10 Functions

Section-A : JEE Advanced/ IIT-JEE

A 1. $\left[0, \frac{3}{\sqrt{2}}\right]$ 2. 0, 1 3. $[-2, -1]\cup[1, 2]$ 4. $n^n, n!$

5. (–2, 1), [–1, 1] 6. $x+1$ and $-x+1$ 7. $\frac{-3\pm\sqrt{5}}{2}, \frac{3\pm\sqrt{5}}{2}$ 8. 1

B 1. T 2. T 3. F

C 1. (d) 2. (b) 3. (d) 4. (c) 5. (d) 6. (c) 7. (a)
8. (d) 9. (b) 10. (c) 11. (d) 12. (b) 13. (a) 14. (d)
15. (a) 16. (d) 17. (d) 18. (a) 19. (b) 20. (a) 21. (c)
22. (d) 23. (b) 24. (a) 25. (d) 26. (d) 27. (c) 28. (c)
29. (a) 30. (a) 31. (d) 32. (a) 33. (b)

D 1. (a, d) 2. (b, c) 3. (a, c) 4. (b) 5. (a) 6. (a, b) 7. (a, b)
8. (a, b) 9. (a, b, c) 10. (b, d) 11. (a, b, c)

E 1. R, [0, 1) ; f is not one to one 3. 3

5. domain = $\{x : x \in R, 16x^4 + 81x^2 - 2025 \le 0\}$; range = $\{y : y \in R,\ y \ge \frac{4x^2}{9}\}$; $R \cap R'$ is not a function.

6. y 8. $a=3$ 9. $-\frac{5}{3}, 0, \frac{5}{3}$ 10. $2 \le \alpha \le 14$, No

F 1. (A) - q; (B) - r 2. (A) -p, r, s ; (B) - q, s ; (C) - q, s ; (D) -p, r, s

I 1. 3

Section-B : JEE Main/ AIEEE

1. (a) 2. (c) 3. (a) 4. (a) 5. (d) 6. (d) 7. (a)
8. (b) 9. (b) 10. (d) 11. (c) 12. (a) 13. (b) 14. (d)
15. (b) 16. (b) 17. (b)

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. For the given function to be defined

$\frac{\pi^2}{16} - x^2 \ge 0 \Rightarrow -\pi/4 \le x \le \pi/4$

$\therefore D_f = [-\pi/4, \pi/4]$

Now, for $x \in [-\pi/4, \pi/4]$, $\sqrt{\pi^2/16 - x^2} \in [0, \pi/4]$ and sine function increases on $[0, \pi/4]$

$\therefore 0 = \sin 0 \le \sin\sqrt{\frac{\pi^2}{16} - x^2} \le \sin \pi/4 = 1/\sqrt{2}$

$\Rightarrow 0 \le 3\sin\sqrt{\frac{\pi^2}{16} - x^2} \le 3/\sqrt{2}$

$\therefore f(x) = [0, 3/\sqrt{2}]$

2. $f(x) = \begin{cases} \frac{x}{1+e^{1/x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$f'(0^+) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} \frac{\frac{h}{1+e^{1/h}}}{h} = \lim_{h\to 0} \frac{1}{1+e^{1/h}}$

$= \lim_{h\to 0} \frac{e^{-1/h}}{e^{-1/h}+1} = \frac{0}{1} = 0$

$f'(0^-) = \lim_{h\to 0} \frac{f(0) - f(0-h)}{h}$

$$= \lim_{h\to 0} \frac{0 - \frac{-h}{1+e^{-1/h}}}{h} = \lim_{h\to 0} \frac{1}{1+e^{-1/h}} = 1$$

Thus $f'(0^+) = 0$ and $f'(0^-) = 1$

3. To find domain of function $f(x) = \sin^{-1}\left(\log_2 \frac{x^2}{2}\right)$

For $f(x)$ to be defined we should have, $-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$

NOTE THIS STEP:

$\Rightarrow\ 2^{-1} \le \frac{x^2}{2} \le 2^1 \Rightarrow 1 \le x^2 \le 4$

$\Rightarrow -2 \le x \le -1$ or $1 \le x \le 2$

$\Rightarrow x \in [-2, -1] \cup [1, 2]$

4. Set A has n distinct elements.
Then to define a function from A to A, we need to associate each element of set A to any one the n elements of set A. So total number of functions from set A to set A is equal to the number of ways of doing n jobs where each job can be done in n ways. The total number such ways is $n \times n \times n \times .... \times n$ (n - times).
Hence the total number of functions from A to A is n^n.
Now for an onto function from A to A, we need to associate each element of A to one and only one element of A. So total number of onto functions from set A to A is equal to number of ways of arranging n elements in range (set A) keeping n elements fixed in domain (set A). n elements can be arranged in $n!$ ways.
Hence, the total number of functions from A to A is $n!$.

5. The given function is,

$$f(x) = \sin\left[\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right]$$

For ℓn to be defined $\frac{\sqrt{4-x^2}}{1-x} > 0 \Rightarrow 1 - x > 0$

Also $4 - x^2 > 0 \Rightarrow x < 1$ and $-2 < x < 2$

Combining these two inequalities, we get $x \in (-2, 1)$

$\therefore$ Domain of f is $(-2, 1)$

Also $\sin\theta$ always lies in $[-1, 1]$.

$\therefore$ Range of f is $[-1, 1]$

6. **KEY CONCEPT :** Every linear function is either strictly increasing or strictly decreasing. If $f(x) = ax + b$, $D_f = [p, q]$, $R_f = [m, n]$
Then $f(p) = m$ and $f(q) = n$, if $f(x)$ is strictly increasing and $f(p) = n, f(q) = m$, If $f(x)$ is strictly decreasing function.
Let $f(x) = ax + b$ be the linear function which maps $[-1, 1]$ onto $[0, 2]$. then $f(-1) = 0$ and $f(1) = 2$
or $f(-1) = 2$ and $f(1) = 0$
Depending upon $f(x)$ is increasing or decreasing respectively.

$\Rightarrow\ -a + b = 0$ and $a + b = 2$ ....(1)

or $\ -a + b = 2$ and $a + b = 0$ ....(2)

Solving (1), we get $a = 1, b = 1$.
Solving (2), we get $a = -1, b = 1$
Thus there are only two functions i.e., $x + 1$ and $-x + 1$.

7. Given that $f(x) = f\left(\frac{x+1}{x+2}\right)$ and f is an even function

$\therefore f(x) = f(-x) = f\left(\frac{-x+1}{-x+2}\right)$

$\Rightarrow x = \frac{-x+1}{-x+2} \Rightarrow x^2 - 3x + 1 = 0 \Rightarrow x = \frac{3\pm\sqrt{5}}{2}$

Also $f(x) = f\left(\frac{x+1}{x+2}\right) = f(-x)$

$\Rightarrow \frac{x+1}{x+2} = -x \Rightarrow x^2 + 3x + 1 = 0 \Rightarrow x = \frac{-3\pm\sqrt{5}}{2}$

$\therefore$ Four values of x are

$$\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

8. $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$\Rightarrow f(x) = \sin^2 x + \left[\sin\left(x + \frac{\pi}{3}\right)\right]^2 + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$\Rightarrow f(x) = \sin^2 x + \frac{1}{4}(\sin x + \sqrt{3}\cos x)^2$

$+ \frac{1}{2}\cos x(\cos x - \sqrt{3}\sin x)$

$= \frac{5}{4}(\sin^2 x + \cos^2 x) = \frac{5}{4}$

$\therefore\ (gof)x = g[f(x)] = g(5/4) = 1$

B. True/False

1. $f(x) = (a - x^n)^{1/n}, a > 0, n$ is + ve integer
$f(f(x)) = f[(a - x^n)^{1/n}] = [a - \{(a - x^n)^{1/n}\}^n]^{1/n}$
$= (a - a + x^n)^{1/n} = x$

2. **KEY CONCEPT :** A function is one-one if it is strictly increasing or strictly decreasing, other wise it is many one.

$f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18} \Rightarrow f'(x) = \frac{-12[x^2 + 2x - 26]}{(x^2 - 8x + 18)^2}$

$\Rightarrow f'(x) = \frac{-12(x - 3\sqrt{3} + 1)(x + 3\sqrt{3} + 1)}{(x^2 - 8x + 18)^2}$

$\Rightarrow f(x)$ increases on $(-3\sqrt{3} - 1, 3\sqrt{3} - 1)$ and decreases otherwise.

$\Rightarrow f(x)$ is many one.

3. We know that sum of any two functions is defined only on the points where both f_1 as well as f_2 are defined that is $f_1 + f_2$ is defined on $D_1 \cap D_2$.
$\therefore$ The given statement is false.

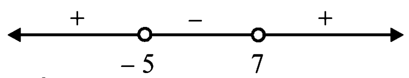
C. MCQs with ONE Correct Answer

1. **(d)** $f(x) = x^2$ is many one as $f(1) = f(-1) = 1$
Also f is into as – ve real number have no pre-image.
$\therefore F$ is neither injective nor surjective.

2. **(b)** $y = x^2 + (k-1)x + 9 = \left(x + \frac{k+1}{2}\right)^2 + 9 - \left(\frac{k-1}{2}\right)^2$

For entire graph to be above x-axis, we should have

$$9 - \left(\frac{k-1}{2}\right)^2 > 0$$



$\Rightarrow \ k^2 - 2k - 35 < 0 \ \Rightarrow \ (k-7)(k+5) < 0$

i.e, $-5 < k < 7$

3. **(d)** $f(x) = |x-1| = \begin{cases} -x+1, & x < 1 \\ x-1, & x \geq 1 \end{cases}$

Consider $f(x^2) = (f(x))^2$

If it is true it should be $\forall\ x$

$\therefore$ Put $x = 2$

LHS $= f(2^2) = |4-1| = 3$ and RHS $= (f(2))^2 = 1$

$\therefore$ (a) is not correct

Consider $f(x+y) = f(x) + f(y)$

Put $x = 2, y = 5$ we get

$f(7) = 6; f(2) + f(5) = 1 + 4 = 5$

$\therefore$ (b) is not correct

Consider $f(|x|) = |f(x)|$

Put $x = -5$ then $f(|-5|) = f(5) = 4$

$|f(-5)| = |-5-1| = 6$

$\therefore$ (c) is not correct.

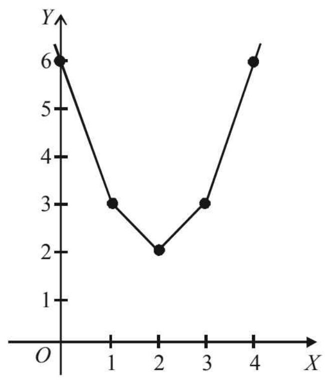
Hence (d) is the correct alternative.

4. **(c)** $|x-1| + |x-2| + |x-3|) \geq 6$

Consider $f(x) = |x-1| + |x-2| + |x-3|$

$$f(x) = \begin{cases} 6-3x, & x < 1 \\ 4-x, & 1 \leq x < 2 \\ x, & 2 \leq x < 3 \\ 3x-6, & x \geq 3 \end{cases}$$

NOTE THIS STEP:



Graph of $f(x)$ shows $f(x) \geq 6$ for $x \leq 0$ or $x \geq 4$

5. **(d)** $f(x) = \cos(\log x)$

$$\therefore \quad f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$$

$= \cos(\log x)\cos(\log y) - \frac{1}{2}$ [cos (log x – log$y$) + cos (log x + log$y$)]

$= \cos(\log x)\cos(\log y) - \frac{1}{2}\,[2\cos(\log x)\cos(\log y)]$

$= 0$

6. **(c)** $y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

$y = f(x) + g(x)$

NOTE THIS STEP : Then domain of given function is $D_f \cap D_g$

Now, for domain of $f(x) = \dfrac{1}{\log_{10}(1-x)}$

We know it is defined only when $1 - x > 0$ and $1 - x \neq 1$

$\Rightarrow x < 1$ and $x \neq 0$

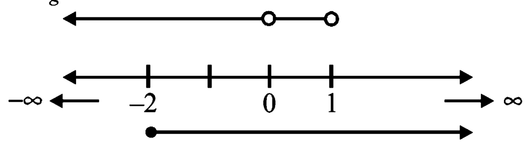
$\therefore D_f = (-\infty, 1) - \{0\}$

For domain of $g(x) = \sqrt{x+2}$

$x + 2 \geq 0$

$\Rightarrow \ x \geq -2$

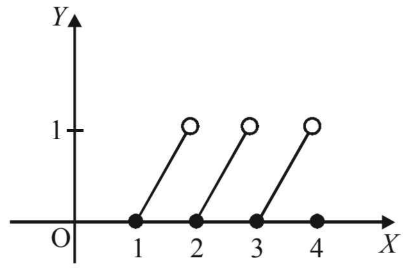
$\therefore D_g = [-2, \infty)$



$\therefore$ Common domain is $[-2, 1) - \{0\}$

7. **(a)** $f(x) = x - [x] = \begin{cases} \ldots \\ x-1, & 1 \leq x < 2 \\ x-2, & 2 \leq x < 3 \\ x-3, & 3 \leq x < 4 \\ \ldots \end{cases}$

graph of function is



Clearly it is a periodic function with period 1.

$\therefore$ (a) is the correct alternative.

8. **(d)** We have $f o g(x) = f(g(x)) = \sin(\ln|x|)$

$\therefore \quad R_1 = \{u : -1 \leq u \leq 1\}$

$(\because -1 \leq \sin\theta \leq 1, \ \forall\ \theta)$

Also $g o f(x) = g(f(x)) = \ln|\sin x|$

$\because \quad 0 \leq |\sin x| \leq 1$

$\therefore \quad -\infty < \ln|\sin x| \leq 0$

$\therefore \quad R_2 = \{v : -\infty < v \leq 0\}$

9. **(c)** $f(x) = f^{-1}(x) \Rightarrow fof(x) = x$

$[(x+1)^2 - 1 + 1]^2 - 1 = x$

$\Rightarrow \quad (x+1)^4 = x + 1$

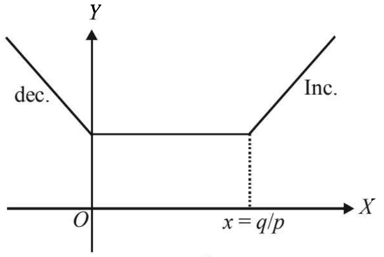
$\therefore \quad x = 0$ or -1

$\therefore \quad$ Req. set is $\{0, -1\}$

10. (c) $f(x) = |px - q| + r|x|$

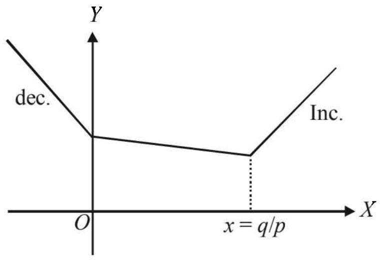
$$= \begin{cases} -px + q - rx, & x \le 0 \\ -px + q + rx, & 0 < x \le q/p \\ px - q + rx, & q/p < x \end{cases}$$

For $r = p, f'(x) < 0$ if $x < 0$
$= 0$ if $0 < x < q/p$
> 0 if $x > q/p$



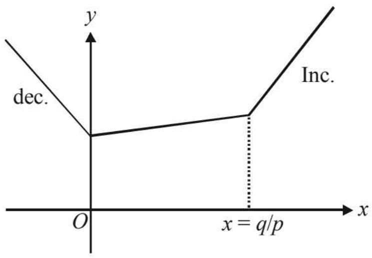
(i)

From graph (i) infinite many points for min value of $f(x)$
for $r < p, f'(x) < 0$ if $x \le 0$
< 0 if $0 < x \le q/p$
> 0 if $x < q/p$



(ii)

From graph (ii) only pt. of min of $f(x)$ at $x = q/p$
For $r > p, f'(x) < 0$ if $x \le 0$
> 0 if $0 < x$



(iii)

From graph (iii) only one pt. of min of $f(x)$ at $x = 0$

11. (d) $f(x)$ is continuous and defined for all $x > 0$ and

$$f\left(\frac{x}{y}\right) = f(x) - f(y)$$

Also $f(e) = 1$
$\Rightarrow$ Clearly $f(x) = \ell n\, x$ which satisfies all these properties
$\therefore f(x) = \ell n\, x$

12. (b) Let $y = 2^{x(x-1)}$
$\Rightarrow x^2 - x - \log_2 y = 0;$

$$x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\log_2 y}\right)$$

Since x is + ive, we choose only + out of $\pm$
(for $y \ge 1, \log_2 y \ge 0$)

$$\therefore\ x = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 y}\right)$$

or $$f^{-1}(x) = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$$

13. (c) Let $h(x) = |x|$ then
$g(x) = |f(x)| = h(f(x))$
Since composition of two continuous functions is continuous, therefore g is continuous if f is continuous.

14. (d) It is given that

$2^x + 2^y = 2 \ \forall\ x, y \in R$

but $2^x, 2^y > 0 \ \forall\ x, y \in R$
Therefore, $2^x = 2 - 2^y < 2$
$\Rightarrow 0 < 2^x < 2 \Rightarrow x < 1$
Hence domain $= (-\infty, 1)$

15. (b) $g(x) = 1 + x - [x];$

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

For integral values of x; $g(x) = 1$
For $x < 0$; (but not integral value) $x - [x] > 0 \Rightarrow g(x) > 1$
For $x > 0$;(but not integral value) $x - [x] > 0 \Rightarrow g(x) > 1$
$\therefore g(x) \ge 1, \forall x$ $\qquad \therefore f(g(x)) = 1, \forall x$

16. (a) $f(x) = x + \dfrac{1}{x} = y \Rightarrow x^2 - yx + 1 = 0$

$$\Rightarrow x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$\therefore x = \frac{y + \sqrt{y^2 - 4}}{2} \qquad (\because\ x \ge 1 \text{ and } y \ge 2)$$

$$\therefore f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

NOTE THIS STEP:

17. (d) For domain of $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$

$x^2 + 3x + 2 \ne 0$ and $x + 3 > 0$
$\Rightarrow x \ne -1, -2$ and $x > -3$
$\therefore D_f = (-3, \infty) - \{-1, -2\}$

18. (a) From E to F we can define, in all, $2 \times 2 \times 2 \times 2 = 16$ functions (2 options for each element of E) out of which 2 are into, when all the elements of E either map to 1 or to 2.
$\therefore$ No. of onto functions $= 16 - 2 = 14$

19. (d) $f(x) = \dfrac{\alpha x}{x+1}, x \ne -1$

$$f(f(x)) = x \Rightarrow \frac{\alpha\left(\dfrac{\alpha x}{x+1}\right)}{\dfrac{\alpha x}{x+1} + 1} = x \Rightarrow \frac{\alpha^2 x}{(\alpha+1)x + 1} = x$$

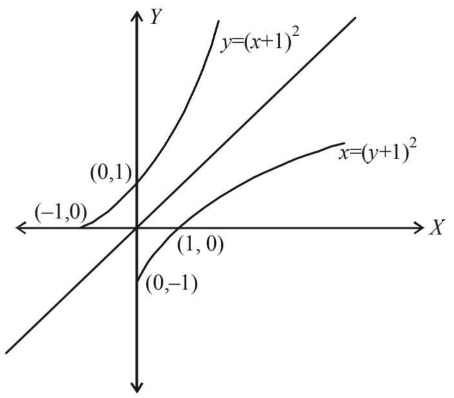
$\Rightarrow (\alpha + 1)x^2 + (1 - \alpha^2)x = 0$ ...(1)
$\Rightarrow \alpha + 1 = 0$ and $1 - \alpha^2 = 0 \Rightarrow \alpha = -1$

20. (d) Given that $f(x) = (x+1)^2, x \geq -1$
Now if $g(x)$ is the reflection of $f(x)$ in the line $y = x$ then it can be obtained by interchanging x and y in $f(x)$
i.e., $y = (x+1)^2$ changes to $x = (y+1)^2$

$\Rightarrow \; y + 1 = \sqrt{x}$

$\left[\begin{array}{l} y + 1 \neq -\sqrt{x}, \text{since } y \geq -1 \\ \text{as in figure.} \end{array} \right.$

$\Rightarrow \; y = \sqrt{x} - 1 \quad$ defined $\forall \; x \geq 0$



$\therefore g(x) = \sqrt{x} - 1 \quad \forall \; x \geq 0$

21. (a) Given that
$f(x) = 2x + \sin x, \;\; x \in R \Rightarrow f'(x) = 2 + \cos x$
But $-1 \leq \cos x \leq 1$
$\Rightarrow 1 \leq 2 + \cos x \leq 3 \Rightarrow 1 \leq 2 + \cos x \leq 3$
$\therefore f'(x) > 0, \; \forall x \in R$
$\Rightarrow f(x)$ is strictly increasing and hence one-one
Also as $x \to \infty, f(x) \to \infty$ and $x \to -\infty, f(x) \to -\infty$
$\therefore$ Range of $f(x) = R =$ domain of $f(x) \Rightarrow f(x)$ is onto.
Thus, $f(x)$ is one-one and onto.

22. (b) Given that $f: [0, \infty) \to [0, \infty)$

Such that $f(x) = \dfrac{x}{x+1}$

Then $f'(x) = \dfrac{1 + x - x}{(1+x)^2} = \dfrac{1}{(1+x)^2} > 0 \; \forall \; x$

$\therefore f$ is an increasing function $\Rightarrow f$ is one-one.
Also, $D_f = [0, \infty)$

And for range let $\dfrac{x}{1+x} = y \Rightarrow x = \dfrac{y}{1-y}$

$x \geq 0 \Rightarrow 0 \leq y < 1$
$\therefore R_f = [0, 1) \neq$ Co-domain
$\therefore f$ is not onto.

23. (a) For $f(x) = \sqrt{\sin^{-1}(2x) + \dfrac{\pi}{6}}$ to be defined and real

if $\sin^{-1} 2x + \pi/6 \geq 0$

$\Rightarrow \sin^{-1} 2x \geq -\dfrac{\pi}{6}$ ...(1)

But we know that
$-\pi/2 \leq \sin^{-1} 2x \leq \pi/2$ ...(2)
Combining (1) and (2), we get

$-\dfrac{\pi}{6} \leq \sin^{-1} 2x \leq \dfrac{\pi}{2}$

$\Rightarrow \sin(-\pi/6) \leq 2x \leq \sin(\pi/2) \Rightarrow -1/2 \leq 2x \leq 1$

$\Rightarrow -1/4 \leq x \leq 1/2 \; \therefore D_f = \left[-\dfrac{1}{4}, \dfrac{1}{2}\right]$

24. (c) We have

$$f(x) = \frac{x^2 + x + 2}{x^2 + x + 1} = \frac{(x^2 + x + 1) + 1}{x^2 + x + 1}$$

$$= 1 + \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

We can see here that as $x \to \infty, f(x) \to 1$ which is the min value of $f(x)$. i.e. $f_{\min} = 1$. Also $f(x)$ is max when

$\left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ is min which is so when $x = -1/2$

i.e. when $\left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4} = \dfrac{3}{4}$.

$\therefore f_{\max} = 1 + \dfrac{1}{3/4} = 7/3$

$\therefore R_f = (1, 7/3]$

25. (d) We have
$f(x) = x^2 + 2bx + 2c^2$; $g(x) = -x^2 - 2cx + b^2$
$\Rightarrow f(x) = (x + b)^2 + 2c^2 - b^2$
and $g(x) = -(x + c)^2 + b^2 + c^2$
$\Rightarrow f_{\min} = 2c^2 - b^2$ and $g_{\max} = b^2 + c^2$
For $f_{\min} > g_{\max} \Rightarrow 2c^2 - b^2 > b^2 + c^2$
$\Rightarrow c^2 > 2b^2 \Rightarrow |c| > |b| \sqrt{2}$

26. (b) $f(x) = \sin x + \cos x, \; g(x) = x^2 - 1$
$\Rightarrow g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly $g(f(x))$ is invertible in $-\dfrac{\pi}{2} \leq 2x \leq \dfrac{\pi}{2}$
[$\because \sin\theta$ is invertible when $-\pi/2 \leq \theta \leq \pi/2$]

$\Rightarrow -\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{4}$

27. (a) We are given that

$f: R \to R$ such that $f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$

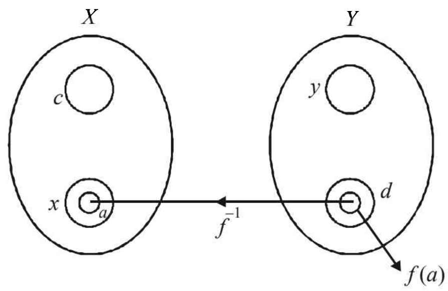
$g: R \to R$ such that $g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$

$\therefore (f - g): R \to R$ such that

$(f - g)(x) = \begin{cases} -x, \text{ if} & x \in \text{rational} \\ x, \text{ if} & x \in \text{irrational} \end{cases}$

Since $f - g: R \to R$ for any x there is only one value of $(f(x) - g(x))$ whether x is rational or irrational. Moreover as $x \in R$, $f(x) - g(x)$ also belongs to R. Therefore, $(f - g)$ is one-one onto.

28. (d) Given that X and Y are two sets and $f: X \to Y$.
$\{f(c)=y; c \subset X, y \subset Y\}$ and
$\{f^{-1}(d)=x : d \subset Y, x \subset X\}$
The pictorial representation of given information is as shown:



Since $f^{-1}(d) = x \Rightarrow f(x) = d$ Now if $a \subset x$
$\Rightarrow f(a) \subset f(x) = d \Rightarrow f^{-1}[f(a)] = a$
$\therefore f^{-1}(f(a)) = a, a \subset x$ is the correct option.

29. (a) $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$

$$\Rightarrow F'(x) = 2f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right).\frac{1}{2} + 2g\left(\frac{x}{2}\right).g'\left(\frac{x}{2}\right).\frac{1}{2}$$

$$= f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) + f'\left(\frac{x}{2}\right).f''\left(\frac{x}{2}\right)$$

$[\because g(x) = f'(x) \Rightarrow g'(x) = f''(x)]$

$$= f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) - f'\left(\frac{x}{2}\right)f\left(\frac{x}{2}\right)$$

$= 0 \qquad [\because f''(x) = -f(x)]$

$\Rightarrow F(x)$ is a constant function.

$\therefore F(x) = F(5) = 5 \; \forall x \in R \Rightarrow F(10) = 5$

30. (a) Given $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ for $n \ge 2$

$$\therefore fof(x) = f[f(x)] = f\left[\frac{x}{(1+x^n)^{1/n}}\right]$$

$$= \frac{\dfrac{x}{(1+x^n)^{1/n}}}{\left[1+\dfrac{x^n}{1+x^n}\right]^{1/n}} = \frac{x}{(1+2x^n)^{1/n}}$$

Further, $fofof(x) = \dfrac{x}{(1+3x^n)^{1/n}}$

Proceeding in the similar manner, we get

$$g(x) = fofof.....of(x) = \frac{x}{(1+nx^n)^{1/n}}$$

(f occurs n times)

Now, $\int x^{n-2} g(x)dx = \int \dfrac{x^{n-1}}{(1+nx^n)^{1/n}}dx$

Let $1 + nx^n = t \Rightarrow n^2x^{n-1}\,dx = dt$

$$\therefore \text{Integral becomes} = \frac{1}{n^2}\int t^{-1/n}dt = \frac{1}{n^2}.\frac{t^{-\frac{1}{n}+1}}{-\frac{1}{n}+1} + K$$

$$= \frac{1}{n}.\frac{t^{1-1/n}}{n-1} + K = \frac{(1+nx^n)^{1-1/n}}{n(n-1)} + K$$

31. (d) $f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right) \ge 0,$

$\forall x \in [0,1]$

$\therefore$ f(x) is an increasing function on $[0,1]$

Hence $f_{max} = f(1) = e + \dfrac{1}{e} = a$

$g(x) = xe^{x^2} + e^{-x^2}$

$\Rightarrow g'(x) = (2x^2+1)e^{x^2} - 2xe^{-x^2} \ge 0, \forall x \in [0,1]$

$\therefore$ $g(x)$ is an increasing function on $[0,1]$

$\therefore g_{\max} = g(1) = e + \dfrac{1}{e} = b$

$h(x) = x^2e^{x^2} + e^{-x^2}$

$h'(x) = 2x\left[e^{x^2}(1+x^2) - e^{-x^2}\right] \ge 0, \forall x \in [0,1]$

$\therefore$ $h(x)$ is an increasing function on $[0,1]$

$\therefore h_{\max} = h(1) = e + \dfrac{1}{e} = c$

Hence $a = b = c$.

32. (a) Given that $f(x) = x^2$ and $g(x) = \sin x$, $\forall x \in R$
Then (gof) $(x) = \sin x^2$
$\Rightarrow$ (gogof) $(x) = \sin(\sin x^2)$
$\Rightarrow$ (fogogof) $(x) = \sin^2(\sin x^2)$
As given that (fogogof) (x) = (gogof) (x)
$\Rightarrow \sin^2(\sin x^2) = \sin(\sin x^2)$
$\Rightarrow \sin(\sin x^2) = 0, 1$

$\Rightarrow \sin x^2 = n\pi$ or $((4n+1)\dfrac{\pi}{2}$ where $n \in Z$

$\Rightarrow \sin x^2 = 0 \quad \because \sin x^2 \in [-1,1] \Rightarrow x^2 = n\pi$

$\Rightarrow x = \pm\sqrt{n\pi}$ where $n \in W$

33. (b) We have $f(x) = 2x^3 - 15x^2 + 36x + 1$
$\Rightarrow f'(x) = 6x^2 - 30x + 36$
$= 6(x^2 - 5x + 6)$
$= 6(x-2)(x-3)$
$\because f'(x) > 0 \;\; \forall x \in [0, 2]$ and $f'(x) < 0 \;\; \forall x \in [2, 3]$
$\therefore$ $f(x)$ is increasing on [0, 2] and decreasing on [2, 3]
$\therefore$ $f(x)$ is many one on [0, 3]
Also $f(0) = 1, f(2) = 29, f(3) = 28$
$\therefore$ Global min = 1 and Global max = 29
i.e., Range of f = [1, 29] = codomain
$\therefore f$ is onto.

D. MCQs with ONE or MORE THAN ONE Correct

1. **(a, d)**

Given that $f(x) = y = \frac{x+2}{x-1}$

Let us check each option one by one.

(a) $f(x) = \frac{x+2}{x-1} = y \Rightarrow x = f(y)$

$\therefore$ (a) is correct

(b) $f(1) \neq 3$ as function is not defined for $x = 1$

$\therefore$ (b) is not correct.

(c) $f'(x) = \frac{x-1-x-2}{(x-1)^2} = \frac{-3}{(x-1)^2}$

$\therefore$ $f'(x) < 0$, if $x \neq 1 \Rightarrow f(x)$ is decreasing if $x \neq 1$

$\therefore$ (c) is not correct.

(d) $f(x) = \frac{x+2}{x-1}$, which is a rational function of x.

2. **(b, c)**

As (0, 0) and (x, $g(x)$) are two vertices of an equilateral triangle; therefore, length of the side of Δ is

$= \sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$

$\therefore$ The area of equilateral $\quad \Delta = \frac{\sqrt{3}}{4}(x^2 + (g(x))^2)$

ATQ, this area $= \frac{\sqrt{3}}{4}$

$\therefore$ We get $\frac{\sqrt{3}}{4}(x^2 + (g(x))^2) = \frac{\sqrt{3}}{4}$

$\Rightarrow$ $(g(x))^2 = 1 - x^2 \Rightarrow g(x) = \pm\sqrt{1-x^2}$

$\therefore$ (b), (c) are the correct answers as (a) is not a function ($\because$ image of x is not unique)

3. **(a, c)**

$f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

We know $9 < \pi^2 < 10$ and $-10 < -\pi^2 < -9$

$\Rightarrow [\pi^2] = 9$ and $[-\pi^2] = -10$

$\Rightarrow \therefore f(x) = \cos 9x + \cos(-10x)$

$f(x) = \cos 9x + \cos 10x$

Let us check each option one by one.

(a) $f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos 5\pi = -1$ **(true)**

(b) $f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ **(false)**

(c) $f(-\pi) = \cos(-9\pi) + \cos(-10\pi)$
$= \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ **(true)**

(d) $f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{5\pi}{2}$

$= \cos\left(2\pi + \frac{\pi}{4}\right) + 0 = \frac{1}{\sqrt{2}}$ **(false)**

Thus (a) and (c) are the correct options.

4. **(b)** $f(x) = 3x - 5$ (given), which is strictly increasing on R, so $f^{-1}(x)$ exists.

Let $y = f(x) = 3x - 5$

$\Rightarrow y + 5 = 3x \Rightarrow x = \frac{y+5}{3}$ ...(1)

and $y = f(x) \Rightarrow x = f^{-1}(y)$ ...(2)

From (1) and (2):

$f^{-1}(y) = \frac{y+5}{3} = f^{-1}(x) = \frac{x+5}{3}$

5. **(a)** Let us check each option one by one.

(a) $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Now, $fog = f(g(x)) = f\left(\sqrt{x}\right) = \sin^2\sqrt{x} = (\sin\sqrt{x})^2$

and $gof(x) = g(f(x)) = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$

(a) is true.

(b) $f(x) = \sin x, g(x) = |x|$

$fog(x) = f(g(x)) = f(|x|) = \sin|x| \neq \left(\sin\sqrt{x}\right)^2$

$\therefore$ (b) is not true

(c) $f(x) = x^2, g(x) = \sin\sqrt{x}$

$fog(x) = f[g(x)] = f\left(\sin\sqrt{x}\right) = \left(\sin\sqrt{x}\right)^2$

and $(gof)(x) = g[f(x)] = g(x^2) = \sin\sqrt{x^2}$
$= \sin|x| \neq |\sin x|$

$\therefore$ (c) is not true.

6. **(a, b)** We have $f(x) = \frac{b-x}{1-bx}$, $0 < b < 1$

Let $f(x_1) = f(x_2) \Rightarrow \frac{b-x_1}{1-bx_1} = \frac{b-x_2}{1-bx_2}$

$\Rightarrow b - b^2x_2 - x_1 + bx_1x_2 = b - x_2 - b^2x_1 + bx_1x_2$

$\Rightarrow x_2(1-b^2) = x_1(1-b^2) \Rightarrow x_1 = x_2$ as $1 - b^2 \neq 0$

$\therefore f$ is one one.

Also $\frac{b-x}{1-bx} = y \Rightarrow b - x = y - bxy \Rightarrow (by-1)x = y - b$

$\Rightarrow x = \frac{y-b}{by-1}$

For $y = \frac{1}{b}$, x is not defined

$\therefore f$ is neither onto nor invertible.

Also $f'(x) = \frac{-1(1-bx)-(-b)(b-x)}{(1-bx)^2} = \frac{b^2-1}{(1-bx)^2}$

$\therefore f'(b) = \frac{1}{b^2-1}$ and $f'(0) = b^2 - 1 \Rightarrow f'(b) = \frac{1}{f'(0)}$

Hence a and b are the correct options.

7. **(a, b)**

Given $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta} = \frac{2\cos^2\theta}{2\cos^2\theta - 1}$

$= \frac{1+\cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$

Let $\cos 4\theta = \frac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \frac{1}{3} \Rightarrow \cos 2\theta = \pm\sqrt{\frac{2}{3}}$

$\therefore f(\cos 4\theta) = 1 \pm \sqrt{\frac{3}{2}}$ or $f\left(\frac{1}{3}\right) = 1 \pm \sqrt{\frac{3}{2}}$

8. **(a, b)** For $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$

the critical points can be obtained by solving $|x| = 0$,

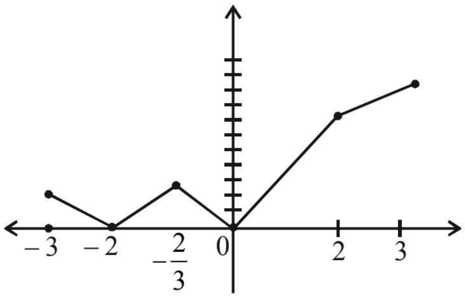
$|x+2| = 0$ and $||x+2| - 2|x|| = 0$

we get $x = 0, -2, 2, -\frac{2}{3}$

Then we can write

$$f(x)\begin{cases} -2x-4, & x \le -2 \\ 2x+4, & -2 < x \le -\frac{2}{3} \\ -4x, & -\frac{2}{3} < x \le 0 \\ 4x, & 0 < x \le 2 \\ 2x+4, & x > 2 \end{cases}$$

The graph of $y = f(x)$ is as follows



From graph $f(x)$ has local minimum at –2 and 0 and local maximum at $-\frac{2}{3}$

9. **(a, b, c)** $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow R$

$f(x) = [\log(\sec x + \tan x)]^3$

$f(-x) = [\log(\sec x - \tan x)]^3$

$$= \left[\log\left(\frac{(\sec x - \tan x)(\sec x + \tan x)}{\sec x + \tan x}\right)\right]^3$$

$$= \left[\log\left(\frac{1}{\sec x + \tan x}\right)\right]^3 = \left[-\log(\sec x + \tan x)\right]^3$$

$$= -\left[\log(\sec x + \tan x)\right]^3 = -f(x)$$

$\therefore f$ is an odd function.

(a) is correct and (d) is not correct.

Also

$$f'(x) = 3\left[\log(\sec x + \tan x)\right]^2 . \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= 3\sec x\left[\log(\sec x + \tan x)\right]^2 > 0 \ \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\therefore f$ is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

We know that strictly increasing function is one one.

$\therefore f$ is one one

$\therefore$ (b) is correct.

Also $\lim_{x \to \frac{\pi}{2}^-} \left[\log(\sec x + \tan x)\right]^3 \to \infty$

and $\lim_{x \to \frac{\pi}{2}^+} \left[\log(\sec x + \tan x)\right]^3 \to -\infty$

$\therefore$ Range of $f = (-\infty, \infty) = R$

$\therefore f$ is an onto function.

$\therefore$ (c) is correct.

10. **(b,d)** $f(x) = x^5 - 5x + a$

$f(x) = 0 \Rightarrow x^5 - 5x + a = 0 \Rightarrow a = 5x - x^5 = g(x)$

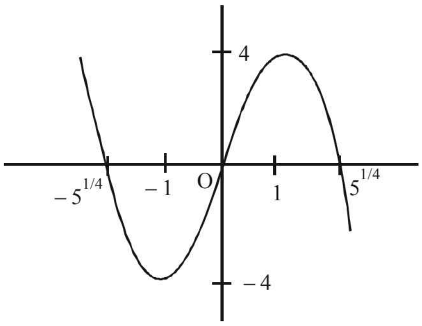
$\Rightarrow g(x) = 0$ when $x = 0, 5^{1/4}\ -5^{1/4}$

and $g'(x) = 0 \Rightarrow x = 1, -1$

Also $g(-1) = -4$ and $g(1) = 4$

$\therefore$ graph of $g(x)$ will be as shown below.

From graph



if $a \in (-4, -4)$

then $g(x) = a$ or $f(x) = 0$ has 3 real roots

If $a > 4$ or $a < -4$

then $f(x) = 0$ has only one real root.

$\therefore$ (b) and (d) are the correct options.

11. **(a, b, c)**

$$f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$$

$$-1 \le \sin x \le 1 \Rightarrow -\frac{\pi}{2} \le \frac{\pi}{2}\sin x \le \frac{\pi}{2}$$

$$\Rightarrow -1 \le \sin\left(\frac{\pi}{2}\sin x\right) \le 1 \Rightarrow \frac{-\pi}{6} \le \frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right) \le \frac{\pi}{6}$$

$$\Rightarrow \frac{-1}{2} \le \sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right] \le \frac{1}{2}$$

$\therefore$ Range of $f = \left[\frac{-1}{2}, \frac{1}{2}\right]$

$$fog(x) = \sin\left[\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right]$$

Range of $fog = \left[\frac{-1}{2}, \frac{1}{2}\right]$

$$\lim_{x\to 0} \frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}$$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)} \times \frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}$$

$= \pi/6$

$$gof(x) = \frac{\pi}{2}\sin\left(\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)$$

$$-\frac{\pi}{2}\sin\left(\frac{1}{2}\right) \le g(f(x)) \le \frac{\pi}{2}\sin\left(\frac{1}{2}\right)$$

$-0.73 \le g(f(x)) \le 0.73$

$\therefore$ $gof(x) \ne 1$ for any $x \in R$.

E. Subjective Problems

1. Since $f(x)$ is defined and real for all real values of x, therefore domain of f is R.

Also $\frac{x^2}{1+x^2} \ge 0$, for all $x \in R$

and $\frac{x^2}{1+x^2} < 1$ $(\because x^2 < 1 + x^2)$ for all $x \in R$

$\therefore$ $0 \le \frac{x^2}{1+x^2} < 1 \Rightarrow 0 \le f(x) < \Rightarrow$ Range of $f = [0, 1)$

Also since $f(1) = f(-1) = 1/2$

$\therefore$ f is not one-to-one.

2. $y = |x|^{1/2}, -1 \le x \le 1$

$\Rightarrow$ $y = \sqrt{-x}$ if $-1 \le x \le 0 = \sqrt{x}$ if $0 \le x \le 1$

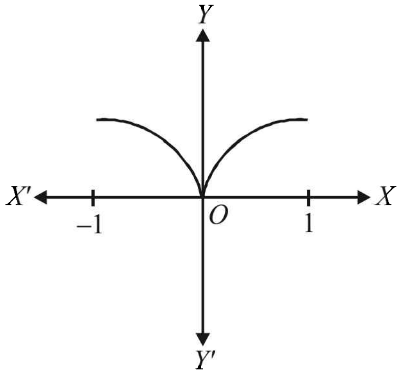
$\Rightarrow$ $y^2 = -x$ if $-1 \le x \le 0$ and $y^2 = x$ if $0 \le x \le 1$

[Here y should be taken always + ve, as by definition y is a + ve square root].

Clearly $y^2 = -x$ represents upper half of left handed parabola (upper half as y is + ve)

and $y^2 = x$ represents upper half of right handed parabola.

Therefore the resulting graph is as shown below :



3. $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$

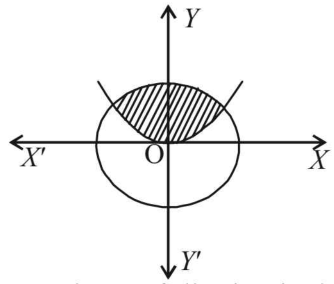
Then $f(6) = 6^9 - 6 \times 6^8 - 2 \times 6^7 + 12 \times 6^6 + 6^4 - 7 \times 6^3 + 6 \times 6^2 + 6 - 3$

$= 6^9 - 6^9 - 2 \times 6^7 + 2 \times 6^7 + 6^4 - 7 \times 6^3 + 6^3 + 6 - 3 = 3$

4. $R = [(x, y); x \in R, y \in R, x^2 + y^2 \le 25]$ which represents all the points inside and on the circle $x^2 + y^2 = 5^2$, with centre (0, 0) and radius = 5

$$R' = \left\{(x, y) : x \in R, y \in R, y \ge \frac{4}{9}x^2\right\}$$

which represents all the points inside and on the upward parabola $x^2 \le \frac{9}{4}y$.



Thus $R \cap R'$ = The set of all points in shaded region.

For domain of $R \cap R'$.

$x^2 + y^2 \le 25$

$\Rightarrow$ $x^2 \le 25 - y^2$ ....(1)

and $y \ge \frac{4}{9}x^2 \Rightarrow \frac{16x^4}{81} \le y^2 \Rightarrow -\frac{16x^4}{81} \ge -y^2$

$\Rightarrow$ $25 - \frac{16x^4}{81} \ge 25 - y^2$ ....(2)

$\therefore$ Combining (1) and (2) $x^2 \le 25 - \frac{16}{81}x^4$

$\Rightarrow$ $16x^4 + 81x^2 - 2025 \le 0$

$\therefore$ Domain of $R \cap R'$ =

$\{x : x \in R, 16x^4 + 81x^2 - 2025 \le 0\}$ and range of $R \cap R'$

$= \{y : y \in R, y \ge \frac{4x^2}{9}, 16x^4 + 81x^2 - 2025 \le 0\}$

$R \cap R'$ is not a function because image of an element is not unique , e..g., (0, 1), (0, 2), (0,3) ..... $\in R \cap R'$.

5. As there is an injective maping from A to B, each element of A has unique image in B. Similarly as there is an injective mapping from B to A, each element of B has unique image in A. So we can conclude that each element of A has unique image in B and each element of B has unique image in A or in other words there is one to one mapping from A to B. Thus there is bijective mapping from A to B.

6. f is one one function,

$D_f = \{x, y, z\}$; $R_f = \{1, 2, 3\}$

Eaxctly one of the following is true :

$f(x) = 1, f(y) \ne 1, f(z) \ne 2$

To determine f^{-1} (1):

Case I: $f(x) = 1$ is true.

$\Rightarrow$ $f(y) \ne 1, f(z) \ne 2$ are false.

$\Rightarrow$ $f(y) = 1, f(z) = 2$ are true.

But $f(x) = 1, f(y) = 1$ are true, is not possible as f is one to one.

$\therefore$ This case is not possible.

Case II: $f(y) \neq 1$ is true.

$\Rightarrow f(x) = 1$ and $f(z) \neq 2$ are false

$\Rightarrow f(x) \neq 1$ and $f(z) = 2$ are true

Thus, $f(x) \neq 1, f(y) \neq 1, f(z) = 2$

$\Rightarrow$ Either $f(x)$ or $f(y) = 2$. So, f is not one to one

$\therefore$ This case is also not possible.

$\therefore f(z) \neq 2$ is true

$\therefore f(x) = 1$ and $f(y) \neq 1$ are false.

$\Rightarrow f(x) \neq 1$ and $f(y) = 1$ are true.

$\Rightarrow f^{-1}(1) = y$

7. Since $|f(x) - f(y)| \leq |x - y|^3$ is true $\forall\ x, y \in R$

We have for $x \neq y$, $\dfrac{|f(x) - f(y)|}{|x - y|} \leq |x - y|^2$

$$\Rightarrow \lim_{y \to x}\left|\frac{f(x) - f(y)}{x - y}\right| \leq \lim_{y \to x}|x - y|^2$$

$$\Rightarrow \left|\lim_{y \to x}\frac{f(x) - f(y)}{x - y}\right| \leq 0$$

$\Rightarrow |f'(x)| \leq 0 \Rightarrow f'(x) = 0$

$\Rightarrow f(x)$ is a constant function. Hence Proved.

8. Given that $f(x + y) = f(x)f(y)\ \forall\ x, y \in N$ and $f(1) = 2$

To find 'a' such that,

$$\sum_{k=1}^{n} f(a + k) = 16(2^n - 1) \quad ...(1)$$

For this we start with

$f(1) = 2$ ...(2)

Then $f(2) = f(1 + 1) = f(1)f(1)$

$\Rightarrow f(2) = 2^2$ Using (2)

Similarly we get, $f(3) = 2^3$,

$f(4) = 2^4, ..., f(n) = 2^n$

Now eq. (1) can be written as

$f(a + 1) + f(a + 2) + f(a + 3) + ... + f(a + n) = 16(2^n - 1)$

$\Rightarrow f(a)f(1) + f(a)f(2) + f(a)f(3) + ... + f(a)f(n) = 16(2^n - 1)$

$\Rightarrow f(a)f(1) + f(a)f(2) + f(a)f(3) + ... + f(a)f(n) = 16(2^n - 1)$

$\Rightarrow f(a)[f(1) + f(2) + f(3) + ... + f(n)] = 16[2^n - 1]$

$\Rightarrow f(a)[2 + 2^2 + 2^3 + ... + 2^n] = 16[2^n - 1]$

$$\Rightarrow f(a)\left[\frac{2(2^n - 1)}{2 - 1}\right] = 16[2^n - 1]$$

$\Rightarrow f(a) = 8 = 2^3 = f(3) \Rightarrow a = 3$

9. Given that $4\{x\} = x + [x]$

Where $[x]$ = greatest integer $\leq x$

$\{x\}$ = fractional part of x

$\therefore x = [x] + \{x\}$ for any $x \in R$

$\therefore$ Given eq^n becomes

$4\{x\} = [x] + \{x\} + [x] \Rightarrow 3\{x\} = 2[x]$

$\Rightarrow [x] = \dfrac{3}{2}\{x\}$ ....(1)

Now $-1 < \{x\} < 1$

$\Rightarrow -\dfrac{3}{2} < \dfrac{3}{2}\{x\} < \dfrac{3}{2}$

$\Rightarrow -\dfrac{3}{2} < [x] < \dfrac{3}{2}$ [Using eqn (1)]

$\Rightarrow [x] = -1, 0, 1$

If $[x] = -1$

$\Rightarrow -1 = \dfrac{3}{2}\{x\}$ [Using eqn (1)]

$\Rightarrow \{x\} = -\dfrac{2}{3}$

$\therefore x = [x] + \{x\}$

$\Rightarrow x = -1 + (-2/3) = -5/3$

If $[x] = 0$

$\Rightarrow \dfrac{3}{2}\{x\} = 0$ [Using eq^n (1)]

$\Rightarrow \{x\} = 0$

$\therefore x = 0 + 0 = 0$

If $[x] = 1$

$\Rightarrow \dfrac{3}{2}\{x\} = 1$ [Using eq^n (1)]

$\Rightarrow \{x\} = 2/3 \Rightarrow x = 1 + 2/3 = 5/3$

Thus, $x = -5/3, 0, 5/3$

10. Let us put $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow (\alpha + 6x - 8x^2)y = \alpha x^2 + 6x - 8$

$\Rightarrow (\alpha + 8y)x^2 + 6(1 - y)x - (8 + \alpha y) = 0$

Since x is real, $D \geq 0$

$\Rightarrow 36(1 - y)^2 + 4(\alpha + 8y)(8 + \alpha y) \geq 0$

$\Rightarrow 9(1 - 2y + y^2) + [8\alpha + (64 + \alpha^2)y + 8\alpha y^2] \geq 0$

$\Rightarrow y^2(9 + 8\alpha) + y(46 + \alpha^2) + (9 + 8\alpha) \geq 0$ ...(1)

For (1) to hold for each $y \in R$, $9 + 8\alpha > 0$

and $(46 + \alpha^2)^2 - 4(9 + 8\alpha)^2 \leq 0 \Rightarrow \alpha > -9/8$

and $[46 + \alpha^2 - 2(9 + 8\alpha)][46 + \alpha^2 + 2(9 + 8\alpha)] \leq 0$

$\Rightarrow \alpha > -9/8$

and $(\alpha^2 - 16\alpha + 28)(\alpha^2 + 16\alpha + 64) \leq 0 \Rightarrow \alpha > -9/8$

and $(\alpha - 2)(\alpha - 14)(\alpha + 8)^2 \leq 0 \Rightarrow \alpha > -8/9$

and $(\alpha - 2)(\alpha - 14) \leq 0$ $[\because (\alpha + 8)^2 \geq 0]$

$\Rightarrow \alpha > -8/9$ and $2 \leq \alpha \leq 14 \Rightarrow 2 \leq \alpha \leq 14$

Thus, $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ will be onto if $2 \leq \alpha \leq 14$.

When $\alpha = 3$

$$f(x) = \frac{3x^2 + 6x - 8}{3 + 6x - 8x^2}$$

In this case $f(x) = 0$ implies, $3x^2 + 6x - 8 = 0$

$\Rightarrow x = \dfrac{-6 \pm \sqrt{36+96}}{6} = \dfrac{-6 \pm \sqrt{132}}{6} = \dfrac{-6 \pm 2\sqrt{33}}{6}$

$= \dfrac{1}{3}(-3 \pm \sqrt{33})$

This shows that

$f\left[\dfrac{1}{3}(-3+\sqrt{33})\right] = f\left[\dfrac{1}{3}(-3-\sqrt{33})\right] = 0$

Therefore, f is not one-to-one at $\alpha = 3$.

11. Suppose $f(x) = Ax^2 + Bx + C$ is an integer whenever x is an integer.

$\therefore$ $f(0), f(1), f(-1)$ are integers

$\Rightarrow$ $C, A+B+C, A-B+C$ are integers.

$\Rightarrow$ $C, A+B, A-B$ are integers

$\Rightarrow$ $C, A+B, (A+B)+(A-B) = 2A$ are integers.

Conversely suppose $2A$, $A + B$ and C are integers.

Let x be any integer.

We have

$f(x) = Ax^2 + Bx + C$

$= 2A\left[\dfrac{x(x-1)}{2}\right] + (A+B)\,x + C$

Since x is an integer x, x $(x-1)/2$ is an integer.

Also $2A$, $A + B$ and C are integers.

We get $f(x)$ is an integer for all integer x.

F. Match the Following

1. (A) $f(x) = 1 + 2x, D_f = (-\pi/2, \pi/2)$

The given function represents a straight line so it is one one.

But $f_{min} = 1 - \pi = f\left(-\dfrac{\pi}{2}\right)$, $f_{max} = 1 + \pi = f\left(\dfrac{\pi}{2}\right)$

$\therefore$ Range $f = (1-\pi, 1+\pi) \in (-\infty, \infty)$

$\therefore f$ is not onto. Hence (A) $\rightarrow$ (q).

(B) $f(x) = \tan x$

It is an increasing function on $(-\pi/2, \pi/2)$ and its range $= (-\infty, \infty)$ = co-domain of f.

$\therefore f$ is one one onto.

$\therefore$ (B) $\rightarrow$ r

2. We have $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6} = \dfrac{(x-5)(x-1)}{(x-2)(x-3)}$

(A) If $-1 < x < 1$ then $f(x) = \dfrac{(-ve)(-ve)}{(-ve)(-ve)} = +ve$

$\therefore$ $f(x) > 0$ (r)

Also $f(x) - 1 = \dfrac{-x-1}{x^2 - 5x + 6} = -\dfrac{(x+1)}{(x-2)(x-3)}$

For $-1 < x < 1$, $f(x) - 1 = \dfrac{-(+ve)}{(-ve)(-ve)} = -ve$

$\Rightarrow f(x) - 1 < 0 \Rightarrow f(x) < 1$ (s)

$\therefore$ $0 < f(x) < 1$ (p)

(B) If $1 < x < 2$ then $f(x) = \dfrac{(-ve)(+ve)}{(-ve)(-ve)} = -ve$

$\therefore$ $f(x) < 0$ (q) and so $f(x) < 1$ (s)

(C) If $3 < x < 5$ then $f(x) = \dfrac{(-ve)(+ve)}{(+ve)(+ve)} = -ve$

$\therefore$ $f(x) < 0$ (q) and so $f(x) < 1$ (s)

(D) For $x > 5$, $f(x) > 0$ (r)

Also $f(x) - 1 = \dfrac{-(x+1)}{(x-2)(x-3)} < 0$

For $x > 5$, $f(x) < 1$ (s)

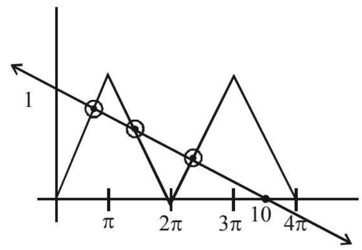
$\therefore$ $0 < f(x) < 1$ (p)

I. Integer Value Correct Type

1. **(3)** We have $f : [0, 4\pi] \rightarrow [0, \pi]$

$f(x) = \cos^{-1}(\cos x)$

and $g(x) = \dfrac{10-x}{10} = 1 - \dfrac{x}{10}$



The graph of $y = f(x)$ and $y = g(x)$ are as follows.

Clearly $f(x) = g(x)$ has 3 solutions.

Section-B JEE Main/ AIEEE

1. **(a)** $f(x)=\sin^{-1}\left(\log_3\left(\frac{x}{3}\right)\right)$ exists

if $-1 \le \log_3\left(\frac{x}{3}\right) \le 1 \Leftrightarrow 3^{-1} \le \frac{x}{3} \le 3^1$

$\Leftrightarrow 1 \le x \le 9$ or $x \in [1, 9]$

2. **(c)** $f(x) = \log(x+\sqrt{x^2+1})$

$f(-x) = \log\left\{-x+\sqrt{x^2+1}\right\} = \log\left\{\frac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right\}$

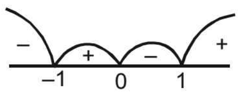
$= -\log(x+\sqrt{x^2+1}) = -f(x)$

$\Rightarrow f(x)$ is an odd function.

3. **(a)** $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3-x)$

$4-x^2 \ne 0;\ x^3 - x > 0;$

$x \ne \pm\sqrt{4}$ and $-1 < x < 0$ or $1 < x < \infty$



$\therefore D = (-1, 0) \cup (1, \infty) - \left\{\sqrt{4}\right\}$

$D = (-1, 0) \cup (1, 2) \cup (2, \infty)$.

4. **(a)** $f(x+y) = f(x) + f(y)$.

Function should be $f(x) = mx$

$f(1) = 7;\ \therefore m = 7,\ f(x) = 7x$

$\sum_{r=1}^{n} f(r) = 7\sum_{1}^{n} r = \frac{7n(n+1)}{2}$

5. **(d)** We have $f: N \to I$

If x and y are two even natural numbers,

then $f(x) = f(y) \Rightarrow \frac{-x}{2} = \frac{-y}{2} \Rightarrow x = y$

Again if x and y are two odd natural numbers then

$f(x) = f(y) \Rightarrow \frac{x-1}{2} = \frac{y-1}{2} \Rightarrow x = y$

$\therefore f$ is onto.

Also each negative integer is an image of even natural number and each positive integer is an image of odd natural number.

$\therefore f$ is onto.

Hence f is one one and onto both.

6. **(d)** ${}^{7-x}P_{x-3}$ is defined if

$7-x \ge 0, x-3 \ge 0$ and $7-x \ge x-3$

$\Rightarrow 3 \le x \le 5$ and $x \in \mathbf{I}$

$\therefore x = 3, 4, 5$

$\therefore f(3) = {}^{7-3}P_{3-3} = {}^{4}P_0 = 1$

$\therefore f(4) = {}^{7-4}P_{4-3} = {}^{3}P_1 = 3$

$\therefore f(5) = {}^{7-5}P_{5-3} = {}^{2}P_2 = 2$

Hence range $= \{1, 2, 3\}$

7. **(a)** $f(x)$ is onto $\therefore S =$ range of $f(x)$

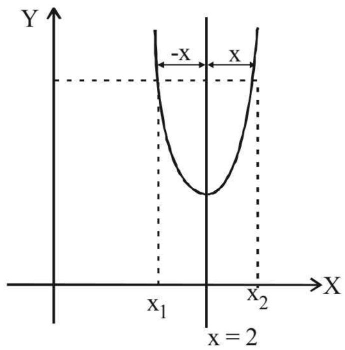
Now $f(x) = \sin x - \sqrt{3}\cos x + 1 = 2\sin\left(x - \frac{\pi}{3}\right) + 1$

$\because -1 \le \sin\left(x - \frac{\pi}{3}\right) \le 1$

$-1 \le 2\sin\left(x - \frac{\pi}{3}\right) + 1 \le 3$

$\therefore f(x) \in [-1, 3] = S$

8. **(b)** Let us consider a graph symm. with respect to line $x = 2$ as shown in the figure.



From the figure

$f(x_1) = f(x_2)$, where $x_1 = 2 - x$ and $x_2 = 2 + x$

$\therefore f(2-x) = f(2+x)$

9. **(b)** $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is defined

if (i) $-1 \le x - 3 \le 1 \Rightarrow 2 \le x \le 4$

and (ii) $9 - x^2 > 0 \Rightarrow -3 < x < 3$

Taking common solution of (i) and (ii),

we get $2 \le x < 3$ $\therefore$ Domain = [2, 3)

10. **(d)** Given $f(x) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = 2\tan^{-1}x$ for $x \in (-1, 1)$

If $x \in (-1, 1) \Rightarrow \tan^{-1} x \in \left(\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right)$

$\Rightarrow 2\tan^{-1} x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

Clearly, range of $f(x) = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

For f to be onto, codomain = range

$\therefore$ Co-domain of function $= B = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

11. **(c)** Clearly function $f(x) = 3x^2 - 2x + 1$ is increasing when

$f'(x) = 6x - 2 \ge 0 \Rightarrow \quad x \in [1/3, \infty)$

$\therefore f(x)$ is incorrectly matched with $\left(-\infty, \dfrac{1}{3}\right)$

12. **(a)** $f(2a-x) = f(a-(x-a))$

$= f(a)f(x-a) - f(0)f(x) = f(a)f(x-a) - f(x)$

$= -f(x)$

$[\because x = 0, y = 0,\ f(0) = f^2(0) - f^2(a)$

$\Rightarrow f^2(a) = 0 \Rightarrow f(a) = 0]$

$\Rightarrow f(2a-x) = -f(x)$

13. **(b)** $f(x) = 4^{-x^2} + \cos^{-1}\left(\dfrac{x}{2} - 1\right) + \log(\cos x)$

$f(x)$ is defined if $-1 \le \left(\dfrac{x}{2} - 1\right) \le 1$ and $\cos x > 0$

or $\quad 0 \le \dfrac{x}{2} \le 2$ and $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

or $\quad 0 \le x \le 4$ and $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$\therefore \quad x \in \left[0, \dfrac{\pi}{2}\right)$

14. **(d)** Clearly f is one one and onto, so invertible

Also $f(x) = 4x + 3 = y \Rightarrow x = \dfrac{y-3}{4}$

$\therefore\ g(y) = \dfrac{y-3}{4}$

15. **(b)** Given that $f(x) = (x+1)^2 - 1,\ x \ge -1$

Clearly $D_f = [-, \infty)$ but co-demain is not given. Therefore $f(x)$ need not be necessarily onto.

But if $f(x)$ is onto then as $f(x)$ is one one also, $(x+1)$ being something +ve, $f^{-1}(x)$ will exist where

$(x+1)^2 - 1 = y$

$\Rightarrow x + 1 = \sqrt{y+1}$ (+ve square root as $x + 1 \ge 0$)

$\Rightarrow x = -1 + \sqrt{y+1} \Rightarrow f^{-1}(x) = \sqrt{x+1} - 1$

Then $f(x) = f^{-1}(x) \Rightarrow (x+1)^2 - 1 = \sqrt{x+1} - 1$

$\Rightarrow (x+1)^2 = \sqrt{x+1} \Rightarrow (x+1)^4 = (x+1)$

$\Rightarrow (x+1)[(x+1)^3 - 1] = 0 \Rightarrow x = -1, 0$

$\therefore$ The statement-1 is correct but statement-2 is false.

16. **(b)** Given that $f(x) = x^3 + 5x + 1$

$\therefore\ f'(x) = 3x^2 + 5 > 0,\ \forall x \in R$

$\Rightarrow$ $f(x)$ is strictly increasing on R

$\Rightarrow$ $f(x)$ is one one

$\therefore$ Being a polynomial $f(x)$ is cont. and inc.

on R with $\lim\limits_{x\to\infty} f(x) = -\infty$

and $\lim\limits_{x\to\infty} f(x) = \infty$

$\therefore$ Range of $f = (-\infty, \infty) = R$

Hence f is onto also. So, f is one one and onto R.

17. **(b)** $f(x) = \dfrac{1}{\sqrt{|x| - x}}$, define if $|x| - x > 0$

$\Rightarrow |x| > x, \Rightarrow x < 0$

Hence domain of $f(x)$ is $(-\infty, 0)$