CHAPTER

6 APPLICATIONS OF DERIVATIVES

Syllabus

Applications of derivatives : rate of change of bodies, increasing/decreasing functions, tangents & normals, use of derivatives in approximation, maxima and minima (first derivative test motivated geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

Chapter Analysis

	2016		2017		2018
	Delhi	OD	Delhi	OD	Delhi/OD
Rate of Change of Bodies	–	–	1 Q. (2 marks)	1 Q. (2 marks)	2 Q. (2 marks)
Equation of Tangents and Normals	1 Q. (4 marks)	1 Q. (4 marks)	–	–	2 Q. (4 marks)
Increasing & Decreasing Functions	1 Q. (6 marks)	1 Q. (6 marks)	1 Q. (2 marks)	1 Q. (1 marks)	2 Q. (4 marks)
Maxima and Minima	1 Q. (6 marks)	1 Q. (6 marks)	1 Q. (6 marks)	1 Q. (6 marks)	2 Q. (4 marks)

TOPIC-1
Rate of Change of Bodies

Revision Notes

1. Interpretation of $\frac{dy}{dx}$ as a rate measure :

If two variables x and y are varying with respect to another variable say t, *i.e.*, if $x = f(t)$ and $y = g(t)$, then by the Chain Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \frac{dx}{dt} \neq 0.$$

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x both with respect to t.

Also, if y is a function of x and they are related as $y = f(x)$ then, $f'(\alpha)$ *i.e.*, $\left[\frac{dy}{dx}\right]_{at\ x=\alpha}$ represents the rate of change of y with respect to x at the instant when $x = \alpha$.

Objective Type Questions

(1 mark each)

Q. 1. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is :

(a) $10\ cm^2/s$ **(b)** $\sqrt{3}\ cm^2/s$

(c) $10\sqrt{3}\ cm^2/s$ **(d)** $\frac{10}{3}\ cm^2/s$

[NCERT Exemp. Ex. 6.3, Q. 35, Page 138]

Ans. Correct option : (c)

Explanation : Let the side of an equilateral triangle be x cm.

$\therefore$ Area of equilateral triangle, $A = \frac{\sqrt{3}}{4}x^2$ ...(i)

Also, $\frac{dx}{dt} = 2$ cm/s

On differentiating equation (i) with respect to t, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}.2x.\frac{dx}{dt}$$

$$= \frac{\sqrt{3}}{4}.2.10.2$$

$$\left[\because x = 10 \text{ and } \frac{dx}{dt} = 2\right]$$

$$= 10\sqrt{3}\ cm^2/s$$

Q. 2. A ladder, 5-meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metre from the wall is :

(a) $\frac{1}{10}$ **radian/sec** **(b)** $\frac{1}{20}$ **radian/sec**

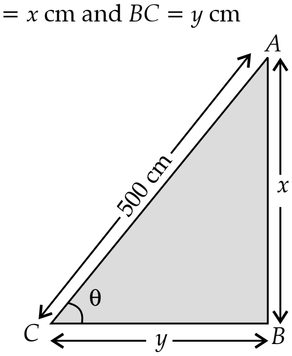
(c) 20 radian/sec **(d) 10 radian/sec**

[NCERT Exemp. Ex. 6.3, Q. 36, Page 138]

Ans. Correct option : (b)

Explanation : Let the angle between floor and the ladder be θ.

Let $AB = x$ cm and $BC = y$ cm



$\therefore \quad \sin\theta = \frac{x}{500}$ and $\cos\theta = \frac{y}{500}$

$\Rightarrow \quad x = 500\sin\theta$ and $y = 500\cos\theta$

Also, $\frac{dx}{dt} = 10$ cm/s

$\Rightarrow 500.\cos\theta.\frac{d\theta}{dt} = 10$ cm/s

$\Rightarrow \quad \frac{d\theta}{dt} = \frac{10}{500\cos\theta} = \frac{1}{50\cos\theta}$

For $y = 2$ m $= 200$ cm,

$$\frac{d\theta}{dt} = \frac{1}{50.\frac{y}{500}} = \frac{10}{y}$$

$$= \frac{10}{200} = \frac{1}{20} \text{ rad/s}$$

Very Short Answer Type Questions

(1 mark each)

Q. 1. The contentment obtained after eating x-units of a new dish at a trial function is given by the function $f(x) = x^3 + 6x^2 + 5x + 3$. If the marginal contentment is defined as the rate of change of $f(x)$ with respect to the number of units consumed at an instant, then find the marginal contentment when three units of dish are consumed.

R&U **[S.Q.P. 2012]**

Sol. $f(x) = x^3 + 6x^2 + 5x + 3$

$\frac{df(x)}{dx} = 3x^2 + 12x + 5$ ½

At $x = 3$,

Marginal contentment

$= 3 \times (3)^2 + 12 \times 3 + 5$

$= 27 + 36 + 5$

$= 68$ units. ½

Short Answer Type Questions (2 marks each)

AI Q. 1. The volume of a cube is increasing at the rate of 9 cm^3/sec. How fast is the surface area increasing when the length of an edge is 10 cm. **R&U [O.D. Set I 2017] [NCERT]**

Sol. Let V be the volume of cube, then $\frac{dV}{dt} = 9$ cm^3/s.

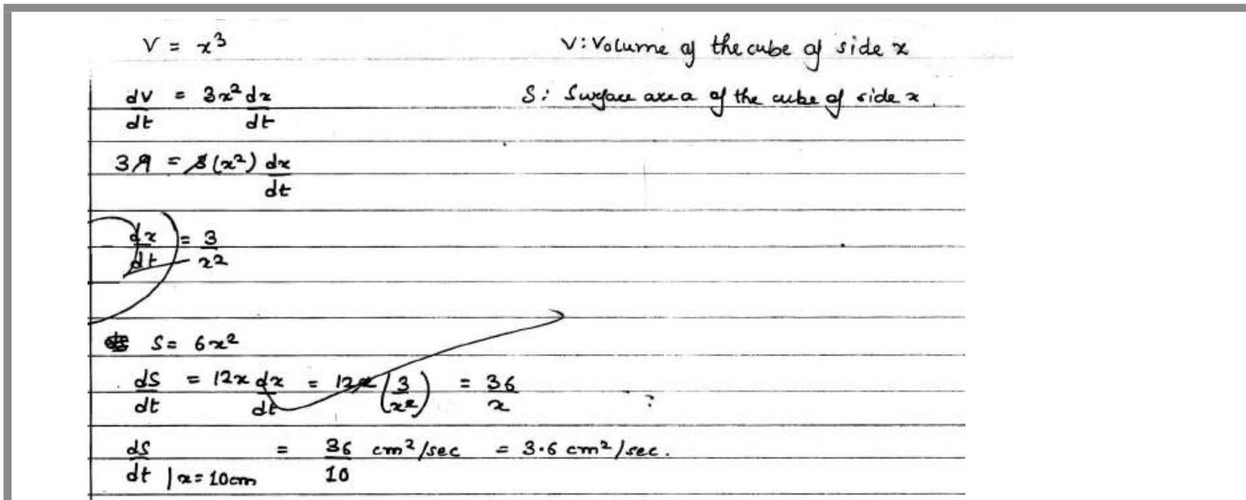
Surface area (S) of cube $= 6x^2$, where x is the side.

then $V = x^3$ or $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$ or $\frac{dx}{dt} = \frac{1}{3x^2}.\frac{dV}{dt}$ **1**

$S = 6x^2$ or $\frac{dS}{dt} = 12x\frac{dx}{dt} = 12x.\frac{1}{3x^2}\frac{dV}{dt}$ **½**

$= 4.\frac{1}{10}.9 = 3.6$ cm^2/s **[CBSE Marking Scheme 2017] ½**

OR


$V = x^3$ V: Volume of the cube of side x

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$ S: Surface area of the cube of side x.

$3\,9 = 3(x^2)\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{3}{x^2}$

$S = 6x^2$

$\frac{dS}{dt} = 12x\frac{dx}{dt} = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}$

$\left.\frac{dS}{dt}\right|_{x=10cm} = \frac{36}{10}$ cm^2/sec $= 3.6$ cm^2/sec.

[Topper's Answer 2017]

Q. 2. The length x, of a rectangle is decreasing at the rate of 5 cm/minute and the width y, is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle. **R&U [NCERT] [O.D. 2017, O.D. Comptt., 2009]**

Sol. Given, $\frac{dx}{dt} = -5$ cm/m,

$\frac{dy}{dt} = 4$ cm/m

$A = x \times y$

$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$ **1**

$= 8(4) + 6(-5) = 2$

$\therefore$ Area is increasing at the rate of 2 cm^2/minute. **1**

[CBSE Marking Scheme 2017]

AI Q. 3. The volume of a sphere is increasing at the rate of 8 cm^3/sec. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm. **R&U [O.D. 2017]**

Sol. $\frac{dV}{dt} = 8$ cm^3/s, where V is the volume of sphere

i.e., $V = \frac{4}{3}\pi r^3$

or $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

or $\frac{dr}{dt} = \frac{1}{4\pi r^2}\cdot\frac{dV}{dt}$ **1**

$S = 4\pi r^2$

or $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$= 8\pi r \cdot \frac{1}{4\pi r^2}\cdot 8$ ½

or $= \frac{2\times 8}{12}$

$= \frac{4}{3}$ cm²/sec ½

[CBSE Marking Scheme 2017]

Q. 4. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. R&U **[Delhi 2017]**

Sol. $V = \frac{4}{3}\pi r^3$

or $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ or $\frac{dr}{dt} = \frac{3}{4\pi r^2}$ **1**

$S = 4\pi r^2$

or $\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$ ½

$\frac{dS}{dt} = 8\pi r \cdot \frac{3}{4\pi r^2}$

or $\left.\frac{dS}{dt}\right|_{r=2} = 3\text{cm}^2/\text{s}$ ½

[CBSE Marking Scheme 2017]

Q. 5. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then find the rate of change of the slope of the curve when $x = 3$. R&U **[Delhi 2017]**

Sol. Given curve is $y = 5x - 2x^3$

or $\frac{dy}{dx} = 5 - 6x^2$

or $m = 5 - 6x^2 \left[\text{slope}(m) = \frac{dy}{dx}\right]$ **1**

$\frac{dm}{dt} = -12x\frac{dx}{dt} = -24x$

$\left.\frac{dm}{dt}\right|_{x=3} = -72$ **1**

Rate of Change of the slope is decreasing by 72 units/s. **[CBSE Marking Scheme 2017]**

Q. 6. The radius r of a right circular cone is decreasing at the rate of 3 cm/minute and the height h is increasing at the rate of 2 cm/minute. When r = 9 cm and h = 6 cm, find the rate of change of its volume. R&U **[Delhi Comptt., 2017]**

Sol. $\frac{dr}{dt} = -3$ cm/min,

$\frac{dh}{dt} = 2$ cm/min ½

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right]$ **1**

$\left(\frac{dV}{dt}\right)_{\text{at } r=9,\, h=6} = -54\,\pi$ cm³/min ½

or Volume is decreasing at the rate 54π cm³/min.

[CBSE Marking Scheme 2017]

Q. 7. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which y-coordinate is changing 2 times as fast as x-coordinate.

R&U**[O.D. Comptt., 2017]**

Sol. $6y = x^3 + 2$ or $6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$ **1**

Given : $\frac{dy}{dt} = 2\frac{dx}{dt}$ or $12 = 3x^2$ or $x = \pm 2$ ½

∴ The points are (2, 5/3), (–2, –1) ½

[CBSE Marking Scheme 2017]

Q. 8. The radius r of a right circular cylinder is increasing at the rate of 5 cm/min and its height h, is decreasing at the rate of 4 cm/min. When r = 8 cm and h = 6 cm, find the rate of change of the volume of cylinder. R&U **[O.D. Comptt., 2017]**

Sol. $\frac{dr}{dt} = 5$ cm/min, $\frac{dh}{dt} = -4$ cm/min

$V = \pi r^2 h$ ½

$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right)$ **1**

$\frac{dV}{dt} = \pi\,[64 \times (-4) + 2 \times 6 \times 8 \times 5]$

$\left.\frac{dV}{dt}\right)_{r=8,\, h=6} = 224\,\pi$ cm³/min ½

∴ Volume is increasing at the rate of 224π cm³/min. **[CBSE Marking Scheme 2017]**

Q. 9. A balloon, which always remains spherical, has a variable diameter $\frac{2}{3}(3x+1)$. Find the rate of change of its volume with respect to x.

R&U**[O.D. Comptt., 2017]**

Sol. Radius $= \frac{1}{3}(3x+1)$

$V = \frac{4}{3}\pi\frac{(3x+1)^3}{27}$ **1**

$$\frac{dV}{dx} = \frac{12\pi \times 3}{81}(3x+1)^2$$

$$= \frac{4\pi}{9}(3x+1)^2 \text{ square units} \quad \mathbf{1}$$

[CBSE Marking Scheme 2017]

Q. 10. The radius r of a right circular cylinder is decreasing at the rate of 3 cm/min. and its height h is increasing at the rate of 2 cm/min. When r = 7 cm and h = 2 cm, find the rate of change of the volume of cylinder.

[Use $\pi = \frac{22}{7}$] R&U **[Foreign 2017]**

Sol. $V = \pi r^2 h$ or $\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2r\frac{dr}{dt}h\right)$ **1**

$= \frac{22}{7}[49 \times (+2) + 14(2)(-3)] = 44 \text{ cm}^3/\text{min}$ **1**

$\therefore$ Volume is increasing at the rate of 44 cm^3/min.

[CBSE Marking Scheme 2017]

Q. 11. The radius r of a right circular cylinder is increasing uniformly at the rate of 0.3 cm/s and its height h is decreasing at the rate of 0.4 cm/s. When r = 3.5 cm and h = 7 cm, find the rate of change of the curved surface area of the cylinder.

[Use $\pi = \frac{22}{7}$] R&U **[Foreign 2017]**

Sol. CSA of cylinder, A = $2\pi rh$

or $\frac{dA}{dt} = 2\pi\left[r\frac{dh}{dt} + h\frac{dr}{dt}\right]$ **1**

$= 2 \times \frac{22}{7}[3.5 \times (-0.4) + 7(0.3)] = 4.4 \text{ cm}^2/\text{s}$ **1**

$\therefore$ CSA is increasing at the rate of 4.4 cm^2/s.

[CBSE Marking Scheme 2017]

Q. 12. The radius r of the base of a right circular cone is decreasing at the rate of 2 cm/min. and height h is increasing at the rate of 3 cm/min. When r = 3.5 cm and h = 6 cm, find the rate of change of the volume of the cone.

[Use $\pi = \frac{22}{7}$] R&U **[Foreign 2017]**

Sol. $V = \frac{1}{3}\pi r^2 h$

or $\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2\frac{dh}{dt} + h.2r\frac{dr}{dt}\right]$ **1**

$= \frac{1}{3} \times \frac{22}{7}[(3.5)^2 \times 3 - 2(3.5)(6)(2)]$

$= -49.5 \text{ cm}^3/\text{min}$ **1**

$\therefore$ Volume of decreasing at the rate of 49.5 cm^3/min. **[CBSE Marking Scheme 2017]**

Q. 13. The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$ in rupees. Find the marginal revenue when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

R&U **[Comptt. set I, II, III 2018]**

Sol. $R'(x) = 6x + 36$ **1**

$R'(5) = 66$ **1**

[CBSE Marking Scheme 2018]

Long Answer Type Questions-I

(4 marks each)

Q. 1. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm ? R&U **[Delhi, 2015]**

Sol. Let x be the side of an equilateral triangle

$\therefore \quad \frac{dx}{dt} = 2 \text{ cm/s}$ **1**

Area (A) $= \frac{\sqrt{3}}{4}x^2$ **1**

or $\frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt}$ **1**

or $\frac{dA}{dt} = \frac{\sqrt{3}}{2}(20) \times (2)$

[$\because$ Side of triangle = 20 cm]

$= 20\sqrt{3} \text{ cm}^2/\text{s}$ **1**

Hence its area increasing at the rate of $20\sqrt{3}$ cm^2/s.

[CBSE Marking Scheme 2015]

Q. 2. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing, when the foot of the ladder is 4 m away from the wall ?

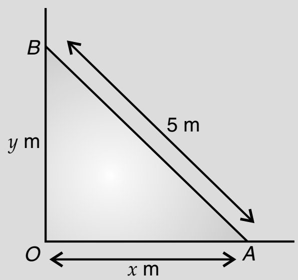
R&U **[Delhi Set I, II, III, 2012]**

Sol. Let AB be the ladder and OB be the wall.

At any instant, let $OA = x$ m and $OB = y$ m.

Also $AB = 5$ m, (given)

$\therefore \quad x^2 + y^2 = 5^2 = 25$ ...(i) **1**



Differentiating (i) w.r.t. 't', we get

$$2x\frac{dx}{dt}+2y\frac{dy}{dt} = 0$$

or $$x\frac{dx}{dt}+y\frac{dy}{dt} = 0 \quad ...(ii)$$

When $x = 4$, then from (i), we get

$$y^2 = 25 - x^2$$
$$= 25 - 16 = 9$$

$\therefore$ $$y = 3 \quad 1$$

Also, given $\frac{dx}{dt} = 2$ cm/sec $= 0{\cdot}02$ m/sec

Using these values in (ii), we get

$$4\times 0{\cdot}02+3\frac{dy}{dt} = 0 \quad 1$$

or $$\frac{dy}{dt} = -\frac{0{\cdot}08}{3}\text{ m/sec} = -\frac{8}{3}\text{ cm/sec} \quad ½$$

Thus, the height of the ladder on the wall is decreasing at the rate of $\frac{8}{3}$ cm/sec. ½

[CBSE Marking Scheme 2012]

Q. 3. Sand is pouring from the pipe at the rate of 12 cm^3/s. The falling sand forms a cone on a ground in such a way that the height of cone is always one-sixth of radius of the base. How fast is the height of sand cone increasing when the height is 4 cm ? R&U **[Delhi 2011] [NCERT]**

Sol. Let V be the volume of cone, h be the height and r be the radius of base of the cone.

Given, $$\frac{dV}{dt} = 12\text{cm}^3/\text{s} \quad ...(i)½$$

Also, height of cone = 1/6 (radius of base of cone)

$\therefore$ $$h = \frac{1}{6}r \text{ or } r = 6h \quad ...(ii)½$$

We know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h \quad ...(iii)$$

On putting $r = 6h$ from Eq. (ii) in Eq. (iii), we get

$$V = \frac{1}{3}\pi(6h)^2.h \text{ or } V = \frac{\pi}{3}.36h^3 \text{ or } V = 12\pi h^3 \quad 1$$

On differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = 12\pi\times 3h^2.\frac{dh}{dt}$$

or $$\frac{dV}{dt} = 36\pi h^2.\frac{dh}{dt} \quad 1$$

On putting $\frac{dV}{dt} = 12\text{cm}^3/\text{s}$ and $h = 4$cm, we get

$$12 = 36\pi \times 16 \times \frac{dh}{dt}$$

or $$\frac{dh}{dt} = \frac{12}{36\pi\times 16}$$

$\therefore$ $$\frac{dh}{dt} = \frac{1}{48\pi}\text{cm/s}$$

Hence, the height of sand cone is increasing at the rate of 1/48π cm/s. 1

Commonly Made Error

- Many candidates consider surface area as function instead of volume. Also few candidates do not apply proper sign through it.

Answering Tips

- Understand the difference between rate of increase and rate of decrease.

Q. 4. The total cost associated with provision of free mid-day meals to x students of a school in primary classes is given by $C(x) = 0{\cdot}005x^3 - 0{\cdot}02x^2 + 30x + 50$. If the marginal cost is given by rate of change $\frac{dc}{dx}$ of total cost, write the marginal cost of food for 300 students.

R&U **[Delhi Set I, II, III Comptt. 2013]**

Sol. We have $C(x) = 0{\cdot}005x^3 - 0{\cdot}02x^2 + 30x + 50$ 1

$C'(x) = 0{\cdot}015x^2 - 0{\cdot}04x + 30 + 0$ 1

$C'(300) = 0{\cdot}015(300)^2 - 0{\cdot}04(300) + 30$

$= 1{,}368$ 1

So the marginal cost of food for 300 students is ₹ 1,368. 1

[CBSE Marking Scheme 2013]

Q. 5. The total expenditure (in ₹) required for providing the cheap edition of a book for poor and deserving students is given by $R(x) = 3x^2 + 36x$, where x is the number of set of books. If the marginal expenditure is defined as $\frac{dR}{dx}$, write the marginal expenditure required for 1200 such sets. R&U **[O.D. Set I, II, III Comptt. 2013]**

Sol. Here, $R(x) = 3x^2 + 36x$, where x is the number of sets of book

$$\frac{dR}{dx} = 6x + 36 \quad \textbf{1+1}$$

$$\left(\frac{dR}{dx}\right)_{x=1200} = 6(1{,}200) + 36 = ₹\ 7{,}236 \quad \textbf{1+1}$$

Q. 6. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $p(x) = 0{\cdot}005x^3 + 0{\cdot}02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added. R&U **[Delhi Set I, II, III, 2013]**

Sol. Here, pollution content is given by $p(x) = 0{\cdot}005x^3 + 0{\cdot}02x^2 + 30x$, where x is the number of diesel vehicles

$$\frac{dp}{dx} = 0{\cdot}015x^2 + 0{\cdot}04x + 30 \quad \textbf{1½}$$

$$\left(\frac{dp}{dx}\right)_{x=3} = 0{\cdot}015(3)^2 + 0{\cdot}04(3) + 30 \quad \textbf{1½}$$

$$= 30{\cdot}255 \quad \textbf{1}$$

[CBSE Marking Scheme 2013]

Q. 7. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$. R&U **[O.D. Set I, II, III 2013] [All India 2012]**

Sol. Total revenue is given by $R(x) = 3x^2 + 36x + 5$ **1**

$$\text{Marginal revenue} = \frac{dR}{dx} = 6x + 36 \quad \textbf{1½}$$

$$\left(\frac{dR}{dx}\right)_{x=5} = 6 \times 5 + 36 = 66 \quad \textbf{1½}$$

[CBSE Marking Scheme 2013]

Q. 8. If $C = 0{\cdot}003x^3 + 0{\cdot}02x^2 + 6x + 250$ gives the amount of carbon pollution in air in an area on the entry of x number of vehicles, then find the marginal carbon pollution in the air, when 3 vehicles have entered in the area. R&U **[Foreign Set I, II, III, 2013]**

Sol. Given $C = 0{\cdot}003x^3 + 0{\cdot}02x^2 + 6x + 250$ **1**

$$\frac{dC}{dx} = 0{\cdot}009x^2 + 0{\cdot}04x + 6 \quad \textbf{1½}$$

$$\left(\frac{dC}{dx}\right)_{x=3} = 0{\cdot}009(3)^2 + 0{\cdot}04(3) + 6$$

$$= 0{\cdot}081 + 0{\cdot}12 + 6$$

$$= 6{\cdot}201 \quad \textbf{1½}$$

[CBSE Marking Scheme 2013]

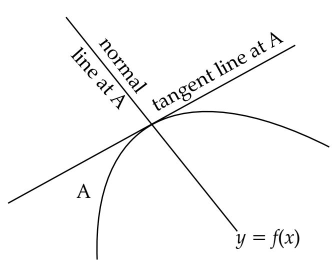
TOPIC-2
Tangents and Normals

Revision Notes

1. Slope or gradient of a line :

If a line makes an angle θ with the positive direction of X-axis in anti-clockwise direction, then tan θ is called the slope or gradient of the line. [Note that θ is taken as positive or negative accordingly as it is measured in anti-clockwise (*i.e.*, **from positive direction of X-axis to the positive direction of *Y*-axis)** or clockwise direction respectively.]

2. Pictorial representation of tangent & normal :



3. Facts about the slope of a line :

(a) If a line is parallel to x-axis (or perpendicular to y-axis), then its slope is 0 (zero).

(b) If a line is parallel to y-axis (or perpendicular to x-axis), then its slope is $\frac{1}{0}$ *i.e.*, not defined.

(c) It two lines are perpendicular, then product of their slopes equals – 1 *i.e.*, $m_1 \times m_2 = -1$. Whereas, for two parallel lines, their slopes are equal *i.e.*, $m_1 = m_2$. (Here in both the cases, m_1 and m_2 represent the slopes of respective lines).

4. Equation of Tangent at (x_1, y_1) :

$(y - y_1) = m_T(x - x_1)$, where m_T is the slope of tangent such that $m_T = \left[\frac{dy}{dx}\right]_{at\,(x_1,\,y_1)}$

5. Equation of Normal at (x_1, y_1) :

$(y - y_1) = m_N(x - x_1)$, where m_N is the slope of normal such that $m_N = \dfrac{-1}{\left[\frac{dy}{dx}\right]_{at\,(x_1,\,y_1)}}$

Note that $m_T \times m_N = -1$ which is obvious because tangent and normal are perpendicular to each other. In other words, the tangent and normal lines are inclined at right angle to each other.

6. Acute angle between the two curves whose slopes m_1 and m_2 are known :

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1.m_2}\right| \text{ or } \theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1.m_2}\right|.$$

It is absolutely sufficient to find one angle **(generally the acute angle)** between the two curves. Other angle between the curve is given by $\pi - \theta$.

Note that if the curves cut **orthogonally** (*i.e.,* they cut each other at right angles), then it means $m_1 \times m_2 = -1$, where m_1 and m_2 represent the slopes of the tangents of curves at the intersection point.

7. Finding the slope of a line $ax + by + c = 0$:

STEP 1 : Express the given line in the standard slope-intercept form $y = mx + c$ *i.e.*, $y = \left(-\frac{a}{b}\right)x - \frac{c}{b}$.

STEP 2 : By comparing to the standard form $y = mx + c$, we can conclude $-\frac{a}{b}$ is the slope of given line $ax + by + c = 0$.

Objective Type Questions

(1 mark each)

Q. 1. The curve $y = x^{1/5}$ has at (0, 0)

(a) a vertical tangent (parallel to y-axis)

(b) a horizontal tangent (parallel to x-axis)

(c) an oblique tangent

(d) no tangent

[NCERT Exemp.]

Ans. Correct option : (a)

Explanation : Given that, $y = x^{1/5}$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-4/5}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{5} \times (0)^{-4/5} = \infty$$

So, the curve $y = x^{1/5}$ has a vertical tangent at (0, 0), which is parallel to y-axis.

Q. 2. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(a) $3x - y = 8$ **(b) $3x + y + 8 = 0$**

(c) $x + 3y \pm 8 = 0$ **(d) $x + 3y = 0$**

[NCERT Exemp.]

Ans. Correct option : (c)

Explanation : We have, the equation of the curve is $3x^2 - y^2 = 8$ ….(i)

Also, the given equation of the line is $x + 3y = 8$.

$\Rightarrow 3y = 8 - x$

$\Rightarrow \quad y = -\frac{x}{3} + \frac{8}{3}$

Thus, slope of the line is $-\frac{1}{3}$ which should be equal to slope of the equation of normal to the curve.

On differentiating equation (i) with respect to x, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y} = \text{Slope of the curve}$$

Now, slope of **n**ormal to the curve

$$= -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{\left(\frac{3x}{y}\right)} = -\frac{y}{3x}$$

$$\therefore \quad -\left(\frac{y}{3x}\right) = -\frac{1}{3}$$

$$\Rightarrow \quad -3y = -3x$$

$$\Rightarrow \quad y = x$$

On substituting the value of the given equation of the curve, we get

$$3x^2 - x^2 = 8$$
$$\Rightarrow \quad 2x^2 = 8$$
$$\Rightarrow \quad x^2 = 4$$
$$\Rightarrow \quad x = \pm 2$$

For $x = 2$

$$3(2)^2 - y^2 = 8$$
$$\Rightarrow \quad y^2 = 4$$
$$\Rightarrow \quad y = \pm 2$$

and for $x = -2$,

$$3(-2)^2 - y^2 = 8$$
$$\Rightarrow \quad y^2 = 4$$
$$\Rightarrow \quad y = \pm 2$$

So, the points at which normal is parallel to the given line are (±2, ±2).

Hence, the equation of normal at (±2, ±2) is

$$\Rightarrow \quad y - (\pm 2) = -\frac{1}{3}[x - (\pm 2)]$$
$$\Rightarrow 3[y - (\pm 2)] = -[x - (\pm 2)]$$
$$\therefore \quad x + 3y \pm 8 = 0$$

Q. 3. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is :

(a) 1 **(b) 0**

(c) –6 **(d) 6**

[NCERT Exemp.]

Ans. Correct option : (d)

Explanation : Given that, $ay + x^2 = 7$ and $x^3 = y$

On differentiating with respect to x in both equations, we get

$$a.\frac{dy}{dx} + 2x = 0 \text{ and } 3x^2 = \frac{dy}{dx}$$
$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a} \text{ and } \frac{dy}{dx} = 3x^2$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{a} = m_1 \text{ and } \left(\frac{dy}{dx}\right)_{(1,1)} = 3.1 = 3 = m_2$$

Since, the curve cuts orthogonally at (1, 1).

$$\therefore \quad m_1 m_2 = -1$$
$$\Rightarrow \left(\frac{-2}{a}\right).3 = -1$$
$$\therefore \quad a = 6$$

Q . 4. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is :

(a) $x + 5y = 2$ **(b) $x - 5y = 2$**

(c) $5x - y = 2$ **(d) $5x + y = 2$**

[NCERT Exemp.]

Ans. Correct option : (a)

Explanation : Given that the equation of curve is $y(1 + x^2) = 2 - x$ **....(i)**

On differentiating with respect to x, we get

$$\therefore \quad y.(0 + 2x) + (1 + x^2).\frac{dy}{dx} = 0 - 1$$
$$\Rightarrow \quad 2xy + (1 + x^2)\frac{dy}{dx} = -1$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1 - 2xy}{1 + x^2} \quad \text{...(ii)}$$

Since, the given curve passes through x-axis, i.e., $y = 0$

$$\therefore \quad 0(1 + x^2) = 2 - x \quad \text{[By using Eq. (i)]}$$
$$\Rightarrow \quad x = 2$$

So, the curve passes through the point (2, 0).

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(2,\,0)} = \frac{-1 - 2\times 0}{1 + 2^2} = -\frac{1}{5}$$
$$= \text{Slope of the curve}$$

$\therefore$ Slope of tangent to the curve $= -\frac{1}{5}$

$\therefore$ Equation of tangent of the curve passsing through (2, 0) is

$$y - 0 = -\frac{1}{5}(x - 2)$$
$$\Rightarrow y + \frac{x}{5} = +\frac{2}{5}$$
$$\Rightarrow 5y + x = 2$$

Q. 5. The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are :

(a) (2, –2), (–2, –34) **(b) (2, 34), (–2, 0)**

(c) (0, 34), (–2, 0) **(d) (2, 2), (–2, 34)**

[NCERT Exemp.]

Ans. Correct option : (d)

Explanation : The equation of the curve is given by

$$y + x^3 = 12x + 18$$

On differentiating with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 12$$

So, the slope of line parallel to the x-axis,

$$\therefore \quad \frac{dy}{dx} = 0$$
$$\Rightarrow \quad 3x^2 - 12 = 0$$
$$\Rightarrow \quad x^2 = \frac{12}{3} = 4$$
$$\therefore \quad x = \pm 2$$

For $x = 2$,

$$y = 2^3 - 12 \times 2 + 18 = 2$$

and for $x = -2$,

$$y = (-2)^3 - 12 \times (-2) + 18 = 34$$

So, the points are (2, 2) and (−2, 34).

Q. 6. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis at :

(a) (0, 1) **(b) $\left(-\frac{1}{2}, 0\right)$**

(c) (2, 0) **(d) (0, 2)**

[NCERT Exemp.]

Ans. Correct option : (b)

Explanation : The equation of the curve is given by $y = e^{2x}$

Since, it passes through the point (0, 1).

$$\therefore \quad \frac{dy}{dx} = e^{2x}.2 = 2.e^{2x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,1)} = 2.e^{2.0} = 2$$

= Slope of tangent to the curve.

$\therefore$ Equation of tangent is

$$y - 1 = 2(x - 0)$$

$$\Rightarrow \quad \boldsymbol{y} = 2x + 1$$

$$\therefore \quad \frac{dy}{dx} = e^{2x}.2 = 2.e^{2x}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,1)} = 2.e^{2.0} = 2$$

= Slope of tangent to the curve.

$\therefore$ Equation of tangent is

$$y - 1 = 2(x - 0)$$

$$\Rightarrow \quad \boldsymbol{y} = 2x + 1$$

Since, tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis, *i.e.*, $y = 0$.

$$\therefore \quad 0 = 2x + 1$$

$$\Rightarrow \quad x = -\frac{1}{2}$$

So, the required point is $\left(-\frac{1}{2}, 0\right)$.

Very Short Answer Type Questions **(1 mark each)**

Q. 1. Write the equation of tangent drawn to the curve $y = \sin x$ at the point (0, 0).
R&U **[Delhi Comptt., 2017]**

Sol. $\frac{dy}{dx} = \cos x$ ½

Slope of tangent at (0, 0) = $\left[\frac{dy}{dx}\right]_{(0,0)} = \cos 0° = 1$

Equation of tangent is $y = x$ ½

[CBSE Marking Scheme 2017]

Q. 2. Find the slope of tangent to the curve $y = 3x^2 - 6$ at the point on it whose x-coordinate is 2.
R&U **[All India 2009C]**

Sol. Given, $y = 3x^2 - 6$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6x \quad ½$$

Now, slope of tangent

$$= \left(\frac{dy}{dx}\right)_{x=2} = 6(2) = 12 \quad ½$$

Hence, required slope is 12.

[CBSE Marking Scheme 2009]

Q. 3. Find the slope of tangent of the curve $y = 3x^2 - 4x$ at point whose x-coordinate is 2.
R&U **[Delhi 2009 C]**

Sol. Try Yourself [Similar to Q. 2 Very Short type Questions]

Short Answer Type Questions **(2 marks each)**

Q. 1. Find the slope of tangent and normal to the curve $x^2 + 2y + y^2 = 0$ at (– 1, 2). R&U

Sol. Given, $x^2 + 2y + y^2 = 0$

$$2x + 2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2 + 2y) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2(1+y)} = -\frac{x}{1+y} \quad \mathbf{1}$$

Slope of tangent at (– 1, 2)

$$\left[\frac{dy}{dx}\right]_{(-1,2)} = \frac{-(-1)}{1+2} = \frac{1}{3} \quad ½$$

Slope of normal at (– 1, 2)

$$\frac{-1}{\left[\frac{dy}{dx}\right]_{(-1,2)}} = -\frac{3}{1} = -3. \quad ½$$

Long Answer Type Questions-I **(4 marks each)**

Q. 1. Find the equations of the normal to the curve $y = 4x^3 - 3x + 5$ which are perpendicular to the line $9x - y + 5 = 0$.
R&U **[SQP 2018-19]**

Sol. The given curve is

$$y = 4x^3 - 3x + 5$$

Let the required normal be at (x_1, y_1)

Slope of the tangent = $\frac{dy}{dx} = 12x^2 - 3$

$$m_1 = \text{slope of the normal} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

$$= \frac{-1}{12x_1^2 - 3}$$ **1**

m_2 = Slope of the line = 9

Since normal is perpendicular to the line.

Therefore, $m_1.m_2 = -1$

$$\frac{-1}{12x_1^2-3} \times 9 = -1$$

$$\Rightarrow \quad 12x_1^2 - 3 = 9$$

$$\Rightarrow \quad x_1 = \pm 1$$

Hence, the points are (1, 6) and (–1, 4) **1**

Equations of normals are: **1**

$$y - 6 = -\frac{1}{9}(x-1) \text{ i.e., } x + 9y = 55$$

and $$y - 4 = -\frac{1}{9}(x+1) \text{ i.e., } x + 9y = 35$$

[CBSE Marking Scheme 2018-19]

Commonly Made Error

- Sometimes students get confused in condition of slopes of **parallel** and **perpendicular** lines.

Answering Tip

- When two lines are perpendicular, the slopes, $m_1 \cdot m_2 = -1$, and when two are parallel the slopes, $m_1 = m_2$

Q. 2. Show that the equation of normal at any point t on the curve $x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is $4(y\cos^2 t - x\sin^3 t) = 3\sin 4t$

R&U [Delhi Set I, II, III 2016][NCERT Exemplar]

Sol. $x = 3\cos t - \cos^3 t$

$$\frac{dx}{dt} = -3\sin t + 3.\cos^2 t.\sin t$$

$$= -3\sin t(1-\cos^2 t) = -3\sin^3 t$$

Again, $y = 3\sin t - \sin^3 t$

or $$\frac{dy}{dt} = 3\cos t - 3\sin^2 t.\cos t$$ ½

$$= 3\cos t(1-\sin^2 t) = 3\cos^3 t$$ ½

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{\cos^3 t}{\sin^3 t}$$

$$\therefore \text{ Slope of normal } = -\frac{dx}{dy} = \frac{\sin^3 t}{\cos^3 t}$$ **1**

$\therefore$ Equation of normal is

$$y - (3\sin t - \sin^3 t) = \frac{\sin^3 t}{\cos^3 t}\{x - (3\cos t - \cos^3 t)\}$$ **1**

or $y\cos^3 t - 3\sin t\cos^3 t + \sin^3 t\cos^3 t$

$= x\sin^3 t - 3\cos t\sin^3 t + \sin^3 t\cos^3 t$

or $y\cos^3 t - x\sin^3 t = 3\sin t\cos^3 t - 3\cos t\sin^3 t$

or $y\cos^3 t - x\sin^3 t = 3\sin t\cos t(\cos^2 t - \sin^2 t)$

or $y\cos^3 t - x\sin^3 t = 3\sin t\cos t\cos 2t$

Multiply both sides by 4, we get

$4(y\cos^3 t - x\sin^3 t) = 3 \times 4\sin t\cos t\cos 2t$

$= 3 \times 2\sin 2t\cos 2t$ **1**

$= 3\sin 4t$ **Hence proved**

Q. 3. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a and b.

R&U [Delhi Set I, II, III 2016]

Sol. $y^2 = ax^3 + b$

Differentiate with respect to x,

$$2y\frac{dy}{dx} = 3ax^2$$

or $$\frac{dy}{dx} = \frac{3ax^2}{2y}$$

or $$\frac{dy}{dx} = \frac{3ax^2}{\pm 2\sqrt{ax^3+b}} \quad [\because y^2 = ax^3 + b]$$

or $$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3a(2)^2}{\pm 2\sqrt{a(2)^3+b}}$$ **1**

$$= \frac{12a}{\pm 2\sqrt{8a+b}}$$

$$= \frac{6a}{\pm\sqrt{8a+b}}$$

Since (2, 3) lies on the curve

$y^2 = ax^3 + b$

or $9 = 8a + b$ ...(i) **1**

Also from equation of tangent

$y = 4x - 5$

slope of the tangent = 4

$$\therefore \quad \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{6a}{\pm\sqrt{8a+b}} \text{ becomes}$$

$$4 = \frac{6a}{\pm\sqrt{9}} \quad \{\text{from (i)}\}$$

$$\therefore \quad 4 = \frac{6a}{\pm 3}$$ **1**

$$\therefore \quad 4 = \frac{6a}{3} \text{ or } 4 = \frac{6a}{-3}$$

or $a = 2$ or $a = -2$

For $a = 2$, $9 = 8(2) + b$

or $b = -7$

and for $a = -2$, $9 = 8(-2) + b$

or $b = 25$

$\therefore$ $a = 2$ and $b = -7$

or $a = -2$ and $b = 25$ **1**

Q. 4. Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

R&U [O.D. Set I, II, III, 2014]

Sol. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or $\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = \frac{b^2x}{a^2y}$ **1**

Slope of tangent at $(\sqrt{2}a, b)=\frac{\sqrt{2}b}{a}$ ½

Slope of normal at $(\sqrt{2}a, b)=-\frac{a}{\sqrt{2}b}$ ½

Equation of tangent is

$$y-b = \frac{\sqrt{2}b}{a}(x-\sqrt{2}a)$$

i.e., $\sqrt{2}bx-ay = ab$ **1**

and equation of normal is

$$y-b = -\frac{a}{\sqrt{2}b}(x-\sqrt{2}a)$$

i.e., $ax+\sqrt{2}by = \sqrt{2}(a^2+b^2)$ **1**

[CBSE Marking Scheme 2014]

Q. 5. For the curve $y = 4x^3 - 2x^5$, find all those points at which the tangent passes through the origin.

R&U **[Delhi Set I, II, III Comptt. 2014][NCERT]**

Sol. $y = 4x^3 - 2x^5$

or $\frac{dy}{dx} = 12x^2 - 10x^4$

[As, $(y - y_0) = m(x - x_0)$]

Tangent at (x_1, y_1) is given by :

$$y - y_1 = (12x_1^2 - 10x_1^4)(x - x_1)$$

As it passes through the origin,

$$0 - y_1 = (12x_1^2 - 10x_1^4)(0 - x_1)$$

or $y_1 = 12x_1^3 - 10x_1^5$ ...(i) **1**

We know that (x_1, y_1) lies on the curve

or $y_1 = 4x_1^3 - 2x_1^5$ ...(ii) ½

From eq. (i) and eqn. (ii),

$8x_1^3 - 8x_1^5 = 0$

or $x_1^3(1 - x_1^2) = 0$

If $x_1^3 = 0$, then $x_1 = 0$

If $1 - x_1^2 = 0$, then $x_1 = 1$ or $x_1 = -1$ **1**

Substituting these values of x_1 in eq. (i), we get

$y_1 = 0$ (when $x_1 = 0$)

$y_1 = 12 - 10 = 2$ (when $x_1 = 1$)

$y_1 = [12(-1) - 10(-1)]$ **1**

(when $x = -1$)

$= -2$

So, required points : (0, 0), (1, 2) and (– 1, – 2). ½

Q. 6. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

R&U **[Delhi/OD 2018]**

Sol. $x_1 = 2 \Rightarrow y_1 = 3$ $(\because y_1 > 0)$ ½

Differentiating the given equation, we get, ½

$$\frac{dy}{dx} = \frac{-16x}{9y}$$

Slope of tangent at (2, 3) $= \left.\frac{dy}{dx}\right|_{(2,3)} = -\frac{32}{27}$ ½

Slope of Normal at (2, 3) $= \frac{27}{32}$ ½

Equation of tangent: $32x + 27y = 145$ **1**

Equation of Normal: $27x - 32y = -42$ **1**

[CBSE Marking Scheme, 2018]

Detailed Solution :

Given, $16x^2 + 9y^2 = 145$ ...(1)

It passes through (x_1, y_1)

$$16x_1^2 + 9y_1^2 = 145$$

put $x_1 = 2$

$$16 \times 4 + 9y_1^2 = 145$$

$\Rightarrow$ $9y_1^2 = 81 \Rightarrow y_1 = 3$ $(\because y_1 > 0)$

Diff. equation (1) with respect to x.

$$32x + 18y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{32x}{18y}$$

Slope of the tangent at (2, 3)

$$= \left[\frac{dy}{dx}\right]_{(2,3)} = \frac{-32\times 2}{18\times 3} = \frac{-32}{27}$$

The equation of tangent at (2, 3) is

$$y-3 = -\frac{32}{27}.(x-2)$$

$\Rightarrow$ $27y - 81 = -32x + 64$

$\Rightarrow$ $32x + 27y - 145 = 0$

Slope of Normal at (2, 3) $= -\frac{1}{\left[\frac{dy}{dx}\right]_{(2,3)}} = \frac{27}{32}$

Now, equation of Normal is given by

$$y-3 = \frac{27}{32}(x-2)$$

$\Rightarrow$ $32y - 96 = 27x - 54$

$\Rightarrow$ $27x - 32y + 42 = 0$

Q. 7. Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$, $y = b\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

R&U **[Delhi Set I, II, III Comptt. 2013] [NCERT]**

Sol. $\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$,

$\frac{dy}{d\theta} = -3b\cos^2\theta\sin\theta$

or $\frac{dy}{dx} = \frac{-b}{a}\cot\theta$ **1**

At $\theta = \frac{\pi}{4}$,

Slope : $\left[\dfrac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = -\dfrac{b}{a}\cot\dfrac{\pi}{4} = -\dfrac{b}{a}$ **1**

when $\theta = \dfrac{\pi}{4}$, $x = \dfrac{a}{2\sqrt{2}}, y = \dfrac{b}{2\sqrt{2}}$.

[as $x = a\sin^3\theta$ and $y = b\cos^3\theta$]

Equation of tangent at $\theta = \dfrac{\pi}{4}$ is :

$$y - \frac{b}{2\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{2\sqrt{2}}\right)$$

or $\sqrt{2}(bx + ay) = ab$ **1**

Slope of normal $= -\left(\dfrac{1}{-\frac{b}{a}}\right) = \dfrac{a}{b}$

Equation of normal at $\theta = \dfrac{\pi}{4}$ is :

$$y - \frac{b}{2\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$2\sqrt{2}y - b = \frac{a}{b}(2\sqrt{2}x - a)$$

or $2\sqrt{2}(by - ax) = b^2 - a^2$ **1**

Q. 8. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

R&U **[O.D. 2016, Foreign Set I, II, III, 2012] [NCERT]**

Sol. $ay^2 = x^3$

Diff. w.r.t. 'x',

$$2ay\frac{dy}{dx} = 3x^2 \quad ½$$

$$\left(\frac{dy}{dx}\right)_{(am^2,\, am^3)} = \frac{3(am^2)^2}{2a(am^3)} \quad ½$$

$$= \frac{3a^2m^4}{2a^2m^3} = \frac{3}{2}m \quad ½$$

Slope of normal $= -\dfrac{2}{3m}$ ½

Eq. of normal at (x_1, y_1) is

$$y - y_1 = -\frac{2}{3m}(x - x_1) \quad ½$$

Eq. of normal at (am^2, am^3)

$$y - am^3 = -\frac{2}{3m}(x - am^2) \quad ½$$

or $3my - 3am^4 = -2x + 2am^2$

or $2x + 3my = 3am^4 + 2am^2$ ½

or $2x + 3my = am^2(3m^2 + 2)$ ½

[CBSE Marking Scheme, 2016]

OR

Sol.

$x = am^2$

$ay^2 = (am^2)^3$

$ay^2 = a^3m^6$

$y^2 = a^2m^6$

$\therefore y = \pm am^3$

Considering $(x, y) = (am^2, am^3)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dm}}{\frac{dx}{dm}}$$

$$\frac{dy}{dm} = \frac{d}{dm}(am^3) = 3am^2$$

$$\frac{dx}{dm} = \frac{d}{dm}(am^2) = 2am$$

$\therefore \left.\dfrac{dy}{dx}\right]_{(am^2, am^3)} = \dfrac{3am^2}{2am} = \dfrac{3m}{2}$ - Slope of tangent at (am^2, am^3)

Slope of normal at $(am^2, am^3) = \dfrac{-1}{\left.\dfrac{dy}{dx}\right]_{(am^2, am^3)}} = \dfrac{-1}{\frac{3m}{2}} = \dfrac{-2}{3m}$

∴ Equation of normal to the curve at (am^2, am^3):

$$y - am^3 = \frac{-2}{3m}(x - am^2)$$

$$3my - 3am^4 = -2x + 2am^2$$

$3my + 2x = 3am^4 + 2am^2$ is the required equation.

[Topper's Answer 2016]

Q. 9. Show that the equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$.

R&U **[O.D. Set I, II, III, Comptt. 2012]**

Sol. $y^2 = 4ax$

Differentiating w.r.t. x

$$2y\frac{dy}{dx} = 4a$$

or $$\frac{dy}{dx} = \frac{2a}{y}$$

This gives the slope of the line tangent to the parabola at (x_1, y_1).

∴ Slope at (x_1, y_1), $m = \dfrac{2a}{y_1}$ **1**

∴ Required line is $y - y_1 = m(x - x_1)$

$$= \frac{2a}{y_1}(x - x_1)$$ **1**

or $yy_1 - y_1^2 = 2ax - 2ax_1$

But (x_1, y_1) lies on $y^2 = 4ax$

∴ $y_1^2 = 4ax_1$ **1**

Substituting this value of y_1^2, we get

$$yy_1 - 4ax_1 = 2ax - 2ax_1$$

or $yy_1 = 2a(x + x_1)$ **1**

This is the required equation.

Q. 10. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$.

R&U **[NCERT] [S.Q.P. Dec. 2016-17]**
[Delhi Set I, II, III, 2012, Delhi Comptt., 2012]

Sol. Slope of the tangent $y = x - 11$ is 1,

$y = x^3 - 11x + 5$ ½

or $\dfrac{dy}{dx} = 3x^2 - 11$ ½

If the point is (x_1, y_1), then

$3x_1^2 - 11 = 1$

or $x_1 = \pm 2$ **1**

When $x_1 = 2$, $y_1 = 8 - 22 + 5 = -9$

When $x_1 = -2$, $y_1 = -8 + 22 + 5 = 19$ **1**

Since $(-2, 19)$ does not lie on the tangent $y = x - 11$ ½

∴ Required point is $(2, -9)$. ½

[CBSE Marking Scheme 2012]

Commonly Made Error

- Few candidates make mistakes while finding the pint of contact at which the equation of tangent to the curve where tangent is parallel to the x-axes.

Answering Tips

- Learn all concepts of application of derivatives and practice number of problems based on derivatives.

Q. 11. Find the points on the curve $y = x^3 - 3x^2 - 9x + 7$ at which the tangent to the curve is parallel to the x-axis. R&U **[Delhi Set I, II, III Comptt. 2016]**

[NCERT]

Sol. Since tangents to $y = x^3 - 3x^2 - 9x + 7$ are parallel to x-axis

∴ $\dfrac{dy}{dx} = 0$

or $3x^2 - 6x - 9 = 0$ **2**

⇒ $x^2 - 2x - 3 = 0$

⇒ $(x - 3)(x + 1) = 0$

or $x = 3$, $x = -1$ and points on curve are $(3, -20)$ & $(-1, 12)$ **2**

[CBSE Marking Scheme 2016]

Q.12. Find the equations of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).

R&U **[Delhi Set I, II, III Comptt. 2016] [NCERT]**

Sol. $x^2 = 4y$

or $\dfrac{dy}{dx} = \dfrac{x}{2}$

∴ Slope of tangent at $(x_1, y_1) = \dfrac{x_1}{2}$ **1**

∴ Slope of Normal $= -\dfrac{2}{x_1}$ **1**

Since normal passes through (1, 2)

$\therefore$ Slope of normal line $= \frac{y_1 - 2}{x_1 - 1} = \frac{\frac{x_1^2}{4} - 2}{x_1 - 1}$ **1**

or $-\frac{2}{x_1} = \frac{x_1^2 - 8}{4(x_1 - 1)}$

$x_1^3 - 8x_1 = -8x_1 + 8$

or $x_1 = 2$

$x_1 = 2$, and $y_1 = 1$

$\therefore$ Point of contact is (2, 1)

Equation of normal is

$y - 1 = -\frac{2}{2}(x - 2)$ or $x + y = 3$ **1**

[CBSE Marking Scheme 2016]

Q. 13. Find the equation(s) of the tangent(s) to the curve $y = (x^3 - 1)(x - 2)$ points where the curve intersects the x-axis. R&U **[SQP 2017-18]**

Sol. When $y = 0$, we have $(x - 1)(x^2 + x + 1)(x - 2) = 0$, *i.e.*, $x = 1$ or 2. ½

$\frac{dy}{dx} = x^3 - 1 + (x - 2)\,3x^2 = 4x^3 - 6x^2 - 1$ ½

$\left(\frac{dy}{dx}\right)_{(1,0)} = -3$ ½

$\left(\frac{dy}{dx}\right)_{(2,0)} = 7$ ½

The required equations of the tangents are $y - 0 = -3(x - 1)$ or, $y = -3x + 3$ and $y - 0 = 7(x - 2)$ or, $y = 7x - 14$. **2**

[CBSE Marking Scheme 2017]

Q. 14. Find the equation of tangent to the curve $y = \cos(x + y), -2\pi \le x \le 0$, that is parallel to the line $x + 2y = 0$. R&U **[NCERT] [NCERT Exemplar] [Delhi Comptt., 2017, Foreign 2016]**

Sol. Differentiating $y = \cos(x + y)$ wrt x we get

$\frac{dy}{dx} = \frac{-\sin(x+y)}{1+\sin(x+y)}$ **1**

Slope of given line is $\frac{-1}{2}$ ½

As tangent is parallel to line $x + 2y = 0$

$\therefore \frac{-\sin(x+y)}{1+\sin(x+y)} = \frac{-1}{2}$

or $\sin(x + y) = 1$

or $x + y = n\pi + (-1)^n\frac{\pi}{2}, n \in Z$ ...(i) **1**

Putting (1) in $y = \cos(x + y)$

we get $y = 0$

or $x = n\pi + (-1)^n\frac{\pi}{2}, n \in Z$

$x = \frac{-3\pi}{2} \in [-2\pi, 0]$ ½

$\therefore$ Required equation of tangent is

$y = \frac{-1}{2}\left(x + \frac{3\pi}{2}\right)$

or $2y + x + \frac{3\pi}{2} = 0$ **1**

[CBSE Marking Scheme 2017]

Q. 15. Find the equation of tangents to the curve $y = x^3 + 2x - 4$ which are perpendicular to the line $x + 14y - 3 = 0$. R&U **[All India 2016]**

Sol. Given, equation of curve is

$y = x^3 + 2x - 4$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx} = 3x^2 + 2$

$\therefore$ The slope of required tangent is

$m_1 = \frac{dy}{dx} = 3x^2 + 2$ ½

Now, slope of line $x + 14y - 3 = 0$

or $y = -\frac{x}{14} + \frac{3}{14}$

is $m_1 = -\frac{1}{14}$ ½

Since, the required tangent is perpendicular to the line $x + 14y - 3 = 0$.

$\therefore m_1 m_2 = -1$

or $(3x^2 + 2) \times \left(-\frac{1}{14}\right) = -1$

or $3x^2 + 2 = 14$

or $3x^2 = 12$ or $x^2 = 4$

or $x = \pm 2$ **1**

When $x = 2$, then

$y = 2^3 + 2 \times 2 - 4$

$= 8 + 4 - 4 = 8$

When $x = -2$, then

$y = (-2)^3 + 2 \times (-2) - 4$

$= -8 - 4 - 4$

$= -16$

Points of contact are (2, 8) and (– 2, – 16). ½

Now, equation of tangent at point (2, 8) is

$y - 8 = \left(\frac{dy}{dx}\right)_{(2,8)}(x - 2)$

or $y - 8 = (3 \times 2^2 + 2)(x - 2)$

or $y - 8 = 14(x - 2)$

or $y - 8 = 14(x - 2)$

$y = 14x - 20$ ½

and equation of tangent at point (– 2, – 16) is

$y + 16 = \frac{dy}{dx}_{(-2,-16)}(x + 2)$

or $y + 16 = [3(-2)^2 + 2](x + 2)$

or $y + 16 = 14(x + 2)$

$\therefore 14x - y - 12 = 0$ **1**

Q. 16. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which tangent is parallel to x-axis.

R&U **[Delhi 2011] [NCERT]**

Sol. Given, equation of curve is

$$x^2 + y^2 - 2x - 3 = 0 \quad ...(i)$$

On differentiating both side of Eq. (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

or $$2y\frac{dy}{dx} = 2 - 2x$$

or $$\frac{dy}{dx} = \frac{2-2x}{2y} \text{ or } \frac{dy}{dx} = \frac{1-x}{y} \quad \mathbf{1}$$

We know that, when a tangent to the curve is parallel to x-axis, then $\frac{dy}{dx} = 0$. **1**

$$\frac{dy}{dx} = 0$$

$$\frac{1-x}{y} = 0$$

$$1 - x = 0 \Rightarrow x = 1$$

from equ. (i)

$$(1)^2 + y^2 - 2(1) - 3 = 0$$

$$y^2 - 4 = 0$$

$$y = \pm 2$$

Hence points on the curve at which tangents are parallel to x-axis is (1, 2) and (1 – 2) **2**

[CBSE Marking Scheme 2011]

Q. 17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to y-coordinate of the point. R&U **[Foreign 2011] [NCERT]**

Sol. Given, equation of curve is $y = x^3$.

On differentiating both sides *w.r.t.* x, we get

$$\frac{dy}{dx} = 3x^2$$

$\therefore$ Slope of tangent at any point (x, y) is

$$\frac{dy}{dx} = 3x^2 \quad \mathbf{1}$$

Now, given that

Slope of tangent = y-coordinate of the point

or $$\frac{dy}{dx} = y$$

or $$3x^2 = y \quad \left[\because \frac{dy}{dx} = 3x^2\right]$$

or $$3x^2 = x^3 \quad [\because y = x^3]$$

or $$3x^2 - x^3 = 0 \text{ or } x^2(3 - x) = 0$$

or Either $x^2 = 0$ or $3 - x = 0$

$$\therefore \quad x = 0, 3 \quad \mathbf{1}$$

Now, on putting $x = 0$ and 3 in Eq. (i), we get

$$y = (0)^3 = 0 \text{ [at } x = 0]$$

and $$y = (3)^3 = 27 \text{ [at } x = 3] \quad \mathbf{1}$$

Hence, the required points are (0, 0) and (3, 27) **1**

[CBSE Marking Scheme 2011]

Answering Tip

- Learn with the geometrical interpretation of differentiation with ample examples.

Q. 18. Find the equation of tangent to curves

$$x = \sin 3t,\ y = \cos 2t \text{ at } t = \frac{\pi}{4}.$$

R&U **[All India 2011C, 2008]**

Sol. We know that, the equation of tangent at the points (x_1, y_1) is $y - y_1 = m(x - x_1)$ ...(i)

where, $$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \quad \tfrac{1}{2}$$

Now, given $x = \sin 3t$ ...(ii)

$$\therefore \quad \frac{dx}{dt} = 3\cos 3t \quad \text{[differentiate w.r.t. } t]$$

and $y = \cos 2t$ ...(iii)

$$\therefore \quad \frac{dy}{dt} = -2\sin 2t$$

[differentiate w.r.t. t]

Then, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{3\cos 3t} \quad \mathbf{1}$$

On putting $t = \frac{\pi}{4}$, we get

$$m = \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{-2\sin\frac{\pi}{2}}{3\cos\frac{3\pi}{4}} = \frac{-2}{-\frac{3}{\sqrt{2}}}$$

$$\left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\right]$$

or $$m = \frac{2\sqrt{2}}{3} \quad \mathbf{1}$$

Also, to find (x_1, y_1), we put

$$t = \frac{\pi}{4} \text{ in}$$

From Eqs. (ii) and (iii), we get

$$x_1 = \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and $$y_1 = \cos\frac{\pi}{4} = 0$$

$$\therefore \quad (x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right) \quad \tfrac{1}{2}$$

Now, on putting $(x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$ and $m = \frac{2\sqrt{2}}{3}$ in Eq. (i). we get

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

or $$3y = 2\sqrt{2}x - \frac{2}{3}$$

Hence, required equation of tangent is

$$6\sqrt{2x} - 9y - 2 = 0 \quad \textbf{1}$$

Q. 19. Find the equations of tangents to the curve $y = (x^2 - 1)(x - 2)$ at the points, where the curve cuts the X-axis. R&U **[All India 2011C]**

Sol. Given, equation of the curve is

$$y = (x^2 - 1)(x - 2) \quad ...(i)$$

Since, the curve cuts the X-axis, so at that point y-coordinate will be zero.

So, on putting $y = 0$, we get

$$(x^2 - 1)(x - 2) = 0$$

or $x^2 = 1$ or $x = 2$

$\therefore$ $x = \pm 1$ or 2 or $x = -1, 1, 2$

Thus, the given curve cuts the X-axis at points $(-1, 0)$, $(1, 0)$ and $(2, 0)$. **1**

$$\frac{dy}{dx} = (x^2 - 1) . 1 + (x - 2) . 2x$$

[by using product rule of derivative]

or $$\frac{dy}{dx} = x^2 - 1 + 2x^2 - 4x$$

or $$\frac{dy}{dx} = 3x^2 - 4x - 1 \quad \textbf{1}$$

Now, slope of tangent at $(-1, 0)$ is

$$m_1 = \left(\frac{dy}{dx}\right)_{(-1,0)} = 3(-1)^2 - 4(-1) - 1$$
$$= 3 + 4 - 1 = 6$$

Slope of tangent at $(1, 0)$ is

$$m_2 = \left(\frac{dy}{dx}\right)_{(1,0)} = 3(1)^2 - 4(1) - 1$$
$$= 3 - 4 - 1 = -2$$

Slope of tangent at $(2, 0)$ is

$$m_3 = \left(\frac{dy}{dx}\right)_{(2,0)} = 3(2)^2 - 4(2) - 1$$
$$= 12 - 8 - 1 = 3 \quad \textbf{1}$$

We know that, equation of tangent at the point is (x_1, y_1) is given by $y - y_1 = m(x - x_1)$.

Here, we get three equations of tangents.

Equation of tangent at point $(-1, 0)$ having slope $(m_1) = 6$, is

$$y - 0 = 6(x + 1)$$

or $y = 6x + 6$ or $6x - y = -6$

Equation of tangent at point $(1, 0)$ having slope $(m_2) = -2$, is

$$y - 0 = -2(x - 1)$$

or $y = -2x + 2$ or $2x + y = 2$

and equation of tangent at point $(2, 0)$ having slope $(m_3) = 3$, is

$$y - 0 = -3(x - 2) \text{ or } y = 3x = 6$$

$\therefore$ $3x - y = 6$ **1**

[CBSE Marking Scheme 2011]

Q. 20. Find the equation of tangent to the curve $4x^2 + 9y^2 = 36$ at the point $(3\cos\theta, 2\sin\theta)$. R&U **[Delhi 2011C]**

Sol. Given, equation of curve is

$$4x^2 + 9y^2 = 36$$

On differentiating both sides w.r.t. x, we get

$$8x + 18y\frac{dy}{dx} = 0$$

or $$18y\frac{dy}{dx} = -8x$$

or $$\frac{dy}{dx} = -\frac{8x}{18y}$$

or $$\frac{dy}{dx} = \frac{-4x}{9y} \quad ...(i) \textbf{1}$$

But given that, tangent passes through the point $(3\cos\theta, 2\sin\theta)$.

On putting $x = 3\cos\theta$, $y = 2\sin\theta$ in Eq. (i), we get

$$\frac{dy}{dx} = \frac{-12\cos\theta}{18\sin\theta} \text{ or } \frac{dy}{dx} = \frac{-2\cos\theta}{3\sin\theta}$$

$\therefore$ Slope of the tangent, $m = \dfrac{-2\cos\theta}{3\sin\theta}$ **1**

$$\left[\because m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right]$$

Now, equation of tangent at the point $(3\cos\theta, 2\sin\theta)$ having slope

$$m = -\frac{2\cos\theta}{3\sin\theta} \text{ is}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2\sin\theta = -\frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta) \quad \textbf{1}$$

or $3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$

or $2x\cos\theta + 3y\sin\theta - 6(\sin^2\theta + \cos^2\theta) = 0$

$\therefore$ $2x\cos\theta + 3y\sin\theta - 6 = 0$ $[\because \sin^2\theta + \cos^2\theta = 1]$

which is the required equation of tangent. **1**

Q. 21. Find the equation of the tangent to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at point $x = 1, y = 0$. R&U **[Delhi 2011C]**

Sol. Given, equation of curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 - 26x - 10 \quad \textbf{1}$$

Slope of a tangent at point $(1, 0)$ is

$$m = \left[\frac{dy}{dx}\right]_{x=1} = 4 - 18 + 26 - 10 = 2 \quad \textbf{1}$$

$\therefore$ Equation of tangent at point $(1, 0)$ having slope 2 is **1**

$$y - 0 = 2(x - 1)$$

or $y = 2x - 2$

Hence, required equation of tangent is $2x - y = 2$ **1**

Q. 22. Find the equation of tangent to the curve $y = \dfrac{x - 7}{x^2 - 5x + 6}$ at the point, where it cuts the x-axis.

R&U **[All India 2010C, 2010] [NCERT]**

Sol. Given, equation of curve is

$$y = \frac{x-7}{x^2-5x+6}. \qquad ...(i)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2-5x+6).1-(x-7)(2x-5)}{(x^2-5x+6)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\right]$$

or $$\frac{dy}{dx} = \frac{\left[(x^2-5x+6)-y(x^2-5x+6)(2x-5)\right]}{(x^2-5x+6^2)}$$

$$\left[\because \quad y = \frac{x-7}{x^2-5x+6} \quad \therefore \quad (x-7) = y(x^2-5x+6)\right]$$

or $$\frac{dy}{dx} = \frac{1-(2x-5)y}{x^2-5x+6} \qquad ...(ii)$$ **1**

[dividing numerator and denominator by $x^2 - 5x + 6$]

Also, given that curve cuts x-axis, so its y-coordinate is zero.

Put $y = 0$ in Eq. (i) we get

$$\frac{x-7}{x^2-5x+6} = 0$$

or $x = 7$ **1**

So, curve passes through the point (7, 0).

Now, slope of tangent at (7, 0) is

$$m = \left(\frac{dy}{dx}\right)_{(7,0)} = \frac{1-0}{49-35+6} = \frac{1}{20}$$ **1**

Hence, the required equation of tangent passing through the point (7, 0) having slope 1/20 is

$$y - 0 = \frac{1}{20}(x-7)$$

or $20y = x - 7$

$\therefore$ $x - 20y = 7$ **1**

AI Q. 23. Find the equations of the normal to the curve $y = x^3 + 2x + 6$, which are parallel to line $x + 14y + 4 = 0$. R&U **[Delhi 2010] [NCERT]**

Sol. Given, equation of curve is

$$y = x^3 + 2x + 6 \qquad ...(i)$$

and the given equation of line is

$$x + 14y + 4 = 0$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 + 2$$

$$\therefore \text{ Slope of normal } = \frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-1}{3x^2+2}$$

Also, slope of the line $x + 14y + 4 = 0$ is $-\frac{1}{14}$. **1**

$\left[\because \text{slope of the line } Ax+By+C=0 \text{ is } -\frac{A}{B}\right]$

We know that, if two lines are parallel, then their slopes are equal.

$$\therefore \quad -\frac{1}{3x^2+2} = -\frac{1}{14}$$

or $3x^2 + 2 = 14$

or $3x^2 = 12$ or $x^2 = 4$

or $x = \pm 2$ **1**

When, $x = 2$, then from Eq. (i)

$$y = (2)^3 + 2(2) + 6 = 8 + 4 + 6 = 18$$

and, when $x = -2$, then from Eq. (i),

$y = (-2)^3 + 2(-2) + 6 = -8 - 4 + 6 = -6$

$\therefore$ Normal passes through (2, 18) and (–2, –6).

Also, slope of normal = $\frac{-1}{14}$.

Hence, equation of normal at point (2, 18) is

$$y - 18 = \frac{-1}{14}(x-2)$$

or $14y - 252 = -x + 2$

or $x + 14y = 254$ **½**

and equation of normal at point (– 2, – 6) is

$$y + 6 = -\frac{1}{14}(x+2)$$

or $14y + 84 = -x - 2$

or $x + 14y = -86$ **½**

Hence, the two equations of normal are

$x + 14y = 254$ and $x + 14y = -86$ **1**

Q. 24. Find the equation of tangent to the curve $x^2 + 3y = 3$, which is parallel to line $y - 4x + 5 = 0$. R&U **[Delhi 2009C]**

Sol. Given, equation of curve is

$$x^2 + 3y = 3 \qquad ...(i)$$ **½**

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x + 3\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{2x}{3}$$

$$\therefore \text{ Slope of tangent } (m) = -\frac{2x}{3}$$ **½**

Given equation of the line is

$y - 4x + 5 = 0$ or $y = 4x - 5$

which is of the from $y = mx + c$.

$\therefore$ Slope of line = 4

$\therefore$ Slope of tangent = Slope of line

or $-\frac{2x}{3} = 4$ or $-2x = 12$

or $x = -6$ **1**

On putting $x = -6$ in Eq. (i), we get

$(-6)^2 + 3y = 3$ or $3y = 3 - 36$

or $3y = -33$ or $y = -11$ **1**

So, the tangent is passing through point (– 6, – 11) and it has slope 4.

Hence, the required equation of tangent is

$y + 11 = 4(x + 6)$

or $y + 11 = 4x + 24$

or $4x - y = -13$ **1**

Q. 25. Find the angle of intersection of the curves $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$, at the point in the first quadrant. R&U **[Delhi Comptt. Set I, II, III, 2018]**

Sol. Point of intersection = $\left(1, \sqrt{3}\right)$ **1**

$$x^2 + y^2 = 4 \Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\left.\frac{dy}{dx}\right|_{\left[1,\sqrt{3}\right]} = -\frac{1}{\sqrt{3}}$$ ½ + ½

$$(x-2)^2 + y^2 = 4 \Rightarrow 2(x-2) + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(x-2)}{y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right]_{\left[1,\sqrt{3}\right]} = \frac{1}{\sqrt{3}}$$ ½ + ½

So, $\tan\phi = \dfrac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 - 1/3} = \sqrt{3} \Rightarrow \phi = \dfrac{\pi}{3}$ **1**

[CBSE Marking Scheme 2018]

Long Answer Type Questions-II

(6 marks each)

Q. 1. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is
(i) parallel to the line $2x - y + 9 = 0$
(ii) perpendicular to the line $5y - 15x = 13$.
R&U **[NCERT] [Delhi Set I Comptt. 2014]**

Sol. Slope of tangent = $\dfrac{dy}{dx} = 2x - 2$ **1**

(i) Tangent parallel to $2x - y + 9 = 0$,

Slope of line = m_1(say)

$\therefore \quad m_1 = 2$

$\because$ They are parallel

$\therefore \quad \dfrac{dy}{dx} = m_1$

$\therefore \quad 2x - 2 = 2, \qquad \left(\text{As, } \dfrac{dy}{dx} = 2\right)$

$x = 2, y = 7$ **1½**

Equation of tangent through (2, 7) and parallel to the given line is $y - 7 = 2(x - 2)$ or $y = 2x + 3$ **1**

(ii) Tangent perpendicular to $5y - 15x = 13$

Slope of line = m_2(say)

$m_2 = 3$

$\because$ They are perpendicular

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{m_2}$

$\therefore \quad (2x - 2)\cdot 3 = -1$

$\therefore \quad x = \dfrac{5}{6}, y = \dfrac{217}{36}$ **1½**

Equation of tangent through $\left(\dfrac{5}{6}, \dfrac{217}{36}\right)$ and perpendicular to the line is

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right)$$

or $$y = \frac{-x}{3} + \frac{227}{36}$$

or $12x + 36y = 227$ **1**

Q. 2. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. R&U **[Delhi Set II Comptt. 2013] [NCERT] [Delhi 2009]**

Sol. Given $y = \sqrt{3x-2}$

Diff. w.r.t. 'x',

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$$ ½

Since the tangent is parallel to the line

$4x - 2y + 5 = 0$, slope $m = 2$ ½

$\therefore \quad \dfrac{dy}{dx} = 2$

$\therefore \quad \dfrac{3}{2\sqrt{3x-2}} = 2$ ½

or $3 = 4\sqrt{3x-2}$

or $\sqrt{3x-2} = \dfrac{3}{4}$ ½

On squaring both sides, we get

or $3x - 2 = \dfrac{9}{16}$

or $3x = \dfrac{9}{16} + 2$ ½

or $3x = \dfrac{41}{16}$

or $x = \dfrac{41}{48}$ ½

When $x = \dfrac{41}{48}, y = \dfrac{3}{4}$

$\therefore$ The point is $\left(\dfrac{41}{48}, \dfrac{3}{4}\right)$ ½

The eqn. of tangent is

or $y - \dfrac{3}{4} = 2\left(x - \dfrac{41}{48}\right)$ ½

or $\dfrac{4y-3}{4} = \dfrac{2(48x-41)}{48}$ ½ or $24y - 18 = 48x - 41$ ½

or $48x - 24y = 23$ ½

$48x - 24y - 23 = 0$ ½

Q. 3. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. R&U **[NCERT]**

[Delhi Set III Comptt. 2013]

Sol. $x = y^2$ ...(i)

and $xy = k$ ...(ii)

From (ii) $y^2y = k$

or $y^3 = k$

or $y = k^{\frac{1}{3}}$ ½

or $x = k^{\frac{2}{3}}$ ½

The point of intersection is $(k^{\frac{2}{3}}, k^{\frac{1}{3}})$.

Diff. (i) & (ii), we get

From (i) or $1 = 2y\dfrac{dy}{dx}$ ...(iii)

From (ii) or $x\dfrac{dy}{dx} + y = 0$ ...(iv) **1**

From (iii), we have,

$$\left(\frac{dy}{dx}\right)_{(k^{\frac{2}{3}},\, k^{\frac{1}{3}})} = \frac{1}{2k^{\frac{1}{3}}} = m_1 \quad 1$$

From (iv), we have

$$\left(\frac{dy}{dx}\right)_{(k^{\frac{2}{3}},\, k^{\frac{1}{3}})} = -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = m_2 \quad \mathbf{1}$$

$$m_1m_2 = \frac{1}{2k^{\frac{1}{3}}} \times \frac{-k^{\frac{1}{3}}}{k^{\frac{2}{3}}}$$

$$= -\frac{k^{-\frac{2}{3}}}{2} \quad \mathbf{1}$$

Since $8k^2 = 1$

or $k^2 = \dfrac{1}{8}$

or $\therefore m_1m_2 = -\dfrac{(8)^{\frac{1}{3}}}{2}$

$$= -\frac{2}{2} = -1 \quad \mathbf{1}$$

$\therefore$ (i) & (ii) cut at right angles.

[CBSE Marking Scheme 2013]

Q. 4. Find the equation of the normal at a point on the curve $x^2 = 4y$ which passes through the point (1, 2). Also find the equation of the corresponding tangent. R&U **[Delhi Set I, II, III, 2013]**

Sol. Let the point at which the normal on the curve $x^2 = 4y$ passes through the point (1, 2) be $P(x_1, y_1)$

Now $x^2 = 4y$ ...(i)

Since $P(x_1, y_1)$ lies on (i), so

$x_1^2 = 4y_1$ ...(ii)

Differentiate (i) w.r.t. to x both sides,

$$2x_1 = 4\frac{dy_1}{dx_1}$$

or $\dfrac{dy_1}{dx_1} = \dfrac{2x_1}{4} = \dfrac{x_1}{2}$ **1**

$\therefore$ Slope of normal, $m_N = -\dfrac{2}{x_1}$ **1**

Now, equation of normal :

$y - y_1 = m_N(x - x_1)$

Since the normal passes through (1, 2)

So, $2 - y_1 = \left(-\dfrac{2}{x_1}\right)(1 - x_1)$ **1**

Using eqn. (ii),

$$\left(2 - \frac{x_1^2}{4}\right) = \left(2 - \frac{2}{x_1}\right) x_1^3 = 8$$

$x_1 = 2, y_1 = 1$

So the point is $P(2, 1)$. **1**

Hence, equation of normal is :

$$y - 1 = \left(-\frac{2}{2}\right)(x - 2)$$

or $y - 1 = -x + 2$

or $x + y = 3$ **1**

and the equation of tangent is :

$$y - 1 = \frac{2}{2}(x - 2)$$

or $y - 1 = x - 2$

or $x - y = 1$. **1**

Q. 5. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3}, 0\right)$.

[O.D. Set I, II, III, 2013]

Sol. $3x^2 - y^2 = 8$ ...(i)

Diff. w.r.t. 'x'

$$6x - 2y\frac{dy}{dx} = 0$$

or $\dfrac{dy}{dx} = \dfrac{3x}{y}$

$$\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{3x_1}{y_1} \quad \mathbf{1}$$

$\therefore$ The eq. of tangent at (x_1, y_1) is

$$y - y_1 = \frac{3x_1}{y_1}(x - x_1) \quad ½$$

Tangent passes through the point $\left(\dfrac{4}{3}, 0\right)$

$\therefore$ $0 - y_1 = \dfrac{3x_1}{y_1}\left(\dfrac{4}{3} - x_1\right)$ ½

or $-y_1^2 = 3x_1\left(\dfrac{4}{3} - x_1\right)$

or $-y_1^2 = 4x_1 - 3x_1^2$

Using eqn. (i), $8 - 3x_1^2 = 4x_1 - 3x_1^2$

or $x_1 = 2$ ½

$$\therefore \quad 3(2)^2 - y_1^2 = 8$$

$$\text{or} \quad 12 - y_1^2 = 8$$

$$\text{or} \quad y_1^2 = 4$$

$$\text{or} \quad y_1 = \pm 2 \qquad \tfrac{1}{2}$$

∴ The points are (2, 2) and (2, – 2). **1**

∴ The eq. of tangent at (2, 2) is :

$$y - 2 = 3(x - 2)$$

$$\text{or} \quad y = 3x - 4 \qquad \mathbf{1}$$

The eq. of tangent at (2, – 2) is :

$$y + 2 = -3(x - 2)$$

$$\text{or} \quad y = -3x + 4 \qquad \mathbf{1}$$

$$3x + y = 4$$

AI Q. 6. Show that the normal at any point θ to the curve $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.

R&U [O.D. Comptt., 2013, O.D. Comptt., 2017] [NCERT] [Delhi Set I, II, III Comptt. 2013]

Sol.

$$x = a\cos\theta + a\theta\sin\theta$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta$$

$$= a\theta\cos\theta \qquad \mathbf{1}$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta \qquad \mathbf{1}$$

$$= a\theta\sin\theta$$

$$\text{Or} \quad \frac{dy}{dx} = \frac{a\,\theta\sin\theta}{a\,\theta\cos\theta} = \tan\theta \qquad \tfrac{1}{2}$$

Equation of tangent is :

$$y - (a\sin\theta - a\,\theta\cos\theta) = \tan\theta\,(x - a\cos\theta - a\theta\sin\theta) \qquad \mathbf{1}$$

Equation of normal is

$$y - (a\sin\theta - a\,\theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta) \qquad \mathbf{1}$$

or $y\sin\theta + x\cos\theta = a$

distance of normal from origin ½

$$= \frac{|-a|}{\sqrt{\sin^2\theta + \cos^2\theta}}$$

$$= |a| \text{ (constant)} \qquad \mathbf{1}$$

[CBSE Marking Scheme 2017]

Alternative Method :

$$\frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta \qquad \mathbf{1}$$

$$= a\theta\cos\theta \qquad \mathbf{1}$$

$$\text{and} \quad \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta$$

$$= a\theta\sin\theta \qquad \mathbf{1}$$

$$\text{Or} \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \tan\theta$$

$$\therefore \text{ Slope of normal} = -\frac{1}{\tan\theta} = -\cot\theta$$

∴ Equation of normal is :

$$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}[x - a(\cos\theta + \theta\sin\theta)] \qquad \mathbf{1}$$

Simplifying it to get $x\cos\theta + y\sin\theta - a = 0$ **1**

Length of perpendicular from origin

$$= \frac{|a|}{\sqrt{\sin^2\theta + \cos^2\theta}}$$

$$= |a| \text{ (constant)} \qquad \mathbf{1}$$

Q. 7. Find the angle of intersection of the curves $y^2 = 4ax$ and $x^2 = 4by$. A [NCERT Exemplar] [Foreign 2016]

Sol. Given, equations of curves are

$$y^2 = 4ax \qquad \text{...(i)}$$

$$\text{and} \quad x^2 = 4by \qquad \text{...(ii)}$$

Clearly, the angle of intersection of curves (i) and (ii) is the angle between the tangents to the curves at the point of intersection. ½

So, let us first find the intersection point of given curves.

On substituting the value of y from Eq. (ii) in Eq. (i), we get

$$\left(\frac{x^2}{4b}\right)^2 = 4ax$$

$$\text{or} \quad \frac{x^4}{16b^2} = 4ax \text{ or } x^4 = 64ab^2x$$

$$\text{or} \quad x^4 - 64ab^2x = 0$$

$$\text{or} \quad x(x^3 - 64ab^2) = 0$$

$$\text{or} \quad x = 0 \text{ or } x = 4a^{1/3}b^{2/3}$$

Clearly, when $x = 0$, then from Eq. (i), $y = 0$ and when $x = 4a^{1/3}b^{2/3}$, then from Eq.(i),

$$\text{or} \quad y^2 = 16a^{4/3}b^{2/3} \text{ or } y = 4a^{2/3}b^{1/3}$$

Thus the points of intersection are (0, 0) and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$. **1**

Now, let us find the angle of intersection at (0, 0) and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$. Let m_1 be the slope of tangent to the curve (i) and m_2 be the slope of tangent of the curve (ii). ½

Angle of intersection at (0, 0)

$$\text{Now,} \quad m_1 = \left(\frac{dy}{dx}\right)_{\text{at }(0,0)} = \left(\frac{2a}{y}\right)_{\text{at }(0,0)} = \infty$$

$$\text{and} \quad m_2 = \left(\frac{dy}{dx}\right)_{\text{at }(0,0)} = \left(\frac{x}{2b}\right)_{\text{at }(0,0)} = 0 \qquad \mathbf{1}$$

or Tangent to the curve (i) is parallel to Y-axis and tangent to the curve (ii) is parallel to X-axis.

$\therefore$ Angle between these two curves is $\frac{\pi}{2}$.

or The angle of intersection of the curves is $\frac{\pi}{2}$. **1**

Angle of intersection at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

Here, $m_1 = \frac{2a}{4a^{2/3}b^{1/3}} = \frac{1}{2}.\frac{a^{1/3}}{b^{1/3}} = \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}$

and $m_2 = \frac{4a^{1/3}b^{2/3}}{2b} = 2\left(\frac{a}{b}\right)^{1/3}$ ½

Let θ be the angle between the tangents. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$ ½

$$= \left|\frac{2\left(\frac{a}{b}\right)^{\frac{1}{3}} - \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}}{1 + 2\left(\frac{a}{b}\right)^{\frac{1}{3}}\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}}\right|$$

$$= \left|\frac{\frac{3}{2}\left(\frac{a}{b}\right)^{1/3}}{1 + \left(\frac{a}{b}\right)^{2/3}}\right| = \left|\frac{\frac{3}{2}\left(\frac{a}{b}\right)^{1/3}.b^{2/3}}{b^{2/3} + a^{2/3}}\right|$$

$$= \left|\frac{3(ab)^{1/3}}{2(a^{2/3} + b^{2/3})}\right|$$

or $\theta = \tan^{-1}\left\{\frac{3(ab)^{1/3}}{2(a^{2/3} + b^{2/3})}\right\}$

Hence, the angles of intersection of the curves are $\frac{\pi}{2}$ and $\tan^{-1}\left\{\frac{3(ab)^{1/3}}{2(a^{2/3} + b^{2/3})}\right\}$. **1**

Q. 8. Find the value of p for which the curves $x^2 = 9p(9 - y)$ and $x^2 = p(y + 1)$ cut each other at right angles.

A [All India 2015]

Sol. Given, equations of curves are

$x^2 = 9p(9 - y)$ ...(i)

and $x^2 = p(y + 1)$ ...(ii)

As, these curves cut each other at right angle, therefore their tangent at point of intersection are perpendicular to each other.

So, let us first find the point of intersection and slope of tangents to the curves. **1**

From Eqs. (i) and (ii), we get

$9p(9 - y) = p\,(y + 1)$

$9\,(9 - y) = y + 1$

[$\because$ $p \neq 0$, as if $p = 0$, them curves becomes straight, which will be parallel]

or $81 - 9y = y + 1$ or $80 = 10y$ or $y = 8$ **1**

On substituting the value of y in Eq. (i), we get

$x^2 = 9p$ or $x = \pm 3\sqrt{p}$

Thus, the point of intersection are $\left(3\sqrt{p}, 8\right)$ and $\left(-3\sqrt{p}, 8\right)$.

Now, consider Eq. (i), we get

$\frac{x^2}{9p} = 9 - y$ or $y = 9 - \frac{x^2}{9p}$ **1**

On differentiating both sides *w.r.t.* x, we get

$\frac{dy}{dx} = \frac{-2x}{9p}$ ...(iii)

From Eq. (ii), we get $\frac{x^2}{p} = y + 1$

or $y = \frac{x^2}{p} - 1$

On differentiating both sides w.r.t x, we get

$\frac{dy}{dx} = \frac{2x}{p}$ ...(iv) **1**

Now, for intersection point $\left(3\sqrt{p}, 8\right)$, we have slope of tangent to the first curve

$= \frac{-2(3\sqrt{p})}{9p} = \frac{-6\sqrt{p}}{9p}$

[using Eq. (iii)]

and slope of tangent to the second curve

$= \frac{2(3\sqrt{p})}{p} = \frac{6\sqrt{p}}{p}$ [using Eq. (iv)]

$\because$ Tangents are perpendicular to each other.

Then,

Slope of first curve × Slope of second curve = – 1

$\therefore \frac{-6\sqrt{p}}{9p} \times \frac{6\sqrt{p}}{p} = -1$ or $\frac{4}{p} = 1$ or $p = 4$ **1**

Hence, the value of p is 4. **1**

Q. 9. Find the equation of tangent and normal to the curve $x = 1 - \cos\theta$, $y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

R&U [All India 2010]

Sol. Given curves are $x = 1 - \cos\theta$ and $y = \theta - \sin\theta$.

On differentiating both sides w.r.t. θ, we get

$\frac{dx}{d\theta} = \frac{d}{d\theta}(1 - \cos\theta) = \sin\theta$

and $\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta - \sin\theta) = 1 - \cos\theta$

$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1 - \cos\theta}{\sin\theta}$

At $\theta = \frac{\pi}{4}, \left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{4}} = \frac{1 - \cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1 - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

$= \sqrt{2} - 1$ **1**

Also, at $\theta = \frac{\pi}{4}$, $x_1 = 1 - \cos\frac{\pi}{4}$

$$= 1-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-1}{\sqrt{2}}$$

and $$y_1 = \frac{\pi}{4}-\sin\frac{\pi}{4}=\frac{\pi}{4}-\frac{1}{\sqrt{2}}$$ **1**

We know that, equation of tangent at (x_1, y_1) having slope m, is given by

$$y-y_1 = m(x-x_1)$$

$$\therefore \quad y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right) = (\sqrt{2}-1)\left[x-\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\right]$$

or $$y-\frac{\pi}{4}+\frac{1}{\sqrt{2}} = x(\sqrt{2}-1)-\frac{(\sqrt{2}-1)^2}{\sqrt{2}}$$ **1**

or $$y-\frac{\pi}{4}+\frac{1}{\sqrt{2}} = x(\sqrt{2}-1)-\frac{(2+1-2\sqrt{2})}{\sqrt{2}}$$

$[\because (a-b)^2 = a^2+b^2-2ab]$

or $$\left(y-\frac{\pi}{4}+\frac{1}{\sqrt{2}}\right) = x(\sqrt{2}-1)-\frac{(3-2\sqrt{2})}{\sqrt{2}}$$

or $$x(\sqrt{2}-1)-y = \frac{3-2\sqrt{2}}{\sqrt{2}}-\frac{\pi}{4}+\frac{1}{\sqrt{2}}$$

Hence, the equation of tangent is

$$x(\sqrt{2}-1)-y = \frac{12-8\sqrt{2}-\sqrt{2}\pi+4}{4\sqrt{2}}$$

or $$x(8-4\sqrt{2})-4\sqrt{2}y = (16-\sqrt{2}\pi-8\sqrt{2})$$ **1**

Also, the equation of normal at (x_1, y_1) having slope $-\frac{1}{m}$ is given by

$$y-y_1 = -\frac{1}{m}(x-x_1)$$

$$y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}-1}\left(x-\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

or $$y(\sqrt{2}-1)-\left(\frac{\sqrt{2}\pi-4}{4\sqrt{2}}\right)(\sqrt{2}-1) = -x+\frac{\sqrt{2}-1}{\sqrt{2}}$$ **1**

or $$y(\sqrt{2}-1)-\left(\frac{2\pi-\sqrt{2}\pi-4\sqrt{2}+4}{4\sqrt{2}}\right) = \frac{-\sqrt{2}x+\sqrt{2}-1}{\sqrt{2}}$$

or $$4\sqrt{2}y\,(\sqrt{2}-1)-2\pi+\sqrt{2}\pi+4\sqrt{2}-4 = -4\sqrt{2}x+4\sqrt{2}-4$$

or $$4\sqrt{2}x+4\sqrt{2}y\,(\sqrt{2}-1) = 2\pi-\sqrt{2}\pi$$

or $$4\sqrt{2}x+y(8-4\sqrt{2}) = 2\pi-\sqrt{2}\pi$$

$$\therefore \quad 4\sqrt{2}x+(8-4\sqrt{2})y = \pi(2-\sqrt{2})$$ **1**

TOPIC-3
Approximate Values, Differentials & Errors

Revision Notes

1. Approximate change in the value of functions $y = f(x)$:

Given $$y = f(x)$$

From the definition of derivatives, $$\underset{\delta x\to 0}{Lt}\frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$\therefore$ by definition of limit as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$

$\therefore$ δx is very near to zero, then we have $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ (approximately).

Therefore, $\delta y = \frac{dy}{dx}\cdot\delta x$, where δy represents the approximate change in y.

In case $dx = \delta x$ is relatively small when compared with x, dy is a good approximation of δy and we denote it by $dy \approx \delta y$.

2. Approximate value :

By the definition of derivatives (first principle),

$$f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$\therefore$ by the definition of limit when $h \to 0$,

We have $$\frac{f(x+h)-f(x)}{h} \to f'(x).$$

$\therefore$ if h is very near to zero, then we have

$$\frac{f(x+h)-f(x)}{h} = f'(x) \text{ (approximately)}$$

or $$f(x+h) = f(x) + hf'(x) \text{ (approximately as } h \to 0)$$

Objective Type Questions **(1 mark each)**

Q. 1. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y

(a) 0.32 **(b) 0.032**

(c) 5.68 **(d) 5.968**

[NCERT Exemp. Ex. 6.3, Q. 40, Page 139]

Ans. Correct option : (a)

Explanation : Given that,

$$y = x^4 - 10$$

On differentiating with respect to x, we get

$$\Rightarrow \frac{dy}{dx} = 4x^3$$

and $\Delta x = 2.00 - 1.99 = 0.01$

$$\therefore \quad \Delta y = \frac{dy}{dx} \times \Delta x$$

$$= 4x^3 \times \Delta x$$

$$= 4 \times 2^3 \times 0.01$$

$$= 32 \times 0.01$$

$$= 0.32$$

So, the approximate change in y is 0.32.

Q. 2. If $f(x) = 3x^2 + 15x + 5$, **then the approximate value of $f(3.02)$ is**

(a) 47.66 **(b) 57.66**

(c) 67.66 **(d) 77.66**

[NCERT Ex. 6.4, Q. 8, Page 216]

Ans. Correct option : (d)

Explanation : Let $x = 3$ and $\Delta x = 0.02$. Then, we have

$$f(3.02) = f(x+\Delta x) = 3(x+\Delta x)^2 + 15(x+\Delta x) + 5$$

Now, $\Delta y = f(x+\Delta x) - f(x)$

$\Rightarrow f(x+\Delta x) = f(x) + \Delta y$

$\approx f(x) + f'(x)\Delta x$ (As $dx = \Delta x$)

$\Rightarrow f(3.02) \approx (3x^2 + 15x + 5) + (6x+15)\Delta x$

$= [3(3)^2 + 15(3) + 5] + [6(3) + 15](0.02)$

[As $x = 3,\ \Delta x = 0.02$]

$= (27 + 45 + 5) + (18 + 15)(0.02)$

$= 77 + (33)(0.02)$

$= 77 + 0.66$

$= 77.66$

Therefore, the approximate value of f(3.02) is 77.66.

Q. 3. The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is

(a) 0.06 x^3 m^3 **(b) 0.6 x^3 m^3**

(c) 0.09 x^3 m^3 **(d) 0.9 x^3 m^3**

[NCERT Ex. 6.4, Q. 9, Page 216]

Ans. Correct option : (c)

Explanation : The volume of a cube (V) of side x is given by $V = x^3$.

$$\therefore dV = \left(\frac{dV}{dx}\right)\Delta x$$

$$= (3x^2)\Delta x$$

$= (3x^2)(0.03x)$ [As 3% of x is $0.03x$]

$$= 0.09x^3 \text{ m}^3$$

Hence, the approximate change in the volume of the cube is $0.09x^3\ m^3$.

Very Short Answer Type Question **(1 mark each)**

Q. 1. Using derivative, find the approximate percentage increase in the area of a circle if its radius is increased by 2%. A **[S.Q.P. 2015-16]**

Sol. The area of a circle having radius (r) is given by :

$$A = \pi r^2$$

$\therefore \quad dA = \frac{dA}{dr}\cdot(\Delta r)$ ½

$$= \pi\cdot(2r)\cdot\left(\frac{2}{100}r\right)$$

$$= \pi(2r)(0\cdot02\ r) = 0\cdot04\ \pi r^2$$

$\therefore$ Approximate percentage increase in area

$\frac{dA}{A} \times 100\% = \lim_{h\to 0}\frac{h^2 - 5h}{-h} = 4\%$ ½

[CBSE Marking Scheme 2015]

Short Answer Type Questions **(2 marks each)**

Q. 1. If x changes from 4 to 4·01, then find the approximate change in log x.

R&U **[S.Q.P. 2016-17]**

Sol. Let $y = \log x,\ x = 4,\ \delta x = 4.01 - 4 = 0\cdot01$

$\frac{\delta y}{\delta x} = \left(\frac{dy}{dx}\right)_{x=4}$ ½

$$\delta y = \left(\frac{dy}{dx}\right)_{x=4} \times \delta x \qquad ½$$

$$\delta y = \left[\frac{d}{dx}(\log x)\right]_{x=4} \times \delta x \qquad ½$$

$$\delta y = \left(\frac{1}{x}\right)_{x=4} \times \delta_x$$

$$\delta y = \frac{1}{4} \times 0.01 = \frac{1}{400}$$

$$= \frac{1}{400} = 0{\cdot}0025 \qquad 1$$

[CBSE Marking Scheme 2016]

Long Answer Type Questions-I

(4 marks each)

Q. 1. Using differentials, find the approximate value of $(3{\cdot}968)^{3/2}$. R&U **[Delhi Set I Comptt. 2014] [NCERT]**

Sol. Let $y = f(x) = x^{3/2}$, $x = 4$,

$x + \Delta x = 3{\cdot}968$

$\therefore \quad \Delta x = -0{\cdot}032$ **2**

$$\Delta y = \left[\frac{dy}{dx}\right]_{x=4} \times \Delta x$$

or $$\Delta y = \left[\frac{3}{2}x^{1/2}\right]_{x=4} \times \Delta x$$

or $$\Delta y = \frac{3}{2} \times 2(-0{\cdot}032) = -0{\cdot}096 \qquad 1$$

$(3{\cdot}968)^{3/2} = f(x + \Delta x)$

$= f(x) + \Delta y = 8 - 0{\cdot}096$

$= 7{\cdot}904$ **1**

[CBSE Marking Scheme 2014]

Q. 2. Find the approximate value of $f(3{\cdot}02)$, up to 2 places of decimal, where $f(x) = 3x^2 + 5x + 3$.

R&U **[NCERT] [Foreign Set I, II, III, 2014]**

Sol. Given $f(x) = 3x^2 + 5x + 3$

$f(3) = 3(3)^2 + 5(3) + 3 = 45$

or $f'(x) = 6x + 5$,

Let $x = 3, \Delta x = 0{\cdot}02$ **1**

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$f'(3) = 6 \times 3 + 5 = 23$

or $f(x + \Delta x) = (\Delta x)f'(x) + f(x)$ **1½**

$\therefore \quad f(3{\cdot}02) = (0{\cdot}02)f'(3) + f(3)$

$= (0{\cdot}02)(23) + 45$

$= 45{\cdot}46$ **1½**

[CBSE Marking Scheme 2014]

Q. 3. Using differentials, find the approximate value of $\sqrt{49{\cdot}5}$. R&U **[Delhi Set I, II, III, 2012] [NCERT]**

Sol. Let $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$\therefore \quad y + \Delta y = \sqrt{x + \Delta x}$ **½**

or $$y + \frac{dy}{dx}\cdot\Delta x = \sqrt{x+\Delta x} \qquad ½$$

or $$\sqrt{x} + \frac{1}{2\sqrt{x}}\cdot\Delta x = \sqrt{x+\Delta x} \qquad ½$$

Putting $x = 49$ and $\Delta x = 0{\cdot}5$, we get **1**

$$\sqrt{49} + \frac{1}{2\sqrt{49}}\cdot(0{\cdot}5) = \sqrt{49.5} \qquad ½$$

or $$\sqrt{49.5} = 7 + \frac{1}{28} = 7{\cdot}0357 \qquad 1$$

[CBSE Marking Scheme 2012]

Commonly Made Error

- Many candidates do not have an idea of approximation concept as application of differentiation and a few candidates make mistakes in writing the final answer in the correct form.

Answering Tips

- Learn the concept of approximation as one of the applications of differentiation and give enough practice.

Q. 4. If the radius of sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

R&U **[All India 2011] [NCERT]**

Sol. Let S be the surface area, r be the radius of the sphere.

Given, $r = 9$ cm

Then, dr = Approximate error in radius r

and dS = Approximate error in surface area **1**

Now, we know that surface area of sphere is given by

$$S = 4\pi r^2$$

On differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r \qquad 1$$

or $dS = 8\pi r \times dr$

or $dS = 8\pi \times 9 \times 0.03$

$[\because r = 9 \text{ cm and } dr = 0.03 \text{ cm}]$ **1**

or $dS = 72 \times 0.03\pi$

or $dS = 2.16\pi \text{ cm}^2/\text{cm}$

Hence, approximate error in surface area is 2.16π cm^2/cm. **1**

TOPIC-4
Increasing/Decreasing Functions

Revision Notes

1. A function $f(x)$ is said to be an increasing function in $[a, b]$, if as x increases, $f(x)$ also increases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta$, $f(\alpha) > f(\beta)$.

   If $f'(x) \geq 0$ lies in (a, b), then $f(x)$ is an increasing function in $[a, b]$, provided $f(x)$ is continuous at $x = a$ and $x = b$.

2. A function $f(x)$ is said to be a **decreasing function** in $[a, b]$, if, as x increases, $f(x)$ decreases *i.e.*, if $\alpha, \beta \in [a, b]$ and $\alpha > \beta \Rightarrow f(\alpha) < f(\beta)$.

   If $f'(x) \leq 0$ lies in (a, b), then $f(x)$ is a decreasing function in $[a, b]$ provided $f(x)$ is continuous at $x = a$ and $x = b$.

➲ A function $f(x)$ is a **constant function** in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

➲ By **monotonic function** $f(x)$ in interval I, we mean that f is either **only increasing** in I or **only decreasing** in I.

3. Finding the intervals of increasing and/or decreasing of a function :

ALGORITHM

STEP 1 : Consider the function $y = f(x)$.

STEP 2 : Find $f'(x)$.

STEP 3 : Put $f'(x) = 0$ and solve to get the critical point(s).

STEP 4 : The value(s) of x for which $f'(x) > 0$, $f(x)$ is increasing; and the value(s) of x for which $f'(x) < 0$, $f(x)$ is decreasing.

Objective Type Questions **(1 mark each)**

Q. 1. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is :

(a) $[-1, \infty)$ **(b)** $[-2, -1]$

(c) $(-\infty, -2]$ **(d)** $[-1, 1]$

[NCERT Exemp.]

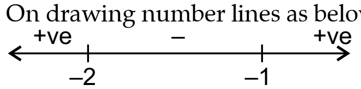
Ans. Correct option : (b)

Explanation : Given that,

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$
$$f'(x) = 6x^2 + 18x + 12$$
$$= 6(x^2 + 3x + 2)$$
$$= 6(x + 2)(x + 1)$$

So, $f'(x) \leq 0$, for decreasing.

On drawing number lines as below :



We see that $f'(x)$ is decreasing in $[-2, -1]$.

Q. 2. $y = x(x - 3)^2$ decreases for the values of x given by :

(a) $1 < x < 3$ **(b)** $x < 0$

(c) $x > 0$ **(d)** $0 < x < \frac{3}{2}$

[NCERT Exemp.]

Ans. Correct option : (a)

Explanation : Given that,

$$y = x(x-3)^2$$
$$\therefore \frac{dy}{dx} = x.2(x-3).1 + (x-3)^2.1$$
$$= 2x^2 - 6x + x^2 + 9 - 6x$$
$$= 3x^2 - 12x + 9$$
$$= 3(x^2 - 3x - x + 3)$$
$$= 3(x - 3)(x - 1)$$

So, $y = x(x - 3)^2$ decreases for (1, 3).

[Since, $y' < 0$ for all $x \in (1, 3)$, hence y is decreasing on (1, 3)].

Q. 3. The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(a) increasing in $\left(P, \frac{3P}{2}\right)$

(b) decreasing in $\left(\frac{P}{2}, P\right)$

(c) decreasing in $\left(\frac{-P}{2}, \frac{P}{2}\right)$

(d) decreasing in $\left(0, \frac{P}{2}\right)$

[NCERT Exemp.]

Ans. Correct option : (b)

Explanation : Given that,

$$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$$

On differentiating with respect to x, we get

$f'(x) = 12\sin^2 x.\cos x - 12\sin x.\cos x + 12\cos x$

$12[\sin^2 x.\cos x - \sin x.\cos x + \cos x]$

$12\cos x[\sin^2 x - \sin x + 1]$

$\Rightarrow f'(x) = 12\cos x[\sin^2 x + 1(1 - \sin x)]$

$1 - \sin x \geq 0$ and $\sin^2 x \geq 0$

$\sin^2 x + 1 - \sin x \geq 0$

Hence, $f'(x) > 0$, when $\cos x > 0$, *i.e.*, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

So, $f(x)$ is increasing when $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

and $f'(x) < 0$, when $\cos x < 0$, *i.e.*, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Hence, $f'(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Since,

$\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

Q. 4. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$.

(a) sin 2*x* **(b) tan *x***

(c) cos *x* **(d) cos 3*x***

[NCERT Exemp.]

Ans. Correct option : (c)

Explanation : In the given interval $\left(0, \frac{\pi}{2}\right)$

$f(x) = \cos x$

On differentiating with respect to x, we get

$f'(x) = -\sin x$

which gives $f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

Hence, $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

Q. 5. The function $f(x) = \tan x - x$

(a) always increases

(b) always decreases

(c) never increases

(d) sometimes increases and sometimes decreases

[NCERT Exemp.]

Ans. Correct option : (a)

Explanation : We have,

$f(x) = \tan x - x$

On differentiating with respect to x, we get

$f'(x) = \sec x - 1$

$\Rightarrow f'(x) > 0, \forall x \in F$

So, $f(x)$ always increases.

Short Answer Type Questions

(2 marks each)

Q. 1. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on *R*. R&U **[NCERT] [O.D. Set I 2017]**

Sol.

$f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$ ½

$= 3[x^2 - 2x + 2] = 3[(x-1)^2 + 1]$ **1**

since $f'(x) > 0; x \in R$

$\therefore f(x)$ is increasing on R ½

[CBSE Marking Scheme 2017]

OR

$f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 2)$

discriminant of the formed quadratic = $b^2 - 4ac = (-2)^2 - 4(1)(2)$

$= 4 - 8 = -4$

$b^2 - 4ac < 0$

but $a > 0$ $[a = 1]$

Hence $x^2 - 2x + 2 > 0$ for all $x \in R$

$\therefore 3(x^2 - 2x + 2) > 0$; $x \in R$

$f'(x) > 0$; $x \in R$

Hence $f(x)$ is increasing on R.

[Topper's Answer 2017]

Q. 2. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing or R. R&U **[Delhi 2017]**

Sol. $f(x) = 4x^3 - 18x^2 + 27x - 7$

$f'(x) = 12x^2 - 36x + 27$ ½

$= 3(2x-3)^2 \geq 0;\ x \in R$ 1

$\therefore f(x)$ is increasing on R. ½

[CBSE Marking Scheme 2017]

Q. 3. Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$ is decreasing for all $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

R&U **[Foreign 2017]**

Sol. $f(x) = \dfrac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}$ 1

$1 + (\sin x + \cos x)^2 > 0;\ x \in R$

and $\frac{\pi}{4} < x < \frac{\pi}{2}$ or $\cos x < \sin x$ or $\cos x - \sin x < 0$

½

or $f'(x) < 0$ or $f(x)$ is decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. ½

[CBSE Marking Scheme 2017]

Long Answer Type Questions-I

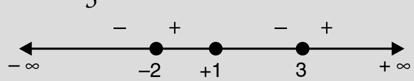
(4 marks each)

Q. 1. Find the intervals in which $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is (a) strictly increasing (b) strictly decreasing.

R&U **[NCERT] [O.D. Set I, II, III Comptt. 2014]**

Sol. $f'(x) = \frac{12}{10}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$ ½

$= \frac{6}{5}(x-1)(x+2)(x-3)$ 1½



$f'(x) = 0$ at $x = 1, -2, 3$ ½

$\therefore$ Intervals are $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$ ½

$\because f'(x) > 0$ for $(-2, 1) \cup (3, \infty)$

$f(x)$ is strictly increasing in $(-2, 1)$ and $(3, \infty)$ ½

$\because f'(x) < 0$ for $(-\infty, -2)$ and $(1, 3)$

$f(x)$ is strictly decreasing in $(-\infty, -2)$ and $(1, 3)$ ½

[CBSE Marking Scheme 2014]

Q. 2. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is (a) strictly increasing (b) strictly decreasing.

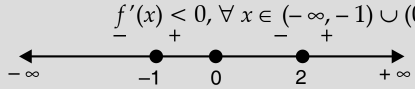
R&U **[Delhi Set I, II, III, 2014]**
[Delhi Set I, II, III Comptt. 2013]

Sol. $f'(x) = 12x^3 - 12x^2 - 24x$

$= 12x(x+1)(x-2)$ 1½

$f'(x) > 0,\ \forall x \in (-1, 0) \cup (2, \infty)$ 1

$f'(x) < 0,\ \forall x \in (-\infty, -1) \cup (0, 2)$ 1



$\therefore f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$

and strictly decreasing in $(-\infty, -1) \cup (0, 2)$ ½

[CBSE Marking Scheme, 2014]

Q. 3. Find the intervals in which the function $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ is :

(i) strictly increasing

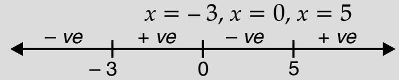
(ii) strictly decreasing. R&U **[Foreign Set II, 2014]**

Sol. $f(x) = 6x^3 - 12x^2 - 90x$ 1

$= 6x(x-5)(x+3)$ 1

Equating $f'(x) = 0$

or $x = -3, x = 0, x = 5$



$f'(x) > 0,\ \forall x \in (-3, 0) \cup (5, \infty)$ ½

Or Strictly increasing

$f'(x) < 0,\ \forall x \in (-\infty, -3) \cup (0, 5)$

Or Strictly decreasing **1+½**

[CBSE Marking Scheme 2014]

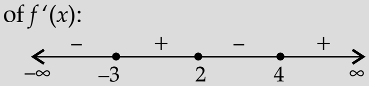
Q. 4. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (a) strictly increasing, (b) strictly decreasing. R&U **[Delhi/OD 2018]**

Sol. $f'(x) = x^3 - 3x^2 - 10x + 24$ ½

$= (x-2)(x-4)(x+3)$ 1

$f'(x) = 0 \Rightarrow x = -3, 2, 4.$ ½

sign of $f'(x)$:



$\therefore f(x)$ is strictly increasing on $(-3, 2) \cup (4, \infty)$ 1

and $f(x)$ is strictly decreasing on $(-\infty, -3) \cup (2, 4)$ 1

[CBSE Marking Scheme, 2018]

Detailed Solution :

Given $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

Differentiating with respect to x.

$f'(x) = x^3 - 3x^2 - 10x + 24$

put $f'(x) = 0$

$x^3 - 3x^2 - 10x + 24 = 0$

$\Rightarrow x^2(x-2) - x(x-2) - 12(x-2) = 0$

$\Rightarrow (x-2)(x^2 - x - 12) = 0$

$\Rightarrow (x-2)(x+3)(x-4) = 0$

$\Rightarrow \therefore x = -3, 2$ and 4

$\therefore$ The intervals are $(-\infty, -3)$, $(-3, 2)$ $(2, 4)$ and $(4, \infty)$.

$$\underset{-\infty}{\longleftarrow} \;-\; \underset{-3}{\bullet} \;+\; \underset{2}{\bullet} \;-\; \underset{4}{\bullet} \;+\; \underset{\infty}{\longrightarrow}$$

(a) $f'(x) > 0\ \forall\ x \in (-3, 2) \cup (4, \infty)$

$\therefore f(x)$ is strictly increasing $\forall\ x \in (-3, 2) \cup (4, \infty)$

(b) $f'(x) < 0\ \forall\ x \in (-\infty, -3) \cup (2, 4)$

$\therefore f(x)$ is strictly decreasing $\forall\ x \in (-\infty, -3) \cup (2, 4)$

Q. 5. Separate the intervals $\left[0, \frac{\pi}{2}\right]$ into sub intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing. R&U **[S.Q.P. 2013]**

Sol. $f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$

$= -\sin 4x$ **1**

Or $f'(x) = 0$ or $x = \frac{\pi}{4}$ **1**

In the interval	Sign of $f'(x)$	Conclusion
$\left(0, \frac{\pi}{4}\right)$	-ve as $0 < 4x < \pi$	f is strictly decreasing in $\left[0, \frac{\pi}{4}\right]$
$\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$	+ve as $\pi < 4x < 2\pi$	f is strictly increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

1

[CBSE Marking Scheme, 2013]

Alternative Method :

$f(x) = \sin^4 x + \cos^4 x$

or $f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$

$= -4\sin x \cos x[-\sin^2 x + \cos^2 x]$

$= -2\sin 2x \cos 2x = -\sin 4x$

1

On equating,

$f'(x) = 0$ or $-\sin 4x = 0$

or $4x = 0, \pi, 2\pi, \ldots\ldots\ldots$, $x = 0, \frac{\pi}{4}, \frac{\pi}{2}$. **1**

Sub-intervals are $\left[0, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

or $f'(x) < 0$ in $\left[0, \frac{\pi}{4}\right]$

or $f(x)$ is decreasing in $\left[0, \frac{\pi}{4}\right]$ **1**

and, $f'(x) > 0$ in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$\therefore f'(x)$ is increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. **1**

AI **Q. 6. Find the intervals in which the function given by $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$ is**

(i) increasing

(ii) decreasing. R&U **[NCERT][O.D. 2009]**

[Delhi Set I, II, III Comptt. 2012]

[Foreign 2011, Delhi Comptt., 2017]

Sol. $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$ **½**

$f'(x) = \cos x - \sin x$ **1**

$f'(x) = 0$ or $\cos x = \sin x$ **1**

$\therefore$ $x = \frac{\pi}{4}, \frac{5\pi}{4}$ **1**

Sign of $f'(x)$

$$\underset{0}{\vdash} \;+ve\; \underset{\frac{\pi}{4}}{+} \;-ve\; \underset{\frac{5\pi}{4}}{+} \;+ve\; \underset{2\pi}{\dashv}$$

So $f(x)$ is strictly increasing in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

and strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ **1**

[CBSE Marking Scheme 2017]

Alternative Method :

$f(x) = \sin x + \cos x$

or $f'(x) = \cos x - \sin x$

Now, $f'(x) = 0$ gives $\sin x = \cos x$ which gives

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2\pi$ **1**

$\tan x = 1$

The points $x = \frac{\pi}{4}, \frac{5\pi}{4}$ divides the interval $[0, 2\pi]$ into 3 disjoint intervals, $\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$. **1**

Note that $f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ or $f(x)$ is strictly increasing in intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

Also $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ **1**

or $f(x)$ is strictly decreasing in this interval.

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right)$	+ ve	f is strictly increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	– ve	f is strictly decreasing
$\left(\frac{5\pi}{4}, 2\pi\right]$	+ ve	f is strictly increasing

1

Q. 7. Find the value (s) of x for which $y = [x(x-2)]^2$ is an increasing function. R&U **[Delhi 2010]**

[NCERT] [O.D Set I, II, III, 2014]

Sol. $y = [x(x-2)]^2$

$= [x^2 - 2x]^2$

$\therefore \quad \frac{dy}{dx} = 2(x^2-2x)(2x-2)$ **1**

or $\quad \frac{dy}{dx} = 4x(x-1)(x-2)$ **1**

On equating $\frac{dy}{dx} = 0$,

$4x(x-1)(x-2) = 0 \Rightarrow x = 0, x = 1, x = 2$

$\therefore$ Intervals are $(-\infty, 0), (0, 1), (1, 2), (2, \infty)$ **1**

Since, $\frac{dy}{dx} > 0$ in $(0, 1)$ or $(2, \infty)$

$\therefore f(x)$ is increasing in $(0, 1) \cup (2, \infty)$ **1**

[CBSE Marking Scheme 2014]

Q. 8. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.

R&U **[NCERT] [Foreign Set I, II, III, 2012]**

Sol. $y = \log(1+x) - \frac{2x}{2+x}, x > -1$

Diff. w.r.t. 'x',

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{[(2+x)(2)-2x]}{(2+x)^2} \quad ½$$

$$= \frac{1}{1+x} - \frac{[4+2x-2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2} \quad ½$$

$$= \frac{(2+x)^2 - 4(1+x)}{(2+x)^2(1+x)}$$

$$= \frac{4+x^2+4x-4-4x}{(2+x)^2(1+x)}$$

$$= \frac{x^2}{(2+x)^2(1+x)} \quad ½$$

For increasing function,

$$\frac{dy}{dx} \geq 0 \quad ½$$

or $\quad \frac{x^2}{(2+x)^2(x+1)} \geq 0$

or $\quad \frac{(2+x)^2(x+1)x^2}{(2+x)^4(x+1)^2} \geq 0$

or $\quad (2+x)^2(x+1)x^2 \geq 0$ **1**

When $x > -1$,

$\frac{dy}{dx}$ is always greater than zero.

$\therefore y = \log(1+x) - \frac{2x}{2+x}$ is always increasing throughout its domain. **1**

Q. 9. Determine for what values of x, the function $f(x) = x^3 + \frac{1}{x^3}$ $(x \neq 0)$ is strictly increasing or strictly decreasing. A **[S.Q.P. Dec. 2016-17] [NCERT]**

Sol. Here, $f'(x) = 3x^2 - 3x^{-4}$

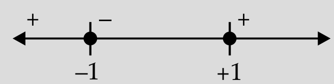
$$= \frac{3(x^6-1)}{x^4} \quad 1$$

$$= \frac{3(x^4+x^2+1)}{x^4}(x+1)(x-1)$$

Critical points are -1 and 1. **1**

or $f'(x) > 0$ if $x > 1$ or $x < -1$, and $f'(x) < 0$ if $-1 < x < 1$

$\left(\because \frac{3(x^4+x^2+1)}{x^4} \text{ always + ve}\right)$



Hence, $f(x)$ is strictly increasing for $x > 1$ **1**
or $x < -1$; and strictly decreasing in
$(-1, 0) \cup (0, 1)$ **1**

[CBSE Marking Scheme 2016]

Q. 10. Find the intervals in which the functions $f(x) = -3\log(1+x) + 4\log(2+x) - \frac{4}{2+x}$ is strictly decreasing. R&U **[SQP 2017-18]**

Sol. Domain $f = (-1, \infty)$

$$f'(x) = \frac{-3}{1+x} + \frac{4}{(2+x)} + \frac{4}{(2+x)^2}$$

$$= \frac{x(x+4)}{(1+x)(2+x)^2}. \quad 1$$

$f'(x) = 0$ or $x = 0$ $[x \neq -4$ as $-4 \notin (-1, \infty)]$. **1**

In $(-1, 0)$, $\quad f'(x) = \frac{(\text{-ve})(\text{+ve})}{(\text{+ve})(\text{+ve})} = \text{–ve}$.

Therefore, f is strictly decreasing in $(-1, 0)$. **1**

In $(0, \infty)$, $\quad f'(x) = $ +ve.

Therefore, f is strictly increasing in $(0, \infty)$. **1**

Q. 11. Find the intervals in which the functions given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is
(i) Strictly increasing
(ii) Strictly decreasing R&U **[O.D. Comptt., 2017]**

Sol. $f'(x) = 6x^2 - 6x - 36$ ½

$= 6(x^2 - x - 6)$

$= 6(x-3)(x+2)$

$f'(x) = 0$ or $x = -2, x = 3$ **1**

$\therefore$ the intervals are $(-\infty, -2), (-2, 3), (3, \infty)$ ½

getting $f'(x)$ + ve in $(-\infty, -2) \cup (3, \infty)$

and –ve in $(-2, 3)$ **1½**

$\therefore f(x)$ is strictly increasing in $(-\infty, -2) \cup (3, \infty)$, and strictly decreasing in $(-2, 3)$ ½

[CBSE Marking Scheme 2017]

Q. 12. Find the intervals in which the function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is (i) Strictly increasing (ii) Strictly decreasing R&U **[O.D Comptt set I, II, III 2018]**

Sol. $f(x) = -2x^3 - 9x^2 - 12x + 1$

Now, $f'(x) = -6x^2 - 18x - 12$

$= -6[x^2 + 3x + 2]$

$= -6[x^2 + 2x + x + 2]$

$$f'(x) = -6(x + 1)(x + 2)$$ **1**

Put, $f'(x) = 0 \Rightarrow x = -2, x = -1$ ½

$\Rightarrow$ Intervals are $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$ ½

Getting $f'(x) > 0$ in $(-2, -1)$ and $f'(x) < 0$ in $(-\infty, -2) \cup (-1, \infty)$ **1**

$\Rightarrow f(x)$ is strictly increasing in $(-2, -1)$

and strictly decreasing in $(-\infty, -2) \cup (-1, \infty)$ **1**

[CBSE Marking Scheme 2018]

Q.13. Find the intervals in which the function given by $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is (i) increasing, (ii) decreasing. R&U **[All Delhi 2012C]**

Sol. Given, function is

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$$

On differentiating both sides w.r.t. x, we get

$$\begin{aligned} f'(x) &= 4x^3 - 24x^2 + 44x - 24 \\ &= 4(x^3 - 6x^2 + 11x - 6) \\ &= 4(x - 1)(x^2 - 5x + 6) \\ &= 4(x - 1)(x - 2)(x - 3) \end{aligned}$$ **1**

Put $f'(x) = 0$

or $4(x - 1)(x - 2)(x - 3) = 0$ or $x = 1, 2, 3$

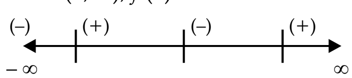
So, the possible intervals are $(-\infty, 1)$, $(1, 2)$, $(2, 3)$ and $(3, \infty)$ **1**

For interval $(-\infty, 1)$, $f'(x) < 0$

For interval $(1, 2)$, $f'(x) > 0$

For interval $(2, 3)$, $f'(x) < 0$

For interval $(3, \infty)$, $f'(x) > 0$.



Also, as $f(x)$ is a polynomial function, so it is continuous at $x = 1, 2, 3$. Hence,

(i) function increases in $[(1, 2)$ and $(3, \infty)]$. **1**

(ii) function decreases in $(-\infty, 1)$ and $(2, 3)$. **1**

Q. 14. Find the intervals in which the function $f(x) = (x - 1)^3 (x - 2)^2$ is (i) increasing, (ii) decreasing. R&U **[All Delhi 2011C]**

Sol. Given, $f(x) = (x - 1)^3 (x - 2)^2$

On differentiating both sides w.r.t. x, we get

$$f'(x) = (x-1)^3 . \frac{d}{dx}(x-2)^2 + (x-2)^2 . \frac{d}{dx}(x-1)^3 \quad \left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

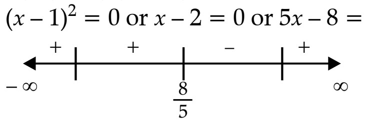
$$\begin{aligned} f'(x) &= (x - 1)^3 . 2(x - 2) + (x - 2)^2 . 3(x - 1)^2 \\ &= (x - 1)^2 (x - 2)[2(x - 1) + 3(x - 2)] \\ &= (x - 1)^2 (x - 2)(2x - 2 + 3x - 6) \end{aligned}$$

or $f'(x) = (x - 1)^2 (x - 2)(5x - 8)$

Now, put $f'(x) = 0$

or $(x - 1)^2 (x - 2)(5x - 8) = 0$

Either $(x - 1)^2 = 0$ or $x - 2 = 0$ or $5x - 8 = 0$



$\therefore \quad x = 1, \frac{8}{5}, 2$ **1**

Now, we find intervals and check in which interval $f(x)$ is strictly increasing and strictly decreasing.

Interval	$f'(x) = (x - 1)^2 (x - 2)(5x - 8)$	Sign of $f'(x)$
$x < 1$	$(+)(-)(-)$	+ve
$1 < x < \frac{8}{5}$	$(+)(-)(-)$	+ve
$\frac{8}{5} < x < 2$	$(+)(-)(+)$	–ve
$x > 2$	$(+)(+)(+)$	+ve

1

We know that, a function $f(x)$ is said to be an strictly increasing function, if $f'(x) > 0$ and strictly decreasing if $f'(x) < 0$ So, the given function $f(x)$ is increasing on the intervals $(-\infty, 1)$ $\left(1, \frac{8}{5}\right)$ and $(2, \infty)$ and decreasing on $\left(\frac{8}{5}, 2\right)$. **1**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 1, \frac{8}{5}, 2.$ Hence, $f(x)$ is

(i) increasing on intervals $\left(-\infty, \frac{8}{5}\right]$ and $[2, \infty)$

(ii) decreasing on interval $\left[\frac{8}{5}, 2\right]$ **1**

Q. 15. Show that the function $f(x) = x^3 - 3x^2 + 3x, x \in R$ is increasing on R. R&U **[All India 2011C]**

Sol. We know that, a function $y = f(x)$ is said to be increasing on R, if $\frac{dy}{dx} > 0, \forall x \in R$ **1**

Given, $y = x^3 - 3x^2 + 3x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^3 - 6x + 3$$

or $$\frac{dy}{dx} = 3(x^2 - 2x + 1)$$

or $$\frac{dy}{dx} = 3(x - 1)^2$$ **1**

Now, $3(x - 1)^2 \geq 0$ for all real values of x, *i.e,.* $\forall x \in R$

$\therefore \quad \frac{dy}{dx} \geq 0, \forall x \in R$

Hence, the given function is increasing on R. **2**

Q. 16. Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x + 20$ is (i) increasing. (ii) decreasing. R&U **[Delhi 2011C]**

Sol. Similar to Ques.11, LATQ-I

(i) increasing on intervals $(-\infty, -2]$ and $[-1, \infty)$.

(ii) decreasing on interval $[-2, -1]$.

Q. 17. Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x - 15$ is (i) increasing (ii) decreasing. R&U **[Delhi 2011C]**

Sol. Similar to Ques.11, LATQ-I

(i) The function increasing on intervals $(-\infty, -1]$ and $[-2, \infty)$.

(ii) The function decreasing on interval $[-1, -2]$.

Q. 18. Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 17$ is (i) increasing (ii) decreasing. R&U **[All India 2010C]**

Sol. Similar to Ques.11, LATQ-I

(i) The function increasing on $(-\infty, 2]$ and $[3, \infty)$ and **(ii)** decreasing on interval $[2, 3]$.

Q. 19. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is (i) increasing (ii) decreasing. R&U **[All India 2010C, 2008, 2008C]**

Sol. Similar to Question 11, LATQ-I

(i) The function increasing on $(-\infty, 1]$ and $[2, \infty)$.

(ii) The function decreasing on $[1, 2]$.

Q. 20. Find the intervals in which the function $f(x) = (x-1)(x-2)^2$ is increasing or decreasing. R&U **[All India 2009C]**

Sol. Similar to Question 11, LATQ-I

The function is increasing on intervals $\left(-\infty, \frac{4}{3}\right]$ and $[2, \infty)$ and decreasing on interval $\left[\frac{4}{3}, 2\right]$.

Q. 21. Find the intervals in which the function $f(x) = x^3 - 12x^2 + 36x + 17$ is increasing or decreasing function. R&U **[Delhi 2009C]**

Sol. Similar to Question 11, LATQ-I

Increasing on $(-\infty, 2]$ and $[6, \infty)$ and decreasing on $[2, 6]$.

AI **Q. 22. Find the intervals in which the function $f(x) = 20 - 9x + 6x^2 - x^3$ is (i) strictly increasing (ii) strictly decreasing.** R&U **[All India 2010]**

Sol. Similar to Question 11, LATQ-I

(i) Strictly increasing on the interval $(1, 3)$,

(ii) Strictly decreasing on the intervals $(-\infty, 1)$ and $(3, \infty)$.

Q. 23. Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function of θ on $\left[0, \frac{\pi}{2}\right]$. R&U **[Outside Delhi 2016]**

[NCERT]

Sol. Getting $\frac{dy}{d\theta} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$ **1**

Equating $\frac{dy}{d\theta}$ to 0 and getting critical point as $\cos\theta = 0$ *i.e.*, $\theta = \frac{\pi}{2}$ **1**

For all $\theta, 0 \le \theta \le \frac{\pi}{2}, \frac{dy}{d\theta} \ge 0$ **1**

Hence, y is an increasing function of θ on $\left[0, \frac{\pi}{2}\right]$. **1**

[CBSE Marking Scheme 2016]

Commonly Made Error

- Sometimes candidates do not have any basic knowledge of applications of derivatives while few candidates do not able to differentiate the functions with respect to θ.

Answering Tips

- Give adequate practice on problems based on applications of derivatives.

Alternative Method :

$$f(\theta) = \frac{4\sin\theta}{2+\cos\theta} - \theta$$

Now, $$f'(\theta) = \frac{(2+\cos\theta)4\cos\theta - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

or $$f'(\theta) = \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$ **1**

or $$f'(\theta) = \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$ **1**

or $$f'(\theta) = \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

or $$f'(\theta) = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

∴ $$f'(\theta) = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

or Here $f'(\theta)$ is increasing when $f'(\theta) \ge 0$ **1**

i.e., $$\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \ge 0$$

or $$\cos\theta \ge 0$$

$$\left[\because \frac{4-\cos\theta}{(2+\cos\theta)^2} \ge 0 \ \forall\ \theta \in R\right]$$

or $$\theta \in \left[0, \frac{\pi}{2}\right]$$ **1**

$(4 - \cos\theta)$ is always greater than 0.

Since $-1 \le \cos\theta \le 1$, $(2 + \cos\theta)^2 > 0$.

Long Answer Type Questions-II

(6 marks each)

Q.1. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing. R&U **[Delhi Set I, II, III, 2016]**

Sol. $f(x) = \sin 3x - \cos 3x, \quad 0 < x < \pi$

or $f'(x) = 3\cos 3x + 3\sin 3x$

$= 3(\cos 3x + \sin 3x)$ 1

Put $f'(x) = 0$

or $\cos 3x + \sin 3x = 0$

or $\sin 3x = -\cos 3x$

or $-\tan 3x = 1$

or $\tan 3x = -1$ 1

As $0 < x < \pi$, $0 < 3x < 3\pi$

$\therefore$ $\tan 3x$ is negative for the following values :

$$3x = \frac{3\pi}{4}$$

or $$x = \frac{\pi}{4}$$

$$3x = \pi + \frac{3\pi}{4} = \frac{7\pi}{4}$$

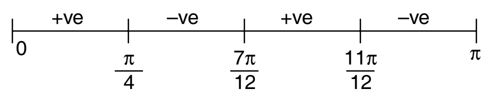
or $$x = \frac{7\pi}{12}$$

$$3x = \frac{7\pi}{4} + \pi = \frac{11\pi}{4}$$

or $$x = \frac{11\pi}{12}$$ ½

Hence we have intervals :

Intervals	Sing of $f'(x)$	Nature of function	
$\left(0, \frac{\pi}{4}\right)$	Positive	Increasing	½
$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$	Negative	Decreasing	½
$\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$	Positive	Increasing	½
$\left(\frac{11\pi}{12}, \pi\right)$	Negative	Decreasing	½



½

Hence, $f(x) = \sin 3x - \cos 3x$ is strictly increasing in the intervals $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ and strictly decreasing in intervals $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$ 1

Q. 2. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence find the intervals in which $f(x)$ is :

(i) strictly increasing

(ii) strictly decreasing.

R&U **[Delhi Set II Comptt. 2014]**

Sol. $f'(x) = 2x - 1$ 1

$f'(x) > 0, \forall x \in \left(\frac{1}{2}, 1\right)$ 1

$f'(x) < 0, \forall x \in \left(-1, \frac{1}{2}\right)$ 1

$\therefore$ $f(x)$ is neither increasing nor decreasing in $(-1, 1)$. 1

$f(x)$ is strictly increasing on $\left(\frac{1}{2}, 1\right)$ 1

and $f(x)$ is strictly decreasing on $\left(-1, \frac{1}{2}\right)$. 1

[CBSE Marking Scheme 2014]

Q. 3. Find the intervals in which the function f given by $f(x) = \sin x - \cos x$, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing. R&U **[Delhi 2010]**

Sol. Given, function is $f(x) = \sin x - \cos x$, $0 \le x \le 2\pi$

On differentiating both sides w.r.t. x we get $f'(x) = \cos x + \sin x$

On putting $f'(x) = 0$, we get

$\cos x + \sin x = 0$ or $\sin x = -\cos x$

or $$\frac{\sin x}{\cos x} = -1 \text{ or } \tan x = -1$$ 1

For $$x \in [0, 2\pi],\ \tan x = \frac{3\pi}{4} \text{ or } x = \frac{3\pi}{4}$$

or $$\tan x = \tan\frac{7\pi}{4}$$

or $$x = \frac{7\pi}{4} \therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

[$\because$ tan θ is negative in 2nd quadrant and 4th quadrant] **2**

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	Test value	$f'(x) = \cos x + \sin x$	Sign of $f'(x)$
$0 < x < \frac{3\pi}{4}$	At $x = \frac{\pi}{2}$	$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$	+ve
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	At $x = \frac{5\pi}{6}$	$f'\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}$ $= \cos\left(\pi - \frac{\pi}{6}\right) + \sin\left(\pi - \frac{\pi}{6}\right)$ $= -\cos\frac{\pi}{6} + \sin\frac{\pi}{6} = \frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{-\sqrt{3}+1}{2}$	– ve
$\frac{7\pi}{4} < x < 2\pi$	At $x = \frac{23\pi}{12}$	$f'\left(\frac{23\pi}{12}\right) = \cos\frac{23\pi}{12} + \sin\frac{23\pi}{12}$ $= \cos\left(2\pi - \frac{\pi}{12}\right) + \sin\left(2\pi - \frac{\pi}{12}\right) = \cos\frac{\pi}{12} - \sin\frac{\pi}{12} > 0$	+ve

2

We know that, a function $f(x)$ is said to be strictly increasing in an interval when $f'(x) > 0$ and it is said to be strictly decreasing when $f'(x) < 0$. So, the given function $f(x)$ is strictly increasing in intervals $\left[0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right]$ and it is strictly decreasing in the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$. **1**

TOPIC-5
Maxima and Minima

Revision Notes

1. Understanding maxima and minima :

Consider $y = f(x)$ be a well defined function on an interval I, then

(a) f is said to have a **maximum value** in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in I$.
The value corresponding to $f(c)$ is called the maximum value of f in I and the point c is called the **point of maximum value of** f **in** I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in I$.
The value corresponding to $f(c)$ is called the minimum value of f in I and the point c is called the **point of minimum value of** f **in** I**.**

(c) f is said to have an **extreme value** in I, if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.
The value $f(c)$ in this case, is called an extreme value of f in I and the point c called an **extreme point**.

Know the Terms

1. Let f be a real valued function and also take a point c from its domain, then

(i) c is called a point of **local maxima** if there exists a number $h > 0$ such that $f(c) > f(x)$, for all x in $(c - h, c + h)$.
The value $f(c)$ is called the **local maximum value of** f.

(ii) c is called a point of **local minima** if there exists a number $h > 0$ such that $f(c) < f(x)$, for all x in $(c - h, c + h)$.
The value $f(c)$ is called the **local minimum value of** f**.**

2. Critical points

It is a point c (say) in the domain of a function $f(x)$ at which either $f'(x)$ vanishes *i.e.*, $f'(c) = 0$ or f is not differentiable.

3. First Derivative Test :

Consider $y = f(x)$ be a well defined function on an open interval I. Now proceed as have been mentioned in the following algorithm :

STEP 1 : Find $\frac{dy}{dx}$.

STEP 2 : Find the critical point(s) by putting $\frac{dy}{dx}=0$. Suppose $c \in I$ (where I is the interval) be any critical point point and f be continuous at this point c. Then we may have following situations :

- $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through c, then the function attains a **local maximum** at $x = c$.
- $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through c, then the function attains a **local minimum** at $x = c$.
- $\frac{dy}{dx}$ **does not change sign** as x increases through c, then $x = c$ is **neither** a point of **local maximum nor** a point of **local minimum**. Rather in this case, the point $x = c$ is called the **point of inflection**.

4. Second Derivative Test :

Consider $y = f(x)$ be a well defined function on an open interval I and twice differentiable at a point c in the interval. Then we observe that :

- $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.
  The value $f(c)$ is called the local maximum value of f.
- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  The value $f(c)$ is called the local minimum value of f.

This test fails if $f'(c) = 0$ and $f''(c) = 0$. In such a case, we use **first derivative test** as discussed above**.**

5. Absolute maxima and absolute minima :

If f is a continuous function on a **closed interval** I, then f has the absolute maximum value and f attains it atleast once in I. Also f has the absolute minimum value and the function attains it atleast once in I.

ALGORITHM

STEP 1 : Find all the critical points of f in the given interval, *i.e.*, find all the points x where either $f'(x) = 0$ or f is not differentiable.

STEP 2 : Take the end points of the given interval.

STEP 3 : At all these points (*i.e.*, the points found in STEP 1 and STEP 2) calculate the values of f.

STEP 4 : Identify the maximum and minimum value of f out of the values calculated in STEP 3. This maximum value will be the **absolute maximum value** of f and the minimum value will be the **absolute minimum value** of the function f.

Absolute maximum value is also called as **global maximum value** or **greatest value**. Similarly absolute minimum value is called as **global minimum value** or the **least value.**

Objective Type Questions

(1 mark each)

Q. 1. Let the $f : R \to R$ be defined by $f(x) = 2x + \cos x$, then f :

(a) has a minimum at $x = \pi$

(b) has a maximum, at $x = 0$

(c) is a decreasing function

(d) is an increasing function

[NCERT Exemp.]

Ans. Correct option : (d)

Explanation : Given that,

$f(x) = 2x + \cos x$

Differentiating with respect to x, we get

$$f'(x) = 2 + (-\sin x)$$
$$= 2 - \sin x$$

Since, $f'(x) > 0, \forall x \in R$

Hence, $f(x)$ is an increasing function.

Q. 2. If x is real, the minimum value of $x^2 - 8x + 17$ is

(a) –1 **(b) 0**

(c) 1 **(d) 2**

[NCERT Exemp.]

Ans. Correct option : (c)

Explanation : Let,

$f(x) = x^2 - 8x + 17$

On differentiating with respect to x, we get

$$f'(x) = 2x - 8$$

So, $f'(x) = 0$

$\Rightarrow 2x - 8 = 0$

$\Rightarrow \quad 2x = 8$

$\therefore \quad x = 4$

Now, Again on differentiating with respect to x, we get

$f''(x) = 2 > 0, \forall x$

So, $x = 4$ is the point of local minima.

Minimum value of $f(x)$ at $x = 4$

$f(4) = 4\times4-8\times4+17 = 1$

Q. 3. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is

(a) 126 **(b) 0**

(c) 135 **(d) 160**

[NCERT Exemp.]

Ans. Correct option : (b)

Explanation : Given that, the smallest value of polynomial is $x^3 - 18x^2 + 96x$

On differentiating with respect to x, we get

$f'(x) = 3x^2 - 36x + 96$

So,

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2 - 36x + 96 = 0$$

$$\Rightarrow 3(x^2 - 12x + 32) = 0$$

$$\Rightarrow \quad (x-8)(x-4) = 0$$

$$\Rightarrow \quad x = 8, 4 \in [\mathbf{0}, 9]$$

We shall now calculate the value of f at these points and at the end points of the interval [0, 9], *i.e.*, at $x = 4$ and $x = 8$ and at $x = 0$ and at $x = 9$.

$$f(4) = 4^3 - 18.4^2 + 96.4$$
$$= 64 - 288 + 384$$
$$= 160$$
$$f(8) = 8^3 - 18.8^2 + 96.8$$
$$= 128$$
$$f(9) = 9^3 - 18.9^2 + 96.9$$
$$= 729 - 1458 + 864$$
$$= 135$$
$$\text{and } f(0) = 0^3 - 18.0^2 + 96.0$$
$$= 0$$

Thus, we conclude that absolute minimum value of f on [0, 9] is 0 occurring at $x = 0$.

Q. 4. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(a) two points of local maximum

(b) two points of local minimum

(c) one maxima and one minima

(d) no maxima or minima

[NCERT Exemp.]

Ans. Correct option : (c)

Explanation : We have,

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

$$f'(x) = 6x^2 - 6x - 12$$

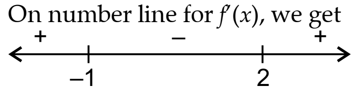
$$\text{Now,} \quad f'(x) = 0$$

$$\Rightarrow \quad 6(x^2 - x - 2) = 0$$

$$\Rightarrow \mathbf{6}(x+1)(x-2) = 0$$

$$\Rightarrow \quad x = -1 \text{ and } x = +2$$

On number line for $f'(x)$, we get



Hence, $x = -1$ is point of local maxima and $x = 2$ is point of local minima.

So, $f(x)$ has one maxima and one minima.

Q. 5. The maximum value of sin x.cos x is

(a) $\frac{1}{4}$ **(b) $\frac{1}{2}$**

(c) $\sqrt{2}$ **(d) $2\sqrt{2}$**

[NCERT Exemp.]

Ans. Correct option : (b)

Explanation : Let us assume that, $f(x) = \sin x.\cos x$

Now, we know that

$$\sin x.\cos x = \frac{1}{2}\sin 2x$$

$$\therefore \quad f'(x) = \frac{1}{2}.\cos 2x.2 = \cos 2x$$

$$\text{Now,} \quad f'(x) = 0$$

$$\Rightarrow \quad \cos 2\boldsymbol{x} = 0$$

$$\Rightarrow \quad \cos 2x = \cos\frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{4}$$

$$\text{Also,} \quad f''(x) = \frac{d}{dx}.\cos 2x = -2.\sin 2x$$

$$\therefore [f''(\boldsymbol{x})]_{\text{at } x=\frac{\pi}{4}} = -2\sin 2.\frac{\pi}{4} = -2\sin\frac{\pi}{2} = -2 < 0$$

At $\frac{\pi}{4}$ is maximum and $\frac{\pi}{\mathbf{4}}$ is point of maxima.

$$f\left(\frac{\pi}{4}\right) = \frac{1}{2}.\sin.2.\frac{\pi}{4} = \frac{1}{2}$$

Q. 6. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is :

(a) 0 **(b) 12**

(c) 16 **(d) 32**

[NCERT Exemp.]

Ans. Correct option : (b)

Explanation : Given that,

$$y = -x^3 + 3x^2 + 9x - 27$$

$$\therefore \quad \frac{dy}{dx} = -3x^2 + 6x + 9 = \text{Slope of the curve}$$

$$\text{and} \quad \frac{d^2y}{dx^2} = -6x + 6 = -6(x-1)$$

$$\therefore \quad \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow -6(x-1) = 0$$

$$\Rightarrow \quad x = 1 > 0$$

$$\text{Now,} \quad \frac{d^3y}{dx^3} = -6 < 0$$

So, the maximu**m** slope of given curve is at $x = 1$.

$$\therefore \left(\frac{dy}{dx}\right)_{(x=1)} = -3.1^2 + 6.1 + 9 = 12$$

Q. 7. The maximum value of $\left(\frac{1}{x}\right)^x$ is :

(a) e **(b) e^e**

(c) $e^{1/e}$ **(d) $\left(\frac{1}{e}\right)^{1/e}$**

[NCERT Exemp. Ex. 6.3, Q. 59, Page 141]

$\Rightarrow \quad \log y = x.\log \frac{1}{x}$

$\therefore \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\frac{1}{x}}.\left(-\frac{1}{x^2}\right) + \log\frac{1}{\boldsymbol{x}}.1$

$= -1 + \log\frac{1}{x}$

Ans. Correct option : (c)

Explanation :

Let $\quad y = \left(\frac{1}{x}\right)^x$

$\Rightarrow \quad \log y = x.\log \frac{1}{x}$

$\therefore \quad \frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\frac{1}{x}}.\left(-\frac{1}{x^2}\right) + \log\frac{1}{\boldsymbol{x}}.1$

$= -1 + \log\frac{1}{x}$

$\therefore \quad \frac{dy}{dx} = \left(\log\frac{1}{x} - 1\right).\left(\frac{1}{x}\right)^x$

Now, $\frac{dy}{dx} = 0$

$\Rightarrow \quad \log\frac{1}{\boldsymbol{x}} = 1 = \log e$

$\Rightarrow \quad \frac{1}{x} = e$

$\Rightarrow \quad x = \frac{1}{e}$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (\boldsymbol{e})^{1/e}$.

Long Answer Type Questions-I

(4 marks each)

AI **Q. 1. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question ?** R&U **[Delhi/OD 2018]**

Sol. Let side of base = x and depth of tank = y

$$V = x^2y \Rightarrow y = \frac{V}{x^2},$$

(V = Quantity of water = constant)

Cost of material is least when area of sheet used is minimum.

A (Surface area of tank) = $x^2 + 4xy = x^2 + \frac{4V}{x}$

$$\frac{dA}{dx} = 2x - \frac{4V}{x^2}$$

$$\frac{dA}{dx} = 0 \qquad ½+½$$

$$\Rightarrow \quad x^3 = 2V,$$

$$y = \frac{x^3}{2x^2} = \frac{x}{2} \qquad ½+½$$

[as $V = x^2y$]

$$\frac{d^2A}{dx} = 2 + \frac{8V}{x^3} > 0, \qquad ½+½$$

$\therefore$ Area is minimum, thus cost is minimum when

$$y = \frac{x}{2}$$

Value: Any relevant value. **1**

[CBSE Marking Scheme 2018]

Commonly Made Error

- In most of the cases, candidates do not read the question attentively which results incorrect variable in area.

Answering Tip

- Give adequate practice on mensuration related concept and problems.

Detailed Answer :

Sol. Let the length, breadth and depth of the open tank be x, x and y, respectively. Length and breadth are same because given tank has a square base. Again, ;let V denotes its volume and S denotes its surface area. Now, given that

$$V = x^2y \qquad ...(i) \ \mathbf{1}$$

Also, we know that the total surface area of the open tank is given by

$$S = x^2 + 4xy \qquad ...(ii)$$

[$\because$ S = area of square base + area of the four walls]

On putting $y = \frac{V}{x^2}$ from Eq. (i) in Eq. (ii) we get

$$S = x^2 + 4x.\frac{V}{x^2}$$

or $\quad S = x^2 + \frac{4V}{x} \qquad \mathbf{1}$

On differentiating both side w.r.t x, we get

$$\frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

For maxima or minima, put $\frac{dS}{dx} = 0$

or $\quad 2x - \frac{4V}{x^2} = 0$

or $\quad 4V = 2x^3 \qquad \mathbf{1}$

or $\quad 4x^2y = 2x^3 \quad [\because V = x^2y$, from Eq. (i)]

or $\quad 2y = x$ or $y = \frac{x}{2}$

So, depth of tank is half of its width. **1**

Also, $\quad \frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{dS}{dx}\right) = \frac{d}{dx}\left(2x - \frac{4V}{x^2}\right)$

$$= 2 + \frac{8V}{x^3}$$

$$= 2 + \frac{8.x^2y}{x^3} \quad \text{[from Eq. (i)]}$$

$$= 2 + \frac{8y}{x} > 0, \text{ as } x > 0 \text{ and } y > 0 \quad \mathbf{1}$$

Thus, $\frac{d^2S}{dx^2} > 0$

or S is minimum.

Hence, total surface area of the tank is least, when depth is half of its width. **Hence proved. 1**

Long Answer Type Questions-II

(6 marks each)

Q. 1. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum and local minimum values.

R&U [NCERT] [Delhi I, II, III, 2015]

Sol. Given $f(x) = \sin x - \cos x, \quad 0 < x < 2\pi$

$f'(x) = \cos x + \sin x, \quad 0 < x < 2\pi$

$\therefore \quad f'(x) = 0$

Or $\cos x + \sin x = 0$ or $\tan x = -1$ **1**

$\therefore \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$ **1**

Getting $f''(x) = \cos x - \sin x$ **1**

$$f''\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} - \sin\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} \quad \mathbf{1}$$

i.e., $f''\left(\frac{3\pi}{4}\right)$ is negative, so at $x = \frac{3\pi}{4}$, $f(x)$ is local maxima **1**

Hence, local maximum value $= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

and $f''\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} - \sin\frac{7\pi}{4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

i.e., $f''\left(\frac{7\pi}{2}\right)$ is positive so $x = \frac{7\pi}{4}$ is local minima

Hence, local minimum value

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} \quad \mathbf{1}$$

[CBSE Marking Scheme 2015]

Q. 2. Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \sin^2 x - \cos x$, $x \in [0, \pi]$

R&U [Outside Delhi Set-II 2015]

Sol. $f(x) = \sin^2 x - \cos x, \quad x \in [0, \pi]$

$f'(x) = 2\sin x \cos x - (-\sin x)$

$f'(x) = 2\sin x \cos x + \sin x$

Put $f'(x) = 0$

Then, $(2\sin x \cos x + \sin x) = 0$

$\sin x\,(2\cos x + 1) = 0$ **1**

$\sin x = 0$ or $\cos x = \frac{-1}{2}$

$x = 0, \pi$ or $x = \frac{5\pi}{6}$ **1**

So, $f(0) = \sin^2 0 - \cos 0 = -1$ **1**

$$f\left(\frac{5\pi}{6}\right) = \sin^2\left(\frac{5\pi}{6}\right) - \cos\left(\frac{5\pi}{6}\right)$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{6}\right) = \left(\frac{1-2\sqrt{3}}{4}\right) \quad \mathbf{1}$$

$f(\pi) = \sin^2\pi - \cos\pi = 1$ **1**

Of these values, the maximum value is 1, and the minimum value is –1.

Thus, the absolute maximum and absolute minimum values of $f(x)$ are 1 and – 1, which is attain at $x \in (0, \pi)$, **1**

[CBSE Marking Scheme 2015]

Q. 3. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 cm^3. If building of tank costs ₹ 100 per square metre for the base and ₹ 50 per square metre for the sides, find the cost of least expensive tank.

R&U [Delhi Set I, II, III Comptt. 2015]

Sol. Let l, b, h be the length, breadth and depth of the tank, respectively.

$\therefore \quad l \times b \times 3 = 75$ **1**

or $\quad l \times b = 25$

Let C be the cost, then

$C = 100(l \times b) + 50 \times 2\,[h(b + l)]$ **1**

$$= 100\left(l \times \frac{25}{l}\right) + 300\left(\frac{25}{l} + l\right)$$

$$= 2500 + 300\left(\frac{25}{l} + l\right) \quad \mathbf{1}$$

Differentiating w.r.t. l,

$$\therefore \quad \frac{dC}{dl} = 0 + 300\left(\frac{-25}{l^2} + 1\right)$$

Putting, $\frac{dC}{dl} = 0$,

or $300\left(-\frac{25}{l^2}+1\right) = 0$

or $l^2 = 25$ or $l = 5$ **1**

Getting, $\frac{d^2C}{dl^2} = 300\left(\frac{50}{l^3}\right)$

or $\left(\frac{d^2C}{dl^2}\right)_{\text{at } l=5} = \frac{15000}{125} > 0$

i.e., C is minimum when $l = 5$

$\Rightarrow$ $b = 5$ **1**

$\therefore$ $C = 100(25) + 300(10)$

$= 2,500 + 3,000$

$= 5,500$

Hence the minimum cost is ₹ 5,500. **1**

[CBSE Marking Scheme 2015]

Q. 4. A jet of enemy is flying along the curve $y = x^2 + 2$ and a soldier is placed at the point (3, 2). Find the minimum distance between the soldier and the jet. R&U **[S.Q.P. 2015]**

Sol. Let $P(x, y)$ be the position of the jet and the soldier is placed at $A(3, 2)$

or $AP = \sqrt{(x-3)^2+(y-2)^2}$ ...(i) **1**

As $y = x^2 + 2$ or $y - 2 = x^2$ ...(ii)

$\therefore$ $AP^2 = (x-3)^2 + x^4 = z$(say) **1**

$\frac{dz}{dx} = 2(x-3) + 4x^3$ **1**

$\frac{dz}{dx} = 0$

$\therefore$ $2x - 6 + 4x^3 = 0$

Put $x = 1$

$2 - 6 + 4 = 0$

$\therefore$ $x - 1$ is a factor

and $\frac{d^2z}{dx^2} = 12x^2 + 2$ **1**

$\frac{dz}{dx} = 0$ or $x = 1$ **1**

and $\frac{d^2z}{dx^2}$ (at $x = 1$) > 0 ½

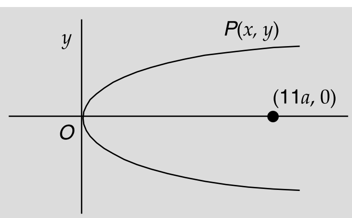
$\therefore$ z is minimum when $x = 1, y = 1 + 2 = 3$ units

$\therefore$ minimum distance $= \sqrt{(1-3)^2+(3-2)^2} = \sqrt{5}$ units ½

Q. 5. Find the point P on the curve $y^2 = 4ax$ which is nearest to the point $(11a, 0)$.

R&U **[O.D. Comptt. Set II, 2014]**

Sol.



1

Let $P(x, y)$ be the nearest point

$\therefore$ $D = \sqrt{(x-11a)^2+y^2}$ ½

or $S = (x-11a)^2 + y^2 = (x-11a)^2 + 4ax$ ½

$\frac{dS}{dx} = 2(x - 11a) + 4a$ (as $y^2 = 4ax$) **1**

$\frac{dS}{dx} = 0$ or $x = 9a$ **1**

$\therefore$ $y = \pm 6a$ **1**

Or $\frac{d^2S}{dx^2} = 2 > 0$

For minimum distance, points are

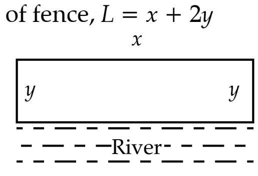
$P(9a, \pm 6a)$ ½+½

[CBSE Marking Scheme 2014]

Q. 6. A given rectangular area is to be fenced off, in a field whose length lies along the river. Show that the least length will be required when length of the field is twice its breadth. R&U **[S.Q.P. 2013]**

Sol. Let length be x m and breadth be y m.

$\therefore$ Length of fence, $L = x + 2y$



Let given area $= a$ or $xy = a$ or $y = \frac{a}{x}$

or $L = x + \frac{2a}{x}$ **1**

or $\frac{dL}{dx} = 1 - \frac{2a}{x^2}$ ½

For maxima or minima,

$\frac{dL}{dx} = 0$ or $x^2 = 2a$, $\therefore$ $x = \sqrt{2a}$ ½

$\frac{d^2L}{dx^2} = \frac{4a}{x^3} > 0$ **1**

For minimum length,

$L = \sqrt{2a} + \frac{2a}{\sqrt{2a}}$ **1**

$= 2\sqrt{2a}$ ½

$x = \sqrt{2a}$, $y = \frac{a}{\sqrt{2a}} = \frac{\sqrt{2a}}{2} = \frac{1}{2}x$ ½

$\therefore$ $x = 2y$ **1**

AI **Q. 7. An open box, with a square base is to be made out of a given quantity of metal sheet of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$.**

R&U **[NCERT Exemplar]**

[O.D. Set I Comptt. 2012] [O.D. Set I, II, III, 2012]

Sol. Let the length, breadth and height of open box be x, x, y units respectively.

$\therefore \quad c^2 = x^2 + 4xy$ ½

or $\quad y = \dfrac{c^2 - x^2}{4x}$ ½

or Volume V of the box $= x^2y = x^2\left(\dfrac{c^2 - x^2}{4x}\right)$ ½

$= \dfrac{c^2}{4}x - \dfrac{x^3}{4}$ **1**

$\dfrac{dV}{dx} = \dfrac{c^2}{4} - \dfrac{3}{4}x^2$

for maxima or minima

$\dfrac{dV}{dx} = 0$

$\dfrac{C^2}{4} - \dfrac{3}{4}x^2 = 0$

or $\quad x = \dfrac{c}{\sqrt{3}}$ **1 + ½**

$\dfrac{d^2V}{dx^2} = \dfrac{-3}{2}x < 0$

or maximum ½

$\therefore \quad x = \dfrac{c}{\sqrt{3}}$

or $\quad y = \dfrac{c}{2\sqrt{3}}$ ½

maximum volume $= x^2y = \dfrac{c^2}{3}\cdot\dfrac{c}{2\sqrt{3}} = \dfrac{c^3}{6\sqrt{3}}$ **1**

[CBSE Marking Scheme 2012]

Q. 8. A cuboidal shaped godown with square base is to be constructed. Three times as much cost per square meter is in curved for constructing the roof as compared to the walls. Find the dimensions of the godown if it is to enclose a given volume and minimize the cost of constructing the roof and the walls. R&U **[SQP 2018-19]**

Sol. Let the length and breadth of the base = x

Also let the height of the godown = y.

Let C be the cost of constructing the godown and V be the given volume. ½

Since cost is proportional to the area, therefore

$C = k[3x^2 + 4xy]$,

where $k > 0$ is constant of proportionality ...(1) 1½

$x^2y = V$(constant) ...(2) ½

$y = \dfrac{V}{x^2}$ ...(3)

Substituting value of y from equation (3), in equation (1), we get

$$C = k\left[3x^2 + 4x\left(\frac{V}{x^2}\right)\right]$$

$$= k\left[3x^2 + \frac{4V}{x}\right]$$

$\dfrac{dC}{dx} = k\left[6x - \dfrac{4V}{x^2}\right]$ ...(4) **1**

For maximum or minimum value of S

$\dfrac{dC}{dx} = 0$

$\Rightarrow \quad 6x - \dfrac{4V}{x^2} = 0$

$\Rightarrow \quad x = \left(\dfrac{2V}{3}\right)^{\frac{1}{3}}$

when, $x = \left(\dfrac{2V}{3}\right)^{\frac{1}{3}}$, $\dfrac{d^2C}{dx^2} = 6 + \dfrac{8V}{x^3}$

$= 18 > 0$

$\therefore$ C is minimum when $x = \left(\dfrac{2V}{3}\right)^{\frac{1}{3}}$ **1**

and $\quad y = \dfrac{(18V)^{\frac{1}{3}}}{2}$ ½

[CBSE Marking Scheme 2018]

AI **Q. 9. A metal box with a square base and vertical sides is to contain 1024 cm³. The material for the top and bottom costs ₹ 5 per cm² and the material for the sides costs ₹ 2·50/cm². Find the least cost of the box.** R&U **[NCERT Exemplar]**

[Delhi Comptt. 2016 Set I, II, III] [Delhi 2017]

Sol. Let side of square base of box $= x$ cm,

height of box $= h$ cm.

Volume of box $= 1024$ cm³

or $\quad x \cdot x \cdot h = 1024$

$\therefore \quad h = \dfrac{1024}{x^2}$ **1**

C(cost of box) $= 5(2x^2) + 2{\cdot}5(4xh)$

$= 10x^2 + 10hx$

$= 10x^2 + \dfrac{10,240}{x}$ **1½**

or $\quad \dfrac{dC}{dx} = 20x - \dfrac{10,240}{x^2}$,

Solving $\quad \dfrac{dC}{dx} = 0$, we get $x^3 = 512 \quad \therefore x = 8$

$\left.\dfrac{d^2C}{dx^2}\right]_{x=8} = \left.20 + \dfrac{2(10240)}{x^3}\right]_{x=8}$ **2**

$= 20 + \dfrac{2(10240)}{8^3} > 0$ ½

Thus, cost of box is least at $x = 8$ and least cost of box is :

$C(8) = 10(8)^2 + \dfrac{10240}{8}$

$= 640 + 1280$

$= ₹\ 1{,}920$ **1**

[CBSE Marking Scheme 2016]

Q. 10. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also, find the corresponding local maximum and local minimum values.

R&U **[Outside Delhi Set I, II, III Comptt. 2016]**
[NCERT Exemplar]

Sol. $f(x) = x^5 - 5x^4 + 5x^3 - 1$

or $f'(x) = 5x^4 - 20x^3 + 15x^2$ **1**

or $f'(x) = 5x^2(x-1)(x-3)$

$f'(x) = 0$ or $x = 0, x = 1, x = 3$ **1½**

$f''(x) = 20x^3 - 60x^2 + 30x$

$= 10x[2x^2 - 6x + 3]$ **1**

$f'(0) = 0, f'(1) = -\text{ve}, f'(3) = +\text{ ve}$

also, $\left.\begin{matrix} f'(>0) = -\text{ve} \\ f'(<0) = -\text{ve} \end{matrix}\right\}$

or $\left\{\begin{matrix} x = 0 \text{ is a point of inflexion} \\ x = 1 \text{ is a point of local maxima} \\ x = 3 \text{ is a point of local minima} \end{matrix}\right.$ **1½**

$\therefore$ Local max. value $= f(1) = 0$

and Local min. value $= f(3) = -28$ **1**

[CBSE Marking Scheme 2016]

Q. 11. A figure consists of a semi-circle with a rectangle on its diameter. Given the perimeter of the figure, find its dimensions in order that the area may be maximum.

OR

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the windows to admit maximum light through the whole opening.

A **[Outside Delhi Set I, II, III Comptt. 2016]**
[O.D. 2017, 2011, Foreign 2014]

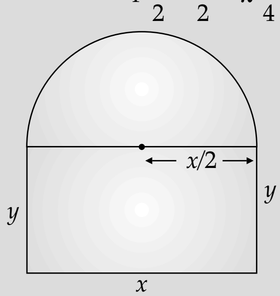
Sol. Let length and breadth of rectangle be x and y. **1**

$\therefore$ $P = 2y + x + \pi\dfrac{x}{2}$ (Given)

$\therefore$ $A = xy + \dfrac{1}{2}\pi\dfrac{x^2}{4}$

$= \dfrac{x}{2}\left[P - x - \dfrac{\pi x}{2}\right] + \dfrac{\pi x^2}{8}$ **1**

$= P\dfrac{x}{2} - \dfrac{x^2}{2} - \pi\dfrac{x^2}{4} + \pi\dfrac{x^2}{8}$



$\dfrac{dA}{dx} = \dfrac{P}{2} - x - \dfrac{\pi x}{2} + \dfrac{\pi x}{4}$

$= \dfrac{P}{2} - x - \dfrac{\pi x}{4}$ **1**

$\dfrac{dA}{dx} = 0$

or $x = \dfrac{2P}{4+\pi}$ and $y = \dfrac{P}{4+\pi}$ **1 + 1**

$\dfrac{d^2A}{dx^2} = -1 - \pi/4 < 0$

or Area is maximum, when length

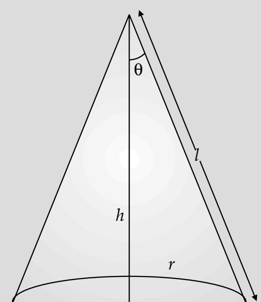
$= \dfrac{2P}{4+\pi}$,

breadth $= \dfrac{P}{4+\pi}$ **1**

[CBSE Marking Scheme 2016]

Q. 12. Show that semi-vertical angle of a cone of maximum volume and given slant height is $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$. A **[Delhi Set I, 2014][O.D. 2016]**

Sol.



$r = l \sin\theta$

$h = l \cos\theta$

$V = \dfrac{1}{3}\pi r^2 h = \dfrac{\pi}{3} l^3 \sin^2\theta \cos\theta$ **1**

or $\dfrac{dV}{d\theta} = \dfrac{\pi}{3} l^3[2\sin\theta\cos^2\theta - \sin^3\theta]$ **1**

For maxima and minima, $\dfrac{dV}{d\theta} = 0$ **1½**

$\sin\theta\,(2\cos^2\theta - \sin^2\theta) = 0$

or $\cos\theta = \dfrac{1}{\sqrt{3}}$ or $\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ **1½**

$\because$ $\dfrac{d^2V}{d\theta^2}$ is negative. 1

Hence, volume of the cone is maximum when semi-vertical angle is $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

[CBSE Marking Scheme 2016]

Commonly Made Error

- Most of the candidates attempt this question incorrectly. Some candidates could not express volume of the cone as a function in mathematical form.

Answering Tip

- Explain exhaustively the concept of maxima and minima and its application. Give practice in problems based on maxima and minima.

Alternative Method :

Let θ be the semi-vertical angle, l be the given slant height, then radius of base = $l \sin\theta$.

height = $l\cos\theta$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(l\sin\theta)^2 l\cos\theta \quad \mathbf{1}$$

or $$V = \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta$$

or $$\frac{dV}{d\theta} = \frac{\pi}{3}l^3\left[2\sin\theta\cos^2\theta + \sin^2\theta(-\sin\theta)\right]$$

$$= \frac{\pi}{3}l^3\sin\theta\left[-(1-\cos^2\theta)+2\cos^2\theta\right]$$

$$= \frac{\pi}{3}l^3\sin\theta(3\cos^2\theta - 1) \quad \mathbf{1}$$

For maximum volume put $\frac{dV}{d\theta} = 0$

or $3\cos^2\theta - 1 = 0$

or $$\cos\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \mathbf{1}$$

or

$$\frac{d^2V}{d\theta^2} = \frac{\pi}{3}l^3\left[\cos\theta(3\cos^2\theta-1)+\sin\theta(6\cos\theta)(-\sin\theta)\right]$$

$$= \frac{\pi}{3}l^3\left[3\cos^3\theta - \cos\theta - 6\cos\theta(1-\cos^2\theta)\right]$$

$$= \frac{\pi}{3}l^3\left[9\cos^3\theta - 7\cos\theta\right] \quad \mathbf{1}$$

$$= \frac{\pi}{3}l^3\cos\theta(9\cos\theta^2 - 7)$$

when $\cos\theta = \frac{1}{\sqrt{3}}$, then

$$\frac{d^2V}{d\theta^2} = \frac{\pi}{3}l^3\left(\frac{1}{\sqrt{3}}\right)(3-7) < 0 \quad \mathbf{1}$$

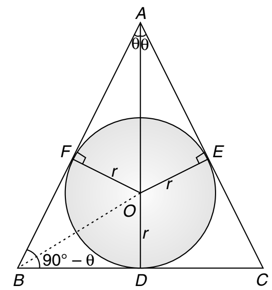
$\therefore$ when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, given volume is maximum.

As r is continuous in $\left(0, \frac{\pi}{2}\right)$, and has only one extreme point in $\left(0, \frac{\pi}{2}\right)$.

$\therefore$ V is absolutely max. for $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. **1**

Q. 13. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}\,r$. A **[O.D. Set I, II, III, 2016]**

Sol. Let ΔABC be an isosceles triangle with $AB = AC$.



Let $\angle BAC = 2\theta$ ½

AO bisects $\angle BAC$.

Join OE, OF, OD,

where O is the centre of the circle and

$$OE = OF = OD = r$$

Also, $OE \perp AC$, $OD \perp BC$ and $OF \perp AB$.

In ΔAOE, $\tan\theta = \frac{r}{AE}$

or $AE = r\cot\theta$

Similarly $AF = r\cot\theta$

In $\angle ABD$, $AD \perp BC$ (ΔABC is isosceles)

$\therefore$ $\angle ABD = 90° - \theta = \frac{\pi}{2} - \theta$ ½

OB bisects $\angle ABD$, $\therefore$ $\angle OBF = \angle OBD = \frac{\pi}{4} - \frac{\theta}{2}$

In ΔOFB,

or $$BF = r\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = r\cot\left(\frac{\pi-2\theta}{4}\right)$$

Similarly $BD = DC = CE = r\cot\left(\frac{\pi-2\theta}{4}\right)$ ½

We have, perimeter of ΔABC

$$P = AB + BC + CA$$
$$= AE + EC + BD + DC + AF + BF$$
$$= 2r\cot\theta + 4r\cot\left(\frac{\pi-2\theta}{4}\right) \quad \mathbf{1}$$

Differentiate with respect to θ,

$$\therefore \quad \frac{dP}{d\theta} = 2r(-\text{cosec}^2\theta) + 4r\left(-\text{cosec}^2\left(\frac{\pi-2\theta}{4}\right)\times -\frac{1}{2}\right)$$

$$= -2r\,\text{cosec}^2\theta + 2r\,\text{cosec}^2\left(\frac{\pi-2\theta}{4}\right)$$

½

or On equating, $\frac{dP}{d\theta} = 0 \Rightarrow 2r\,\text{cosec}^2\left(\frac{\pi-2\theta}{4}\right) = 2r\,\text{cosec}^2\theta$

or $$\sin^2\left(\frac{\pi-2\theta}{4}\right) = \sin^2\theta$$

or $$\sin\left(\frac{\pi-2\theta}{4}\right) = \sin\theta$$

or $$\sin\left(\frac{\pi-2\theta}{4}\right) = -\sin\theta$$

but $0 \le \theta \le \frac{\pi}{2}$ or $\sin 2\theta \ge \sin\theta$ ½

$\therefore \quad \sin\left(\frac{\pi - 2\theta}{4}\right) \neq -\sin\theta$

& $\quad \sin\left(\frac{\pi - 2\theta}{4}\right) = \sin\theta$

or $\quad \frac{\pi - 2\theta}{4} = \theta$

or $\quad \pi - 2\theta = 4\theta \text{ or } \pi = 6\theta$

or $\quad \theta = \frac{\pi}{6} \text{ or } 2\theta = \frac{\pi}{3}$ ½

or ΔABC is an equilateral triangle.

By second derivative test,

$$\frac{d^2P}{d\theta^2} = -2r\{2 \text{ cosec }\theta\,(-\text{cosec }\theta \cot\theta)\} + 2r$$

$$\left[2\text{cosec}\left(\frac{\pi-2\theta}{4}\right)\left\{-\text{cosec}\left(\frac{\pi-2\theta}{4}\right)\cot\left(\frac{\pi-2\theta}{4}\right)\times\left(-\frac{1}{2}\right)\right\}\right]$$

$$= 4r \text{ cosec}^2\theta \cot\theta + \frac{4r}{2}\text{ cosec}^2\left(\frac{\pi-2\theta}{4}\right)\cot\left(\frac{\pi-2\theta}{4}\right)$$ ½

$\therefore \quad \frac{d^2P}{d\theta^2} \geq 0 \text{ at } \theta = \frac{\pi}{3}$ ½

$\therefore$ Perimeter is minimum when $\theta = \frac{\pi}{3}$

$$P = 2r\cot\frac{\pi}{6} + 4r\cot\left(\frac{\pi - 2(\pi/6)}{4}\right)$$

$$= 2r\sqrt{3} + 4r\cot\left(\frac{2\pi}{12}\right)$$

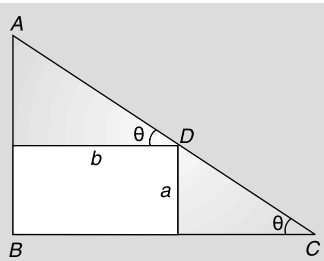
$$= 2r\sqrt{3} + 4r\sqrt{3}$$

$\therefore \quad P = 6\sqrt{3}r.$ 1

Q. 14. A point on the hypotenuse of a right triangle is at distances '*a*' and '*b*' from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$. A **[NCERT]**

[Delhi Set I, II, III Comptt. 2015]

Sol.



$\therefore \quad AD = b\sec\theta \text{ and } DC = a\text{ cosec }\theta$ 1

Let L be the length of the hypotenuse

$\therefore \quad L = AC = AD + DC$

or $\quad L = b\sec\theta + a\text{ cosec }\theta$ 1

or $\quad \frac{dL}{d\theta} = b\sec\theta\tan\theta - a\text{ cosec }\theta\cot\theta$

$\quad \frac{dL}{d\theta} = 0$

or $\quad \tan^3\theta = \frac{a}{b}$ 1

$$\frac{d^2L}{d\theta^2} = b\sec\theta\tan^2\theta + b\sec^3\theta + a\text{ cosec }\theta\cot^2\theta + a\text{ cosec}^3\theta$$ 1

i.e., $\quad \frac{d^2L}{d\theta^2} > 0$ Or minima 1

$\therefore \quad L = b\sec\theta + a\text{ cosec }\theta$

or $\quad L = b\cdot\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} + a\cdot\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}$

or $\quad L = \left(a^{2/3}+b^{2/3}\right)^{3/2}$ 1

[CBSE Marking Scheme 2015]

AI **Q. 15. The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}$, x and $2x$ is constant. Show that the sum of their volumes is minimum if x is equal to the three times the radius of sphere.**

A **[NCERT Exemplar]**

[Foreign 2016] [O.D. Set I, II, III Comptt. 2015]

Sol. Given $\quad S = 4\pi r^2 + 2\left[\frac{x^2}{3} + 2x^2 + \frac{2x^2}{3}\right]$

$\quad S = 4\pi r^2 + 6x^2 \text{ Or } x^2 = \frac{S - 4\pi r^2}{6}$ 1

and $\quad V = \frac{4}{3}\pi r^3 + \frac{2x^3}{3}$

$\therefore \quad V = \frac{4}{3}\pi r^3 + \frac{2}{3}\left(\frac{S-4\pi r^2}{6}\right)^{3/2}$ 1

$\quad \frac{dV}{dr} = 4\pi r^2 + \left(\frac{S-4\pi r^2}{6}\right)^{½}\left(\frac{-8\pi r}{6}\right)$ 1

$\quad \frac{dV}{dr} = 0$ 1

or $\quad r = \sqrt{\frac{S}{54+4\pi}}$ 1

Now $\quad \frac{d^2V}{dr^2} = 8\pi r + \left(\frac{-8\pi}{6}\right)\left(\frac{S-4\pi r^2}{6}\right)^{½} + \frac{1}{2}\left(\frac{S-4\pi r^2}{6}\right)^{-½}\cdot\left(\frac{-8\pi r}{6}\right)$

at $\quad r = \sqrt{\frac{S}{54+4\pi}};\ \frac{d^2V}{dr^2} > 0$

$\therefore$ for $\quad r = \sqrt{\frac{S}{54+4\pi}}$ volume is minimum

i.e., $\quad r^2(54 + 4\pi) = S$

or $\quad r^2(54 + 4\pi) = 4\pi r^2 + 6x^2$

or $\quad 6x^2 = 54r^2$

or $\quad x^2 = 9r^2$

or $\quad x = 3r$ 1

[CBSE Marking Scheme 2015]

Commonly Made Error

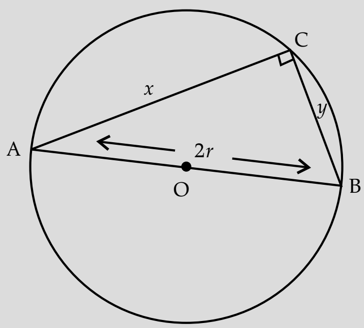
- The required sum of areas of the two parts has to be expressed correctly and in terms of single variable. Sometimes Candidates fail in this.

Answering Tips

- Obtaining the result in the required form needs simplification, cross-multiplication, substitution., etc.

Q. 16. *AB* is a diameter of a circle and *C* is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles. A **[O.D. 2017] [O.D. Set I Comptt. 2014] [NCERT Exemplar]**

Sol.



1

Let the sides of rt. ΔABC be x and y. 1

$\therefore \quad x^2 + y^2 = 4r^2$

and $\quad A = \text{Area of } \Delta = \dfrac{1}{2}xy$ ½

Let, $\quad S = A^2 = \dfrac{1}{4}x^2y^2$

$= \dfrac{1}{4}x^2(4r^2 - x^2)$

$= \dfrac{1}{4}(4r^2x^2 - x^4)$ 1

$\therefore \quad \dfrac{dS}{dx} = \dfrac{1}{4}[8r^2x - 4x^3]$ ½

or $\quad \dfrac{dS}{dx} = 0$ Or $x^2 = 2r^2$ or $x = \sqrt{2}r$ 1

and $\quad y^2 = 4r^2 - 2r^2 = 2r^2$ or $y = \sqrt{2}r$

i.e. $x = y$ and $\dfrac{d^2S}{dx^2} = (2r^2 - 3x^2) = 2r^2 - 6r^2 < 0$ 1

or Area is maximum, when Δ is isosceles.

[CBSE Marking Scheme 2014]

Q. 17. Of all the closed right circular cylindrical cans of volume 128 π cm^3, find the dimensions of the can which has minimum surface area. R&U **[Delhi Set III, 2014]**

Sol. Let r and h be the radius and height of the cylinder, then, Volume of cylinder (V)

$\pi r^2 h = 128\pi$

$\therefore \quad h = \dfrac{128\pi}{\pi r^2} = \dfrac{128}{r^2}$ 1

Surface area of cylinder

$= 2\pi r^2 + 2\pi rh$

$= 2\pi(r^2 + rh)$ 1

$\therefore \quad S = 2\pi\left(r^2 + \dfrac{128}{r}\right)$

$\therefore \quad \dfrac{dS}{dr} = 2\pi\left(2r - \dfrac{128}{r^2}\right)$ 1½

$\dfrac{dS}{dr} = 0$

or $\quad r^3 = 64$

or $\quad r = 4$ ½

At $\quad r = 4; \dfrac{d^2S}{dr^2} = 2\pi\left(2 + \dfrac{256}{r^3}\right)$

$= 2\pi\left(2 + \dfrac{256}{64}\right)$

$= 12\pi > 0$ 1

$\therefore$ Surface area is minimum at $r = 4$ cm;

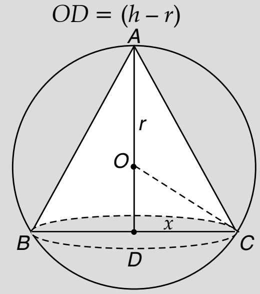
$h = 8$ cm 1

[CBSE Marking Scheme 2014]

AI **Q. 18. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius *r* is $\dfrac{4r}{3}$. Also show that the maximum volume of the cone is $\dfrac{8}{27}$ of the volume of the sphere.** A **[Delhi Comptt., 2010] [O.D. Set I, 2014] [O.D. Comptt. Set II, 2013] [NCERT] [Delhi Set I, II, III 2016] [Delhi Set I, II, III Comptt. 2016]**

Sol. Let radius of cone be x and its height be h.

$\therefore \quad OD = (h - r)$ ½



Volume of cone (V)

$= \dfrac{1}{3}\pi x^2 h$ ...(i) ½

In ΔOCD, $x^2 + (h - r)^2 = r^2$ or $x^2 = r^2 - (h - r)^2$

$\therefore \quad V = \dfrac{1}{3}\pi h\{r^2 - (h - r)^2\}$ 1

$= \dfrac{1}{3}\pi(-h^3 + 2h^2 r)$ 1

or $\quad \dfrac{dV}{dh} = \dfrac{\pi}{3}(-3h^2 + 4hr)$ 1

$\therefore \quad \dfrac{dV}{dh} = 0$ or $h = \dfrac{4r}{3}$

$\dfrac{d^2V}{dh^2} = \dfrac{\pi}{3}(-6h + 4r)$

$$= \frac{\pi}{3}\left(-6\left(\frac{4r}{3}\right)+4r\right)$$

$$= -\frac{4\pi r}{3} < 0 \quad \mathbf{1}$$

$\therefore$ at $h = \frac{4r}{3}$, Volume is maximum

Maximum volume

$$= \frac{1}{3}\pi\cdot\left\{-\left(\frac{4r}{3}\right)^3+2\left(\frac{4r}{3}\right)^2 r\right\}$$

$$= \frac{8}{27}\cdot\left(\frac{4}{3}\pi r^3\right) \quad \mathbf{1}$$

$$= \frac{8}{27} \text{ (volume of sphere)}$$

[CBSE Marking Scheme 2014]

AI **Q. 19. Prove that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.**

A **[NCERT]**
[O.D. Set II Comptt. 2012]
[O.D. Set II, 2014]
[O.D. Set I, II, III Comptt. 2013]
[Delhi Set III Comptt. 2012]

Sol. Let the radius and height of cylinder be r and h respectively

$\therefore$ $V = \pi r^2 h$ ...(i)

But $r^2 = R^2 - \frac{h^2}{4}$ **1**

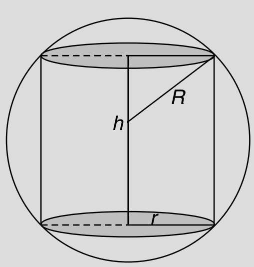
$\therefore$ $\pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\left(R^2 h - \frac{h^3}{4}\right)$ **1**

or $\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$ **½**

For maximum or minimum

$\therefore$ $\frac{dV}{dh} = 0$ or $h^2 = \frac{4R^2}{3}$

or $h = \frac{2R}{\sqrt{3}}$ **½+1**



and $\frac{d^2V}{dh^2} = \pi\left(-\frac{6h}{4}\right) < 0$ **1**

Maximum volume $= \pi\cdot\left[R^2\cdot\frac{2R}{\sqrt{3}} - \frac{1}{4}\left(\frac{2R}{\sqrt{3}}\right)^3\right]$

$= \frac{4\pi R^3}{3\sqrt{3}}$ cubic units **1**

[CBSE Marking Scheme 2014]

Answering Tip

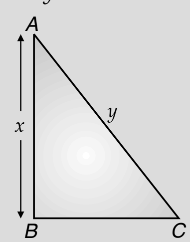
- Differentiation rules for different functions and forms need continuous revision and practice. Obtaining the result in the required form needs simplification, cross multiplication, substitution, etc. hence enough revision and practice is required.

AI **Q. 20. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of triangle is maximum when the angle between them is 60° $\left(i.e., \frac{\pi}{3}\right)$.**

R&U **[Delhi 2017][O.D. Set III, 2014]**
[NCERT Exemplar] [O.D. 2009]

Sol. Let the length of the side AB of rt. ΔABC be x and that of hypotenuse AC be y, and

$x + y = k$ (given) **1**



Area of $\Delta ABC = \frac{1}{2}\sqrt{y^2 - x^2}\cdot x$ **1**

$\therefore$ Let $S = \frac{1}{4}x^2(y^2 - x^2)$

$= \frac{1}{4}x^2[(k-x)^2 - x^2]$

$= \frac{1}{4}[k^2x^2 - 2kx^3]$ **1**

$\frac{dS}{dx} = 0$

or $\frac{1}{4}(2k^2x - 6kx^2) = 0$ or $x = \frac{k}{3}$ **1**

and at $x = \frac{k}{3}$, $\frac{d^2S}{dx^2} = \frac{1}{4}(2k^2 - 12kx)$

$= \frac{1}{4}(2k^2 - 4k^2) < 0$ **1**

$\therefore$ Area of Δ is maximum for

$x = \frac{k}{3}$

and $y = k - \frac{k}{3} = \frac{2k}{3}$

$\therefore \quad \cos\theta = \frac{x}{y} = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{3}$ **1**

[CBSE Marking Scheme 2014]

Q. 21. The sum of the perimeter of a circle and a square is *K*, where *K* is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle. R&U **[NCERT]**

[Delhi Set I Comptt. 2014]

OR

Given the sum of the perimeters of a square and a circle. Show that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

[SQP 2011][Delhi Set III Comptt. 2012]

Sol. Let *r* be the radius and *x* be the side of the square

$S = \pi r^2 + x^2,$

where $2\pi r + 4x = k$ **1**

$$S = \pi r^2 + \left(\frac{k - 2\pi r}{4}\right)^2$$ **1**

$$\frac{dS}{dr} = 2\pi r + \frac{\pi^2 r}{2} - \frac{k\pi}{4}$$ ½

$$\frac{dS}{dr} = 0$$

or $$r = \frac{k}{2(\pi + 4)}$$ ½

$$\frac{d^2S}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0 \text{ Or minima}$$ **1**

Hence *S* is least when

$$r = \frac{k}{2(\pi + 4)}$$ **1**

$$x = \frac{1}{4}\left(k - 2\pi\frac{k}{2(\pi + 4)}\right)$$

or $$x = \frac{k}{\pi + 4} = 2r$$ **1**

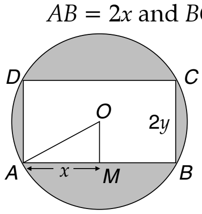
[CBSE Marking Scheme 2014]

Q. 22. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area. A **[NCERT]**

[Delhi Set I, 2013]

Sol. Let *ABCD* be a rectangle inscribed in a given circle with centre at *O* and radius *a*.

Let $AB = 2x$ and $BC = 2y$



1

Then, $OA^2 = OM^2 + AM^2$

or $a^2 = y^2 + x^2$

or $y = \sqrt{a^2 - x^2}$ ...(i)

Let *A* be the area of the rectangle.

$\therefore \quad A = 4xy = 4x\sqrt{a^2 - x^2}$ **1**

$$\frac{dA}{dx} = 4\left[\frac{\sqrt{a^2 - x^2} - x^2}{\sqrt{9^2 - x^2}}\right]$$

or $$\frac{dA}{dx} = 4\left\{\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}\right\}$$

For maximum or minimum value of *A*,

$$\frac{dA}{dx} = 0$$

$$\therefore \quad 4\left\{\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}\right\} = 0 \quad \text{Or } x = \frac{a}{\sqrt{2}}$$ **1**

Now, $$\frac{d^2A}{dx^2} = 4\frac{d}{dx}\left\{(a^2 - 2x^2)(a^2 - x^2)^{-1/2}\right\}$$

or $$\frac{d^2A}{dx^2} = 4[-4x(a^2 - x^2)^{-1/2} + (a^2 - 2x^2)$$

$$\times(-1/2)(a^2 - x^2)^{-3/2}(-2x)]$$ **1**

$$= 4\left[\frac{-4x}{\sqrt{a^2 - x^2}} + \frac{x(a^2 - 2x^2)}{(a^2 - x^2)^{3/2}}\right]$$

$$\therefore \quad \left(\frac{d^2A}{dx^2}\right)_{x = \frac{a}{\sqrt{2}}} = -16 < 0$$ **1**

Thus *A* is maximum, when $x = \frac{a}{\sqrt{2}}$

Putting $x = \frac{a}{\sqrt{2}}$ in (i), $y = \frac{a}{\sqrt{2}}$.

Therefore $x = y = \frac{a}{\sqrt{2}}$

Hence area is maximum, when $x = y$ Or $2x = 2y$

i.e., the rectangle is a square. **1**

Q. 23. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

A **[NCERT Exemplar] [Delhi Set I, II, III, 2013]**

Sol. Let *ABCD* be a rectangle having area *A* inscribed in an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ ...(i)

Let the co-ordinate of *A* be (α, β)

$\therefore$ Co-ordinate of $B \equiv (\alpha, -\beta)$

$C \equiv (-\alpha, -\beta)$ ½

$D \equiv (-\alpha, \beta)$

Now $A = \text{Length} \times \text{Breadth}$

$= 2\alpha \times 2\beta$

or $A = 4\alpha\beta$

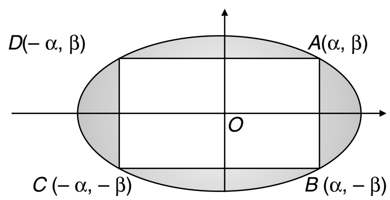
or $A = 4\alpha\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}$ ½

$$\left[\begin{array}{l}\because (\alpha, \beta) \text{ lies on ellipse (i)} \\ \therefore \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \; i.e., \beta=\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}\end{array}\right]$$

or $A^2 = 16\alpha^2\left\{b^2\left(1-\frac{\alpha^2}{a^2}\right)\right\}$ ½

or $A^2 = \frac{16b^2}{a^2}(a^2\alpha^2 - \alpha^4)$

or $\frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha - 4\alpha^3)$ **1**



½

For maximum or minimum value

$$\frac{d(A^2)}{d\alpha} = 0$$

or $2a^2\alpha - 4\alpha^3 = 0$

or $2\alpha(a^2 - 2\alpha^2) = 0$

or $\alpha = 0,\ \alpha = \frac{a}{\sqrt{2}}$ **1**

Again $\frac{d^2(A^2)}{d\alpha^2} = \frac{16b^2}{a^2}(2a^2 - 12\alpha^2)$

or $\left[\frac{d^2(A^2)}{d\alpha^2}\right]_{\alpha=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left(2a^2 - 12\times\frac{a^2}{2}\right) < 0$ **1**

or For $\alpha = \frac{a}{\sqrt{2}}$, A is maximum.

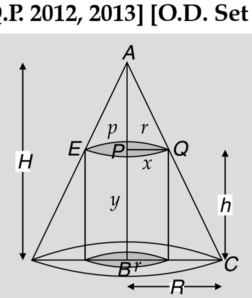
i.e., for greatest area A

$$\alpha = \frac{a}{\sqrt{2}} \text{ and } \beta = \frac{b}{\sqrt{2}} \quad \text{[using (i)]}$$

$\therefore$ Greatest area $= 4\alpha\cdot\beta = 4\frac{a}{\sqrt{2}}\times\frac{b}{\sqrt{2}} = 2ab.$ **1**

Q. 24. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. A **[NCERT]**
[S.Q.P. 2012, 2013] [O.D. Set I, II, III, 2012]

Sol.



1

Triangles APQ and ABC are similar, are

or $\frac{H-h}{r} = \frac{H}{R}$

or $h = \frac{RH - rH}{R} = \frac{R-r}{R}\cdot H$

or $h = \left(1-\frac{r}{R}\right)H$ **1**

$S = 2\pi rh$

S = curved surface is $= 2\pi r\left(1-\frac{r}{R}\right)H$

$= 2\pi H\left(r - \frac{r^2}{R}\right)$ **1½**

or $\frac{dS}{dr} = 2\pi H\left(1-\frac{2r}{R}\right)$

or Putting $\frac{dS}{dr} = 0$ or $r = \frac{R}{2}$ **1 + ½**

Finding $\frac{d^2S}{dr^2}$ as negative or maximum **1**

$\therefore$ for maximum curved surface area of cylinder, its radius is half that of the cone.

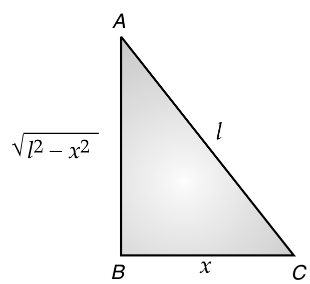
[CBSE Marking Scheme 2013]

Answering Tips

- Give adequate practice on mensuration related concepts and problems.

Q. 25. Prove that area of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles. A **[Delhi Set I Comptt. 2012]**

Sol. Let one of the side of right angled triangle be x and hypotenuse be l, then other side is $\sqrt{l^2 - x^2}$.



1

$\therefore$ Area of triangle $= A(x)$

$= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times x \times \sqrt{l^2 - x^2}$

or $A'(x) = \frac{\sqrt{l^2-x^2}}{2} + \frac{x}{2}(-2x)\times\frac{1}{2}\frac{1}{\sqrt{l^2-x^2}}$ **1**

$= \frac{\sqrt{l^2-x^2}}{2} - \frac{x^2}{2\sqrt{l^2-x^2}}$

$$= \frac{l^2 - x^2 - x^2}{2\sqrt{l^2 - x^2}} = \frac{l^2 - 2x^2}{2\sqrt{l^2 - x^2}}$$

For maxima or minima, put $A'(x) = 0$ **1**

or $\frac{l^2 - 2x^2}{2\sqrt{l^2 - x^2}} = 0$

or $l^2 - 2x^2 = 0$ **1**

or $2x^2 = l^2$

or $x^2 = \frac{l^2}{2}$

$x = \frac{l}{\sqrt{2}}$ (as length cannot be –ve)

other side $AB = \sqrt{l^2 - x^2}$

$= \sqrt{l^2 - \frac{l^2}{2}} = \sqrt{\frac{l^2}{2}}$

$= \frac{l}{\sqrt{2}}$

$$A''(x) = \frac{-4x}{2\sqrt{l^2 - x^2}} + (l^2 - 2x^2) \times (-2x) \times \frac{1}{2}\frac{1}{(\sqrt{l^2 - x^2})^3}$$ **1**

$$[A''(x)]_{x=\frac{l}{\sqrt{2}}} = \frac{-2 \times \frac{l}{\sqrt{2}}}{\frac{l}{\sqrt{2}}} - \left(l^2 - 2 \times \frac{l^2}{2}\right) \times \frac{2l}{\sqrt{2}} \times \frac{1}{2}\frac{1}{\left(l^2 - \frac{l^2}{2}\right)^{3/2}}$$

$= -2 - 0 = -2 < 0$ ½

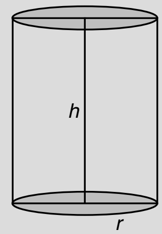
Hence, the area of given right angled triangle is maximum when $x = \frac{l}{\sqrt{2}}$ *i.e.*, when triangle is isosceles. ½

Q. 26. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base.

A [Delhi Set I, II, III, 2012] [NCERT] [Delhi 2010] [Delhi Comptt. 2009]

Sol. Surface area $A = 2\pi rh + 2\pi r^2$

Or $h = \frac{A - 2\pi r^2}{2\pi r}$ ...(i) ½



$V = \pi r^2 h$

$= \pi r^2 \left(\frac{A - 2\pi r^2}{2\pi r}\right)$ **1**

$= \frac{1}{2} \cdot [Ar - 2\pi r^3]$ ½

or $\frac{dV}{dr} = \frac{1}{2}[A - 6\pi r^2]$ **1**

For maximum V,

$\frac{dV}{dr} = 0$ **1**

or $A - 6\pi r^2 = 2\pi rh + 2\pi r^2 - 6\pi r^2 = 0$ **1**

or $4\pi r^2 = 2\pi rh$

or $h = 2r$ = diameter

$\frac{d^2V}{dr^2} = \frac{1}{2}[-12\pi r] < 0$

$\therefore$ $h = 2r$ will give max. volume. **1**

[CBSE Marking Scheme 2012]

Commonly Made Error

- Some candidates consider surface area as function instead of volume. Few candidates do not apply proper sign though it.

Answering Tips

- Emphasise on the difference between condition for maxima and minima.

Q. 27. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. **A [OD Set I, 2017]**

Sol. Let the sides of cuboid be x, x, y

or $x^2 y = k$ and $S = 2(x^2 + xy + xy) = 2(x^2 + 2xy)$ ½+**1**

$\therefore$ $S = 2\left[x^2 + 2x\frac{k}{x^2}\right] = 2\left[x^2 + \frac{2k}{x}\right]$ **1**

$\frac{dS}{dx} = 2\left[2x - \frac{2k}{x^2}\right]$ **1**

$\therefore$ $\frac{dS}{dx} = 0$ Or $x^3 = k = x^2 y$ $x = y$ **1**

$\frac{d^2S}{dx^2} = 2\left[2 + \frac{4k}{x^3}\right] > 0$

$\therefore$ $x = y$ will given minimum surface area **1**

and $x = y$, means sides are equal

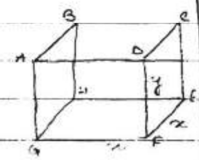
$\therefore$ Cube will have minimum surface area ½

[CBSE Marking Scheme 2017]

OR

Let a closed cuboid have a base of $x \times x$ and a height y

Let its volume be V and surface area be S



$V = x^2 y$ $\qquad S = 2(x^2 + 2xy)$

$y = \frac{V}{x^2}$ $\qquad = 2x^2 + 4xy$

$= 2x^2 + 4x\frac{V}{x^2}$

$= 2x^2 + \frac{4V}{x}$

$\frac{dS}{dx} = \frac{d}{dx}\left(2x^2 + \frac{4V}{x}\right) = 4x + \frac{4V(-1)}{x^2} = 4x - \frac{4V}{x^2}$

$\frac{d^2S}{dx^2} = 4 - \frac{4V(-2)}{x^3} = 4 + \frac{8V}{x^3}$

To maximize or minimize S, $\frac{dS}{dx} = 0$

$4x - \frac{4V}{x^2} = 0$ $\qquad \left.\frac{d^2S}{dx^2}\right|_{x = V^{1/3}} = 4 + \frac{8V}{V} = 12$ which is > 0

$4x = 4V$

$x^3 = V$ $\qquad$ Hence S is minimum at $x = V^{1/3}$

$x = V^{1/3}$

$x = V^{1/3}$

$x^3 = V$

$x^3 = x^2 y$

$x^3 - x^2 y = 0$

$x^2(x - y) = 0$

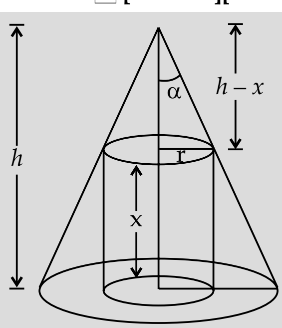
$x = y$

Hence the given cuboid is a cube of side x.

[Topper's Answer, 2017]

Q. 28. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi-vertical angle α, is one-third that of the cone. Hence find the greatest volume of the cylinder. A **[NCERT][Delhi Comptt., 2017]**

Sol.



1

$$\frac{r}{h-x} = \tan\alpha$$

$$r = (h-x)\tan\alpha \quad 1$$

Volume of cylinder

$$V = \pi r^2 x$$

$$V = \pi(h-x)^2 x \tan^2\alpha \quad ½$$

$$\frac{dV}{dx} = \pi\tan^2\alpha\,(h-x)(h-3x)$$

$$\frac{dV}{dx} = 0 \text{ or } h = x \text{ or } h = 3x$$

i.e., $\qquad x = \frac{h}{3}$ $\qquad$ 1½

$$\frac{d^2V}{dx^2} = \pi\tan^2\alpha(6x - 4h)$$

$\therefore \qquad \frac{d^2V}{dx^2} < 0$ at $x = \frac{h}{3}$ $\qquad$ 1

$\therefore$ V is maximum, at $x = \frac{h}{3}$

and maximum volume is $V = \frac{4}{27}\pi h^3 \tan^2\alpha$ $\qquad$ 1

[CBSE Marking Scheme 2017]

Q. 29. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

A **[O.D. Comptt., 2017, Delhi 2011]**
[NCERT]

Sol. Let given volume of cone be, $V = \frac{1}{3}\pi r^2 h$ ...(i)½

$\therefore$ Surface area (curved) $S = \pi rl = \pi r\sqrt{r^2+h^2}$ ½

or $A = S^2 = \pi^2 r^2 (r^2 + h^2)$

$$A = S^2 = \pi^2 r^2\left[r^2 + \left(\frac{3V}{\pi r^2}\right)^2\right]$$

[using (i)]

$$= \pi^2\left[r^4 + \frac{9V^2}{\pi^2 r^2}\right] \quad 1½$$

$$\frac{dA}{dr} = \pi^2\left[4r^3 - \frac{18V^2}{\pi^2 r^3}\right]$$

$$\frac{dA}{dr} = 0 \text{ or } 4\pi^2 r^6 = 18.\frac{1}{9}\pi^2 r^4 h^2 \quad 1$$

or $2r^2 = h^2$ or $h = \sqrt{2}r$ 1½

$$\frac{d^2A}{dr^2} = \pi^2\left[12r^2 + \frac{54V^2}{\pi^2 r^4}\right] > 0 \quad 1$$

or for least curved surface area, height = $\sqrt{2}$ (radius) **[CBSE Marking Scheme 2017]**

Q. 30. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum ? A **[Foreign 2017]**

Sol. Let the length of one piece be x m, then length of the other piece = $(34 - x)$ m

$\therefore$ Side of square is $\frac{x}{4}$ m then width of rectangle will be $\frac{34-x}{6}$ m. 1

Now, Area $(A) = \left(\frac{x}{4}\right)^2 + 2\left(\frac{34-x}{6}\right)^2$ 1

or $\frac{dA}{dx} = \frac{x}{8} - \frac{1}{9}(34-x)$ 1

$\frac{dA}{dx} = 0$ or $x = 16$ 1

also, $\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{1}{9} > 0$

so, A is minimum when $x = 16$ 1

$\therefore$ Lengths of the two pieces are 16 m and 18 m. 1

[CBSE Marking Scheme, 2017]

Q. 31. Find the minimum value of $(ax + by)$, where $xy = c^2$.

A **[Foreign 2015]**

Sol. Let $f(x) = ax + by$, whose minimum value is required.

Then, $f(x) = ax + \frac{bc^2}{x}$

$\left[\text{given, } xy = c^2 \text{ or } y = \frac{c^2}{x}\right]$

On differentiating both sides w.r.t. x, we get

$$f'(x) = a - \frac{bc^2}{x^2} \quad 1$$

For maxima or minima, put $f'(x) = 0$

or $a - \frac{bc^2}{x^2} = 0$

or $a = \frac{bc^2}{x^2}$ or $x^2 = \frac{bc^2}{a}$ or $x = \pm\sqrt{\frac{b}{a}}c$ 1

Now, $f''(x) = 0 + \frac{2bc^2}{x^3}$

At, $x = \sqrt{\frac{b}{a}}c$, $f''(x) = \frac{2bc^2}{\left(\sqrt{\frac{b}{a}}c\right)^3} = +ve$

Hence, $f(x)$ has minimum value at $x = \sqrt{\frac{b}{a}}\cdot c$

and, At $x = -\sqrt{\frac{b}{a}}c$, $f(x) = \frac{2bc^2}{\left(-\sqrt{\frac{b}{a}}c\right)^3} = -ve$

Hence, $f(x)$ has maximum value at $x = -\sqrt{\frac{b}{a}}c$. 1

When $x = \sqrt{\frac{b}{a}}c$, then

$$y = \frac{c^2}{x} = \frac{c^2}{\left(\sqrt{\frac{b}{a}}c\right)} = \sqrt{\frac{a}{b}}c \quad 1$$

$\therefore$ Minimum value of $f(x) = a\sqrt{\frac{b}{a}}.c + b\sqrt{\frac{a}{b}}.c$

$= \sqrt{ab}.c + \sqrt{ab}.c$

$= 2\sqrt{ab}.c$ 1

Q. 32. Show that of all the rectangles with a given perimeter, the square has the largest area.

A **[Delhi 2011]**

Sol. Let x and y be the lengths of two sides of a rectangle. Again, let P denotes its perimeter and A be the area of rectangle.

Then, $P = 2(x + y)$ 1

[$\because$ perimeter of rectangle = $2(l + b)$]

or $P = 2x + 2y$

or $y = \frac{P-2x}{2}$ ...(i) **1**

We know that, area of rectangle is given by

$A = xy$

or $A = x\left(\frac{P-2x}{2}\right)$ [by using Eq. (i)]

or $A = \frac{Px-2x^2}{2}$ **1**

On differentiating both sides w.r.t. x, we get

$$\frac{dA}{dx} = \frac{P-4x}{2}$$ **1**

For maxima or minima, put $\frac{dA}{dx} = 0$

or $\frac{P-4x}{2} = 0$ or $P = 4x$

or $2x + 2y = 4x$ [$\because P = 2x + 2y$]

or $x = y$

So, the rectangle is a square. 1

Also, $\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{P-4x}{2}\right)$

$= -\frac{4}{2} = -2 < 0$

or A is maximum.

Hence, area is maximum, when rectangle is a square. **Hence proved 1**

Q. 33. Show that of all the rectangles of given area, the square has the smallest perimeter. A [Delhi 2011]

Sol. Let x and y be the lengths of sides of a rectangle. Again, let A denote its area and P be the perimeter.

Now, area of rectangle, $A = xy$

or $y = \frac{A}{x}$ ...(i) **1**

And $P = 2(x + y)$

[$\because$ perimeter of rectangle $= 2(l + b)$]

or $P = 2\left(x + \frac{A}{x}\right)$

$\left[\because y = \frac{A}{x}, \text{from Eq. (i)}\right]$ **1**

On differentiating both sides w.r.t. x, we get

$$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$$ **1**

For maxima or minima, put $\frac{dP}{dx} = 0$.

or $2\left(1-\frac{A}{x^2}\right) = 0$ or $1 = \frac{A}{x^2}$ **1**

or $A = x^2$

or $xy = x^2$ [$\because A = xy$]

$\therefore$ $x = y$ **1**

Also, $\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2\left(1-\frac{A}{x^2}\right)\right] = 2\left(\frac{2A}{x^3}\right) = \frac{4A}{x^3} > 0$

Here, x and A being the side and area of rectangle can never be negative. So, P is minimum.

Hence, perimeter of rectangle is minimum, when rectangle is a square. **1**

Q. 34. Find the point on the curve $y^2 = 2x$ which is at a minimum distance from the point (1, 4).

A [All India 2011, 2009 C]

Sol. The given equation of curve is $y^2 = 2x$ and the given point is $Q(1, 4)$.

Let $P(x, y)$ be the point, which is at a minimum distance from point $Q(1, 4)$. **1**

Now, distance between points P and Q is given by

$$PQ = \sqrt{(1-x)^2 + (4-y)^2}$$

$\left[\text{using distance formula } d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}\right]$

or $PQ = \sqrt{1 + x^2 - 2x + 16 + y^2 - 8y}$

$= \sqrt{x^2 + y^2 - 2x - 8y + 17}$

On squaring both sides, we get

$PQ^2 = x^2 + y^2 - 2x + 8y + 7$

or $PQ^2 = \left(\frac{y^2}{2}\right)^2 + y^2 - 2\left(\frac{y^2}{2}\right) - 8y + 17$

$\left[\text{given, } y^2 = 2x \text{ or } x = \frac{y^2}{2}\right]$

$\therefore$ $PQ^2 = \frac{y^4}{4} + y^2 - y^2 - 8y + 17$

or $PQ^2 = \frac{y^4}{4} - 8y + 17$ **1**

Let $PQ^2 = Z$

Then, $Z = \frac{y^4}{4} - 8y + 17$

On differentiating both sides w.r.t. y, we get

$$\frac{dZ}{dy} = \frac{4y^3}{4} - 8 = y^3 - 8$$ **1**

For maxima or minima, put $\frac{dz}{dy} = 0$

or $y^3 - 8 = 0$ or $y^3 = 8$

or $y = 2$ **1**

Also, $\frac{d^2Z}{dy^2} = \frac{d}{dy}(y^3 - 8) = 3y^2$

On putting $y = 2$, we get

$$\left(\frac{d^2Z}{dy^2}\right)_{y=2} = 3(2)^2 = 12 > 0$$

$\because \quad \frac{d^2Z}{dy^2} > 0$

$\therefore$ Z is minimum and therefore PQ is also minimum as $Z = PQ^2$. **1**

On putting $y = 2$ in the given equation, *i.e.*, $y^2 = 2x$, we get

$$(2)^2 = 2x$$

or $\quad 4 = 2x$

or $\quad x = 2$

Hence, the point which is at a minimum distance from point (1, 4) is $P(2, 2)$. **1**

Q. 35. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, with its vertex at one end of the major axis.

A [Delhi 2010 C] [NCERT]

Sol. Given equation of ellipse is

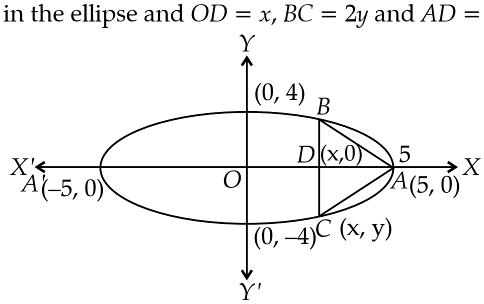
$$\frac{x^2}{25}+\frac{y^2}{16} = 1.$$

Here, $a = 5, b = 4$

$\therefore \quad a > b$

So, major axis is along X-axis.

Let ΔABC be the isosceles triangle which is inscribed in the ellipse and $OD = x$, $BC = 2y$ and $AD = 5 - x$.



Let A denotes the area of triangle. Then, we have

$$A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times BC \times AD$$

or $\quad A = \frac{1}{2} .2y\,(5-x)$ or $A = y(5-x)$ **1**

On squaring both sides, we get

$$A^2 = y^2(5-x)^2 \qquad ...(i)$$

Now, $\quad \frac{x^2}{25}+\frac{y^2}{16} = 1$

or $\quad \frac{y^2}{16} = 1-\frac{x^2}{25}$

or $\quad y^2 = \frac{16}{25}(25-x^2)$

On putting value of y^2 in Eq. (i). we get

$$A^2 = \frac{16}{25}(25-x^2)(5-x)^2$$

Let $\quad A^2 = Z$

Then, $\quad Z = \frac{16}{25}(25-x^2)(5-x)^2$ **1**

On differentiating both sides w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{16}{25}[(25-x^2)2(5-x)(-1)+(5-x)^2(-2x)]$$

[by using product rule of derivative]

$$= 16\,(-2)\,(5-x)\,\{25-x^2+5x-x^2\}$$

$$= \frac{16}{25}(-2)(5-x)^2(2x+5)$$

$$= -\frac{32}{25}(5-x)^2(2x+5)$$ **1**

For maxima or minima, put $\frac{dZ}{dx} = 0$

or $\quad -\frac{32}{25}(5-x)^2(2x+5) = 0$ or $x = 5$ or $-\frac{5}{2}$ **1**

Now, when $x = 5$, then

$$Z = \frac{16}{25}(25-25)(5-5)^2 = 0$$

which is not possible.

So, $x = 5$ is rejected.

$\therefore \quad x = -\frac{5}{2}$

Now, $\quad \frac{d^2Z}{dx^2} = \frac{d}{dx}\left(\frac{dZ}{dx}\right)$

$$= \frac{d}{dx}\left(-\frac{32}{25}(5-x)^2(2x+5\right)$$

$$= -\frac{32}{25}[(5-x)^2.2-(2x+5)2(5-x)]$$

$$= -\frac{64}{25}(5-x)(-3x) = \frac{192x}{25}(5-x)$$

$\therefore$ At $x = \frac{-5}{2}, \left(\frac{d^2Z}{dx^2}\right)_{x=-\frac{5}{2}} < 0$

or Z is maximum. 1

$\therefore$ Area A is maximum, when $x = -\frac{5}{2}$ and $y = 12$

Also, the maximum area is

$$Z = A^2 = \frac{16}{25}\left(25-\frac{25}{4}\right)\left[5+\frac{5}{2}\right]^2$$

$$= \frac{16}{25}\times\frac{75}{4}\times\frac{225}{4} = 3 \times 225$$

Hence, the maximum area, $A = \sqrt{3\times 225}$

$$= 15\sqrt{3} \text{ sq units.}$$ **1**

Q. 36. A manufacturer can sell x items at a price of ₹ $\left(5-\frac{x}{100}\right)$ each. The cost price of x items is ₹ $\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to reach maximum profit. A **[NCERT] [All India 2009]**

Sol. Given that manufacturer sells x items at a price of ₹ $\left(5-\frac{x}{100}\right)$ each.

∴ Total revenue obtained

$$= ₹\left[x\left(5-\frac{x}{100}\right)\right] = ₹\left(5x-\frac{x^2}{100}\right) \quad \mathbf{1}$$

Also, cost price of x items = ₹ $\left(\frac{x}{5}+500\right)$

Let $P(x)$ be the profit function.

We know that,

$$\text{Profit} = \text{Revenue} - \text{cost} \quad \mathbf{1}$$

$$\therefore \quad P = \left(5x-\frac{x^2}{100}\right)-\left(\frac{x}{5}+500\right)$$

$$\text{or} \quad P = \frac{-x^2}{100}+\frac{24x}{5}-500 \quad \mathbf{1}$$

On differentiating both sides w.r.t x, we get

$$\frac{dP}{dx} = -\frac{2x}{100}+\frac{24}{5}$$

For maxima or minima, put $\frac{dP}{dx}=0$

$$\text{or} \quad -\frac{2x}{100}+\frac{24}{5} = 0 \text{ or } x = 240 \quad \mathbf{1}$$

$$\text{Also,} \quad \frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}\left(-\frac{2x}{100}+\frac{24}{5}\right)$$

$$= -\frac{2}{100} = -\frac{1}{50} < 0$$

Thus, at $x = 240$, $\frac{d^2P}{dx^2} < 0$ or P is maximum. **1**

Hence, number of items sold to have maximum profit is 240. **1**

Q. 37. A tank with rectangular base and rectangular sides, open at the top is to be constructed, so that its depth is 2 m and volume is 8 m³. If building of tank cost ₹ 70 per sq m for the base and ₹ 45 per sq m for sides. What is the cost of least expensive tank ? A **[Delhi 2009] [NCERT]**

Sol. Let x m be the length, y m be the breadth and $h = 2$ m be the depth of the tank. Let ₹ H be the total cost for building the tank. Now, given that $h = 2$ m and volume of tank = 8 m^3.

Also, area of the rectangular base of the tank

$$= \text{length} \times \text{breadth} = xy \text{ m}^2 \quad \mathbf{1}$$

and the area of the four rectangular sides

$$= 2\,(\text{length} + \text{breadth}) \times \text{height}$$

$$= 2\,(x + y) \times 2 = 4(x + y) \text{ m}^2 \quad \mathbf{1}$$

∴ Total cost, $H = 70 \times xy + 45 \times 4(x + y)$

or $\quad H = 70xy + 180(x + y) \quad$ ...(i)

Also, volume of tank = 8 m^3

or $\quad l \times b \times h = 8$ or $x \times y \times 2 = 8$

or $\quad y = \frac{4}{x} \quad$ ...(ii) **1**

$$y = 280 + 180\left(x+\frac{4}{x}\right)$$

For maxima or minima, put $\frac{dH}{dx}=0$

or $180\left(1-\frac{4}{x^2}\right) = 0$ or $1-\frac{4}{x^2}=0$ or $\frac{4}{x^2}=1$

or $\quad x^2 = 4$

or $\quad x = 2 \quad [\because x > 0]$ **1**

$$\text{Also,} \quad \frac{d^2H}{dx^2} = \frac{d}{dx}\left(\frac{dH}{dx}\right) = \frac{d}{dx}\left[180\left(1-\frac{4}{x^2}\right)\right]$$

$$= \frac{8}{x^3}\times 180$$

$$\text{At } x = 2, \left[\frac{d^2H}{dx^2}\right]_{x=2} = \frac{8}{2^3}\times 180 = 180 > 0$$

$$\because \quad \frac{d^2H}{dx^2} > 0$$

or H is least at $x = 2$. **1**

Also, the least cost $= 280 + 180\left(2+\frac{4}{2}\right)$

[put $x = 2$ in Eq. (iii) to get least cost H]

= 280 + 180 × 4 = 280 + 720 = ₹ 1000

Hence, the cost of least expensive tank is ₹ 1000. **1**

Q. 38. Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.[OD Set III 2013] AE **[Delhi 2011] [NCERT]**

OR

Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. AE **[Delhi 2014, 2009]**

Sol. Let ABC be right-circular cone having radius 'r' and height 'h'. If V and S are its volume and surface area (curved) respectively, then

$$S = \pi rl$$

$$\text{or} \quad S = \pi r\sqrt{h^2+r^2} \quad \text{...(i)}$$

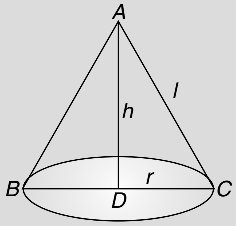
$$\text{Also,} \quad V = \frac{1}{3}\pi r^2 h \text{ or } h = \frac{3V}{\pi r^2}$$

Putting the value of h in (i), we get

$$S = \pi r\sqrt{\frac{9V^2}{\pi^2 r^4}+r^2}$$

or $$S^2 = \pi^2 r^2\left(\frac{9V^2+\pi^2 r^6}{\pi^2 r^4}\right) \quad \mathbf{1}$$

[Maxima or Minima is same for S or S^2]



or $$S^2 = 9\frac{V^2}{r^2}+\pi^2 r^4$$

Differentiating with respect to 'r' **1**

$$(S^2)' = \frac{-18V^2}{r^3}+4\pi^2 r^3 \quad \text{...(ii)}$$

Now, put $(S^2)' = 0$

or $$\frac{-18V^2}{r^3}+4\pi^2 r^3 = 0$$

or $$4\pi^2 r^6 = 18V^2$$

Putting value of V,

$$4\pi^2 r^6 = 18\times\frac{1}{9}\pi^2 r^4 h^2$$

or $$2r^2 = h^2$$

or $$r = \frac{h}{\sqrt{2}} \quad \mathbf{1}$$

Differentiating (ii) with respect to 'r', again

$$(S^2)'' = \frac{54V^2}{r^4}+12\pi^2 r^2$$

or $$(S^2)''\Big]_{r=\frac{h}{\sqrt{2}}} > 0 \quad \text{(for any value of } r)$$

Hence, S^2 *i.e.*, S is minimum for

$$r = \frac{h}{\sqrt{2}}$$

or $$h = \sqrt{2}r.$$

i.e., for least curved surface, altitude is equal to $\sqrt{2}$ times the radius of the base.

Then,

If θ is the semi-vertical angle of cone, then

$$\cot\theta = \frac{h}{r} = \frac{\sqrt{2}r}{r}$$

or $$\cot\theta = \sqrt{2}$$

or $$\theta = \cot^{-1}\left(\sqrt{2}\right) \quad \mathbf{1}$$

[CBSE Marking Scheme 2014]

Q. 39. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window. AE **[(O.D.) 2011]**

Sol. Let x and y be the dimensions of rectangular part of window and x be side of equilateral part.

If A be the total area of window, then

$$A = xy+\frac{\sqrt{3}}{4}x^2 \quad \text{...(i)}$$

Also, $x + 2y + 2x = 12$

or $3x + 2y = 12$

or $$y = \frac{12-3x}{2} \quad \mathbf{1}$$

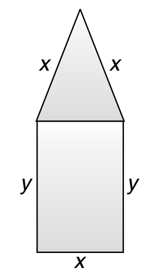
$$\therefore \quad A = x\cdot\frac{(12-3x)}{2}+\frac{\sqrt{3}}{4}x^2 \quad \text{...[From (i)]}$$

or $$A = 6x-\frac{3x^2}{2}+\frac{\sqrt{3}}{4}x^2$$

or $$A' = 6-3x+\frac{\sqrt{3}}{2}x \quad \mathbf{1}$$

[Differentiating with respect to x]

Now, for maxima or minima,



$$A' = 0$$

$$6-3x+\frac{\sqrt{3}}{2}x = 0$$

or $$x = \frac{12}{6-\sqrt{3}}$$

Again $$A'' = -3+\frac{\sqrt{3}}{2} < 0 \text{ (for any value of } x)$$

$$A''\Big]_{x=\frac{12}{6-\sqrt{3}}} = 0$$

i.e., A is maximum if

$$x = \frac{12}{6-\sqrt{3}}$$

and $$y = \frac{12-3\left(\frac{12}{6-\sqrt{3}}\right)}{2}. \quad \mathbf{1}$$

i.e., for largest area of window, dimensions of rectangle are

$$x = \frac{12}{6-\sqrt{3}} \text{ and } y = \frac{18-6\sqrt{3}}{6-\sqrt{3}}. \quad \mathbf{1}$$

AI **Q. 40. If length of three sides of a trapezium other than base are equal to 10 cm, then find the area of the trapezium when it is maximum.**

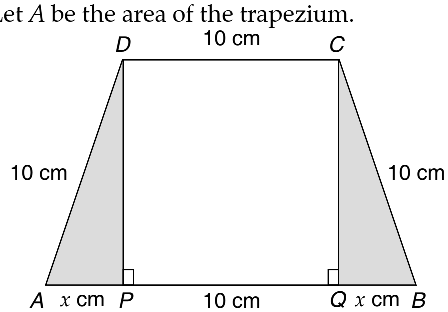
AE **[NCERT]**

[O.D. 2010, O.D. Comptt., 2014, Delhi Comptt., 2013]

Sol. The required trapezium is as given in figure. Draw perpendiculars DP and CQ on AB. Let $AP = x$ cm. Note that $\Delta APD \cong \Delta BQC$. Therefore, $QB = x$ cm. Also, by

Pythagoras theorem $DP = QC = \sqrt{100 - x^2}$.

Let A be the area of the trapezium.



Then, $A \equiv A(x)$ **1**

$$= \frac{1}{2}(\text{sum of parallel sides}) \times (\text{height})$$

$$= \frac{1}{2}(2x + 10 + 10)\sqrt{100 - x^2}$$

$$= (x + 10)\sqrt{100 - x^2}$$

or $A'(x) = (x+10)\dfrac{(-2x)}{2\sqrt{100-x^2}} + (\sqrt{100-x^2})$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$ **1**

Now, $A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$,

i.e., $x = 5$ and $x = -10$.

So, $x = 5$.

Now, $A''(x) =$

$$\frac{\sqrt{100-x^2}(-4x-10) - (-2x^2 - 10x + 100)\dfrac{(-2x)}{2\sqrt{100-x^2}}}{100 - x^2}$$

$$= \frac{2x^3 - 300x - 1000}{(100 - x^2)^{\frac{1}{2}}}$$

(on simplification)

or $A''(5) = \dfrac{2(5)^3 - 300(5) - 1{,}000}{(100 - (5)^2)^{\frac{3}{2}}}$ **1**

$$= \frac{-2{,}250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and the maximum area is given by

$$A(5) = (5 + 10)\sqrt{100 - (5)^2}$$

$$= 15\sqrt{75} = 75\sqrt{3} \text{ cm}^2$$ **1**

[AI] Q. 41. Find the maximum and minimum values of $f(x) = \sec x + \log \cos^2 x$, $0 < x < 2\pi$. [HOTS][O.D. Set II 2016]

Sol. Maximum & Minimum values of $f(x) = \sec x + \log\cos^2 x$

$f'(x) = 0$

$\therefore \sec x \tan x + \dfrac{2}{\cos x}(-\sin x) = 0$ $\quad x \neq \dfrac{\pi}{2}$

$\therefore \tan x(\sec x - 2) = 0$

$\therefore x = \pi$ or $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

Now $f''(x) < 0$ for maximum

$f''(x) > 0$ for minimum

$f''(x) = \sec^2 x(\sec x - 2) + \tan^2 x \sec x$

$f''(\pi) = 1 \times -3 + 0 = -3 < 0$

$f''\left(\dfrac{\pi}{3}\right) = f''\left(\dfrac{5\pi}{3}\right) = 0 + 3 \times 2 = 6 > 0$

Function attains maximum value at $x = \pi$ & minimum value at $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$f(\pi) = -1$

$f\left(\dfrac{\pi}{3}\right) = 2 - 2\log 2$

But when $x = \dfrac{\pi}{2}$ ∴ function becomes undefined.

~~Minimum & Maximum do not exist as Minimum~~

[Topper's Answer 2016]

Commonly Made Error

- Sometimes students forget to differentiate again the function which leads incorrect result.

Answering Tip

- Give ample practice to the problems based on maxima and minima.

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