\beta)^3 = 0$

$\Rightarrow x\left(x - \alpha^2\beta\right) - \alpha\beta^2\left(x + \alpha^2\beta\right) = 0$

$a^3x^2 + abcx + c^3 = 0$

$\Rightarrow x = \alpha^2\beta, \alpha\beta^2$ which is the required answer.

Sol 4: Since $\alpha + \beta = -\dfrac{b}{a}, \alpha\beta = \dfrac{c}{a}$ and

$\alpha + \delta + \beta + \delta = -\dfrac{B}{A}, (\alpha+\delta)(\beta+\delta) = \dfrac{C}{A}$

Now, $\alpha - \beta = (\alpha+\delta) - (\beta+\delta)$

$\Rightarrow (\alpha-\beta)^2 = \left[(\alpha+\delta) - (\beta+\delta)\right]^2$

$\Rightarrow (\alpha+\beta)^2 - 4\alpha\beta = \left[(\alpha+\delta) - (\beta+\delta)\right]^2 - 4(\alpha+\delta).(\beta+\delta)$

$\Rightarrow \left(-\dfrac{b}{a}\right)^2 - \dfrac{4c}{a} = \left(-\dfrac{B}{A}\right)^2 - \dfrac{4C}{A}$

$\Rightarrow \dfrac{b^2}{a^2} - \dfrac{4c}{a} = \dfrac{B^2}{A^2} - \dfrac{4C}{A} \Rightarrow \dfrac{b^2 - 4ac}{a^2} = \dfrac{B^2 - 4AC}{A^2}$

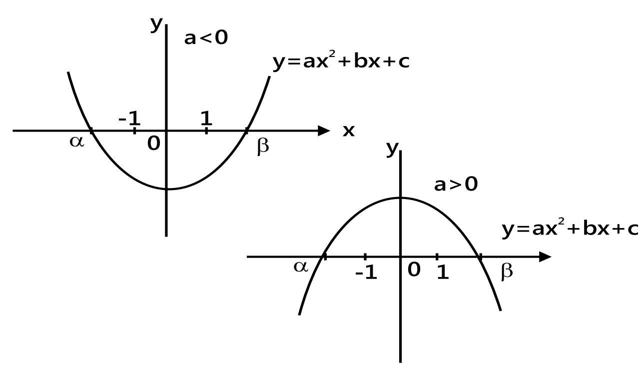
Sol 5: From figure it is clear that if a > 0, then f(– 1) < 0 and f(1) < 0, if a < 0, f(–1) > 0 and f(1) > 0. In both cases, af(–1)< 0 and af(–1) < 0

$\Rightarrow$ a (a – b + c) < 0 and a (a + b + c) < 0

On dividing by a^2, we get

$1 - \dfrac{b}{a} + \dfrac{c}{a} < 0$ and $1 + \dfrac{b}{a} + \dfrac{c}{a} < 0$

On combining both, we get



$1 \pm \dfrac{b}{a} + \dfrac{c}{a} < 0 \Rightarrow 1 + \left|\dfrac{b}{a}\right| + \dfrac{c}{a} < 0$

Sol 6: (B) Given $x^2 + 2px + q = 0$

$\therefore \alpha + \beta = -2p$ ... (i)

$\alpha\beta = q$ ... (ii)

and $ax^2 + 2bx + c = 0$

$\therefore\ \alpha + \frac{1}{\beta} = -\frac{-2b}{a}$ ... (iii)

and $\frac{\alpha}{\beta} = \frac{c}{a}$ ... (iv)

Now, $\left(p^2 - q\right)\left(b^2 - ac\right)$

$$= \left[\left(\frac{\alpha+\beta}{-2}\right)^2 - \alpha\beta\right]\left[\left(\frac{\alpha+\frac{1}{\beta}}{-2}\right)^2 - \frac{\alpha}{\beta}\right]a^2$$

$$= \frac{\alpha^2}{16}\left(\alpha-\beta\right)^2\left(\alpha - \frac{1}{\beta}\right)^2 \ge 0$$

$\therefore$ Statement-I is true.

Again now $pa = -\frac{a}{2}(\alpha+\beta)$

and $b = -\frac{a}{2}\left(\alpha + \frac{1}{\beta}\right)$ Since, $pa \ne b$

$\Rightarrow \alpha + \frac{1}{\beta} \ne \alpha + \beta\ \Rightarrow \beta^2 \ne 1,\ \beta \ne \{-1, 0, 1\}$

which is correct. Similarly, if $c \ne qa$

$\Rightarrow a\frac{\alpha}{\beta} \ne a\ \alpha\beta; \Rightarrow \alpha\left(\beta - \frac{1}{\beta}\right) \ne 0\ \Rightarrow \alpha \ne 0$

and $\beta - \frac{1}{\beta} \ne 0 \Rightarrow \beta \ne \{-1,0,1\}$

Statement-II is true.

Sol 7: Given, $|x-2|^2 + |x-2| - 2 = 0$

Case I when $x \ge 2$

$\Rightarrow (x-2)^2 + (x-2) - 2 = 0$

$\Rightarrow\ x^2 + 4 - 4x + x - 2 - 2 = 0$

$\Rightarrow\ x^2 - 3x = 0 \Rightarrow x(x-3) = 0$

$\Rightarrow\ x = 0, 3$ (0 is rejected)

$\Rightarrow\ x = 3$ ... (i)

Case II when $x < 2$.

$\Rightarrow \{-(x-2)\}^2 - (x-2) - 2 = 0$

$\Rightarrow (x-2)^2 - x + 2 - 2 = 0\ \Rightarrow\ x^2 + 4 - 4x - x = 0$

$\Rightarrow\ x^2 - 4x - 1(x-4) = 0 \Rightarrow x(x-4) - 1(x-4) = 0$

$\Rightarrow\ (x-1)(x-4) = 0$

$\Rightarrow\ x = 1, 4$ (4 is rejected)

$\Rightarrow\ x = 1$ ... (ii)

Hence, the sum of the roots is 3 + 1 = 4

Alternate solution

Given $|x-2|^2 + |x-2| - 2 = 0$

$\Rightarrow \left(|x-2| + 2\right) + \left(|x-2| - 1\right) = 0$

$\therefore |x-2| = -2, 1$ (neglecting – 2)

$\Rightarrow |x-2| = 1$

$\Rightarrow\ x = 3, 1$

$\Rightarrow$ Sum of roots = 4

Sol 8: (B) If $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$

Have a common real root, then

$\Rightarrow \left(a_1c_2 - a_2c_1\right)^2 = \left(b_1c_2 - b_2c_1\right)\left(a_1b_2 - a_2b_1\right)$

$\therefore \left.\begin{matrix} x^2 + bx - 1 = 0 \\ x^2 + x + b = 0 \end{matrix}\right\}$ have a common root.

$\Rightarrow (1+b)^2 = \left(b^2+1\right)(1-b)$

$\Rightarrow b^2 + 2b + 1 = b^2 - b^3 + 1 - b$

$\Rightarrow b^3 + 3b = 0$

$\therefore b\left(b^2 + 3\right) = 0$

$\Rightarrow b = 0, \pm\sqrt{3}\,i$

Sol 9: (D) The equation $x^2 - px + r = 0$ has roots α, β and the equation

$x^2 - qx + r = 0$ has roots $\frac{\alpha}{2}, 2\beta$.

$\Rightarrow r = \alpha\beta$ and $\alpha + \beta = p$, and $\frac{\alpha}{2} + 2\beta = q$

$\Rightarrow \beta = \frac{2q-p}{3}$ and $\alpha = \frac{2(2p-q)}{3}$

$\Rightarrow \alpha\beta = r = \frac{2}{9}(2q-p)(2p-q)$

Sol 10: (A) Let the roots of $x^2 + px + q = 0$ be α and α^2.

$\Rightarrow \alpha + \alpha^2 = -p$; and $\alpha^3 = q$

$\Rightarrow \alpha(\alpha+1) = -p$

$\Rightarrow \alpha^3\left\{\alpha^3+1+3\alpha\left(\alpha+1\right)\right\}=-p^3$ (cubing both sides)

$\Rightarrow$ q (q + 1 – 3p) = $-p^3$

$\Rightarrow p^3-\left(3p-1\right)q+q^2=0$

Sol 11: (B) As we know $ax^2+bx+c>0$ for all $x\in R$, if a > 0 and D < 0

Given equation is

$x^2+2ax+\left(10-3a\right)>0, \forall x\in R$ Now,

$\Rightarrow 4a^2-4\left(10-3a\right)<0$

$\Rightarrow 4\left(a^2+3a-10\right)<0$

$\Rightarrow$ (a + 5) (a – 2) < 0

$\Rightarrow$ a $\in$ (–5,2)

Sol 12: (B) Given $x^2-\left|x+2\right|+x>0$ ... (i)

Case I when x + 2 $\geq$ 0

$\therefore x^2-x-2+x>0$

$\Rightarrow x^2-2>0$

$\Rightarrow x<-\sqrt{2}$ or $x>\sqrt{2}$

$\Rightarrow x\in x\left[-2,-\sqrt{2}\right)\cup\left(\sqrt{2},\infty\right)$ ... (ii)

Case II when x + 2 < 0

$\therefore x^2+x+2+x>0$

$\Rightarrow x^2+2x+2>0$

$\Rightarrow \left(x+1\right)^2+1>0$

Which is true for all x.

$\therefore x\leq -2$ or $x\in\left(-\infty,-2\right)$ ... (iii)

From Eqs. (ii) and (iii), we get

$x\in\left(-\infty,-\sqrt{2}\right)\cup\left(\sqrt{2},\infty\right)$

Sol 13: (B) Given

$\log_4\left(x-1\right)=\log_2\left(x-3\right)=\log_{4^{1/2}}\left(x-3\right)$

$\Rightarrow \log_4\left(x-1\right)=2\log_4\left(x-3\right)$

$\Rightarrow \log_4\left(x-1\right)=\log_4\left(x-3\right)^2$

$\Rightarrow \left(x-3\right)^2=x-1 \Rightarrow x^2-7x+10=0$

$\Rightarrow \left(x-2\right)\left(x-5\right)=0 \Rightarrow$ x = 2 or x = 5

$\Rightarrow$ x = 5 [$\therefore$ x = 2 make log (x – 3) undefined].

Hence, one solution exits.

Sol 14: (B) Given c < 0 < b

Since $\alpha+\beta=-b$ ... (i)

and $\alpha\beta=c$ ... (ii)

From Eq. (ii), c < 0 $\Rightarrow \alpha\beta<0$

$\Rightarrow$ Either α is – ve, β is + ve or α is + ve,

Or β is – ve

From Eq. (i), b > 0 $\Rightarrow -b<0 \Rightarrow \alpha+\beta<0$

$\Rightarrow$ The sum is negative.

$\Rightarrow$ Modulus of negative quantity is > modulus of positive quantity but $\alpha<\beta$ is given.

Therefore, it is clear that α is negative and β is positive and modulus of α is greater than

Modulus of

$\beta \Rightarrow \alpha<0<\beta<\left|\alpha\right|$

Note: This question is not on the theory of interval in which root lie, which appears looking at

First sight. It is new type and first time asked in the paper. It is important for future. The actual

Type is interval in which parameter lie.

Sol 15: (A) Since $\sqrt{x+1}-\sqrt{x-1}=\sqrt{4x-1}$

$\Rightarrow \left(x+1\right)+\left(x-1\right)-2\sqrt{x^2-1}=4x-1$

$\Rightarrow 1-2x=2\sqrt{x^2-1} \Rightarrow 1+4x^2-4x=4x^2-4$

$\Rightarrow 4x=5 \Rightarrow x=\frac{5}{4}$

But it does not satisfy the given equation.

Hence, no solution exists.

Sol 16: (D) Let α and 4β be roots of $x^2 - 6x + a = 0$ and α, 3β be the roots of $x^2 - cx + 6 = 0$, then

$\alpha + 4\beta = 6$ and $4\alpha\beta = a$

$\alpha + 3\beta = c$ and $3\alpha\beta = 6$.

We get $\alpha\beta = 2 \Rightarrow a = 8$

So the first equation is $x^2 - 6x + 8 = 0 \Rightarrow x = 2, 4$

If $\alpha = 2$ and $4\beta = 4$ then $3\beta = 3$

If $\alpha = 4$ and $4\beta = 2$, then $3\beta = 3/2$ (non-integer)

$\therefore$ Common root is $x = 2$.

Sol 17: (C) $bx^2 + cx + a = 0$

Roots are imaginary $\Rightarrow c^2 - 4ab < 0 \Rightarrow c^2 < 4ab$

$\Rightarrow c^2 > -4ab$

$3b^2x^2 + 6bcx + 2c^2$

since $3b^2 > 0$

Given expression has minimum value

$$\text{Minimum value} = \frac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)}$$

$$= -\frac{12b^2c^2}{12b^2} = -c^2 > -4ab$$

Sol 18: (C) $x^2 - 6x - 2 = 0$

$a_n = \alpha^n - \beta^n$

$$\frac{a_{10} - 2a_8}{2a_9} = \frac{\alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{2(\alpha^9 - \beta^9)} = \frac{6\alpha^9 - 6\beta^9}{2(\alpha^9 - \beta^9)} = \frac{\alpha + \beta}{2} = \frac{6}{2} = 3$$

Sol 19: (B) $\because$ p, q, r are in AP

$2q = p + r$ ... (i)

Also, $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 4$

$\Rightarrow \dfrac{\alpha + \beta}{\alpha\beta} = 4$

$$= \frac{-\frac{q}{p}}{\frac{r}{p}} = 4 \Rightarrow q = -4r \quad \text{....(ii)}$$

From (i)

$2(-4r) = p + r$

$p = -9r$

$q = -4r$

$r = r$

Now $|\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

$$= \sqrt{\left(\frac{-q}{p}\right)^2 - \frac{4r}{p}} = \frac{\sqrt{q^2 - 4pr}}{|p|}$$

$$= \frac{\sqrt{16r^2 + 36r^2}}{|-9r|} = \frac{2\sqrt{13}}{9}$$

Sol 20: (A) $x^2 + 2x + 3 = 0$ ... (i)

$ax^2 + bx + c = 0$ ... (ii)

Since equation (i) has imaginary roots

So equation (ii) will also have both roots same as (i).

Thus, $\dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{3} \Rightarrow a = \lambda, b = 2\lambda, c = 3\lambda$

Hence 1 : 2 : 3

JEE Advanced/Boards

Exercise 1

Sol 1: $f(x) = x^2 + ax + b$

One root is $\dfrac{4 + 3\sqrt{3}}{2 + \sqrt{3}} = \left(4 + 3\sqrt{3}\right)\left(2 - \sqrt{3}\right) = -1 + 2\sqrt{3}$

$\therefore$ The other root is $-1 - 2\sqrt{3}$

Sum of roots = $-a = -2$

$\Rightarrow a = 2$

Product $= \dfrac{b}{1} = (-1 + 2\sqrt{3})(-1 - 2\sqrt{3}) = 1 - 12 = -11$

$\therefore g(x) = x^4 + 2x^3 - 10x^2 + 4x - 10$

$= x^4 + 2x^3 - 11x^2 + x^2 + 2x - 11 + 1 + 2x$

$= x^2 f(x) + f(x) + 2x + 1$

$$g\left(\frac{4 + 3\sqrt{3}}{2 + \sqrt{3}}\right) = x^2 \times 0 + 0 + 2\left(-1 + 2\sqrt{3}\right) + 1 = 4\sqrt{3} - 1$$

$\therefore c = 4$ & $d = -1$

Sol 2: $f(x) = \dfrac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$

$x^2 - 8x + 32$ is always positive as $a > 0$ & $b^2 - 4ac < 0$

$\therefore$ For f(x) to be always negative

$ax^2 + 2(a+1)x + (9a + 4) < 0$ for all x

$\Rightarrow a < 0$ & $b^2 - 4ac < 0$

$\therefore \left[2(a+1)\right]^2 - 4a(9a+4) < 0$

$4(a^2+2a+1) - 36a^2 - 16a < 0$

$\Rightarrow 32a^2 + 8a - 4 > 0 \Rightarrow 8a^2 + 2a - 1 > 0$

$\Rightarrow 8a^2 + 4a - 2a - 1 > 0 \Rightarrow (4a-1)(2a+1) > 0$

$\therefore a \in \left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{1}{4}, \infty\right)$ but a is $a < 0$

$\therefore a \in \left(-\infty, \frac{-1}{2}\right)$

Sol 3: $x^2 + 2(a+b)x + (a-b+8) = 0$

Since the equation has unequal roots

$b^2 - 4ac > 0$

$\Rightarrow 4(a+b)^2 - 4(a-b+8) > 0$

$\Rightarrow a^2 + 2ab + b^2 - a + b - 8 > 0$

$a^2 + (2b-1)a + (b^2+b-8) > 0$

Now the quadratic in a always > 0

Discriminant should be less than 0

$\therefore (2b-1)^2 - 4(b^2+b-8) < 0$

$-4b + 1 - 4b + 32 < 0$

$\Rightarrow b > \frac{33}{8}$

$\therefore$ The smallest natural number for b is 5

Sol 4: When $y^2 + my + 2$ is divided by $(y-1)$ the remainder $= f(1) = 1 + m + 2 = 3 + m$

Similarly $R_2 = g(-1) = 3 - m$

if $R_1 = R_2 \Rightarrow m = 0$

Sol 5: $x^2 - 11x + m = 0$ and $x^2 - 14x + 2m = 0$

Let α be the common root

Let $\alpha^2 - 11\alpha + m = 0$ and $\alpha^2 - 14\alpha + 2m = 0$

$\therefore 3\alpha - m = 0 \Rightarrow \alpha = \frac{m}{3}$. Substituting

$\Rightarrow \frac{m^2}{9} - \frac{11m}{3} + m = 0 \Rightarrow \frac{m^2}{9} - \frac{8m}{3} = 0$

for m = 0, 24 the equations have common roots.

Sol 6: $p(x) = ax^2 + bx + c$ (α & -2 are roots)

$Q(x) = ax^2 + cx + b$ (β & 3 are roots)

$\Rightarrow \alpha - 2 = \frac{-b}{a}$ & $-2\alpha = \frac{c}{a}$

and $\beta + 3 = \frac{-c}{a}$ & $3\beta = \frac{b}{a}$

$\therefore 3\beta = 2 - \alpha$ and $3 + \beta = 2\alpha$

$\Rightarrow \beta = \frac{1}{7}$ and $\alpha = 2 - \frac{3}{7} = \frac{11}{7} \Rightarrow \frac{\alpha}{\beta} = 11$

Sol 7: $(\log_{|x+6|} 2)\log_2(x^2 - x - 2) \geq 1$

$\Rightarrow \log_{|x+6|}(x^2 - x - 2) \geq 1$

$|x+6| \neq 1 \Rightarrow x \neq -5, -7$

When $|x+6| > 1 \Rightarrow x \in (-\infty, -7) \cup (-5, \infty)$

$x^2 - x - 2 > |x+6|$

when $x \in (-5, \infty)$

$\Rightarrow x^2 - x - 2 > x + 6 \Rightarrow x^2 - 2x - 8 > 0$

$\Rightarrow x \in (-\infty, -2) \cup (4, \infty)$

$\therefore x \in (-5, -2) \cup (4, \infty)$

when $x \in (-\infty, -7)$

$\Rightarrow x^2 - x - 2 > -x - 6 \Rightarrow x^2 + 4 > 0$

$\Rightarrow x \in (-\infty, -7)$

when $x \in (-7, -5)$

$x^2 - x - 2 \leq |x+6|$

$x^2 - 2x - 8 \leq 0$ when $x \in (-5, -6)$

$\Rightarrow x \in (-2, +4) \Rightarrow$ no possible value of x

When $x \in (-7, -5)$

$x^2 - x - 2 \leq -x - 6 \Rightarrow x^2 + 4 \leq 0 \Rightarrow$ not possible

$\therefore x \in (-7, -\infty) \cup (-5, -2) \cup (4, \infty)$

Sol 8: $\vec{V}_1 = \sin\theta\hat{i} + \cos\theta\hat{j}$

$\vec{V}_2 = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$

angel between $\vec{V}_1$ & $\vec{V}_2 = \alpha = \pi/3$

$\cos\alpha = \frac{\sin\theta + \cos\theta}{1 \times 2}$

$\Rightarrow \frac{\sin\theta + \cos\theta}{1 \times 2} = \frac{1}{2}$

$\Rightarrow \sin\theta + \cos\theta = 1$

The value of $\theta \in [0, 2\pi]$ are $0, \frac{\pi}{2}, 2\pi$

$\therefore$ No. of values of θ are 3

Sol 9: (a) A function is symmetric it when we replace α by β & β by α the function remains same

(i) $f(\beta,\alpha) = \beta^2 - \alpha \neq f(\alpha,\beta)$ (not symmetric)

(ii) $f(\beta,\alpha) = \beta^2\alpha + \beta\alpha^2 = \alpha^2\beta + \beta^2\alpha = f(\alpha,\beta)$

(iii) $f(\beta,\alpha) = \ell n\frac{\beta}{\alpha} = -\frac{\ell n\alpha}{\beta} \neq f(\alpha,\beta)$

(not symmetric)

(iv) $f(\beta,\alpha) = \cos(\beta - \alpha) = \cos(\alpha - \beta) = f(\alpha,\beta)$

$\therefore$ Symmetric

(b) α & β are roots of $x^2 - px + q$

$\Rightarrow \alpha + \beta = p$ & $\alpha\beta = q$

$\Rightarrow R_1 = (\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$

$\Rightarrow [(\alpha+\beta)^2 - 4\alpha\beta](\alpha+\beta)(\alpha+\beta^2) - \alpha\beta)$

$\Rightarrow (p^2 - 4q)\ p\ (p^2 - q)$

$\Rightarrow R_2 = \alpha^2\beta^2(\alpha+\beta) = q^2p$

$\therefore R_1 + R_2 = q^2p + p(p^2 - 4q)(p^2 - q)$

$= p(p^4 - 5p^2q + 5q^2)$

$R_1 + R_2 = q^2p^2(p^2 - 4q)(p^2 - q)$

The equation is $x^2 - (R_1 + R_2)x + R_1R_2 = 0$

Sol 10: $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$

Let $x^2 + 18x + 30 = t$

$\Rightarrow t = 2\sqrt{t + 15} \Rightarrow t^2 = 4(t + 15)$

$\Rightarrow t^2 - 4t - 60 = 0 \Rightarrow (t - 10)(t + 6) = 0$

$\therefore t = 10$ or $t = -6$

$\Rightarrow x^2 + 18x + 20 = 0$; $d > 0$

or $x^2 + 18x + 36 = 0$; $d > 0$

But also $x^2 + 18x + 45 > 0$

$\Rightarrow x \in (-\infty, -15) \cup (-3, \infty)$ and also $x^2 + 18x + 30 > 0$

$\therefore$ The product of the real roots = 20

Sol 11: $f(x) = \frac{\sqrt{x^2 + ax + 4}}{\sqrt{x^2 + bx + 16}}$

for $f(x) > 0$ both $x^2 + ax + 4 > 0$ & $x^2 + bx + 16 > 0$

$\Rightarrow D \leq 0$ for first and $D < 0$ for second eqn denominator can't be 0.

$a \in [-4, 4]$ & $b \in (-8, 8)$

$\therefore$ The possibly integral solution of (a, b) are $9 \times 15 = 135$

Sol 12: $f(0).f(1) < 0$

$f(x) = 9x - 12ax + 4 - a^2$

$f(0) = 4 - a^2$

$f(1) = 13 - 12a - a^2$

$f(0)f(1) = (a - 2)(a + 2)(a + 13)(a - 1) < 0$

$a \in (-13, -2) \cup (1, 2)$

Number of integers = 10

Sol 13: (a) $\left(x - \frac{1}{x}\right)^{1/2} + \left(1 - \frac{1}{x}\right)^{1/2} = x$ ..... (i)

and $\frac{x - 1}{\left(x - \frac{1}{x}\right)^{1/2} - \left(1 - \frac{1}{x}\right)^{1/2}} = x$ (factorizing)

Let $x - \frac{1}{x} = m$ & $1 - \frac{1}{x} = n$

$m^{1/2} = \frac{m + 1}{2}$

$\therefore 4m = (m + 1)^2$

$\therefore (m - 1)^2 = 0 \Rightarrow m = 1$

$\therefore x - \frac{1}{x} = 1 \Rightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$

now in equation (i) LHS >0 $\therefore$ $x > 0$

$\therefore x = \frac{1 + \sqrt{5}}{2}$ only possible solution

(b) Let $\left(x + \frac{1}{x}\right)^3 = m$ & $x^3 + \frac{1}{x^3} = n$

$$\Rightarrow \frac{\left(x+\frac{1}{x}\right)^6-\left(x^3+\frac{1}{x^3}\right)^2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}=\frac{m^2-n^2}{m+n}$$

$$=m-n=\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right) \;=\; 3\left(x+\frac{1}{x}\right)$$

The minimum value of $x+\frac{1}{x}=2$

$\therefore\ f(x)_{min}=3\times 2=6$

Sol 14: Given that

$X^2 + 2mx + 7m - 12 = 0$ .... (i)

$4x^2 - 4mx + 5m - 6 = 0$

For equation (i), D > 0

$(2m)^2 - 4(7m - 12) > 0$

$\Rightarrow 4m^2 - 28m + 48 > 0$

$$\Rightarrow m = \frac{28 \pm \sqrt{(28)^2 - 4\times 4\times 48}}{8}$$

$$= \frac{28 \pm \sqrt{784-768}}{8}$$

$$= \frac{28 \pm 4}{8} = 4, 3$$

For equation (i), D > 0

$16m^2 - 4 \times 4 \times (5m - 6) > 0$

$\Rightarrow 16m^2 - 16(5m - 6) > 0$

$\Rightarrow 16m^2 - 80m - 96 > 0$

$$\Rightarrow m = \frac{80 \pm \sqrt{(80)^2 - 4\times 16\times 96}}{32}$$

$$\Rightarrow m = \frac{21}{8}, \frac{19}{8}$$

Minimum value of m = $\frac{19}{8}$

Maximum value of m = 4

Then, a + b= $\frac{19}{8}$ + 4

$$= \frac{19+32}{8} = \frac{51}{8}$$

Sol 15: (a) $4x^2-(5p+1)x+5p=0$

$\beta = 1+\alpha$

$$\Rightarrow \alpha(1+\alpha)=\frac{5p}{4} \ \ \&\ 2a+1=\frac{5p+1}{4}$$

or $\alpha = \frac{5p-3}{8}$

$$\Rightarrow \frac{5p-3}{8}\times\frac{5(p+1)}{8}=\frac{5}{4}p$$

$5p^2-14p-3=0$

$5p^2-15p+p-3=0$

$\Rightarrow p=3$ or $p=\frac{-1}{5}$

p = 3 is the integral value

(b) $y=\frac{x^2-3x+4}{x^2+3x+4}$

$\Rightarrow x^2(y-1)+3x(y+1)+4(y-1)=0$

$x \in R$

$\therefore 9(y+1)^2-16(y-1)^2>0$

$\Rightarrow -7y^2+50y-7>0$

$\Rightarrow 7y^2-50y+7<0$

$\Rightarrow y \in \left(\frac{1}{7},7\right)$

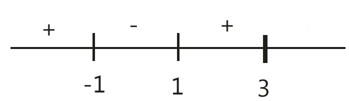
Integers lying in range are 1,2,3,4, or option Q R S T are correct.

(c) $\frac{x+1}{x-1} \ge \frac{x+5}{x+1}$

$$\Rightarrow \frac{x+1}{x-1}-\left(\frac{x+5}{x+1}\right)\ge 0$$

$$\Rightarrow \frac{x^2+2x+1-x^2-4x+5}{(x-1)(x+1)}\ge 0$$

$$\frac{2(3-x)}{(x-1)(x+1)}\ge 0$$



$x \ne 1$ as $x-1$ is in denominator the positive integral values of x are 2 & 3 Ans (R) (S)

(d) $\sin\frac{2\pi}{4}\sin\frac{4\pi}{7}+\sin\frac{4\pi}{7}\sin\frac{8\pi}{7}+\sin\frac{8\pi}{7}\sin\frac{2\pi}{7}=f(\text{say})$

Let $\frac{2\pi}{7}$ = A $\frac{4\pi}{7}$ = B and $\frac{8\pi}{7}$ = C

$$f=\frac{1}{2}\left(\begin{array}{l}\cos\left(\frac{2\pi}{7}-\frac{4\pi}{7}\right)-\cos\left(\frac{6\pi}{7}\right)+\cos\left(\frac{-4\pi}{7}\right)\\-\cos\left(\frac{12\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)-\cos\left(\frac{10\pi}{7}\right)\end{array}\right)$$

$$\frac{1}{2}\left(\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)-\cos\left(\frac{12\pi}{7}\right)-\cos\left(\frac{10\pi}{7}\right)\right)$$

$$\cos\left(\frac{2\pi}{7}\right)=\cos\left(2\pi-\frac{2\pi}{7}\right)=\cos\left(\frac{12\pi}{7}\right)$$

$$\cos\left(\frac{4\pi}{7}\right)=\cos\left(2\pi-\frac{4\pi}{7}\right)=\cos\left(\frac{10\pi}{7}\right)$$

$\therefore\ f=0$

Sol 16: $x^4+2x^3-8x^2-6x+15=p(x)$

$Q(x)=x^3+4x^2-x-10$

By trial one root of $Q(x)=-2$

$\therefore\ Q(x)=(x+2)(x^2+2x-5)$

$\therefore$ The root of x^2+2x-5 should satisfy $p(x)$ x^2+2x-5 has irrational roots and since

Irrational root exist in pairs

x^2+2x-5 should be a factor of p(x)

$\therefore\ p(x)=x^4+2x^3-5x^2-3x^2-6x+15$

$=x^2(x^2+2x-5)-3(x^2+2x-5)\ =(x^2-3)(x^2+2x-5)$

The uncommon real roots are

$x=\sqrt{3}, x=-\sqrt{3}\ \ \&\ x=-2$

$\therefore$ Product = 6

Sol 17: (a) $(x-1)\left|x^2-4x+3\right|+2x^2+3x-5=0$

$(x-1)\left|x^2-4x+3\right|+2x^2+5x-2x-5=0$

$(x-1)\left|x^2-4x+3\right|+(x-1)(2x+5)=0$

$\therefore\ x=1$ is one solution and $\left|x^2-4x+3\right|+(2x+5)=0$

When $x\in(-\infty,1)\cup(3,\infty)$

$x^2-2x+8=0$

D < 0 so not possible

When $x\in(1,3)$

$-x^2+4x-3+2x+5=0\ \Rightarrow x^2-6x-2=0$

$\Rightarrow x=\frac{6\pm\sqrt{40}}{2}\ =3\pm\sqrt{10}$

which doesn't belong to (1,3)

$\therefore\ x=1$ is the only solution

(b) $3\,|x^2+4x+2|=5x-4$

Case I: $x^2+4x+2>0$

$3(x^2+4x+2)=5x-4$

$\Rightarrow 3x^2+12x+6=5x-4$

$\Rightarrow 3x^2+7x+10=0$

$\Rightarrow x=\frac{-7\pm\sqrt{49-120}}{6}$

Which indicates x is imaginary here. So, this is not acceptable.

Case II: $x^2+4x+2<0$

$-3(x^2+4x+2)=5x-4$

$\Rightarrow 3x^2+12x+6=-5x+4$

$\Rightarrow 3x^2+17x+2=0$

$\Rightarrow x=\frac{-17\pm\sqrt{289-24}}{6}=\frac{-17\pm\sqrt{265}}{6}$ ...(i)

Also, $x^2+4x+2<0$

$\left[x-\left(\sqrt{2}-2\right)\right]\left[x-\left(2-\sqrt{2}\right)\right]<0$ ... (ii)

x will be the union of Eq. (i) and Eq. (ii)

(c) For $x\geq-1$

$x^3+x^2-x-1=0$

$x=1\ \&\ x=-1$ are the solutions

$(x+1)(x^2-0x-1)$

$(x+1)(x^2+0x-1)\Rightarrow (x+1)^2(x-1)=0$

for $x<-1$

$\therefore -x^3-1+x^2-x-2=0$

$x^3-x^2+x+3=0$

$(x+1)(x^2-2x+3)=0$

$x=-1$ is only solution

$x=-1, 1$

(d) Same as Example 4 of Solved Examples JEE Advanced.

Sol 18: Given that $x^3-3x^2+1=0$

$\Rightarrow\ \alpha+\beta+\gamma=3,\ \alpha\beta+\beta\gamma+\gamma\alpha=0,\ \alpha\beta\gamma=-1$

Now we have $(\alpha-2)(\beta-2)(\gamma-2)$

$=(\alpha\beta-2\alpha-2\beta+4)(\gamma-2)$

$=\alpha\beta\gamma-2(\alpha\beta+\beta\gamma+\gamma\alpha)+4(\alpha+\beta+\gamma)-8$

$=-1+12-8=3$

Similarly we can find

$$\left(\frac{\alpha}{\alpha-2}\right)\left(\frac{\beta}{\beta-2}\right)\left(\frac{\gamma}{\lambda-2}\right),\ \sum\left(\frac{\alpha}{\alpha-2}\times\frac{\beta}{\beta-2}\right),\ \sum\left(\frac{\alpha}{\alpha-2}\right)$$

$$\left(\frac{\alpha}{\alpha-2}\right)\left(\frac{\beta}{\beta-2}\right)\left(\frac{\lambda}{\lambda-2}\right)=\frac{\alpha\beta\gamma}{(\alpha-2)(\beta-2)(\gamma-2)}=\frac{-1}{3}$$

$$\sum\frac{\alpha}{\alpha-2}\times\frac{\beta}{\beta-2}=\frac{3\alpha\beta\gamma-2(\alpha\beta+\beta\gamma+\gamma\alpha)}{(\alpha-2)(\beta-2)(\gamma-2)}=\frac{-3}{3}=-1$$

$$\sum\frac{\alpha}{\alpha-2}=\frac{4(\alpha+\beta+\gamma)+3(\alpha\beta\gamma)-4(\sum\alpha\beta)}{(\alpha-2)(\beta-2)(\gamma-2)}=\frac{12-3}{3}=3$$

Sol 19: $\dfrac{(-2x^2+5x-10)}{(\sin t)x^2+2(1+\sin t)x+\sin t+4}>0$

The above expansion is always < 0 as D < 0

$\therefore\ (\sin t)x^2+2(1+\sin t)x+9\sin t+4<0$

For all x

$\Rightarrow\ \sin t<0$

and $4(1+\sin t)^2-4\sin t+(9\sin t+4)<0$

$\Rightarrow -32\sin^2 t-8\sin t+4<0$

$\Rightarrow 8\sin^2 t+2\sin t-1>0$

$\Rightarrow 8\sin^2 t+4\sin t-2\sin t-1>0$

$\Rightarrow 4\sin t(2\sin t+1)-1(2\sin t+1)>0$

$\Rightarrow (4\sin t-1)(2\sin t+1)>0$

$\Rightarrow\ \sin t\in\left[-1,\frac{-1}{2}\right)\cup\left(\frac{1}{4},1\right]$

but sin t < 0

$\Rightarrow \sin t\in\left[-1,\frac{-1}{2}\right)\Rightarrow t\in\left(\frac{4\pi}{3},\frac{5\pi}{3}\right)$

$a+b=\dfrac{9\pi}{3}=3\Rightarrow K=3$

Sol 20: Minimum value of quadratic occurs at $x=\dfrac{-b}{2a}=\dfrac{4p}{8}=\dfrac{p}{2}$

When $x_{min}\in[0,2]$

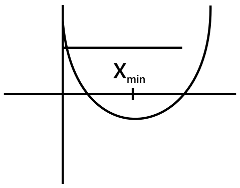
$\therefore\ f(x)_{min}=f(x_{min})=3$

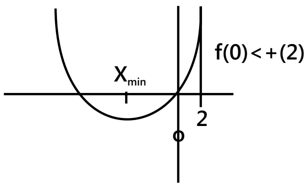
$\Rightarrow\ p\in[0,4]$

$\Rightarrow\ p^2-2p^2+p^2-2p+2=3\ \Rightarrow\ 2p=-1$

$\Rightarrow\ p=\dfrac{-1}{2}$ not true

when $x_{min}<0\Rightarrow p<0$





$\Rightarrow f_{min}$ occurs at $x=0$

$\therefore\ f(0)=p^2-2p+2=3$

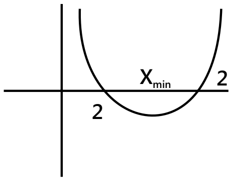
$\Rightarrow\ p^2-2p-1=0$

$p=\dfrac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}$ but $p<0$

$\Rightarrow\ p=1-\sqrt{2}$

When $x_{min}>2\Rightarrow p>4$

$\Rightarrow f_{min}$ occurs at $x=2$



$f(2)=16-8p+p^2-2p+2=3$

$\Rightarrow\ p^2-10p+15=0\ \Rightarrow p=\dfrac{10\pm\sqrt{40}}{2}$

$p=5\pm\sqrt{10}$

But $p>0\Rightarrow p=5+\sqrt{10}$

Sol 21: Since p(x) is a factor of $q(x) = x^4 + 6x^2 + 25$ and $r(x) = 3x^4 + 4x^2 + 28x + 5$, then p(x) will also be a factor of its linear combination.

Now, r(x) - 3q(x) = $x^2 - 2x + 5$

$\therefore\ p(x) = x^2 - 2x + 5$

Sol 22: $f(x) = x^2 - 2x - a^2 + 1 = (x-1)^2 - a^2$

$= (x-1-a)(x-1+a)$

$\therefore\ \alpha = a+1\ \ \&\beta = 1-a$

Now $g(\alpha) < 0\ \&\ g(\beta) < 0$

$\therefore\ (a+1)^2 - 2(a+1)(a+1) + a^2 - a < 0$

$\Rightarrow -a^2 - 2a - 1 + a^2 - a < 0 \Rightarrow\ a > \dfrac{-1}{3}$

and $(1-a)^2 - 2(a+1)(1-a) + a^2 - a < 0$

$\therefore\ 4a^2 - 3a - 1 < 0$

$\Rightarrow\ (4a+1)(a-1) < 0\ \Rightarrow\ a \in \left(-\dfrac{1}{4}, 1\right)$

Sol 23: $\dfrac{x+2}{x-4} \le 0\ \ \Rightarrow\ x \in [-2, 4)$

$x^2 - ax - 4 \le 0$

$\Rightarrow\ x \in \left[\dfrac{a - \sqrt{a^2+16}}{2}, \dfrac{a + \sqrt{a^2+16}}{2}\right]$

$\Rightarrow\ \dfrac{a + \sqrt{a^2+16}}{2} < 4\ \Rightarrow\ a^2 + 16 < (a-8)^2$

$\Rightarrow\ a^2 + 16 < a^2 - 16a + 64 \Rightarrow\ a < 3$

and $\dfrac{a - \sqrt{a^2+16}}{2} \ge -2\ \Rightarrow\ a - \sqrt{a^2+16} \ge -4$

$(a+4)^2 \ge \sqrt{a^2+16}\ \ \Rightarrow a \ge 0$

$\therefore$ The possible integral values of a are 0, 1, 2

Sol 24: Given equations are

$ax^2 + bx - c = 0$ (i)

$ax^2 + cx + b = 0$ (ii)

For sum of roots for (i) and (ii), we can

$\alpha - 2 = \dfrac{-b}{a}$, $\beta + 3 = \dfrac{-c}{a}$

For product of root for (i) and (ii), we can

$-2\alpha = \dfrac{c}{a}$, $3\beta = \dfrac{b}{a}$

We can write here

$\alpha - 2 = -3\beta$ and $\beta + 3 = 2\alpha$

Solving these two equations

$\alpha - 2 = -3\,(2\alpha - 3)$

$\Rightarrow \alpha - 2 = -6\alpha + 9$

$\Rightarrow 7\alpha = 11 \qquad \Rightarrow \alpha = \dfrac{11}{7}$

Therefore, for β, $\beta = 2\left(\dfrac{11}{7}\right) - 3$

$= \dfrac{22}{7} - 3 = \dfrac{1}{7}$

Absolute product of four roots

$= \left|\dfrac{1}{7}\right| \left|\dfrac{11}{7}\right| \left|\dfrac{3}{1}\right| \left|\dfrac{-2}{1}\right| = \dfrac{66}{49}$

Therefore, (p + q) = 66 + 49 = 115

Sol 25: For origin to lie between the roots.

$af(0) < 0$

$\Rightarrow\ (a^2 - 6a + 5)(6a - a^2 - 8) < 0$

$\Rightarrow\ (a-5)(a-1)(a-2)(a-4) > 0$

+	-	+	+
1	2	4	5

$a \in (-\infty, 1) \cup (2, 4) \cup (5, \infty)$

Also $a^2 + 2a \ge 0 \Rightarrow a(a+2) \ge 0$

$\Rightarrow\ a \in (-\infty, -2] \cup [0, \infty)$

$\therefore\ a \in (-\infty, -2] \cup [0, 1) \cup (2, 4) \cup (5, \infty)$

Sol 26: $(\log_2 x)^4 - \left(\log_{1/2} \dfrac{x^5}{4}\right)^2 - 20\log_2 x + 148 < 0$

$\Rightarrow\ (\log_2 x)^4 - (5\log_2 x - 2)^2 - 20\log_2 x + 148 < 0$

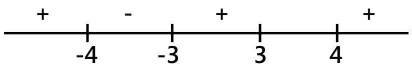
Let $\log_2 x = t$

$\Rightarrow\ \ t^4 - (25t^2 - 20t + 4) - 20t + 148 < 0$

$\Rightarrow\ t^4 - 25t^2 + 144 < 0$

$\Rightarrow\ (t^2 - 16)(t^2 - 9) < 0$

$\Rightarrow\ (t-3)(t+3)(t-4)(t+4) < 0$



$\Rightarrow t \in (-4,-3) \cup (3,4)$

$\therefore x \in \left(\frac{1}{16},\frac{1}{8}\right) \cup (8,16)$

Sol 27: $(\log 100x)^2 + (\log 10x)^2 + \log x \le 14$

$\Rightarrow (2+\log x)^2 + (1+\log x)^2 + \log x \le 14$

$\Rightarrow 2(\log x)^2 + 7\log x + 5 \le 14$

$\Rightarrow 2(\log x)^2 + 7\log x - 9 \le 0$

$\Rightarrow (\log x - 1)(2\log x + 9) \le 0$

$\Rightarrow -\frac{9}{2} \le \log x \le 1 \Rightarrow 10^{-9/2} \le x \le 10$

Sol 28: $\log_{1/2}(x+1) > \log_2(2-x)$

$\Rightarrow \log_2(2-x) + \log_2(x+1) < 0$

$\Rightarrow \log_2(x+1)(2-x) < 0 \Rightarrow (x+1)(2-x) < 1$

$\Rightarrow x^2 - x - 1 > 0$

$x \in \left(-\infty,\frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2},\infty\right)$

Also $x+1>0 \Rightarrow x > -1$ and $x < 2$

$\therefore x \in \left(-1,\frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2},2\right)$

Sol 29: $\log_{1/5}(2x^2+5x+1) < 0$

$\Rightarrow 2x^2 + 5x + 1 > 1$

$x(2x+5) > 0$

$\Rightarrow x \in \left(-\infty,\frac{-5}{2}\right) \cup (0,\infty)$

Exercise 2

Single Correct Choice Type

Sol 1: (C) Let $(r_1+2)(r_2+2)(r_3+2) = f$

$= (r_1 r_2 + 2(r_1+r_2) + 4)(r_3+2)$

$= r_1 r_2 r_3 + 4(r_1+r_2+r_3) + 2(r_1 r_2 + r_2 r_3 + r_1 r_3) + 8$

The equation we have is $x^3 - 2x^2 + 4x + 5074 = 0$

We can write $r_1 + r_2 + r_3 = 2$, $\Sigma r_1 r_2 = +4$, $r_1 r_2 r_3 = -5074$

$\therefore f = -5074 + 4\times 2 + 4\times 2 + 8 = -5050$

Method 2: (We have to find the product of roots of a cubic whose roots are $\alpha+2,\ \beta+2,\ \gamma+2$

$\Rightarrow \alpha + 2 = x \ \therefore \alpha = (x-2)$ Substituting we get

$(x-2)^3 - 2(x-2)^2 + 4(x-2) + 5074$

The constant term = 5050 $\therefore$ Product = -5050

Sol 2: (A) We are given that" after $x \in R$ and the polynomial $x^8 - x^5 + x^2 - x + 1$

When |x| < 1

$\therefore f(x) = x^8 + (x^2 - x^5) + (1-x) > 0$

as $x^2 - x^5 > 0$ & $(1-x) > 0$

When $|x| \ge 1$

$f(x) = (x^8 - x^5) + (x^2 - x) + 1 > 0$

as $x^8 - x^5 > 0$ & $x^2 - x > 0$

$\therefore$ f(x) is always positive.

Sol 3: (B) $a(x^2-2x+1) + b(x^2-3x+2) + x - a^2 = 0$

$\Rightarrow (a+b)x^2 + (1-2a-3b)x + a + 2b - a^2 = 0$

Since this is satisfied by all x

$\Rightarrow a+b=0$, $2a+3b=1$

$\Rightarrow b=1$ & $a=-1$

also $a + 2b - a^2 = 0$

Which is satisfied by (-1, 1)

Sol 4: (D) $y(-1) \ge -4$

$\Rightarrow a - b + c \ge -4$ ... (i)

$y(1) \le 0 \Rightarrow a + b + c \le 0$ ... (ii)

$y(3) \ge 5 \Rightarrow 9a + 3b + c \ge 5$ ... (iii)

From (i) and (iii)

$12a + 4c \ge -7$ ... (iv)

Equation can be written as

$-a - b - c \ge 0$ ... (v)

$\therefore$ From (iv) and (i)

$2a + 2c \ge -4 \Rightarrow a + c \ge -2$ ... (vi)

From (v) and (vi)

$8a \ge 1 \Rightarrow a \ge \frac{1}{8}$

Sol 5: (A) $x = \frac{4\lambda}{1+\lambda^2}$, $y = \frac{2-2\lambda^2}{1+\lambda^2}$

Let $\lambda = \tan\theta$

$\Rightarrow x = 2\sin 2\theta$ & $y = 2\cos 2\theta$

$f = x^2 - xy + y^2$

$= 4 - 4\sin 2\theta\cos 2\theta = 4 - 2\sin 4\theta$

$\therefore$ f lies between 2 and 6 or $f \in [2,6]$

$\therefore a = 2$ & $b = 6$ $\therefore a + b = 8$

Multiple Correct Choice Type

Sol 6: (B, D) $x^2 + abx + c = 0$ & $x^2 + acx + b = 0$ have a common roots lets say α

$\Rightarrow \alpha^2 + ab\alpha + c = 0$ & $\alpha^2 + ac\alpha + b = 0$

$\therefore \alpha = \frac{1}{a}$, $\beta = ac$ & $\gamma = ab$

The other eqn is $x^2 - a(b+c)x + a^2bc = 0$

Sol 7: (C, D) Given α, a^2 are root of the equation

$4x^2 - 15x + 4p = 0$

$\Rightarrow \alpha + \alpha^2 = \frac{15}{4}$ ... (i)

$\Rightarrow \alpha^3 = p$ ... (ii)

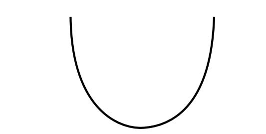
From equation (i)

$4\alpha^2 + 4\alpha - 15 = 0 \Rightarrow 4\alpha^2 + 10\alpha - 6\alpha - 15 = 0$

$\Rightarrow \alpha = \frac{-5}{2}, \alpha = \frac{3}{2}$

$\Rightarrow p = \frac{-125}{8}$ or $p = \frac{27}{8}$

Sol 8: (A, B, C) $f(n) = 4n^2 - 8kn + k$, $f(n) \ge 0$



$\Rightarrow 4n^2 - 8kn + k \ge 0 \Rightarrow \le 0$

$\Rightarrow 64k^2 - 16k \le 0 \Rightarrow (2k+1)(2k-1) \le 0$

$\Rightarrow k \in \left(\frac{-1}{2}, \frac{1}{2}\right]$

$\therefore$ k=0 is the only integral solution

(b) Roots of the equation $f(n) = 0$ are

$\alpha = \frac{8k \pm \sqrt{64k^2 - 16k}}{8} = k \pm \sqrt{k^2 - \frac{k}{4}}$

$\alpha = k - \sqrt{k^2 - \frac{k}{4}}$, $\beta = k + \sqrt{k^2 - \frac{k}{4}}$

If $k < 0$ then $\alpha < 0$, $\beta > 0$ $\left(\because \sqrt{k^2 - \frac{k}{4}} > -k\right)$

Let $-k = p$ $\left(\Rightarrow \sqrt{p^2 + \frac{p}{4}} > p\right)$

(c) $\alpha_1\beta \in (0,1)$

(i) $D \ge 0 \Rightarrow k^2 - \frac{k}{4} \ge 0 \Rightarrow k > \frac{1}{4}, k < 0$

(ii) $af(0) > 0 \Rightarrow 4(k) \ge 0 \Rightarrow k > 0$

$af(1) > 0 \Rightarrow 4(4-7k) > 0 \Rightarrow k < \frac{4}{7}$

(iii) $0 < \frac{8k}{8} < 1 \Rightarrow 0 < k < 1 \Rightarrow k \in \left(\frac{1}{4}, \frac{4}{7}\right)$ c

f(n) min $= 4(k)^2 - 8k^2 + k$

at $n = \frac{-b}{2a} = k - 4k^2$

Sol 9: (A, D) α, α^2 $(\alpha > 0)$ are roots of $x^2 - 30x + b = 0$

$\alpha + \alpha^2 = 30$; $\alpha^3 = b$

$\alpha^2 + \alpha - 30 = 0$

$(\alpha + 6)(\alpha - 5) = 0 \Rightarrow \alpha = -6, \alpha = 5$

$\alpha = 5$ $(\because \alpha > 0)$

$\alpha^2 = 25$

$r = 25, s = 5, b = 125$

$b + r - s = 145$

$b + r + s = 155$

$b - r - s = 95$

$b - r + s = 105$

Comprehension Type

Sol 10: (A) $p(x) = (x - \cos 36^0)(x - \cos 84^0)(x - \cos 156^0)$

co efficient of x^2 is $-(\cos 36^0 + \cos 84^0 + \cos 156^0)$

$= \cos 36^0 + 2\cos(36^0)\cos 120^0 = 0$

Sol 11: (B) Absolute term $= -\cos 36^0 \cos 84^0 \cos 156^0$

$= \frac{-1}{2}(\cos 36^0)(\cos 240^0 + \cos 72^0)$

$= \frac{-1}{2}\cos 36^0\left(\frac{-1}{2} + \cos 72^0\right)$

$= \left(\frac{-1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1-2}{4}\right)$

$= \left(\frac{-1}{2}\right)\left(\frac{1}{16}\right)\left(5-3-2\sqrt{5}\right) = \frac{\sqrt{5}-1}{16}$.

Assertion Reasoning Type

Sol 12: (D) $f(x) = ax^3 + bx + c$ sum of three roots =0

sum is zero only when atleast one of them is negative or all roots are zero.

$\alpha = \beta = \gamma = 0$ is one set to prove assertion as false.

Sol 13: (A) $f(x) = ax^2 + ax + (a+b)$ ...(i)

$g(x) = ax^2 + 3ax + 3a + b = f(x+1)$ ....(ii)

$D(f) = a^2 - 4a(a+b) = -3a^2 - 4ab$

since a and b are of same signs, f is either always positive or always negative depending on a.

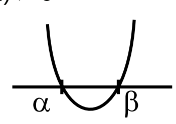
Since g = f(x + 1)

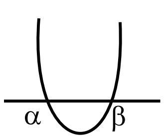
$\therefore$ g(x) will just shift the group of f to 1 unit left. There will be no change along y-axis

$\therefore$ Statement-II is correct explanation of statement-I.

Match the Columns

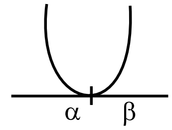
Sol 14: $ax^2 + bx + c = f(x)$, $af(t) > 0$

(A) $a > 0$ & $b^2 > 4ac$ $\therefore f(x) =$ 

$af(x) =$ 

$af(t) > 0$ at $t < \alpha$ or $t > \beta$

& $t \neq \alpha$

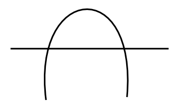
(B) $a > 0$ & $b^2 = 4ac$ $\therefore f(x) =$ 

$af(t) > 0$ at $t < \alpha$, $af(x) =$ & $t \neq \alpha$

$\therefore t < \alpha$ or $t > \alpha$ $c = \beta$ & $t \neq \alpha$

(C) $a < 0$ and $b^2 > 4ac$

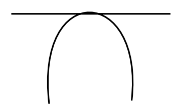
$\therefore af(t) > 0$ for

$t < \alpha$ or $t > \beta$ $t \neq \alpha, \beta$ 

(D) $a < 0$ & $b^2 = 4ac$

$\therefore af(t) > 0$ for

$t < \alpha$ or $t > \beta (= \alpha)$ &

$t \neq \alpha$ 

Sol 15: $f(x) = x^2 - 2px + p^2 - 1$

(A) Both roots of $f(x) = 0$ are less then 4

$\therefore af(4) > 0$ & $\frac{-b}{2a} < 4$

$\therefore 1 \times (16 - 8p + p^2 - 1) > 0$ & $\frac{2p}{p} < 4$

$\Rightarrow (p-3)$ or $(p-5) > 0$ & $P < 4$ ... (i)

$P < 3$ or $p > 5$ ...(ii)

From (i) and (ii) $p \in (-\infty, 3)$

(B) Both roots are greeter then -2

$\therefore af(-2) > 0$ & $\frac{-b}{2a} > -2$

$\Rightarrow 1(4 + 4,p + p^2 - 1) > 0$, $\frac{2p}{2a} > -2 \Rightarrow p > -2$

$\therefore (p+1)(p+3) > 0$, $p > -2$

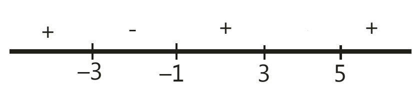
$p < -3$ or $p > -1$ & $p > -2$

$\therefore p \in (-1, \infty)$

(C) Exactly one root lies between (–2, 4)

$\Rightarrow f(-2)f(4) < 0 \Rightarrow (4 + 4p + p^2 - 1)(16 - 8p + p^2 - 1) < 0$

$\Rightarrow (p+1)(p+3)(p-3)(p-5) < 0$



$\therefore\ p \in (-3,-1) \cup (3,5)$

(D) 1 lies between the root

$\therefore\ af(1) < 0$

$\Rightarrow 1(1-2p+p^2-1) < 0 \Rightarrow p(p-2) < 0$

$\Rightarrow P \in (0, 2)$

Sol 16: (A)

$$\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}+2\right)}{\left(x+\frac{1}{x}\right)^3+x^3+\frac{1}{x^3}}=\frac{\left(x+\frac{1}{x}\right)^6-\left(x^3+\frac{1}{x^3}\right)^2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$

Let $\left(x+\frac{1}{x}\right)^3=m\ \&\ x^3+\frac{1}{x^3}=n$

$$=\frac{m^2-n^2}{m+n}=m-n=\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)$$

$$=3x+3\times\frac{1}{x}=3\left(x+\frac{1}{x}\right)$$

The minimum value of $\frac{x+1}{x}=2$

$\therefore\ f(x)_{min}=6$

(B) We want atleast one solution $\therefore$ we want to eliminate the cases when these is no solution

$\therefore$ All c except when

$$1+\log_2\left(2x^2+2x+\frac{7}{2}\right)<\log_2(cx^2+c)$$

For all x

$\Rightarrow\ 4x^2+4x+7 \le (x^2+c)$ for all x

$\Rightarrow\ (c-4)x^2-4x+(c-7) > 0$

$\therefore\ c > 4\ \&\ D < 0$

$\Rightarrow\ 16-4(c-4)(c-7) < 0$

$\therefore\ c \in (-\infty,3) \cup (8,\infty)$

Taking intersection

$\therefore$ The given expansion is not true for any x when $c \in (8,\infty)$

$\therefore$ For $c \in (0,8]$ the given expansion is true for atleast one x.

$[cx^2+c>0 \Rightarrow c>0]$

Sol 17: (A) $K(x^2)+(1-K)x+5=0$

Given, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$

$$\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{4}{5}\ \Rightarrow\ \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{4}{5}$$

$$\therefore\ \frac{(1-K)^2/K^2-2\times 5/K}{+5/K}=\frac{4}{5}$$

$$\Rightarrow\ \frac{(1-K)^2}{K}-10=4$$

$K^2-2K+1-10K=4K$

$\Rightarrow K^2-16K+1=0$

$$\therefore \frac{K_1}{K_2}+\frac{K_2}{K_1}=\frac{(K_1+K_2)^2-2K_1K_2}{K_1K_2}=\frac{(16)^2-2\times 1}{1}=254$$

(B) $y=\frac{x^2+ax+b}{x^2+2x+3}$

$(y-1)x^2+(2y-a)x+(3y-b)=0$

$(2y-a)^2-4(y-1)(3y-b) \ge 0$

$\Rightarrow\ 4y^2+a^2-4ay-4(3y^2-(b+3)y+b) \ge 0$

$\Rightarrow\ 8y^2+4(a-b-3)y+4b-a^2 \le 0$

$2y^2+(a-b-3)y+4b-a^2 \le 0$

Now $-5\ \&\ 4$ are solution of equation

On solving we get $a^2+b^2=277$

(C) $f(x)=x^3+px^2+qx+72$

$x^2+ax+b\ \&\ x^2+bx+a$

Have a common root α

$\Rightarrow\ \alpha^2+a\alpha+b=0$

$\alpha^2+b\alpha+a=0$

$\Rightarrow\ \alpha=1$ common root

Sum of roots $=\beta+\alpha=-a$

$\Rightarrow\ \beta=-(a+1)$

$\gamma=-(b+1)$

$\Rightarrow\ -(a+1)=b$

$\therefore\ a+b=1$

Product of roots = –72

$\therefore\ ab \times 1 = 72$

$a(1-a) = -72$

$a^2 - a - 72 = 0$

$a^2 - 9a + 8a - 72 = 0$

$\therefore\ a = 9 \text{ or } a = -8$

In either case b = –8 or b = 9

Sum of squares of roots = $a^2 + b^2 + (1)^2$

$= 81 + 1 + 64\ = 146$

Previous Years' Questions

Sol 1: Given $3x - y - z = 0$ ... (i)

$-3x + 2y + z = 0$ ... (ii)

and $-3x + z = 0$ ... (iii)

On adding Eqs. (i) and (ii), we get y = 0 So,

3x = z Now, $x^2 + y^2 + z^2 \le 100$

$\Rightarrow x^2 + (3x)^2 + 0 \le 100$

$\Rightarrow 10x^2 \le 100; \Rightarrow x^2 \le 10$

X = –3, – 2, – 1, 0, 1, 2, 3

So, number of such 7 points are possible

Sol 2: Here a +b =10c and c +d =10a

$\Rightarrow (a-c) + (b-d) = 10(c-a)$

$\Rightarrow (b-d) = 11\ (c-a)$ ... (i)

Since 'c' is the root of $x^2 - 10ax - 11b = 0$

$\Rightarrow c^2 - 10ac - 11b = 0$ ....(ii)

Similarly, 'a' is the root of

$x^2 - 10cx - 11d = 0$

$\Rightarrow a^2 - 10ca - 11d = 0$ ... (iii)

On subtracting Eq.(iv) from Eq. (ii) we get

$\left(c^2 - a^2\right) = 11(b-d)$ ... (iv)

$\therefore\ (c+a)(c-a) = 11 \times 11\ (c-a)$

[from Eq. (i)] $\Rightarrow$ c +a = 121

$\therefore$ a + b + c + d = 10c + 10a

= 10(c + a) = 1210

Sol 3: Given $x^2 + (a-b)x + (1-a-b) = 0$ has real and unequal roots

$\Rightarrow$ D > 0

$\Rightarrow (a-b)^2 - 4(1)(1-a-b) > 0$

$\Rightarrow a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

Now, to find values of 'a' for which equation has unequal real roots for all values of b.

i.e, above equation is true for all b.

or $b^2 + b(4-2a)\left(a^2 + 4a - 4\right) > 0$ is true for all b.

$\therefore$ Discriminate, D < 0

$\Rightarrow (4-2a)^2 - 4\left(a^2 + 4a - 4\right) < 0$

$\Rightarrow 16 - 16a + 4a^2 - 4a^2 - 16a + 16 < 0$

$\Rightarrow -32a + 32 < 0 \Rightarrow a > 1$

Sol 4: Let $f(x) = 4x^3 - 3x - p$

Now, $f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) - p$

$= \frac{4}{8} - \frac{3}{2} - p = -1(1+p)$

$f(1) = 4(1)^3 - 3(1) - p = 1 - p$

$\Rightarrow f\left(\frac{1}{2}\right).f(1) = -(1+p)(1-p)$

$= (p + 1)(p - 1) = p^2 - 1$

Which is $\le 0,\ \forall\ p \in [-1, 1]$.

$\therefore$ f(x) has at least one root in $\left[\frac{1}{2}, 1\right]$

Now, f'(x) = $x^2 - 3$

= 3(2x – 1)(2x + 1)

$= \frac{3}{4}\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) > 0$ in $\left[\frac{1}{2}, 1\right]$

$\Rightarrow$ f(x) is an increasing function in $\left[\frac{1}{2}, 1\right]$

Therefore, f(x) has exactly one root in $\left[\frac{1}{2}, 1\right]$ for any p $\in$ [–1, 1].

Now let x = cos θ

$\therefore x \in \left[\frac{1}{2}, 1\right] \Rightarrow \theta \in \left[0, \frac{\pi}{3}\right]$

From Eq. (i),

$4\cos^2\theta - 3\cos\theta = p \Rightarrow \cos 3\theta = p$

$\Rightarrow 3\theta = \cos^{-1} p$

$\Rightarrow \theta = \frac{1}{3}\cos^{-1} p$

$\Rightarrow \cos\theta = \cos\left(\frac{1}{3}\cos^{-1} p\right)$

$\Rightarrow x = \cos\left(\frac{1}{3}\cos^{-1} p\right)$ s

Sol 5: Suppose $f(x) = Ax^2 + Bx + C$ is an integer whenever x is an integer.

$\therefore$ f(0), f(1), f(–1) are integers.

$\Rightarrow$ C, A + B + C, A – B + C are integers.

$\Rightarrow$ C, A + B, A – B are integers.

$\Rightarrow$ C, A + B, (A + B) – (A – B) = 2A are integers.

Conversely suppose 2A, A + B and C are integers.

Let n be any integer. We have,

$$f(n) = An^2 + Bn + C = 2A\left[\frac{n(n-1)}{2}\right] + (A+B)n + C$$

Since, n is an integer, n (n – 1)/2 is an integers. Also 2A, A + B and C are integers.

We get f(n) is an integer for all integer n.

Sol 6: Given $2^{|y|} - \left|2^{y-1} - 1\right| = 2^{y-1} + 1$

Case I when $y \in (-\infty, 0]$

$\therefore 2^{-y} + \left(2^{y-1} - 1\right) = 2^{y-1} + 1$

$\Rightarrow 2^{-y} = 2 \Rightarrow y = -1 \in (-\infty, 0]$ ... (i)

Case II when $y \in (0, 1]$

$\therefore 2^{y} + \left(2^{y-1} - 1\right) = 2^{y-1} + 1$

$\Rightarrow 2^{y} = 2 \Rightarrow y = 1 \in (0, 1]$ ... (ii)

Case III when $y \in (1, \infty)$

$\therefore 2^2 = 2.2^{y-1}$

$\Rightarrow 2^y - 2.2^{y-1} = 0$

$\Rightarrow 2^y - 2^y = 0$ true for all y > 1 ... (iii)

$\therefore$ From Eqs. (i), (ii), and (iii), we get $y \in \{-1\} \cup [1, \infty)$

Sol 7: Given,

$\log_{(2x+3)}\left(6x^2 + 23x + 21\right) = 4 - \log_{(3x+7)}\left(4x^2 + 12x + 9\right)$

$\Rightarrow \log_{(2x+3)}(2x+3)(3x+7) = 4 - \log_{(3x+7)}(2x+3)^2$

$\Rightarrow 1 + \log_{(2x+3)}(3x+7) = 4 - \log_{(3x+7)}(2x+3)$

Put $\log_{(2x+3)}(3x+7) = y \quad \therefore y + \frac{2}{y} - 3 = 0$

$\Rightarrow y^2 - 3y + 2 = 0 \Rightarrow (y-1)(y-2) = 0$

$\Rightarrow$ y = 1 or y = 2

$\log_{(2x+3)}(3x+7) = 1$ or $\log_{(2x+3)}(3x+7) = 2$

$\Rightarrow 3x + 7 = 2x + 3$ or $(3x+7) = (2x+3)^2$

$\Rightarrow$ x = –4 or $3x + 7 = 4x^2 + 12x + 9$

$\Rightarrow$ x = –4 or $4x^2 + 9x + 2 = 0$

$\Rightarrow$ x = –4 or (4x + 1)(x + 2) = 0

$\therefore x = -2, -4, -1/4$ ... (i)

But log exists only when, $6x^2 + 23x + 21 > 0$,

$4x^2 + 12x + 9 > 0$,

2x + 3 > 0 and 3x + 7 > 0

$\Rightarrow x > -\frac{3}{2}$ ... (ii)

$\therefore x = -\frac{1}{4}$ is the only solution.

Sol 8: (B) Let y = x intersect the curve y = ke^x at exactly one point when k ≤ 0.

Sol 9: (**A**) Let $f(x) = ke^x - x$

$f'(x) = ke^x - 1 = 0; \Rightarrow x = -\ln k$

$f''(x) = ke^x; \ \therefore \left[f''(x)\right]_{x=-\ln k} = 1 > 0$

Hence, f(–ln k) = 1 + ln k

For one root of given equation

$1 + \ln k = 0 \,; \Rightarrow k = \dfrac{1}{e}$

Sol 10: (A) For two distinct roots, $1 + \ln k < 0$ $(k > 0)$

$\ln k < -1$, $k < \dfrac{1}{e}$; Hence, $k \in \left(0, \dfrac{1}{e}\right)$

Sol 11: (C) Given $f(x) = 4x^2 + 3x^3 + 2x + 1$

$f'(x) = 2\left(6x^2 + 3x + 1\right); D = 9 - 24 < 0$

Hence, f(x) = 0 has only one real root.

$$f\left(-\frac{1}{2}\right) = 1 - 1 + \frac{3}{4} - \frac{4}{8} > 0$$

$$f\left(-\frac{3}{4}\right) = 1 - \frac{6}{4} + \frac{27}{16} - \frac{108}{64}$$

$\dfrac{64 - 96 + 108 - 108}{64} < 0$ f(x) changes its sign in $\left(-\dfrac{3}{4}, -\dfrac{1}{2}\right)$, hence f(x) = 0 has a root in $\left(-\dfrac{3}{4}, -\dfrac{1}{2}\right)$.

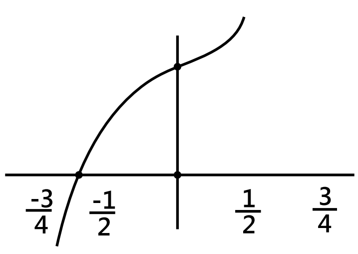
Sol 12: (A) $\int_0^{1/2} f(x)dx < \int_0^{t} f(x)dx < \int_0^{3/4} f(x)dx$

Now, $\int f(x)dx = \int\left(1 + 2x + 3x^2 + 4x^3\right)dx$

$= x + x^2 + x^3 + x^4$;

$$\Rightarrow \int_0^{1/2} f(x)dx = \frac{15}{16} > \frac{3}{4} \quad \int_0^{3/4} f(x)dx = \frac{530}{256} < 3$$

Sol 13: (B) Figure is self explanatory



Sol 14: (C) $\dfrac{a_{10} - 2a_8}{2a_9} = \dfrac{\alpha^{10} - \beta^{10} - 2\alpha^8 + 2\beta^8}{2\left(\alpha^9 - \beta^9\right)}$

$$= \frac{\alpha^8\left(\alpha^2 - 2\right) - \beta^8\left(\beta^2 - 2\right)}{2\left(\alpha^9 - \beta^9\right)}$$

($\because$ α is root of $x^2 - 6x - 2 = 0 \Rightarrow \alpha^2 - 2 = 6\alpha$)

($\because$ Also, β is root of $x^2 - 6x - 2 = 0 \Rightarrow \beta^2 - 2 = 6\beta$)

$$\frac{a_{10} - 2a_{10}}{2a_9} = \frac{\alpha^8(6\alpha) - \beta^8(6\beta)}{2\left(\alpha^9 - \beta^9\right)} = \frac{6\left(\alpha^9 - \beta^9\right)}{9\left(\alpha^9 - \beta^9\right)} = 3$$

Sol 15: (B) Sum of roots = $\dfrac{\alpha^2 + \beta^2}{\alpha\,\beta}$ and product = 1

Given, $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$

$\Rightarrow (\alpha + \beta)\left(\alpha^2 - \alpha\beta + \beta^2\right) = q$

$\therefore \alpha^2 + \beta^2 - \alpha\beta = \dfrac{-q}{p}$ ....(i)

and $(\alpha + \beta)^2 = p^2$

$\Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = p^2$ ....(ii)

From Eqs.(i) and (ii), we get

$\alpha^2 + \beta^2 = \dfrac{p^3 - q}{3p}$ and $\alpha\,\beta = \dfrac{p^3 + q}{3p}$

$\therefore$ Required equation $x^2 - \dfrac{\left(p^3 - 2q\right)}{\left(p^3 + q\right)} + 1 = 0$

$\Rightarrow \left(p^3 + q\right)x^2 - \left(p^3 - 2q\right)x + \left(p^3 + q\right) = 0$

Sol 16: (A) Since, roots are real therefore $D \geq 0$

$\Rightarrow 4(a + b + c)^2 - 12\lambda(ab + bc + ca) \geq 0$

$\Rightarrow (a + b + c)^2 - 3\,\lambda(ab + bc + ca) \geq 0$

$\Rightarrow a^2 + b^2 + c^2 \geq (ab + bc + ca)\,(3\lambda - 2)$

$\Rightarrow 3\lambda - 2 \leq \dfrac{a^2 + b^2 + c^2}{ab + bc + ca}$ ... (i)

Also , $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} < 1$

$\Rightarrow b^2 + c^2 - a^2 < 2bc$

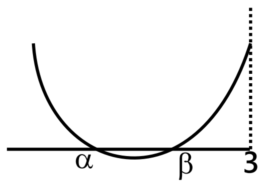
Similarly, $c^2 + a^2 - b^2 < 2ca$ and $a^2 + b^2 - c^2 < 2\,a\,b$

$\Rightarrow a^2 + b^2 - c^2 < 2\,(ab + bc + ca)$

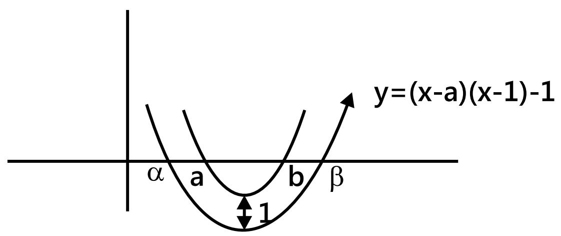
$\Rightarrow \dfrac{a^2 + b^2 + c^2}{ab + bc + ca} < 2$ ... (ii)

$\therefore$ From Eqs. (i) and (ii), we get $3\lambda - 2 < 2$

$\Rightarrow \lambda < \dfrac{4}{3}$



Sol 17: (D) From graph it is clear that one of the roots of



$(x - a)(x - b) - 1 = 0$ lies in $(-\infty, a)$ and other lies in (b, ∞). Therefore,(d) is the answer.

Sol 18: (A) Let $f(x) = x^2 - 2ax + a^2 + a - 3$

Since, both roots are less than 3.

$\Rightarrow \alpha < 3, \beta < 3 \Rightarrow$ Sum, $S = \alpha + \beta < 6$

$\Rightarrow \dfrac{\alpha+\beta}{2} < 3\,; \Rightarrow \dfrac{2\alpha}{2} < 3$

$\Rightarrow a < 3$ ... (i)

Again, product of roots $P = \alpha\,\beta$

$\Rightarrow p < 9; \Rightarrow \alpha\,\beta < 9$

$\Rightarrow a^2 + a - 3 < 9 \;\Rightarrow a^2 + a - 12 < 0$

$\Rightarrow (a - 3)(a + 4) < 0$

$\Rightarrow -4 < a < 3$ ... (ii)

Again, $D = B^2 - 4AC \geq 0$

$\Rightarrow (-2a)^2 - 4.1\,(a^2 + a - 3) \geq 0$

$\Rightarrow 4a^2 - 4a^2 - 4a + 12 \geq 0 \Rightarrow -4a + 12 \geq 0$

$\Rightarrow a \leq 0$ ... (iii)

Again, a f(3) > 0

$\Rightarrow 1\left[(3)^2 - 2a(3) + a^2 + a - 3\right] > 0$

$\Rightarrow 9 - 6a + a^2 + a - 3 > 0 \;\Rightarrow a^2 - 5a + 6 > 0$

$\Rightarrow (a - 2)(a - 3) > 0$

$\therefore a \in (-\infty, 2) \cup (3, \infty)$ ... (iv)

From Eqs. (i), (ii), (iii) and (iv), we get

$a \in (-4, 2)$

Note: There is correction in answer a < 2 should be –4 < a < 2.

Sol 19: (B) Let $f(x) = ax^2 + bx + c > 0$ for all $x \in R$

$\Rightarrow$ a > 0 and $b^2 - 4ac < 0$ ... (i)

$\therefore$ g(x) = f(x) + f′(x) + f″(x)

$\Rightarrow g(x) = ax^2 + bx + c + 2ax + b + 2a$

$\Rightarrow g(x) = ax^2 + x(b + 2a)(c + b + 2a)$

Whose discriminant $= (b + 2a)^2 - 4a(c + b + 2a)$

$= b^2 + 4a^2 + 4ab - 4ac - 4ab - 8a^2 \;= b^2 - 4a^2 - 4ac$

$= (b^2 - 4ac) - 4a^2 < 0$ [from Eq. (i)]

$\therefore$ g(x) > 0 for all x, as a > 0 and discriminate < 0.

Thus, g(x) > 0 for all $x \in R$.

Sol 20: (D) The equation $x^2 - px + r = 0$ has roots (α, β) and the equation

$x^2 - qx + r = 0$ has roots, $\left(\dfrac{\alpha}{2}, \beta\right)$

$\Rightarrow r = \alpha\beta$ and $\alpha + \beta = p$ and $\dfrac{\alpha}{2} + 2\beta = q$

$\Rightarrow \beta = \dfrac{2q - p}{3}$ and $\alpha = \dfrac{2(2p - q)}{3}$

$\Rightarrow \alpha\beta = r = \dfrac{2}{9}(2p - q)(2q - p)$

Sol 21: (B) Suppose roots are imaginary then $\beta = \bar{\alpha}$ and $\dfrac{1}{\beta} = \bar{\alpha} \Rightarrow \beta = \dfrac{1}{\beta}$ not possible

$\Rightarrow$ Roots are real $\Rightarrow (p^2 - q)(b^2 - ac) \geq 0$

$\Rightarrow$ Statement-I is correct.

$\dfrac{-2b}{a} = \alpha + \dfrac{1}{\beta}$ and $\dfrac{\alpha}{\beta} = \dfrac{c}{a}$, $\alpha + \beta = -2p$, $\alpha\beta = q$

If $\beta = 1$, then $\alpha = q \Rightarrow c = qa$(not possible)

also $\alpha + 1 = \dfrac{-2b}{a} \Rightarrow -2p = \dfrac{-2b}{a} \Rightarrow b = ap$ (not possible)

$\Rightarrow$ Statement-II is correct but it is not the correct explanation.

Sol 22: (B) $ax^2 + bx + c = 0 \Rightarrow x^2 + 6x - 7 = 0$

$\Rightarrow \alpha = 1, \beta = -7$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{1}-\frac{1}{7}\right)^n = 7$$

Sol 23: (B) $x^2 + bx - 1 = 0$

$x^2 + x + b = 0$ ... (i)

Common root is

$(b - 1)\,x - 1 - b = 0$

$$\Rightarrow x = \frac{b+1}{b-1}$$

This value of x satisfies equation (i)

$$\Rightarrow \frac{(b+1)^2}{(b-1)^2}+\frac{b+1}{b-1}+b = 0 \Rightarrow b = \sqrt{3}i - \sqrt{3}i, 0$$