MATHEMATICS

PARABOLA

SOLVING PROBLEMS ON FINDING THE VALUES OF TERMS ASSOCIATED TO PARABOLA

NOTES

What you already know

- Basics about conics
- Conditions for different conics
- Analytic interpretation of different conics
- Basic understanding of geometrical figures
- Basic introduction to parabola
- Standard equation of a parabola

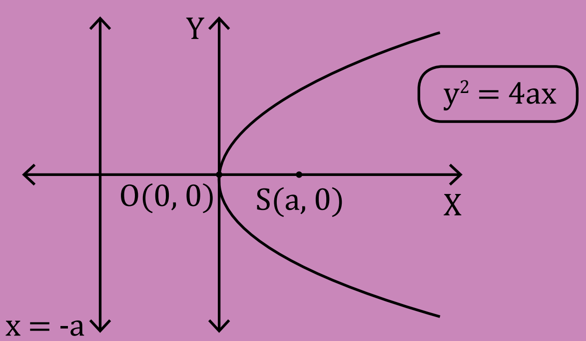
What you will learn

- Translation or shifting of vertex
- Problems based on some mixed concepts

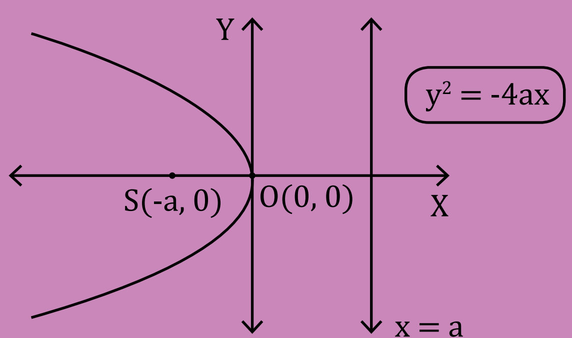
Standard Parabola : Recap

(a) Horizontal Parabola

Rightward opening

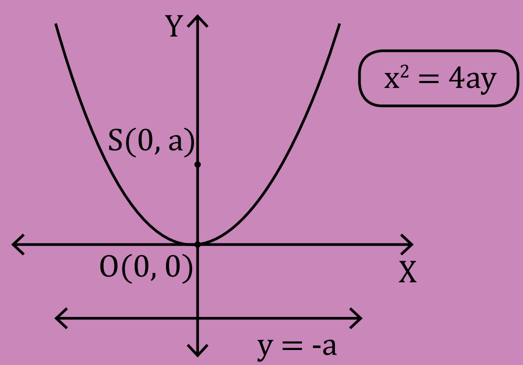


Leftward opening

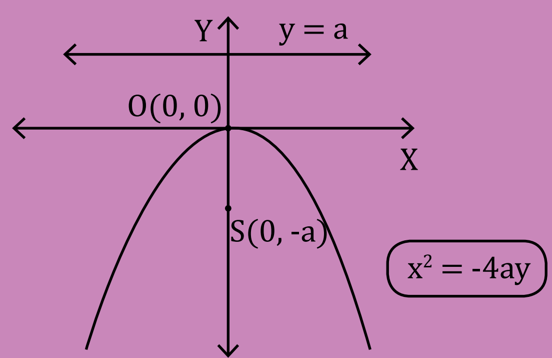


(b) Vertical Parabola

Upward opening



Downward opening



Different parameters associated with the given parabolas are as follows:

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Axis	X - axis	X - axis	Y - axis	Y - axis
Length of LR	$4a$	$4a$	$4a$	$4a$
Focal distance	$x + a$	$-x + a$	$y + a$	$-y + a$

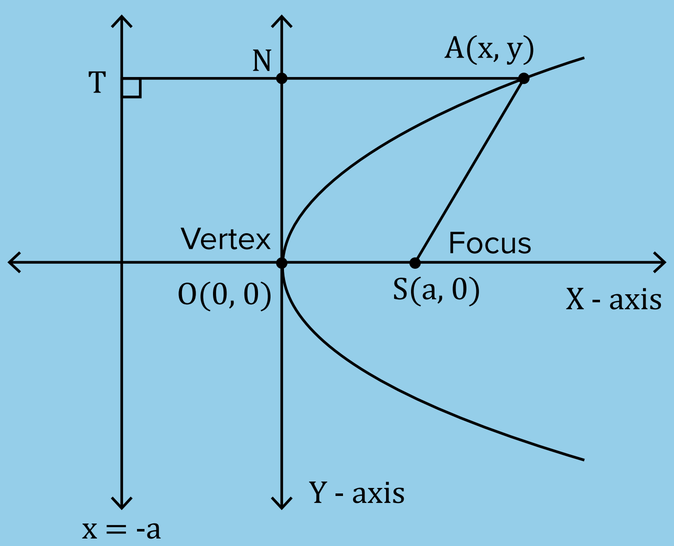
Note

Focal distance of any general point $A(x, y)$ on the parabola $y^2 = 4ax$ is given by $x + a$.

Let $A(x, y)$ be a point on the parabola $y^2 = 4ax$
Then,
Focal distance $= AS$
And
$AS = AT = AN + NT$
$AS = AT = x + a$

As $x \geq 0$, and $a > 0$
therefore $x + a$ always positive.

In a similar way focal distance for other standard parabolas can be obtained.



What is the equation of the parabola if its vertex is at the origin, axis is on the Y-axis, and it is passing through the point (6, -3)?

(a) $y^2 = 12x + 6$ **(b)** $x^2 = 12y$ **(c)** $x^2 = -12y$ **(d)** $y^2 = -12x + 6$

Solution

Step 1 :

Given, vertex ≡ origin (0, 0)

Axis of symmetry ≡ Y - axis
⇔ The parabola is upward or downward opening.

However, as the given parabola is passing through $A(6, -3) \rightarrow 4^{th}$ quadrant,

⇒ The parabola will be downward opening with equation $x^2 = -4ay$

Step 2 :

$(6, -3)$ lies on the parabola.
Therefore, $6^2 = -4a \times (-3) \Rightarrow a = 3$

∴ The equation will become,
$x^2 = -4 \times 3y$
$\Rightarrow x^2 = -12y$

Hence, option (c) is the correct answer.

Concept Check 1

For the parabola $x^2 = 16y$, find the following:

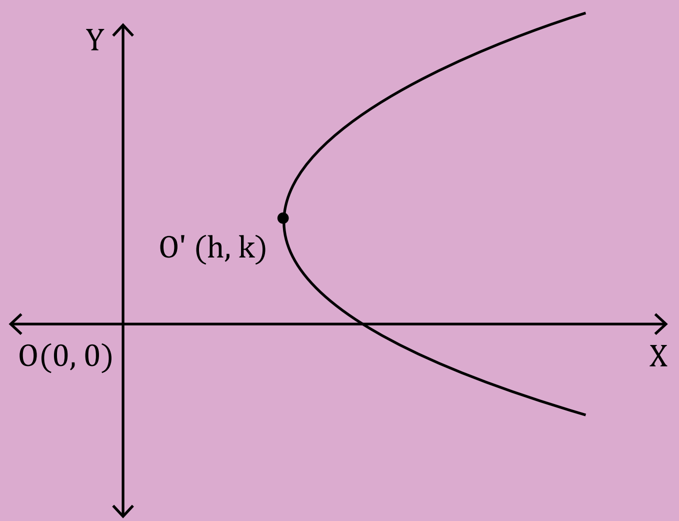
Focus, axis of parabola, equation of directrix, equation of latus rectum (LR), ends of latus rectum (LR), length of latus rectum (LR), and focal distance.

Shifting of Vertex

Let us consider a rightward opening parabola similar to $y^2 = 4ax$, which is translated to a new vertex having coordinates as (h, k).

Now, let us see how different parameters change after the shifting of vertex.
As we know that the rightward opening parabola has equation $y^2 = 4ax$,

After shifting to the new vertex, then the equation becomes $(y')^2 = 4A(x')$



Now, using

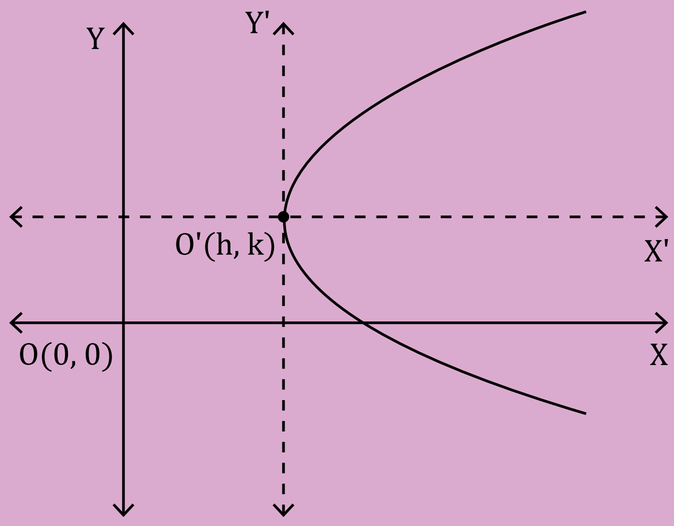
Shifting of origin
XY - plane = $(0, 0) \equiv (x, y)$
And X'Y' - plane = $(h, k) \equiv (x', y')$

Now, relations between the two given coordinates are given by,

$$x' = x - h \text{ and } y' = y - k$$

Therefore, the required equation of the shifted origin will be as follows:

$\Rightarrow (y - k)^2 = 4A\,(x - h)$

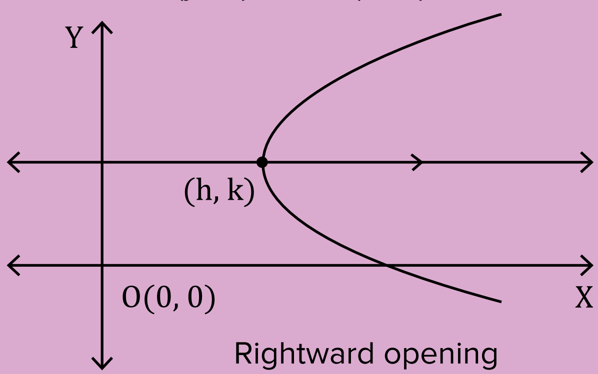


Now, for different standard parabola equations, we have,

1. **Rightward opening**

Equation of standard parabola converts as follows:

$$(y - k)^2 = 4A\,(x - h)$$

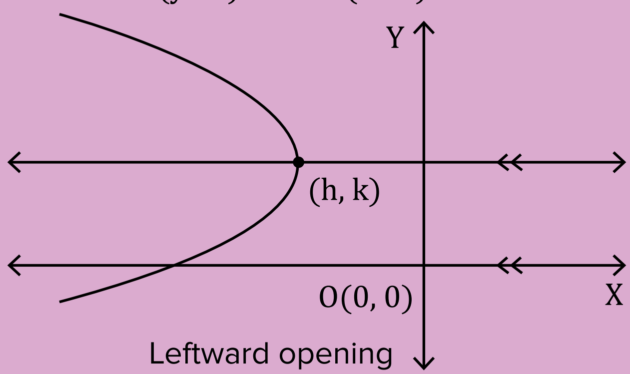


Rightward opening

2. **Leftward opening**

Equation of standard parabola converts as follows:

$$(y - k)^2 = -4A\,(x - h)$$

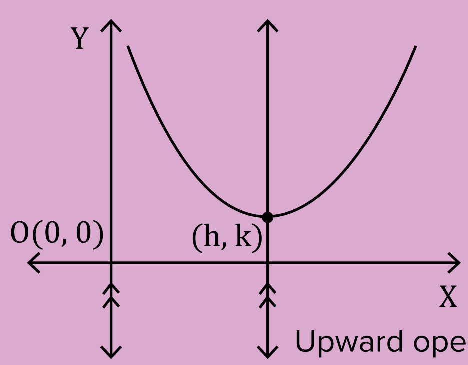


Leftward opening

3. **Upward opening**

Equation of standard parabola converts as follows:

$$(x - h)^2 = 4A\,(y - k)$$

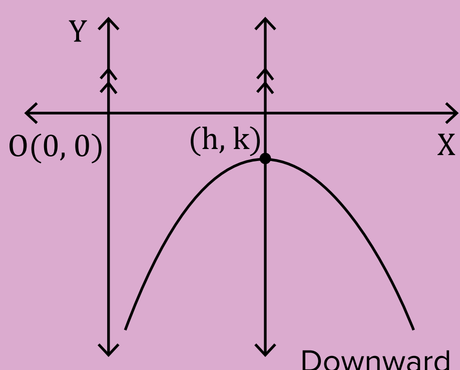


Upward opening

4. **Downward opening**

Equation of standard parabola converts as follows:

$$(x - h)^2 = -4A\,(y - k)$$



Downward opening

Note

- On shifting the parabola, the coordinates and equations change but the distances (length of LR, distance between vertex and focus, distance between directrix and vertex, etc.) do not change.
- The equation $y = Bx^2 + Cx + D$ or, $(x - h)^2 = 4A(y - k)$ represents a vertical parabola.
- The equation $x = By^2 + Cy + D$ or, $(y - k)^2 = 4A(x - h)$ represents a horizontal parabola.

If $y^2 + 2y - x + 5 = 0$ represents a parabola, then find its vertex, axis of symmetry, focus, equation of directrix, equation of LR, length of LR, and extremities of LR.

Solution

Step 1 :

Given, equation of the parabola, $y^2 + 2y - x + 5 = 0$

We can see that the given parabola is a horizontal parabola.

Now, we will convert the given equation into a similar format to the standard equation.

I.e., $y^2 + 2y + 1^2 - 1^2 + 5 = x$

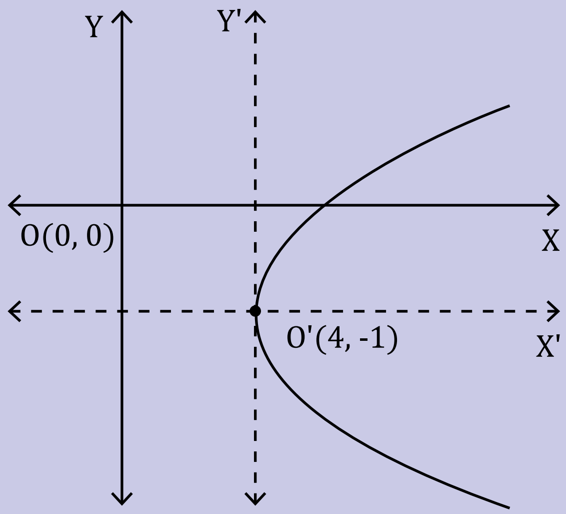
Finally, it becomes, $[y - (-1)]^2 = 1(x - 4)$

As we can see that the given equation is similar to $(y - k)^2 = 4A(x - h)$,

Where $k = -1$, $A = \frac{1}{4}$, and $h = 4$

Thus, $[y - (-1)]^2 = 1(x - 4)$ is a rightward opening parabola having

- Vertex $\equiv (h, k)$
- Axis parallel to x - axis



Or, $(y')^2 = 4A(x')$, where $y' = y + 1, x' = x - 4$, and $A = \frac{1}{4}$

Step 2 :

Now,

$\Rightarrow$ Vertex $\equiv (h, k) \equiv (4, -1)$

$\Rightarrow$ Axis of symmetry

$y' = 0 \Rightarrow y + 1 = 0$ or $y = -1$

$\Rightarrow$ Focus $(A, 0) \equiv (x', y')$

$\Rightarrow x' = A$ and $y' = 0$

Or $x - 4 = \frac{1}{4}$ and $y + 1 = 0$

Focus point is $\left(\frac{17}{4}, -1\right)$

Step 3 :

$\Rightarrow$ Directrix: $x' = -A$

Or, $x - 4 = \frac{-1}{4}$ and directrix is $x = \frac{15}{4}$

$\Rightarrow$ Latus rectum Length $= 4A = 4 \times \frac{1}{4} = 1$ unit

$\Rightarrow$ Extremities of LR: We know that the coordinates $(x', y') \equiv (A, \pm 2A)$

Or, $x - 4 = \frac{1}{4}$ and $y + 1 = \pm \frac{1}{2}$

Or, the coordinates of extremities are $\left(\frac{17}{4}, \frac{-1}{2}\right)$ and $\left(\frac{17}{4}, \frac{-3}{2}\right)$.

$\Rightarrow$ Equation of LR : $x = \frac{17}{4}$

Concept Check 2

If $x^2 + 2x - y - 5 = 0$ represents a parabola, then find its vertex, axis of symmetry, focus, equation of directrix, equation of LR, length of LR, and extremities of LR.

Find the equation of the parabola having its focus at $S(2, 0)$ and one extremity of its LR at $(2, 2)$.

Solution

Step 1 :

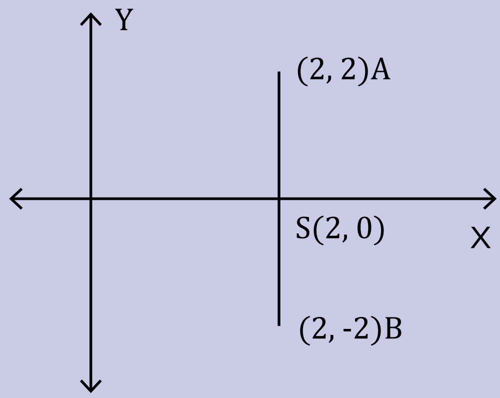
Given, focus $\equiv (2, 0)$ and one extremity of its LR at $(2, 2)$

Clearly, the other extremity of LR is $(2, -2)$.

Because we know that the focus is the midpoint of LR.

Now, $LR = 4A = 4 \Rightarrow A = 1$

$\therefore$ Vertex of parabola is $(1, 0)$ or $(3, 0)$.



Step 2 :

Thus,

When the vertex is $(1, 0)$, its equation becomes,

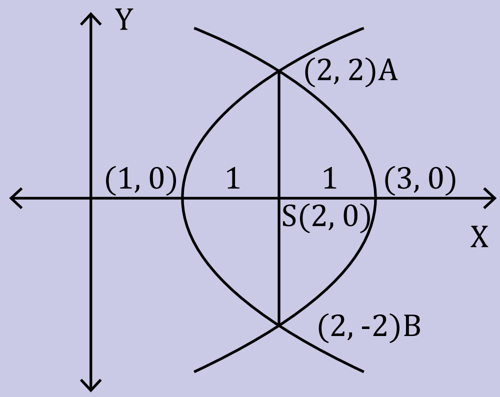
$(y - 0)^2 = 4(1)(x - 1)$

$\Rightarrow y^2 = 4(x - 1)$

When the vertex is $(3, 0)$, its equation becomes,

$(y - 0)^2 = -4(1)(x - 3)$

$\Rightarrow y^2 = -4(x - 3)$



What is the equation of the parabola if its vertex is at $(1, 1)$ and focus at $(3, 1)$?

(a) $(x - 1)^2 = 8(y - 1)$ **(b)** $(y - 1)^2 = 8(x - 1)$ **(c)** $(y - 1)^2 = 8(x - 3)$ **(d)** $(x - 3)^2 = 8(y - 1)$

Solution

Step 1 :

Given, vertex $(V) \equiv (1, 1)$ and Focus $(S) \equiv (3, 1)$

As we can see that the y-coordinate for both vertex and focus is the same,

$\therefore$ The horizontal line joining the vertex and the focus is the axis of symmetry of the parabola with equation $y = 1$

Step 2 :

Here,
A = Distance between focus and vertex

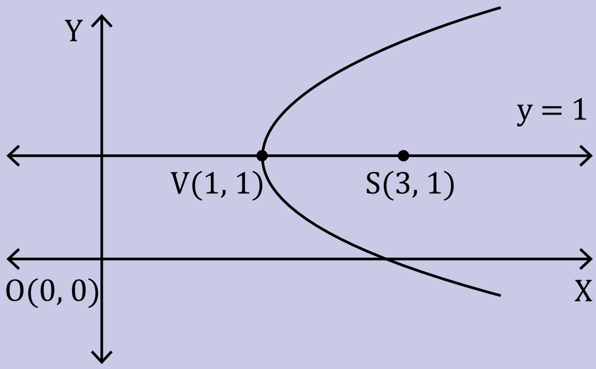
Therefore, A = 3 -1 = 2

Hence, the required equation of the parabola is as follows:

$(y - 1)^2 = 4(2)(x - 1)$

$\Rightarrow (y - 1)^2 = 8(x - 1)$

Option (b) is the correct answer.



Concept Check 3

If the ends of the LR of the parabola are (6, 7) and (6, -1), then find the vertex of the parabola.

(a) (8, 3) (b) (10, 3) (c) (4, -3) (d) (2, 3)

Summary Sheet

Key Results

Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus	(a, 0)	(-a, 0)	(0, a)	(0, -a)
Directrix	x = -a	x = a	y = -a	y = a
Axis	X - axis	X - axis	Y - axis	Y - axis
Length of LR	4a	4a	4a	4a
Focal distance	x + a	-x + a	y + a	-y + a

Shifting of vertex

After shifting to the new vertex, let the equation become $(y')^2 = 4A(x')$
Now, using

Shifting of origin XY - plane = $(0, 0) \equiv (x, y)$

And X'Y' - plane = $(h, k) \equiv (x', y')$

Now, relations between the two given coordinates are given by, $x' = x - h$ and $y' = y - k$

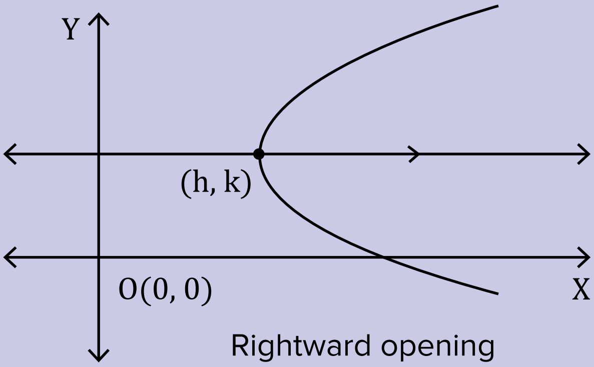
Therefore, the required equation of the shifted origin will be as follows:

$\Rightarrow (y - k)^2 = 4A\,(x - h)$

1. **Rightward opening**

Equation of standard parabola converts as follows:

$$(y - k)^2 = 4A\,(x - h)$$

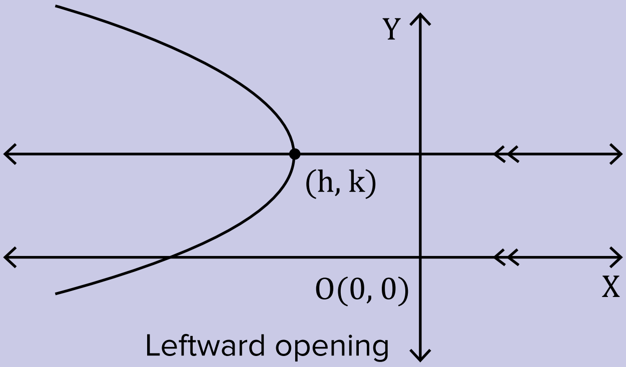


Rightward opening

2. **Leftward opening**

Equation of standard parabola converts as follows:

$$(y - k)^2 = -4A\,(x - h)$$

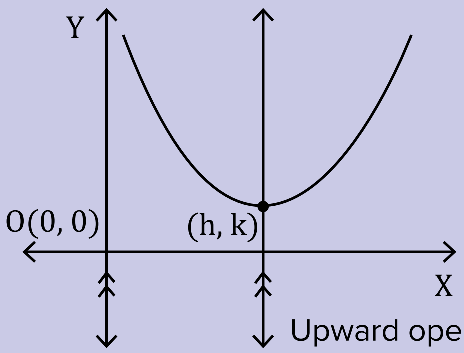


Leftward opening

3. **Upward opening**

Equation of standard parabola converts as follows:

$$(x - h)^2 = 4A\,(y - k)$$

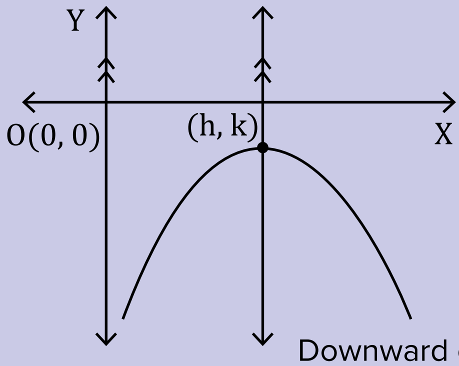


Upward opening

4. **Downward opening**

Equation of standard parabola converts as follows:

$$(x - h)^2 = -4A\,(y - k)$$



Downward opening

- On shifting the parabola, the coordinates and equations change but the distances (length of LR, distance between vertex and focus, distance between directrix and vertex etc.) do not.
- The equation $y = Bx^2 + Cx + D$ or $(x - h)^2 = 4A(y - k)$ represents a vertical parabola.
- The equation $x = By^2 + Cy + D$ or $(y - k)^2 = 4A(x - h)$ represents a horizontal parabola.

Mind map

Terms associated to Parabola

- Translation or shifting of vertex
- Problems based on some mixed concepts

Self-Assessment

1. If M is the foot of the perpendicular from a point P on a parabola $y^2 = 4ax$ to its directrix, and SPM is an equilateral triangle where S is the focus, then find SP.

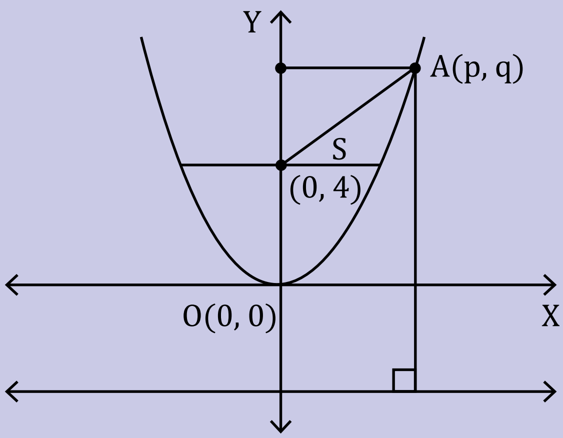
2. Find the equation of the parabola that has its axis parallel to y-axis and passes through points $(0, 2)$, $(-1, 0)$, and $(1, 6)$.

Answers

Concept Check 1

Step 1 :

Given, parabola $x^2 = 16y$
In figure, we consider coordinates of point $A(p, q)$.
As we can see that the given parabola is upward opening, that will be similar to $x^2 = 4ay$



Now, after comparing both the equations, we get,
$4a = 16 \Rightarrow a = 4$

Step 2 :

Now, different parameters will be as follows:

Equation	$x^2 = 4ay$	$x^2 = 16y$
Focus	$(0, a)$	$(0, 4)$
Axis of parabola	$x = 0$	$x = 0$
Equation of directrix	$y = -a$	$y = -4$
Equation of LR	$y = a$	$y = 4$
Ends of LR	$(\pm 2a, a)$	$(\pm 8, 4)$
Length of LR	$4a$	16
Focal distance of point	$y + a$	$q + 4$

Concept Check 2

Step 1 :

Given, equation of parabola: $x^2 + 2x - y - 5 = 0$

We can see that the given parabola is a vertical parabola.

Now, we will transform the given equation in the standard form by converting it into whole square of variable x.

I.e., $x^2 + 2x + 1^2 - 1^2 - 5 = y$

$\Rightarrow [x - (-1)]^2 = 1\,[y - (-6)]$ which is similar to $(x - h)^2 = 4A(y - k)$ or $(x')^2 = 4Ay'$

Having $x' = x + 1$ and $y' = y + 6$

Where $h = -1, k = -6$ and $A = \frac{1}{4}$

Step 2 :

Now, vertex $\equiv (-1, -6)$

Axis of symmetry: $x' = 0 \Rightarrow x = -1$

Focus $\equiv (0, A) \equiv (x', y')$

$\Rightarrow x + 1 = 0$ and $y + 6 = \frac{1}{4}$ or $(-1, \frac{-23}{4})$

Directrix: $y' = -A \Rightarrow y = \frac{-25}{4}$

Length of LR $\equiv 4A = 1$

Step 3 :

Extremities of LR $\equiv (\pm 2A, A)$

$x' = 2A \Rightarrow x + 1 = \pm\frac{1}{2}$ and $y' = A \Rightarrow y + 6 = \frac{1}{4}$

Or extremities are $(\frac{-1}{2}, \frac{-23}{4})$ and $(\frac{-3}{2}, \frac{-23}{4})$

Equation of LR: $y = \frac{-23}{4}$

Concept Check 3

Step 1 :

Given, the ends of the LR of the parabola are

$(6, 7)$ and $(6, -1)$, and we have to find the coordinates of the vertex.

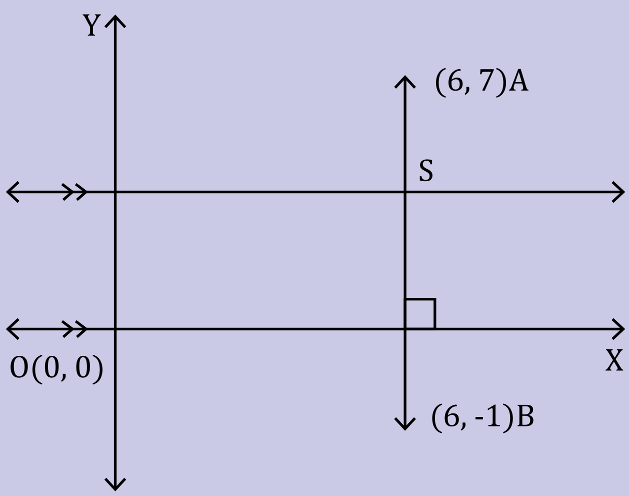
Here, we can see that,

$AB \rightarrow LR \perp x$ - axis

$\Rightarrow$ Axis of symmetry will be horizontal and parallel to the x-axis

As we know that mid point of the LR is the focus,

$\therefore$ Focus $(S) \equiv \left(\frac{6+6}{2}, \frac{7-1}{2}\right) = (6, 3)$



Step 2 :

Vertex $\equiv$ V will certainly lie on the axis of symmetry at a distance of a units from the focus either to its left or to its right.

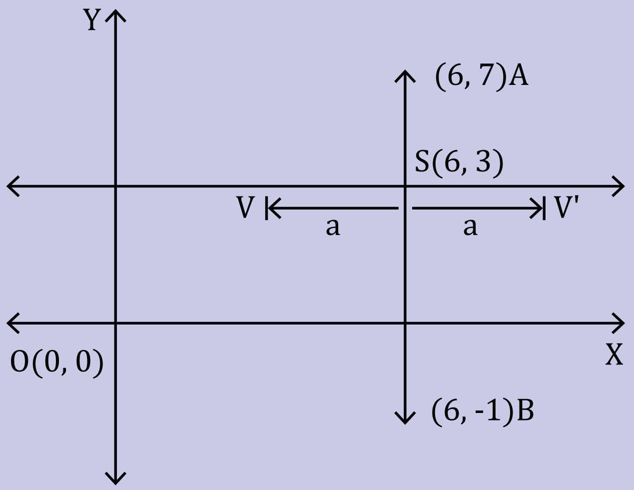
Here,

$LR = 4a = 8 \Rightarrow a = 2$

$\therefore$ Vertex will lie 2 units left or right from the focus.

Thus, vertex can be $(4, 3)$ or $(8, 3)$.

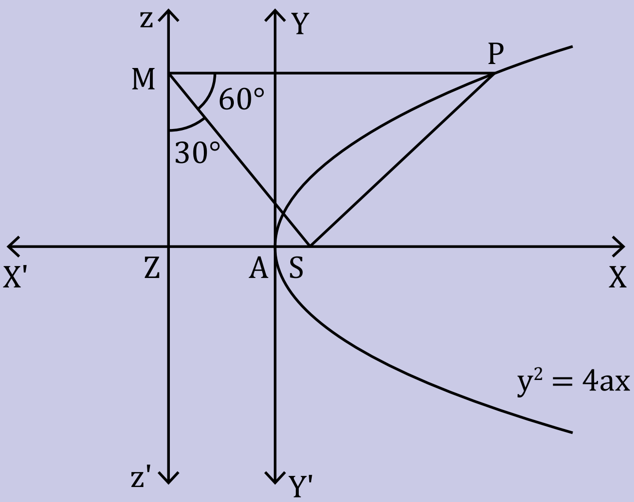
Hence, option (a) is the correct answer.



Self-Assessment 1

Step 1 :

Given, in the figure, M is the foot of the perpendicular from a point P on the parabola $y^2 = 4ax$ to its directrix, and SPM is an equilateral triangle where S is the focus. And we have to find SP.



From the definition of parabola, we know that SPM is an equilateral triangle.

Therefore, $SP = PM = SM$

$\Rightarrow \angle PMS = 60°$

$\Rightarrow \angle SMZ = 30°$

Step 2 :

In SMZ, we have,

$\sin 30° = \frac{SZ}{SM}$

$\Rightarrow \frac{1}{2} = \frac{2a}{SM}$

Or $SM = 4a$

As $SM = SP = 4a$,

Hence, $SP = 4a$

Self-Assessment 2

Step 1 :

Given that the points

$(0, 2)$, $(-1, 0)$, and $(1, 6)$ lie on a parabola whose axis is parallel to the y-axis.

We know that the general equation of a vertical parabola is $y = Ax^2 + Bx + C$

Step 2 :

As the three given points lie on the parabola so they will satisfy the general parabola equation

$C = 2$, $A - B + C = 0$, and $A + B + C = 6$

After solving the given equations, we get, $A = 1$, $B = 3$, and $C = 2$

Or the general equation is as follows:

$y = x^2 + 3x + 2$

MATHEMATICS

PARABOLA

PARAMETRIC COORDINATES OF A PARABOLA AND PROPERTIES OF FOCAL CHORD

NOTES

What you already know

- Basics of conic section
- Equation of a parabola
- Focus, vertex, directrix of a parabola
- Axis of symmetry of a parabola

What you will learn

- Position of a point with respect to a parabola

Homework question

Consider the equation $(x - 1)^2 + (y - 2)^2 = \lambda(x + y + 3)^2$.

(a) It will represent a parabola if λ = ________

(b) Its focus is ________ .

(c) Its directrix is the line ________ .

(d) Its vertex is ________ .

(e) Its axis is the line ________ .

(f) Its latus rectum is the line ________ .

(g) The length of its latus rectum is ________ .

Solution

Step 1

Standard equation of a conic is as follows:

$$\sqrt{(x - \alpha)^2 + (y - \beta)^2} = \frac{e|ax + by + c|}{\sqrt{a^2 + b^2}}$$

Here,

$P(x, y)$: General point on the conic

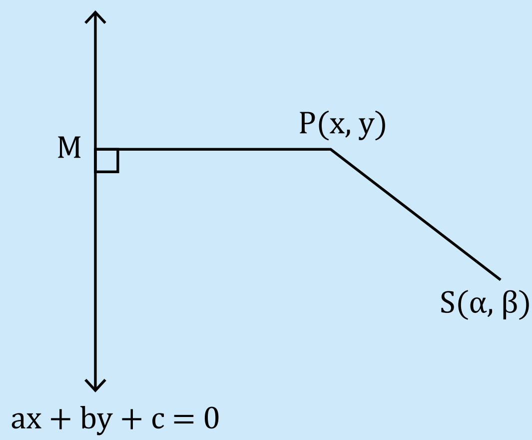
$S(\alpha, \beta)$: Coordinates of focus

$ax + by + c = 0$: Equation of directrix

$$\frac{PS}{PM} = e$$

On squaring both the sides, we get the following:

$$(x - \alpha)^2 + (y - \beta)^2 = \frac{e^2(ax + by + c)^2}{a^2 + b^2}$$



Step 2

Given equation,

$(x - 1)^2 + (y - 2)^2 = \lambda(x + y + 3)^2$

To compare with standard form, we multiply and divide the RHS by $a^2 + b^2$. By comparing numerator of RHS we get $a = b = 1$.

$\Rightarrow (x - 1)^2 + (y - 2)^2 = \dfrac{(1^2 + 1^2)\,\lambda(x + y + 3)^2}{1^2 + 1^2}$

$\Rightarrow (x - 1)^2 + (y - 2)^2 = \dfrac{2\lambda(x + y + 3)^2}{1^2 + 1^2}$

Step 3

On comparing with the standard form, we get the following:

Focus $S(1, 2)$ ...(b)

Equation of directrix: $x + y + 3 = 0$ ...(c)

$e^2 = 2\lambda$

Eccentricity, $e = \sqrt{2\lambda}$ (Since $e \geq 0$)

However, we know that eccentricity of a parabola is 1.

$\Rightarrow \sqrt{2\lambda} = 1$

$\Rightarrow \lambda = \dfrac{1}{2}$ ...(a)

The axis and directrix are perpendicular.

Slope of directrix = -1

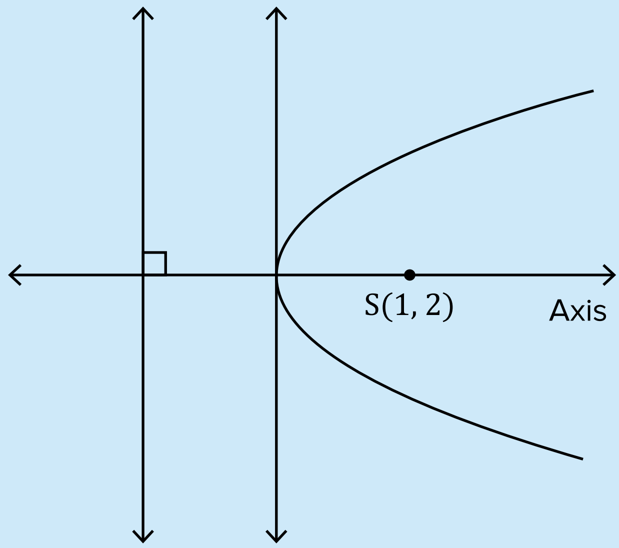
$\Rightarrow$ Slope of axis = 1

Also, the axis passes through the focus $S(1, 2)$.

So, its equation will be the following:

$(y - 2) = 1(x - 1)$

$\Rightarrow y - x = 1$ ...(e)



Step 4

Point of intersection of

Directrix: $x + y + 3 = 0$ and ...(1)

Axis: $y = x + 1$ ...(2)

is M.

The vertex V is the midpoint of S (focus) and M.

Solving equations of the axis and directrix simultaneously:

By substituting the value of y from equation (2) in equation (1), we get the following:

$x + (x + 1) + 3 = 0$

$2x = -4$

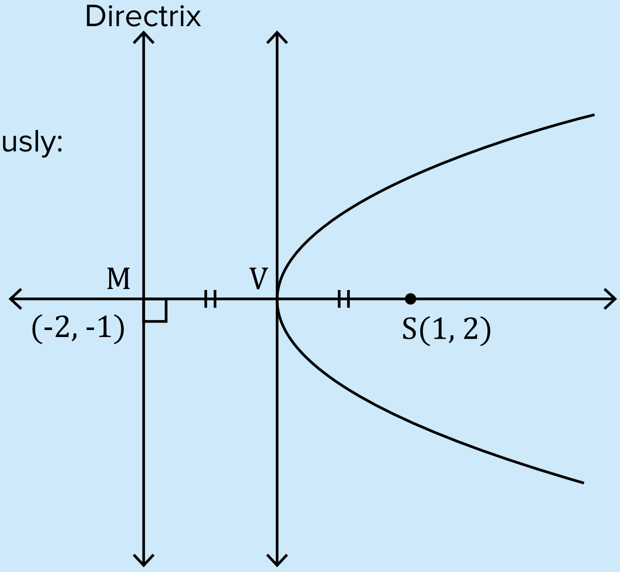
$x = -2$

$y = -2 + 1 = -1$

We get the coordinates of M: (-2, -1)

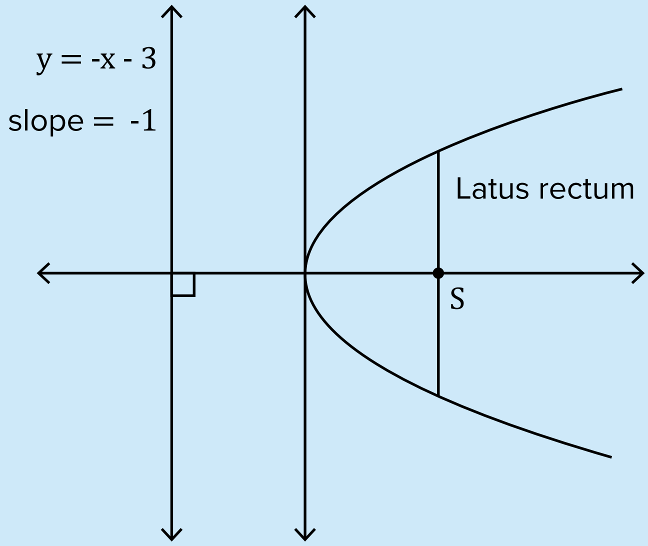
V: $\left(\dfrac{-2 + 1}{2}, \dfrac{-1 + 2}{2}\right)$

The coordinates of vertex $V\left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$ ...(d)



Step 5

Directrix and latus rectum are parallel.
So, slope of latus rectum = Slope of directrix = -1
Also, the latus rectum passes through focus $(1, 2)$.
Equation of latus rectum: $y - 2 = -1(x - 1)$
$\Rightarrow x + y = 3$ ...(f)



Step 6

Length of latus rectum = $4a$
a is the distance between vertex $V\left(-\frac{1}{2}, \frac{1}{2}\right)$ and focus $S(1, 2)$.

$$a = \sqrt{(1 + (\tfrac{1}{2}))^2 + (2 - (\tfrac{1}{2}))^2} = \frac{3\sqrt{2}}{2}$$

Length of latus rectum = $4a$

$$= 4 \times \frac{3\sqrt{2}}{2}$$

$$= 6\sqrt{2} \quad \text{...(g)}$$

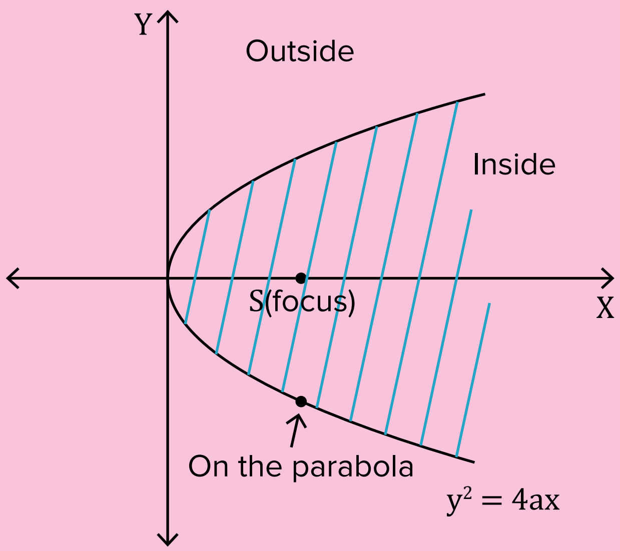
Position of a Point with Respect to a Parabola

Parabola is an open and unbounded figure. It divides the plane containing it into three disjoint parts. The part containing the focus is inside of the parabola.
With respect to a parabola, a point may lie:

(a) Inside the parabola

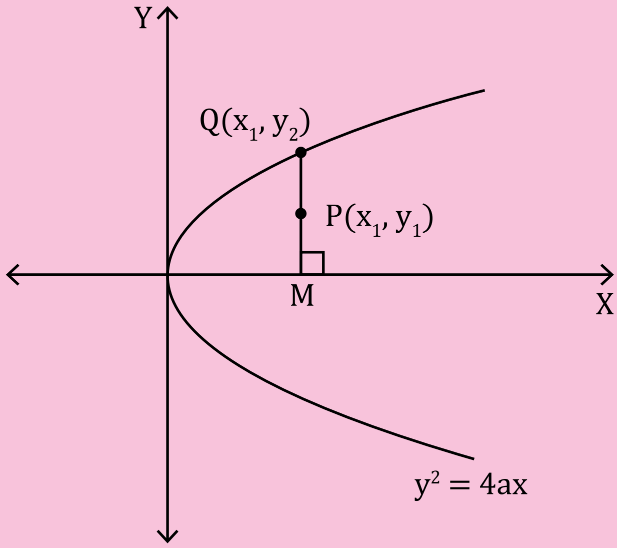
(b) On the parabola

(c) Outside the parabola

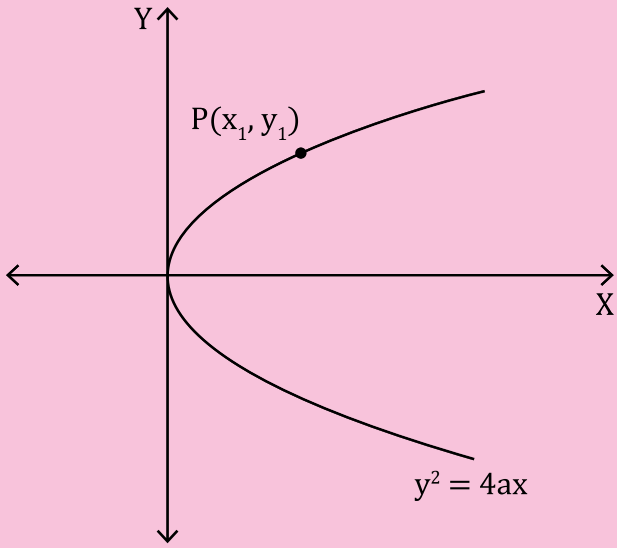


Consider the parabola $y^2 = 4ax$; $a > 0$
Let $P(x_1, y_1)$ be a point.

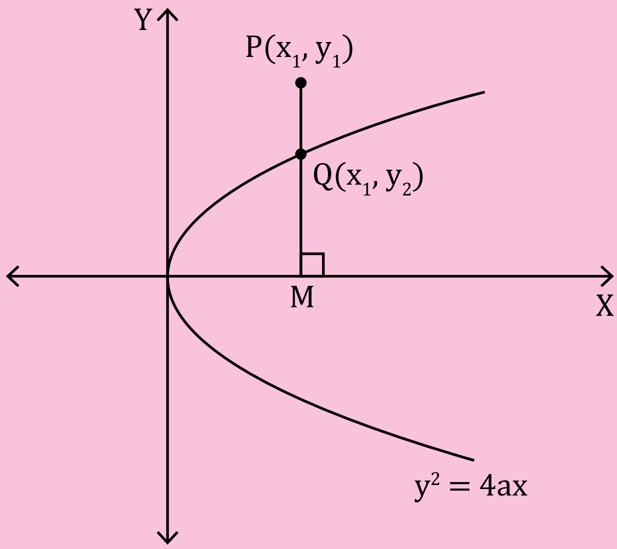
Case 1: $P(x_1, y_1)$ lies inside the parabola.
Condition: $y_1^2 - 4ax_1 < 0$
Given $P(x_1, y_1)$ lies inside the parabola. Let $Q(x_1, y_2)$ be a point on the parabola with the same x coordinate as P. (Choose the point so that $y_2 > 0$). Then, $y_2^2 = 4ax_1$.
Clearly, $y_2 > y_1$
$\Rightarrow y_2^2 > y_1^2$
$\Rightarrow 4ax_1 > y_1^2$
$\Rightarrow y_1^2 - 4ax_1 < 0$



Case 2: $P(x_1, y_1)$ lies on the parabola.
Condition: $y_1^2 - 4ax_1 = 0$
Since $P(x_1, y_1)$ lies on the parabola, it satisfies the equation of the parabola. So, $y_1^2 = 4ax_1$.
$\Rightarrow y_1^2 - 4ax_1 = 0$



Case 3: $P(x_1, y_1)$ lies outside the parabola.
Condition: $y_1^2 - 4ax_1 > 0$.
Given
$P(x_1, y_1)$ lies outside the parabola.
Let $Q(x_1, y_2)$ be a point on the parabola with the same x coordinate as P. (Choose the point so that $y_2 > 0$). Then, $y_2^2 = 4ax_1$.
Clearly, $y_2 < y_1$
$\Rightarrow y_2^2 < y_1^2$
$\Rightarrow 4ax_1 < y_1^2$
$\Rightarrow y_1^2 - 4ax_1 > 0$



Solve

Consider the parabola $y^2 = 4x$. If $P(1, 3)$ and $Q(1, 1)$ are two points lying in the XY plane, then select the correct option.

(a) P and Q are exterior points.
(b) P is interior and Q is an exterior point.
(c) P and Q are interior points.
(d) P is exterior and Q is an interior point.

Solution

Given, $y^2 = 4x$
$y^2 - 4x = 0$
$S \equiv y^2 - 4x$

Step 1

For $P(1, 3)$,
$S_{P(1,3)} = 3^2 - 4(1) = 5 > 0$
$\Rightarrow$ P lies outside the parabola and hence, is an exterior point.

Step 2

For $Q(1, 1)$,
$S_{Q(1,1)} = 1^2 - 4(1) = -3 < 0$
$\Rightarrow$ Q lies inside the parabola and hence, is an interior point.

Option (d) is the correct answer.

Concept Check 1

The ends of a line segment are $P(1, 3)$ and $Q(1, 1)$. R is a point on the line segment PQ such that $PR:QR = 1:\lambda$. If R is an interior point of the parabola $y^2 = 4x$, then which of the following is correct?

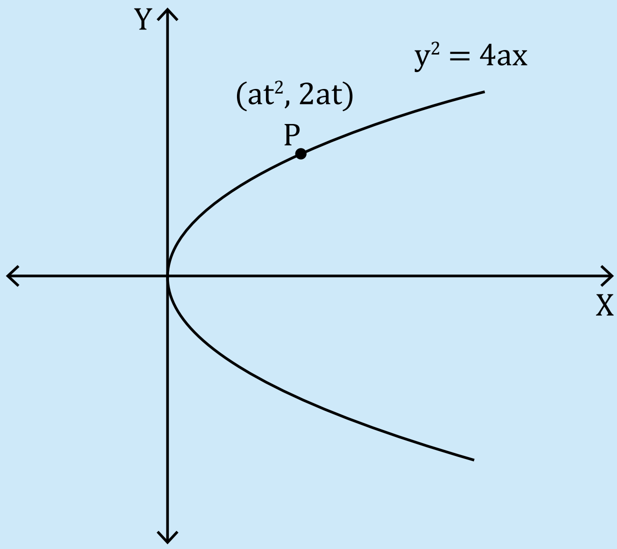
(a) $\lambda \in (0, 1)$
(b) $\lambda \in \left(-\frac{3}{5}, 1\right)$
(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$
(d) None of these

Parametric Coordinates

Let $P(x, y)$ be any point on the parabola $y^2 = 4ax$. The parametric coordinates of P are the coordinates expressed in terms of a real parameter t.
Let the x-coordinate of P be t.
$x = t$
Then the y-coordinate becomes, $y^2 = 4at$
$\Rightarrow y = \pm\sqrt{4at}$.
Since $t \in \mathbb{R}$, the term inside the root may become negative when t is negative, so this is not a valid parametric representation.
Instead if we take $x = at^2$, then $y = \pm\sqrt{4aat^2} = \pm\sqrt{4a^2t^2} = \pm 2at$
We can drop the minus sign here as t is taking all real values.
So, we get $P(at^2, 2at)$ as a valid parametric representation of the parabola $y^2 = 4ax$

Note

You can put $x = at^2$, $y = 2at$ back in $y^2 = 4ax$ to verify that it satisfies the equation for all real values of t.



Parametric Coordinates for Standard Parabolas

t is the parameter.

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric equation	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = at^2$	$x = 2at$ $y = -at^2$

Note

The parametric equations of the parabola $(y - k)^2 = 4a(x - h)$ are:

$x = h + at^2$ and $y = k + 2at$

Proof

Given, $(y - k)^2 = 4a(x - h)$

Let $y' = y - k$ and $x' = x - h$

$(y')^2 = 4ax'$

Parametric coordinates of this parabola are $(at^2, 2at)$.

That is $x' = at^2$ and $y' = 2at$

$\Rightarrow x - h = at^2$ and $y - k = 2at$

$\Rightarrow x = h + at^2$ and $y = k + 2at$

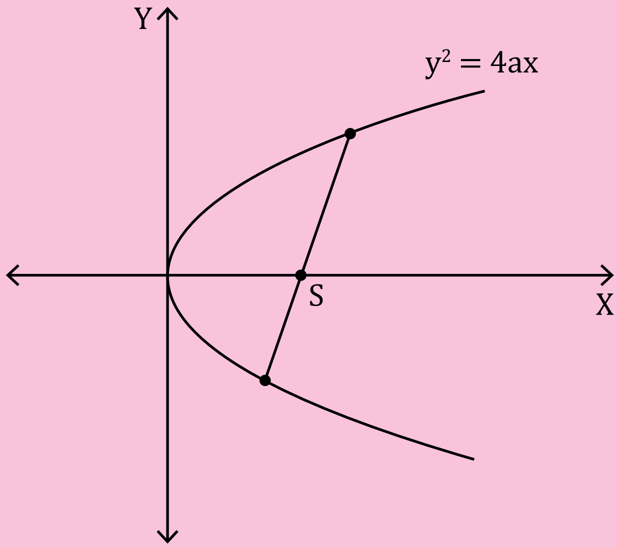
Concept Check 2

The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix:

(a) $x = -a$ (b) $x = a$ (c) $x = 0$ (d) $x = \frac{a}{2}$

Focal Chord

Any chord passing through the focus of the parabola is known as a focal chord.



Properties of Focal Chord

1. If $P(t_1)$ and $Q(t_2)$ are the endpoints of a focal chord, then $t_1t_2 = -1$

Proof

Slope of segment PQ = Slope of segment PS

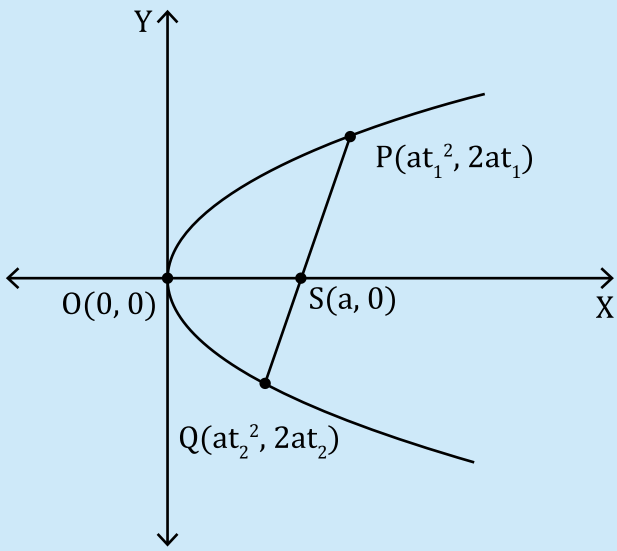
$$\Rightarrow \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2at_1 - 0}{at_1^2 - a}$$

$$\Rightarrow \frac{t_2 - t_1}{t_2^2 - t_1^2} = \frac{t_1}{t_1^2 - 1}$$

$$\Rightarrow \frac{1}{t_2 + t_1} = \frac{t_1}{t_1^2 - 1}$$

$$\Rightarrow t_1^2 - 1 = t_1t_2 + t_1^2$$

$$\Rightarrow t_1t_2 = -1$$



Solve

Find the length of focal chord of point $P(t)$ for $y^2 = 4ax$.

Solution

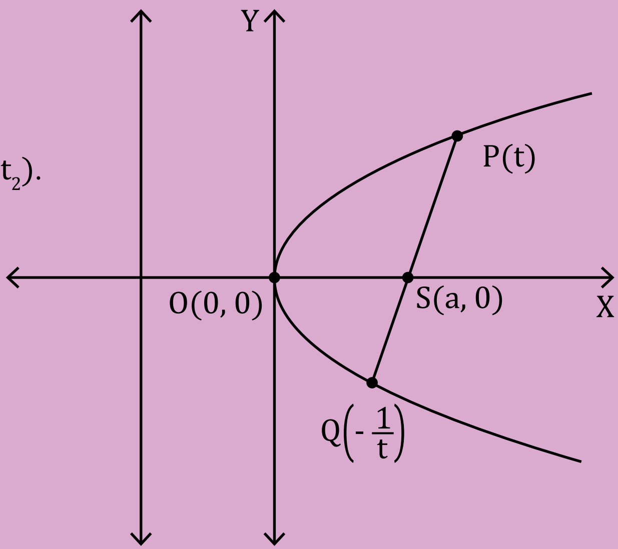
Step 1

Let the endpoints of the focal chord be $P(t_1)$ and $Q(t_2)$.
Given, $t_1 = t$
Also, using the property of focal chord $t_1 t_2 = -1$.

$\Rightarrow t_2 = -\frac{1}{t}$

$\Rightarrow P \equiv (at^2, 2at)$ and $Q \equiv \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

$|PQ| = |PS| + |QS|$



Step 2

$|PQ| = |PS| + |QS| = |PM| + |QN|$

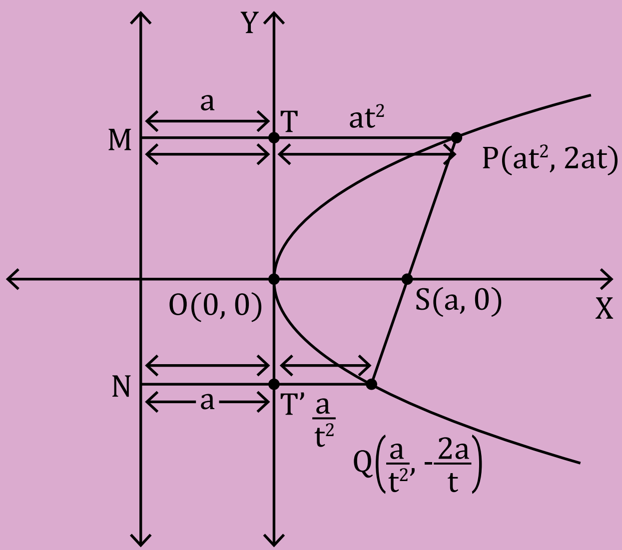
$= |PT| + |TM| + |QT'| + |T'N|$

$= at^2 + a + \frac{a}{t^2} + a$

$= a\left(2 + t^2 + \frac{1}{t^2}\right)$

$= a\left(t + \frac{1}{t}\right)^2$

So, the length of focal chord is $a\left(t + \frac{1}{t}\right)^2$.



Note

The length of focal chord of point $P(t)$ for $y^2 = 4ax$ is $a\left(t + \frac{1}{t}\right)^2$.

2. The length of the focal chord that makes an angle α with the positive direction of X-axis is $4a \operatorname{cosec}^2\alpha$.

Proof

Step 1

Let PQ be the focal chord with $P(t) \equiv (at^2, 2at)$.

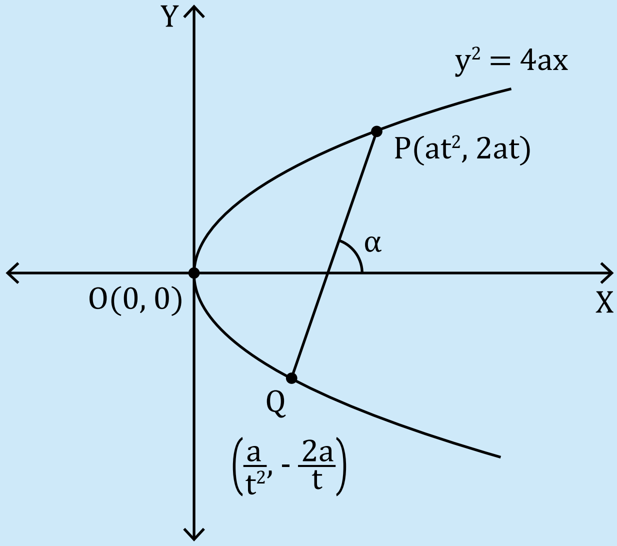
$\Rightarrow Q\left(-\frac{1}{t}\right) \equiv \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

Angle of inclination of PQ = α

Slope of PQ = tan α = $\dfrac{2at + \frac{2a}{t}}{at^2 - \frac{a}{t^2}}$

$$= \frac{2t + \frac{2}{t}}{t^2 - \frac{1}{t^2}} = \frac{2}{t - \frac{1}{t}}$$

$\Rightarrow t - \frac{1}{t} = 2 \cot \alpha$



Step 2

Length of focal chord $= a\left(t + \frac{1}{t}\right)^2$

$$= a\left(\left(t - \frac{1}{t}\right)^2 + 4\right)$$

$$= a(4\cot^2 \alpha + 4)$$

$$= 4a \operatorname{cosec}^2 \alpha$$

Note

The smallest focal chord is the one perpendicular to the axis of symmetry, i.e., latus rectum.

$\alpha = 90°$

Its length is $4a \operatorname{cosec}^2 90°$.

$= 4a$ = Minimum length of a focal chord

Summary Sheet

Key Takeaways

• Let $P(x_1, y_1)$ be a point.

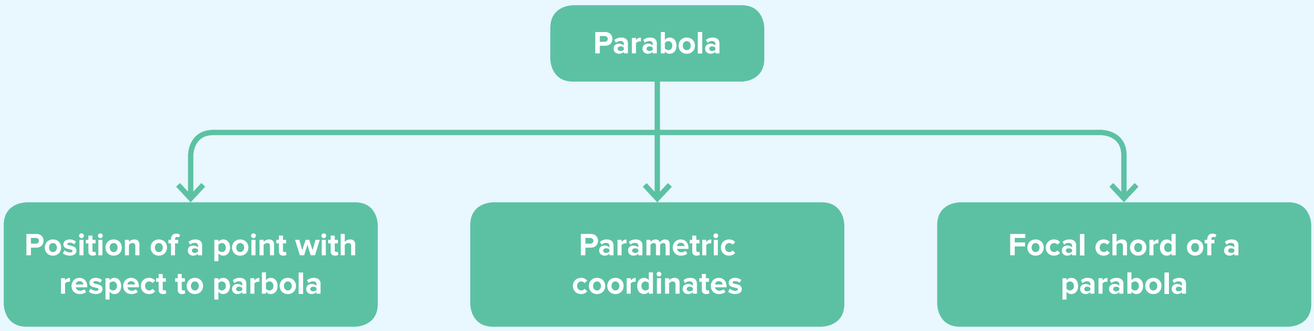
Case 1: $P(x_1, y_1)$ lies inside the parabola.	**Condition:** $y_1^2 - 4ax_1 < 0$
Case 2: $P(x_1, y_1)$ lies on the parabola.	**Condition:** $y_1^2 - 4ax_1 = 0$
Case 3: $P(x_1, y_1)$ lies outside the parabola.	**Condition:** $y_1^2 - 4ax_1 > 0$

- Parametric coordinates of standard parabola

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric equation	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = at^2$	$x = 2at$ $y = -at^2$

- Properties of focal chord
  1. If $P(t_1)$ and $Q(t_2)$ are the endpoints of a focal chord, then $t_1t_2 = -1$
  2. The length of the focal chord of point $P(t)$ for $y^2 = 4ax$ is $a\left(t + \frac{1}{t}\right)^2$.
  3. The length of the focal chord that makes an angle α with the positive direction of the X-axis is $4a \operatorname{cosec}^2 \alpha$.

Mind Map



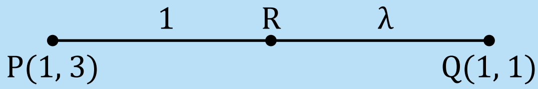
Self-Assessment

1. The parametric equation of a parabola is $x = t^2 + 1$, $y = 2t + 1$. Find the equation of directrix.
2. If $(2, -8)$ is at the end of a focal chord of the parabola $y^2 = 32x$, then find the coordinates of the other end of the chord.

Answers

Concept Check 1

Given,
$P(1, 3)$ and $Q(1, 1)$
$PR : QR = 1 : \lambda$
R is an interior point of $y^2 = 4x$
$S : y^2 - 4x = 0$



Step 1

Let the coordinates of R be (x, y).
By using the section formula, we get the following:

$$x = \frac{1 + \lambda}{1 + \lambda} = 1$$

$$y = \frac{1 + 3\lambda}{1 + \lambda}$$

$$R \equiv \left(1, \frac{1 + 3\lambda}{1 + \lambda}\right)$$

Step 2

R is an interior point, if $S_R < 0$

$$\Rightarrow \left(\frac{1 + 3\lambda}{1 + \lambda}\right)^2 - 4(1) < 0$$

$$\Rightarrow \left(\frac{1 + 3\lambda}{1 + \lambda}\right)^2 - (2)^2 < 0$$

$$\Rightarrow \left(\frac{1 + 3\lambda}{1 + \lambda} - 2\right)\left(\frac{1 + 3\lambda}{1 + \lambda} + 2\right) < 0$$

$$\Rightarrow \left(\frac{1 + 3\lambda - 2 - 2\lambda}{1 + \lambda}\right)\left(\frac{1 + 3\lambda + 2 + 2\lambda}{1 + \lambda}\right) < 0$$

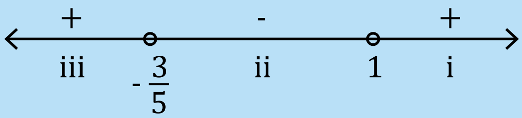
$$\Rightarrow \left(\frac{\lambda - 1}{1 + \lambda}\right)\left(\frac{5\lambda + 3}{1 + \lambda}\right) < 0$$

$$\Rightarrow \frac{(\lambda - 1)(5\lambda + 3)}{(1 + \lambda)^2} < 0$$

Since the denominator is square of a real number, it is positive for all λ. $(\lambda \neq -1)$

$$\Rightarrow (\lambda - 1)(5\lambda + 3) < 0$$

$$\Rightarrow 5(\lambda - 1)\left(\lambda + \frac{3}{5}\right) < 0$$



$$\lambda \in \left(-\frac{3}{5}, 1\right)$$

However, $\lambda > 0$ (As we are talking about the ratios of two distances)

$$\Rightarrow \lambda \in (0, 1)$$

Option (a) is the correct answer.

Concept Check 2

Given,

$y^2 = 4ax$

Let $P \equiv (at^2, 2at)$ be any moving point for some $t \in \mathbb{R}$.

Step 1

$R(h, k)$ is the midpoint of PS.

$\Rightarrow h = \frac{at^2 + a}{2}$ and $k = \frac{2at + 0}{2}$.

By solving both these equations simultaneously to eliminate t, we get the following:

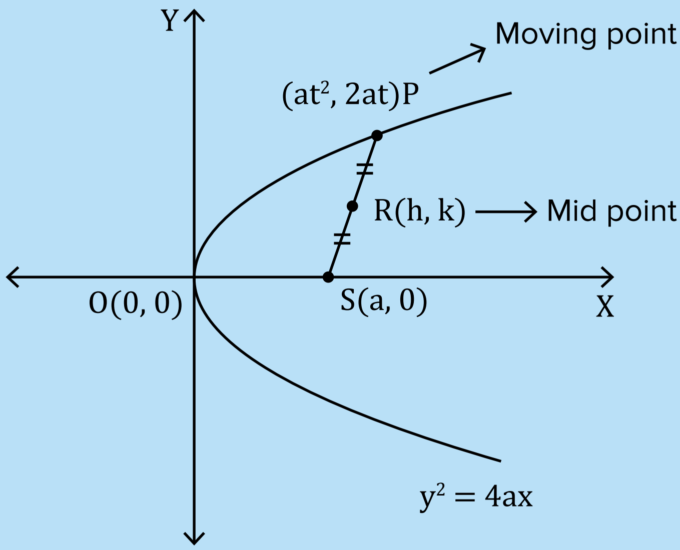
$\Rightarrow h = \frac{a\left(\frac{k}{a}\right)^2 + a}{2}$

$\Rightarrow 2h = \frac{k^2}{a} + a$

$\Rightarrow 2ah = k^2 + a^2$

$\Rightarrow k^2 = 2ah - a^2$

$\Rightarrow k^2 = 2a\left(h - \frac{a}{2}\right)$



Step 2

By replacing h by x and k by y, we get the following:

$\Rightarrow y^2 = 2a\left(x - \frac{a}{2}\right)$

This is the equation of a parabola. Taking $y = y'$, $x - \frac{a}{2} = x'$, $2a = 4A$, we get the following:

$(y')^2 = 4Ax'$

The directrix of this is $x' = -A$

By substituting the original variables, we get the following:

$\Rightarrow x - \frac{a}{2} = -\frac{a}{2}$

$\Rightarrow x = 0$

Option (c) is the correct answer.

Self-Assessment 1

Step 1

Given, $x = t^2 + 1$, $y = 2t + 1$

$\Rightarrow t = \frac{y - 1}{2}$

$\Rightarrow x = \left(\frac{y - 1}{2}\right)^2 + 1$

$\Rightarrow 4(x - 1) = (y - 1)^2$

Step 2

By putting $y - 1 = Y$, $x - 1 = X$, the equation becomes the following: $4X = Y^2$

The equation of directrix is $X = -1$

By putting back the value of X, we get the following:

$x - 1 = -1$

$\Rightarrow x = 0$

Self-Assessment 2

Step 1

Given parabola, $y^2 = 32x$

By comparing this with the standard equation of the parabola, we get the following: $a = 8$

Also, the coordinates of endpoints of the focal chord of this parabola in parametric form will be $P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

I.e., $P(8t^2, 16t)$ and $Q\left(\frac{8}{t^2}, \frac{-16}{t}\right)$.

Step 2

Given, $P \equiv (2, -8)$

By comparing with $(8t^2, 16t)$, we get the following: $t^2 = \frac{1}{4}$ and $t = \frac{-1}{2}$

So, Q becomes $\left(\frac{8}{\left(\frac{1}{4}\right)}, \frac{-16}{\left(\frac{-1}{2}\right)}\right)$.

$Q \equiv (32, 32)$

PARABOLA

PROPERTIES OF FOCAL CHORD AND TANGENTS TO PARABOLA

What you already know

- Basic introduction to parabola
- Standard equation of a parabola
- Translation or shifting of the vertex
- Problems based on some mixed concepts

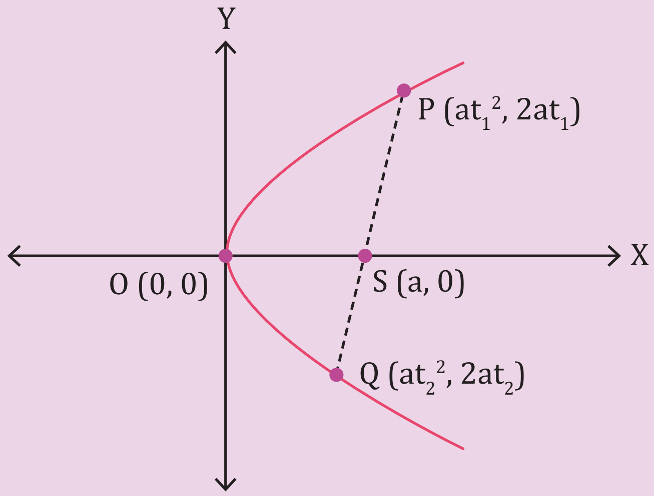
What you will learn

- Properties of a focal chord
- Equation of a tangent to a parabola
- Position of a line with respect to a parabola

Properties of a Focal Chord

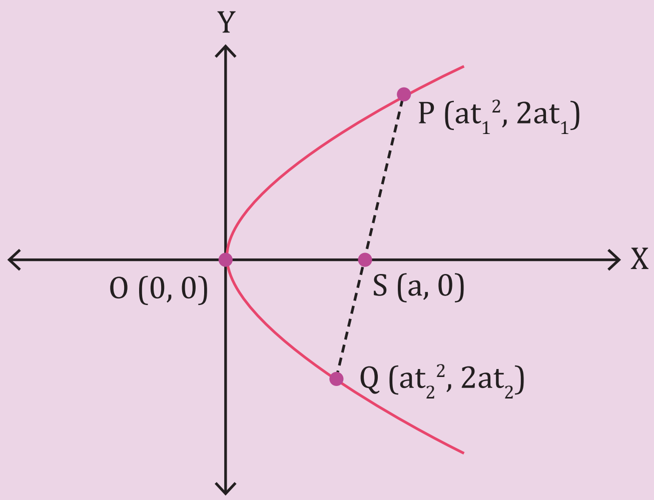
Property (I)

If P(t_1) and Q(t_2) are the end points of a focal chord, then $t_1 t_2 = -1$



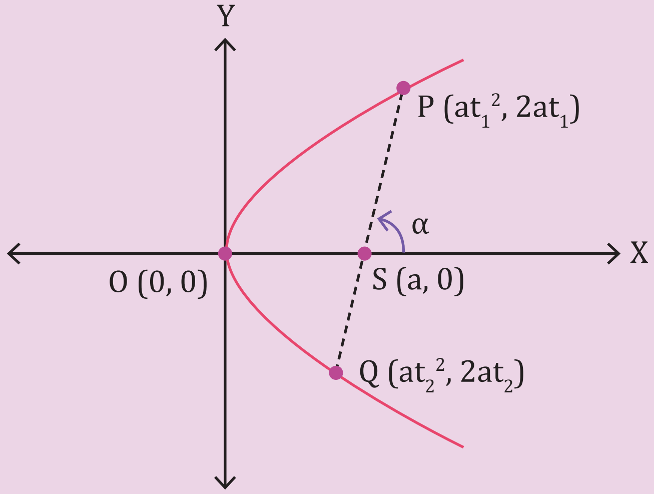
Property (II)

Length of the focal chord PQ in terms of parameter t is equal to $a\left(t + \frac{1}{t}\right)^2$



Property (III)

Length of the focal chord that makes an angle α with the positive direction of the x-axis is $4a\,\mathrm{cosec}^2\alpha$.



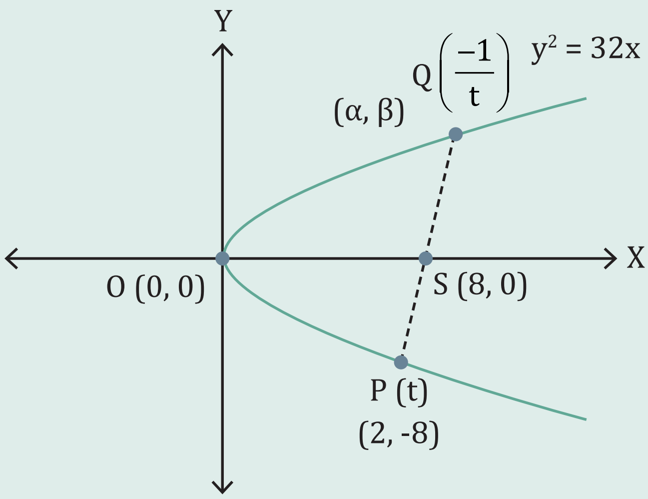
If $(2,-8)$ is at an end of a focal chord of the parabola $y^2 = 32x$ and the other end of the chord is (α, β), then find $\dfrac{\alpha+\beta}{8}$.

(a) 3 (b) 18 (c) 8 (d) 32

Solution

Step 1

Given,
Parabola $y^2 = 32x$, and one end of its focal chord is $(2, -8)$.
As we can see that the given parabola is comparable to $y^2 = 4ax$, which is a rightward-opening parabola,
$\Rightarrow a = 8$
Let the parametric coordinates of point $P \equiv P(at^2, 2at)$
Or $P(8t^2, 16t)$
As the given coordinates of P are $(2, -8)$



Step 2

As we can see,

$P\left(8t^2,\ 16t\right) \equiv (2,\ -8)$

$\Rightarrow 16t = -8 \Rightarrow t = \dfrac{-1}{2}$

Now, the parameter of point P is $t = \dfrac{-1}{2}$

$\Rightarrow$ Parameter of point Q will be $\dfrac{-1}{t} = 2$

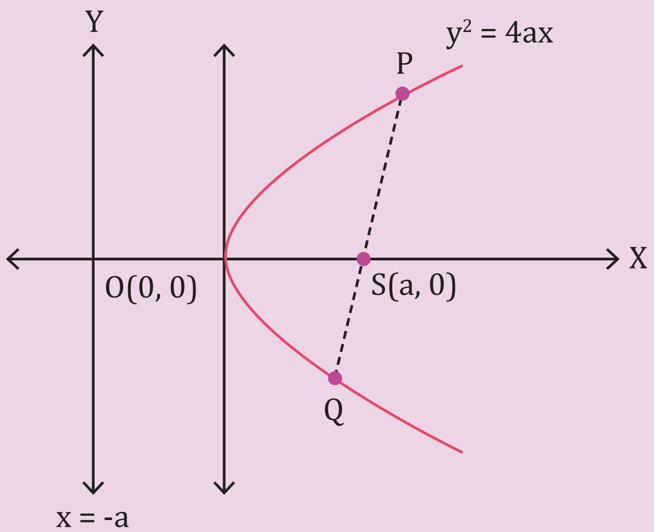
Therefore, point $Q\left(8(2)^2,\ 16(2)\right) \Rightarrow Q(32,\ 32)$

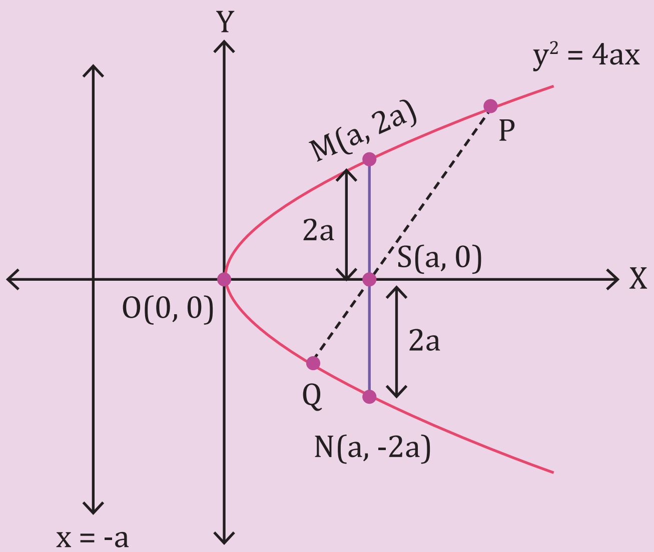
Now, we have, $\alpha = 32$ and $\beta = 32$

Therefore, $\dfrac{\alpha+\beta}{8} = \dfrac{32+32}{8} = 8$

Hence, option (c) is the correct answer.

Basic terminology that will be used further





$$MS = NS = 2a = \frac{1}{2}(4a) = \text{Semi LR}$$

Property (IV)

Semi LR is the harmonic mean (HM) of SP and SQ, where P and Q are the extremities of the focal chord.

2a = HM of |PS| and |QS|

$$a, b, c \text{ are in HP if } b = \frac{2ac}{a+c}$$

$\Rightarrow$ |PS|, 2a, |QS| are in HP,

Iff $2a = 2\left(\frac{|PS| \times |QS|}{|PS| + |QS|}\right)$

Proof

Here,

$|PS|$ = Focal distance = $|PM| = at^2 + a$

$|QS|$ = Focal distance of Q = $|QN| = \frac{a}{t^2} + a$

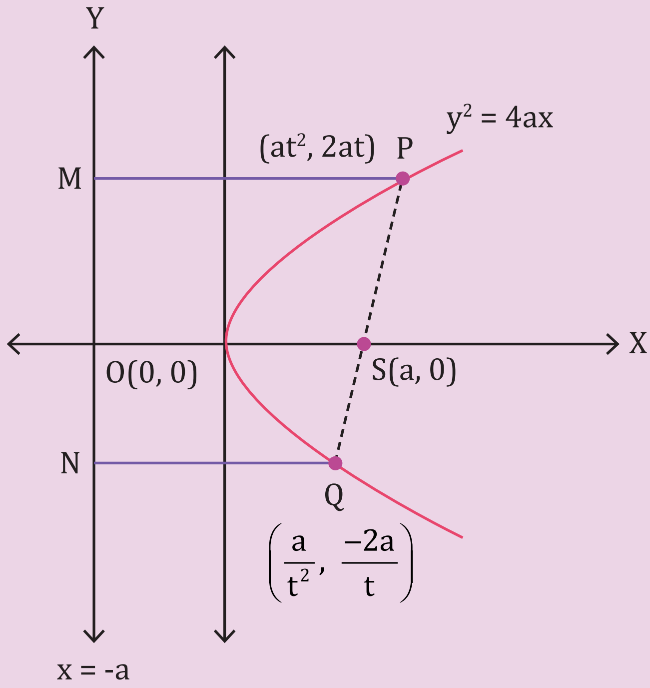
Now, $\frac{1}{|PS|} + \frac{1}{|QS|} = \frac{1}{a + at^2} + \frac{1}{a + \frac{a}{t^2}}$

$$= \frac{(1+t^2)}{a(1+t^2)} = \frac{1}{a}$$

Or $a = \left(\frac{|PS| \times |QS|}{|PS| + |QS|}\right)$

After multiplying both the sides by 2, we get the following

$\Rightarrow 2a = \left(\frac{2 \times |PS| \times |QS|}{|PS| + |QS|}\right)$



If AB is the focal chord of parabola $x^2 - 2x + y - 2 = 0$, whose focus is S, and $|AS| = l$, then find $|BS|$.

Solution

Step 1

Given,
AB is the focal chord of parabola
$x^2 - 2x + y - 2 = 0$, whose focus is S,
and $|AS| = l$

And we have to find the value of $|BS|$.
As we can see that parabola $y = -x^2 + 2x + 2$ is either upward or downward opening,

$y = -x^2 + 2x + 2$

$\Rightarrow y = -[x^2 - 2x - 2]$

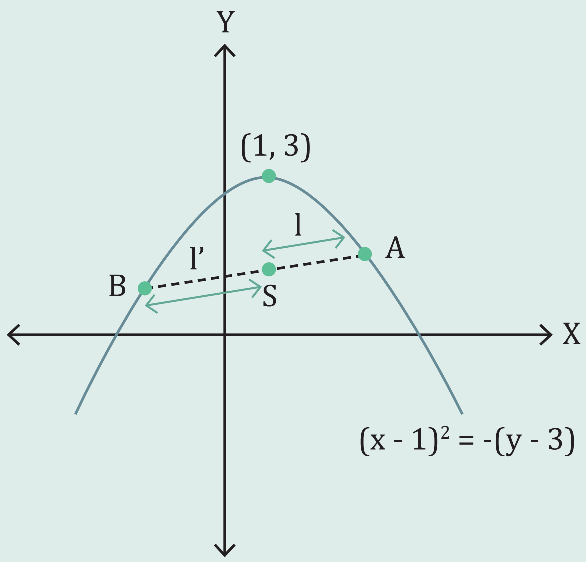
$\Rightarrow y = -[x^2 - 2x + (-1)^2 - (-1)^2 - 2]$

$\Rightarrow y = -[(x - 1)^2 - 3]$

$\Rightarrow y = -(x - 1)^2 + 3$

Or

$(x - 1)^2 = -(y - 3)$



This is comparable to standard equation
$(x - h)^2 = -4A(y - k)$
Thus, the transformed equation of the parabola is $(x - 1)^2 = -(y - 3)$
And it is a downward opening parabola

Step 2

As we know,
$\left[\ |\text{Semi LR}| = \text{HM of } l \text{ and } l'\ \right]$

Length of LR = $4A = 4 \times \frac{1}{4} = 1$ unit

Length of half LR = $2A = \frac{1}{2}$ units

Here, $|AS|, \frac{1}{2}, |BS|$ are in HP.

$$\frac{1}{2} = 2\left(\frac{l \times l'}{l + l'}\right)$$

$\Rightarrow l + l' = 4ll'$

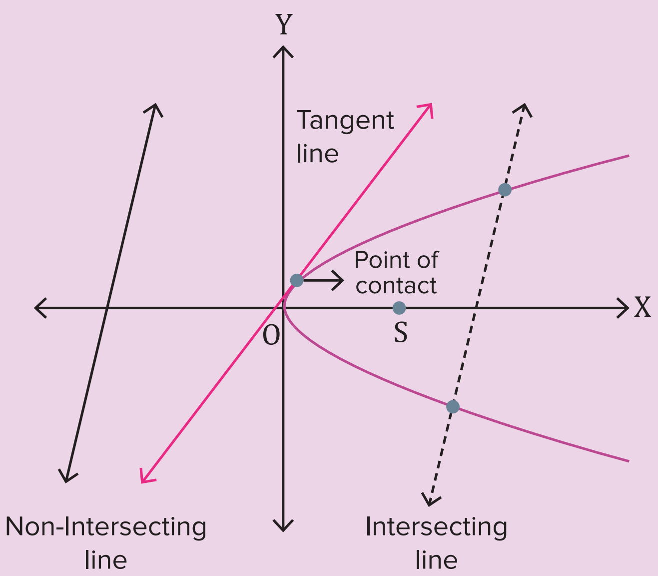
$\Rightarrow 4ll' - l' = l$

Or $l' = \dfrac{l}{4l - 1} = |BS|$

Position of a Line with Respect to a Parabola

Let us consider a parabola, $y^2 = 4ax$
And a line, $y = mx + c$
Solve simultaneously to get the points of intersection.
$\Rightarrow (mx + c)^2 = 4ax$
$\Rightarrow m^2x^2 + c^2 + 2mcx - 4ax = 0$
Or $(m^2)x^2 + (2mc - 4a)x + c^2 = 0$, which is quadratic in x.

Possible cases
- D > 0 ; Intersecting line (Two real and distinct roots)
- D = 0 ; Tangent (Real and repeated roots)
- D < 0 ; Imaginary roots (Non-intersecting line)



Here,

$D = (2mc - 4a)^2 - 4m^2c^2$

$= 4m^2c^2 + 16a^2 - 16amc - 4m^2c^2$

$= 16a(a - mc)$

Now, we will take different cases to get different conditions.

Case 1: Condition for an intersecting line

⇒ Two points of intersection

⇒ Two real and distinct roots

$\Rightarrow D > 0$

$\Rightarrow 16a(a - mc) > 0$

$\Rightarrow a - mc > 0$;[as a is positive]

$\Rightarrow a > mc$

Or $c < \frac{a}{m}$

Case 3: Condition for an non-intersecting line

⇒ No points of intersection

⇒ No real roots

⇒ Two imaginary roots

$\Rightarrow D < 0$

$\Rightarrow 16a(a - mc) < 0$

$\Rightarrow a - mc < 0$;[as a is positive]

Or $c > \frac{a}{m}$

Case 2: Condition for a tangent line

⇒ One point of intersection

⇒ One distinct root

⇒ Two real and repeated roots

$\Rightarrow D = 0$

$\Rightarrow 16a(a - mc) = 0$

$\Rightarrow a - mc = 0$

$\Rightarrow a = mc$

Or $c = \frac{a}{m}$

Note

In $y = mx + c$, on replacing c with $\frac{a}{m}$, the line becomes tangent to the parabola $y^2 = 4ax$.

Concept Check 1

The straight line, $y = 2x + \lambda$, does not meet the parabola $y^2 = 2x$ under which of the following conditions?

(a) $\lambda < \frac{1}{4}$ (b) $\lambda = 4$ (c) $\lambda > \frac{1}{4}$ (d) $\lambda = 1$

Equation of a Tangent to a Parabola

The equation of a tangent divides into the following three forms:

(a) Point form

(b) Parametric form

(c) Slope form

(a) Point form

The equation of a tangent to a parabola, $y^2 = 4ax$, at point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$

Proof

Given,

Parabola $y^2 = 4ax$ and a point on the parabola, $P(x_1, y_1)$

Now, using the concept of differentiation to get the slope at point P

i.e., $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Using the chain rule of differentiation, we get,

$$2y\frac{dy}{dx} = 4a \Rightarrow m = \frac{2a}{y_1}$$

Equation of the tangent line

$$\Rightarrow y - y_1 = \frac{2a}{y_1}(x - x_1)$$

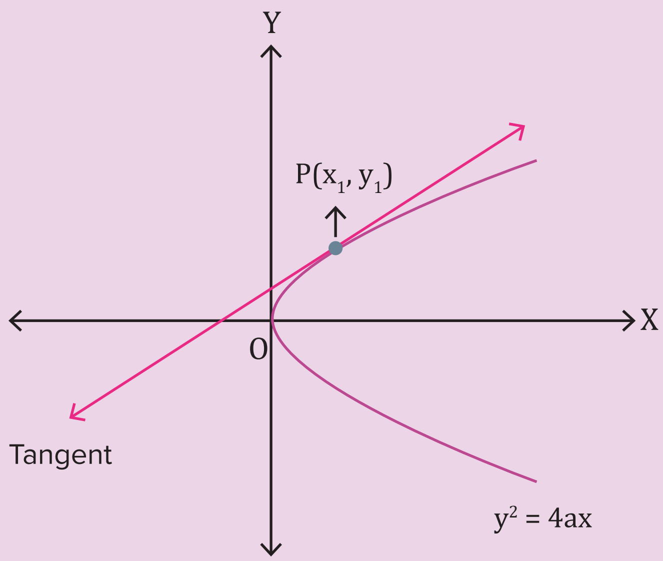
$\Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$

Or $yy_1 - 2ax = y_1^2 - 2ax_1$

As given, $y_1^2 = 4ax_1$

$\Rightarrow yy_1 - 2ax = 4ax_1 - 2ax_1$

Or $yy_1 = 2a(x + x_1)$



Note

- The equation of a tangent is given by $T = 0$,
  Where the T expression is obtained by using the following transformation:

$$x^2 \Rightarrow xx_1,\ y^2 \Rightarrow yy_1,\ x \Rightarrow \frac{x+x_1}{2},\ y \Rightarrow \frac{y+y_1}{2} \text{ and } xy \Rightarrow \frac{xy_1 + x_1y}{2}$$

- The equations of the tangents of all standard parabolas at (x_1, y_1) are as follows:

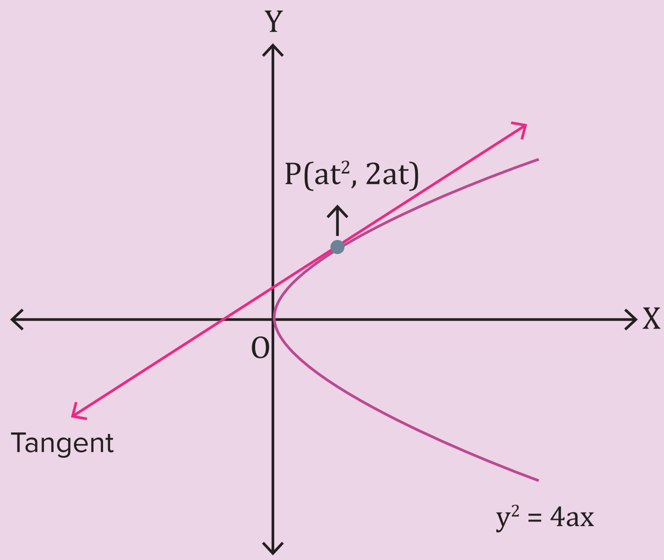
Equation of parabolas	Tangent at (x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

(b) Parametric form

The equation of a tangent to a parabola, $y^2 = 4ax$, at point $P(t) \equiv P(at^2, 2at)$ is $ty = x + at^2$

Proof

We know, by the point form of tangent,
$yy_1 = 2a(x + x_1)$
Now, we know, $(x_1, y_1) \equiv (at^2, 2at)$
Therefore,
$\Rightarrow y(2at) = 2a(x + at^2)$
Or $2aty = 2ax + 2a^2t^2$
$\Rightarrow ty = x + at^2$



Note

The equations of the tangents of all standard parabolas at 't' are as follows:

Equations of parabolas	Parametric coordinates 't'	Tangent at 't'
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

(c) Slope form

The equation of a tangent of slope m to the parabola $y^2 = 4ax$ is given by $y = mx + \frac{a}{m}$

Proof

We know,

$y = mx + c$ is a tangent to the parabola

$y^2 = 4ax$, if $c = \frac{a}{m}$

Now, the equation of the tangent of slope m becomes the following:

$\Rightarrow y = mx + \frac{a}{m}$

Let coordinates of point P be $(at^2, 2at)$

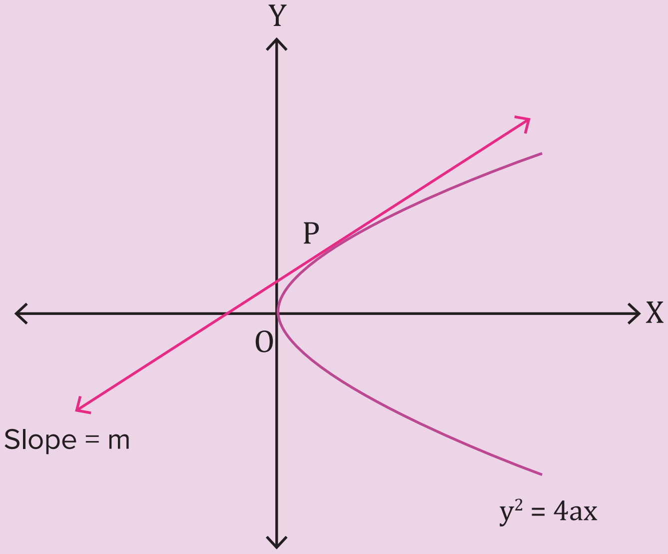
Differentiating the equation of tangent with respect to x, we get,

$$\left(\frac{dy}{dx}\right)_{(at^2,\ 2at)} = m$$

Now, differentiating the equation of parabola using the chain rule, we get

$$\Rightarrow 2y\frac{dy}{dx} = 4a \Rightarrow m = \frac{2a}{2at}$$

Or $m = \frac{1}{t}$, and point P becomes $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$



Note

The equations of the tangents of all standard parabolas in the slope form are as follows:

Equation of parabolas	Points of contact in terms of the slope (m)	Equations of tangents in terms of the slope (m)
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$

If parabola $y^2 = 4ax$ passes through the point $(1, -2)$, then find the tangent at this point.

(a) $x + y - 1 = 0$ (b) $x - y - 1 = 0$ (c) $x + y + 1 = 0$ (d) $x - y + 1 = 0$

Solution

Step 1

Given,

Parabola $y^2 = 4ax$ passes through the point $(1, -2)$ and we have to find the tangent at this point.

As we know,

The equation of the tangent to parabola $y^2 = 4ax$ at point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

We get the following:

$y(-2) = 2a(x + 1)$

$\Rightarrow -2y = 2a(x + 1)$

Or $-y = a(x + 1)$

Point $(1, -2)$ will satisfy this equation

Therefore, $2 = a(1 + 1) \Rightarrow a = 1$

Now, the equation of the tangent is $x + y + 1 = 0$.

Hence, option (c) is the correct answer.

Concept Check 2

Find the equation of the tangent to parabola $y^2 = 16$, which is perpendicular to the line $y = 3x + 7$.

(a) $y - 3x + 4 = 0$ (b) $y - x + 36 = 0$ (c) $3y + x - 36 = 0$ (d) $3y + x + 36 = 0$

Summary Sheet

Key Results

Properties of a focal chord

- **Property (I):** If $P(t_1)$ and $Q(t_2)$ are the endpoints of a focal chord, then $t_1 t_2 = -1$
- **Property (II):** The length of focal chord PQ in terms of parameter t is equal to $a\left(t + \frac{1}{t}\right)^2$.
- **Property (III):** The length of the focal chord that makes angle α with the positive direction of the x-axis is $4a\,\text{cosec}^2\alpha$.
- **Property (IV):** Semi LR is the harmonic mean (HM) of SP and SQ, where P and Q are the extremities of the focal chord and S is the focus of the parabola .

Position of a line with respect to a parabola

Let us consider a parabola, $y^2 = 4ax$, and a line, $y = mx + c$

Case 1: Condition for an intersecting line

When $c < \frac{a}{m}$

Case 2: Condition for a tangent line

When $c = \frac{a}{m}$

Case 3: Condition for a non-intersecting line

When $c > \frac{a}{m}$

Equation of tangent

(a) Point form

The equation of the tangent to the parabola $y^2 = 4ax$ at point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$

(b) Parametric form

The equation of the tangent to the parabola $y^2 = 4ax$ at point $P(t) \equiv P(at^2, 2at)$ is $ty = x + at^2$

(c) Slope form

The equation of the tangent of slope m to the parabola $y^2 = 4ax$ is given by $y = mx + \frac{a}{m}$

Mind Map

- **Parabola**
  - Properties of focal chord
  - Position of a line with respect to a parabola
  - Equation of a tangent to parabola

Self-Assessment

1. If the length of the chord of circle $x^2 + y^2 = 4$ and $y^2 = 4(x - h)$ is the maximum, then find the value of h. .
2. Two tangents are drawn from point $(-2, -1)$ to parabola $y^2 = 4x$. If α is the angle between these tangents, then find the value of $\tan \alpha$. .

Answers

Concept Check 1

Step 1:

Given,

The straight line, $y = 2x + \lambda$, does not meet parabola $y^2 = 2x$

And

We have to find the value or the range for λ.

We can see that parabola $y^2 = 2x$ is comparable to $y^2 = 4ax$,

i.e., $4a = 2 \Rightarrow a = \frac{1}{2}$

As we know, for a non-intersecting line, $c > \frac{a}{m}$

Therefore, $\lambda > \frac{1}{2 \times 2} \Rightarrow \lambda > \frac{1}{4}$

Hence, option (c) is the correct answer.

Concept Check 2

Step 1:

Given,

The equation of the tangent to parabola $y^2 = 16x$, which is perpendicular to the line,

$l : y = 3x + 7$

Let us consider the perpendicular line as l_1, slope of line l as m_l, and slope of line l_1 as m_{l_1}.

Therefore, $m_l \times m_{l_1} = -1$ or $m_{l_1} = \frac{-1}{3}$

Now, the tangent becomes the following:

$$y = \left(\frac{-1}{3}\right)x + c$$

A line $y = mx + c$ is tangent to parabola $y^2 = 4ax$ if $c = \frac{a}{m}$

So, $c = \dfrac{4}{\left(\frac{-1}{3}\right)} = -12$

And the equation will become $y = \frac{-x}{3} - 12$ or $x + 3y + 36 = 0$

Hence, option (d) is the correct answer.

Self-Assessment 1

Step 1:

Given,

The lengths of the chords of circles $x^2 + y^2 = 4$ and $y^2 = 4(x - h)$ are the maximum.

We have to find the value of h.

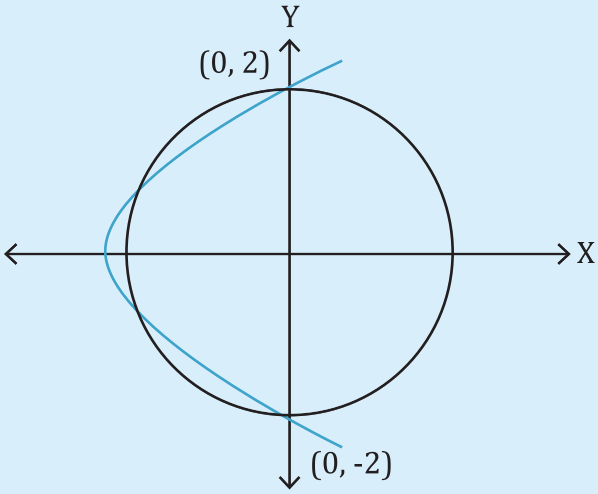
We know that the length of the chord will be the maximum if the parabola passes through (0, 2) and (0, -2).

Hence, from $y^2 = 4(x - h)$,

We have

$4 = 4(0 - h)$

Or $h = -1$.



Self-Assessment 2

Step 1:

Given,

Two tangents are drawn from the point (-2, -1) to the parabola $y^2 = 4x$

We have to find the value of tan α if α is the angle between these tangents.

Here, after comparing with $y^2 = 4ax$, we get,

$a = 1$

Also, any tangent to the parabola $y^2 = 4x$ having slope m is $y = mx + \frac{1}{m}$

It passess through (-2, -1)

$\Rightarrow 2m^2 - m - 1 = 0$

Or $m = 1, \frac{-1}{2}$

Step 2:

Here, $m_1 = 1$ and $m_2 = \frac{-1}{2}$

Now, $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}$

Or $\tan \alpha = 3$

NOTES

MATHEMATICS

PARABOLA

MORE ON TANGENTS TO PARABOLA AND PROPERTIES OF TANGENTS

What you already know

- Equations of different types of standard parabolas
- Properties of focal chord
- Equation of tangent to a parabola

What you will learn

- Equations of tangent in different forms for different types of parabolas
- Properties of tangents

Quick Recap

The equation of tangent of a parabola at (x_1, y_1) is given by $T = 0$, where T is obtained by replacing $x^2 \to xx_1$, $y^2 \to yy_1$, $x \to \frac{x + x_1}{2}$, $y \to \frac{y + y_1}{2}$, $xy \to \frac{xy_1 + yx_1}{2}$ in the equation of given parabola. The equations of tangents for different standard parabolas in point form are given as follows:

Equation of parabolas	Tangent at (x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

The equation of tangent of a parabola in parametric form is also given by $T = 0$. Here, the coordinates of the point of tangency are given in parametric form. The equations of tangents for different standard parabolas in parametric form are given as follows:

Equation of parabolas	Parametric coordinates 't'	Tangent at (t)
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

The equations of tangents of the different standard parabolas in slope form are given as follows:

Equation of parabolas	Point of contact in terms of slope (m)	Equation of tangent in terms of slope (m)
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$

If the tangent to the parabola $y^2 = 16x$ makes an angle of $45°$ with the positive direction of X - axis, then what is the point of contact?

(a) $(8, 8)$ (b) $(4, 4)$ (c) $(8, 4)$ (d) $(4, 8)$

Solution

Step 1:

Given that the tangent makes an angle of $45°$ with the positive direction of X - axis.

So, slope of the tangent = $\tan 45° = 1$

We know that the equation in slope form of the tangent to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

By substituting the values of a and m in the equation, we get the following equation of tangent:

$y = x + 4$ ...(1)

Step 2:

Now, let the point of contact be (x_1, y_1).

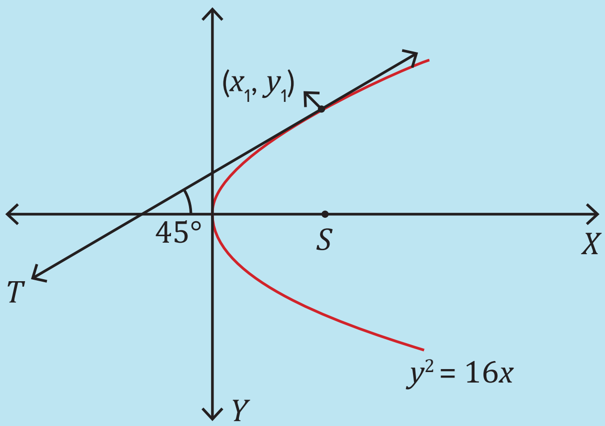
We know that the point form of the tangent to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

By substituting the value of a in the equation, we get the following equation of tangent:

$$yy_1 = 2 \times 4\left(x + x_1\right)$$

$$\Rightarrow yy_1 = 8\left(x + x_1\right)$$

$$\Rightarrow y = \left(\frac{8}{y_1}\right)x + \left(\frac{8x_1}{y_1}\right) \text{...}(2)$$



Step 3:

Now, equation (1) and (2) represent the same tangent. On comparing the coefficients, we get the following:

$$\frac{1}{1} = \frac{y_1}{8} = \frac{4y_1}{8x_1} \quad \Rightarrow y_1 = 8, x_1 = 4$$

Therefore, the point of contact is (4, 8).

Hence, option (d) is the correct answer.

Concept Check

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1. Find the equation of tangent to the parabola $x^2 = 8y$ which makes an angle θ with the positive direction of X - axis.
   (a) $x = y\cot\theta - 2\tan\theta$ (b) $y = x\tan\theta - 2\cot\theta$
   (c) $x = y\cot\theta + 2\tan\theta$ (d) $y = x\tan\theta + 2\cot\theta$

Find the slopes of the tangents drawn from the point $P(4, 10)$ to the parabola $y^2 = 9x$

(a) $\frac{1}{4}, \frac{3}{4}$ **(b)** $\frac{1}{4}, \frac{9}{4}$ **(c)** $\frac{1}{4}, \frac{1}{3}$ **(d)** $\frac{3}{4}, \frac{9}{4}$

Solution

Step 1:

First, we check if the point P lies inside, on, or outside the parabola. For that, we need to find S_P.

$S_P = 10^2 - 9 \times 4 = 64 > 0$

Therefore, point P lies outside the parabola $y^2 = 9x$. So, two tangents can be drawn from P.

We know that the equation of tangent to the parabola $y^2 = 4ax$ in slope form is given as follows:

$$y = mx + \frac{a}{m}$$

So, equation of tangent to the parabola $y^2 = 9x$ in slope form will be $y = mx + \frac{9}{4m}$

Step 2:

Now, the tangent is drawn from the point $P(4, 10)$. So, it should satisfy the equation of tangent.

$$\Rightarrow 10 = 4m + \frac{9}{4m}$$

$$\Rightarrow 16m^2 - 40m + 9 = 0$$

$$\Rightarrow 16m^2 - 4m - 36m + 9 = 0$$

$$\Rightarrow 4m(4m - 1) - 9(4m - 1) = 0$$

$$\Rightarrow (4m - 9)(4m - 1) = 0$$

$$\Rightarrow m = \frac{1}{4}, \frac{9}{4}$$

Here, we got two values of m because two tangents can be drawn from the point P.

Hence, option (b) is the correct answer.

Find the equation of the common tangent to the circle $x^2 + y^2 - 6x = 0$ and the parabola $y^2 = 4x$

(a) $\sqrt{3}y = 3x + 1$ **(b)** $2\sqrt{3}y = -x - 12$

(c) $2\sqrt{3}y = 12x + 1$ **(d)** $\sqrt{3}y = x + 3$

Solution

Step 1:

Consider the circle $x^2 + y^2 - 6x = 0$

Here, the centre is (3, 0) and

radius = $\sqrt{(-3)^2 + 0 - 0} = 3$

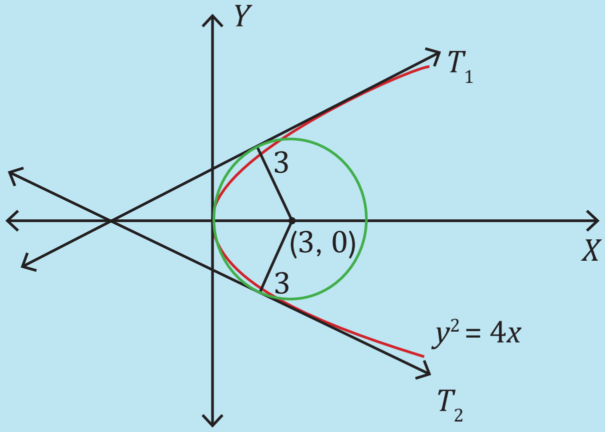
We know that the equation of tangent to the parabola $y^2 = 4ax$ is given by $y = mx + \frac{a}{m}$

So, the equation of tangent to the parabola $y^2 = 4x$ will be

$y = mx + \frac{1}{m} \Rightarrow m^2x - my + 1 = 0$

Now, this line is also tangent to the circle $x^2 + y^2 - 6x = 0$. So, the perpendicular distance from the centre of the circle to the tangent line should be equal to the radius of the circle.

$\Rightarrow \left|\frac{m^2(3) - m(0) + 1}{\sqrt{m^4 + m^2}}\right| = 3$



Step 2:

By squaring both the sides, we get the following:

$\frac{9m^4 + 6m^2 + 1}{m^4 + m^2} = 9$

$\Rightarrow 9m^4 + 6m^2 + 1 = 9m^4 + 9m^2$

$\Rightarrow 3m^2 = 1 \Rightarrow m = \pm\frac{1}{\sqrt{3}}$

Note that as m tends to infinity, the limit of the expression on the left tends to 9. So, we also get a vertical common tangent. As we can observe from the figure, this is the y-axis. It is a common tangent to the circle and the parabola at the point (0, 0), so $x = 0$ is also a common tangent.

By substituting the values of m in the equation of tangent, we get the following:

$y = \frac{1}{\sqrt{3}}x + \sqrt{3} \Rightarrow \sqrt{3}y = x + 3$ and $y = -\frac{1}{\sqrt{3}}x - \sqrt{3} \Rightarrow \sqrt{3}y = -x - 3$

We can see that first equation, $\sqrt{3}y = x + 3$, is there in option (d).

Hence, option (d) is the correct answer.

Concept Check

2. Find the equation of the common tangent of $y^2 = 4ax$ and $x^2 = 4ay$.

Note

Suppose that we have a translated parabola, i.e., the vertex is not at the origin.
Now, let the equation of the translated parabola be $(y - k)^2 = 4a(x - h)$. Then, for this parabola, the equation of tangent in different forms (slope form, point form, parametric form) can be obtained by replacing y with $y - k$ and x with $x - h$ in the equation of tangent of the corresponding standard parabola.
For example, the equation of tangent of the parabola $(y - k)^2 = 4a(x - h)$ in slope form can be obtained by replacing y with $y - k$ and x with $x - h$ in the equation of tangent of the parabola $y^2 = 4ax$ in the slope form, i.e., replace y with $y - k$ and x with $x - h$ in the equation $y = mx + \frac{a}{m}$
So, the equation of tangent in slope form for the parabola $(y - k)^2 = 4a(x - h)$ is as follows: $(y - k) = m(x - h) + \frac{a}{m}$
In a similar way, we can obtain the equation of tangent for different types of translated parabolas.

Find the equations of the common tangents to the parabola $y = x^2$ and $y = -(x - 2)^2$

(a) $y = 4(x - 1)$ (b) $x = 0$ (c) $y = -4(x - 1)$ (d) $y = -30x - 50$

Solution

Step 1:
We have two curves,
$x^2 = y$, $(x - 2)^2 = -y$
We know that the equation of tangent to the parabola $x^2 = 4ay$ in the slope form is given as follows:
$y = mx - am^2$
So, the equation of tangent in the slope form for parabola $x^2 = y$ will be $T_1 : y = mx - \frac{1}{4}m^2$
Similarly, the equation of tangent for the parabola $(x - h)^2 = -4a(y - k)$ in the slope form is given as follows:
$y - k = m(x - h) + am^2$
So, the equation of tangent in the slope form for the parabola $(x - 2)^2 = -y$ will be,
$$T_2 : y - 0 = m(x - 2) + \frac{1}{4}m^2 \Rightarrow T_2 : y = mx - 2m + \frac{1}{4}m^2$$

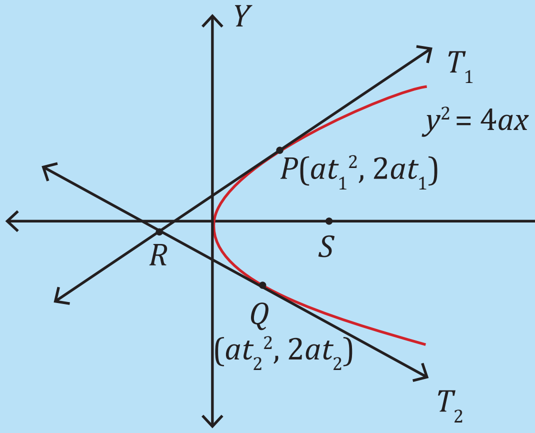
Step 2:
Now, for the common tangent, both T_1 and T_2 should represent the same line. So, the constant terms of both the tangent equations should be equal.
$$\Rightarrow -\frac{1}{4}m^2 = \frac{1}{4}m^2 - 2\text{m} \quad \Rightarrow \frac{1}{2}m^2 - 2m = 0 \quad \Rightarrow m(m - 4) = 0$$
$m = 0, 4$
Therefore, there are two common tangents that are given by the following equations:
$y = 0$
$y = 4x - 4$
Hence, option (a) is the correct answer.

Properties of Tangents

1. **Tangents at $P(t_1)$ and $Q(t_2)$ intersect at $R(at_1t_2, a(t_1 + t_2))$**

Proof

Let us consider the parabola $y^2 = 4ax$
We know that tangent at point $P(t_1)$ to the parabola $y^2 = 4ax$ in parametric form is given as follows: $T_1 : yt_1 = x + at_1^2$. Similarly, tangent at $Q(t_2)$ in the parametric form is given as follows: $T_2 : yt_2 = x + at_2^2$.
Now, by subtracting T_2 from T_1, we get the following:
$(t_1 - t_2)y = a(t_1^2 - t_2^2) \Rightarrow y = a(t_1 + t_2)$
By substituting the value of y in T_1, we get the following:
$a(t_1 + t_2)t_1 = x + at_1^2$
$x = at_1^2 + at_1t_2 - at_1^2 = at_1t_2$
Therefore, the point of intersection of tangents at $P(t_1)$ and $Q(t_2)$ is given by $(at_1t_2, a(t_1+t_2))$.
Hence proved.



Note

The x-coordinate of the point of intersection of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is the geometric mean of the x-coordinates of P and Q, and the y-coordinate of the point of intersection of tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is the arithmetic mean of the y-coordinates of P and Q. Therefore, $R(GM$ of abscissa of P and Q, AM of ordinate of P and $Q) \equiv (at_1t_2, a(t_1 + t_2))$

Find the point of intersection of the tangents to the parabola $y^2 = 4x$ at the points where parameter t has the value 1 and 2.

(a) (3, 8) **(b)** (1, 5) **(c)** (2, 3) **(d)** (4, 6)

Solution

Step 1:
Let the points corresponding to parameter $t_1=1$ and $t_2=2$ be P and Q, respectively.
We know that the tangents at points $P(t_1)$ and $Q(t_2)$ intersect at a point with coordinates $(at_1t_2, a(t_1 + t_2))$.
Now, here $a = 1$, $t_1 = 1$, and $t_2 = 2$
Therefore, the point of intersection will be as follows:
$R \equiv ((1 \times 1 \times 2), 1 \times (1 + 2))$
$R \equiv (2, 3)$
Hence, option (c) is the correct answer.

2. The portion of tangent between the point of contact (P) and the point where it meets the directrix (Q) subtends a right angle at the focus (S).

Proof

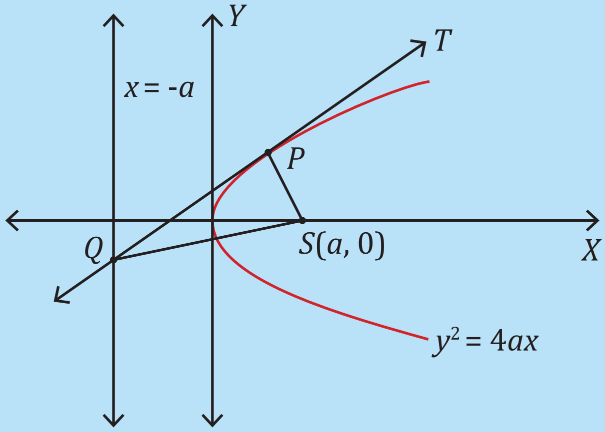
Consider the parabola $y^2 = 4ax$

Let the coordinates of the point where the tangent meets the directrix be $Q \equiv (-a, y_o)$

Equation of tangent at point $P(t)$ is as follows:

$T : yt = x + at^2$

We know that Q lies on the tangent T. So, satisfying the coordinates of Q, we get the following:



$ty_o = -a + at^2 \qquad y_o = \dfrac{-a}{t} + at$

Therefore, the coordinates of Q are $(-a,\ at - \dfrac{a}{t})$.

Now, for angle PSQ to be 90°, the product of slope of PS and QS should be -1.

$$m_{PS} = \frac{2at - 0}{at^2 - a} = \frac{2at}{a\left(t^2 - 1\right)} = \frac{2t}{t^2 - 1}$$

Similarly,

$$m_{QS} = \frac{at - \frac{a}{t} - 0}{-a - a} = \frac{at^2 - a}{-2at} = \frac{t^2 - 1}{-2t}$$

Now, $m_{PS} \times m_{QS}$

$$= \frac{2t}{t^2 - 1} \times \frac{t^2 - 1}{-2t} = -1$$

PS and QS are perpendicular to each other. Hence, PQ subtends the right angle at the focus (S)

3. Tangents drawn at the extremities of the focal chord are perpendicular and intersect on the directrix.

Proof

Let $P(t_1)$ and $Q(t_2)$ be the endpoints of a focal chord.

We know that the tangents drawn at two different points with parameters t_1 and t_2 intersect at a point with coordinates $(at_1t_2,\ a(t_1 + t_2))$.

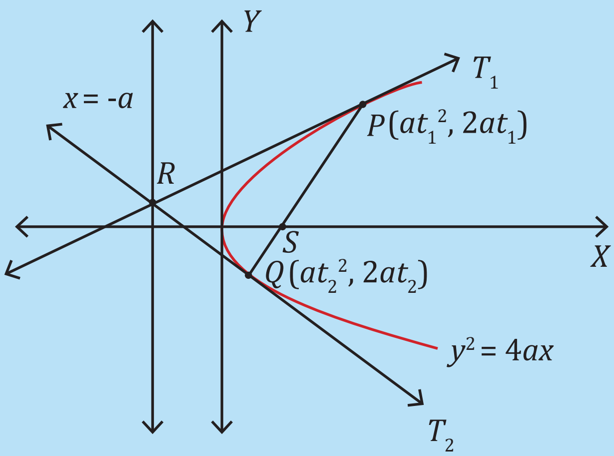
So, $R = (at_1t_2,\ a(t_1 + t_2))$

Since P and Q are the endpoints of a focal chord, product of t_1 and t_2 will be -1.

Therefore, $R = (-a,\ a(t_1 + t_2))$

R lies on the line $x = -a$ which is the directrix of the parabola.

Now, let m_1 and m_2 be the slopes of the tangents at $P(t_1)$ and $Q(t_2)$, respectively.



We know that the parameter associated with a point and the slope of the tangent at that point are inversely related to each other, i.e., $m = \frac{1}{t}$

$$\Rightarrow m_1 = \frac{1}{t_1} \text{ and } m_2 = \frac{1}{t_2}$$

$$\Rightarrow m_1 m_2 = \frac{1}{t_1 t_2}$$

$$\Rightarrow m_1 m_2 = -1 \quad [t_1 t_2 = -1]$$

Hence, the tangents at P and Q are perpendicular to each other and their point of intersection lies on the directrix.

Note

1. The locus of all the points from where two perpendicular tangents can be drawn to a circle is called the director circle of that circle.
2. A pair of tangents drawn from each point on the directrix of a parabola are perpendicular to each other. So, here the directrix plays the similar role as a director circle in case of a circle.

Summary sheet

Key Results

- The tangents at two different points with parameters t_1 and t_2 intersect at a point with coordinates $(at_1t_2, a(t_1 + t_2))$.
- The portion of tangent between the point of contact and the point where the tangent meets the directrix subtends the right angle at the focus.
- The tangents drawn at the extremities of the focal chord are perpendicular and intersect on the directrix.

Mind map

Properties of tangent to parabola

- Intersection point of tangent at t_1 and t_2 is given by $(at_1t_2, a(t_1 + t_2))$
- The segment of the tangent between point of contact and the point where it meets the directrix subtends right angle at the focus
- Tangent drawn at the extremities of a focal chord are perpendicular and intersect on the directrix

Self-Assessment

A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find the points of contact.

Answers

Concept Check 1

Method 1.

Step 1:

We know that a line making an angle θ with the positive direction of X-axis has a slope equal to tan θ. Now, the equation of tangent to the parabola $x^2 = 4ay$ in the slope form is given as follows:

$y = mx - am^2$

Given that $x^2 = 8y$, so $a = 2$

By substituting the values of a and m, we get the following:

$y = x \tan\theta - 2\tan^2\theta$

$\Rightarrow y + 2\tan^2\theta = x\tan\theta$

$\Rightarrow x = \dfrac{y}{\tan\theta} + 2\tan\theta$

$\Rightarrow x = y\cot\theta + 2\tan\theta$

Hence, option (c) is the correct answer.

Method 2.

Step 1:

We know that the equation of tangent to the parabola $x^2 = 4ay$ in the parametric form is given as follows:

$tx - y - at^2 = 0$

Now, the relation between the parameter t and slope (m) of the tangent for parabola $x^2 = 4ay$ is $t = m$

By substituting the values of a and t in the equation of tangent, we get the following:

$mx - y - 2m^2 = 0$

$x = \dfrac{y}{m} + 2m$

By substituting m by tan θ, we get the following:

$x = \dfrac{y}{\tan\theta} + 2\tan\theta$

$\Rightarrow x = y\cot\theta + 2\tan\theta$

Hence, option (c) is the correct answer.

Concept Check 2

Step 1:
We know that the equation of tangent in slope form for the parabola $y^2 = 4ax$ is given by

$$y = mx + \frac{a}{m}$$

Now, this tangent should also be a tangent to the parabola $x^2 = 4ay$, i.e., they should touch each other.
Substituting the value of y from the equation of tangent in the equation $x^2 = 4ay$.

$$\Rightarrow x^2 = 4a\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow mx^2 - 4am^2x - 4a^2 = 0 \qquad ...(1)$$

Step 2:
Now, equation (1) should have equal roots for the tangent to touch the parabola $x^2 = 4ay$.

$\Rightarrow D = 0$

$\Rightarrow 16a^2m^4 - 4m(-4a^2) = 0$

$\Rightarrow m(m^3 + 1) = 0$

$\Rightarrow m = -1 \qquad (m \neq 0)$

Step 3:
Substituting the value of m in the equation of tangent, we get,

$$y = -x + \frac{a}{-1}$$

$$\Rightarrow y + x + a = 0$$

Self-Assessment

Step 1:
Equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is as follows:
$ty = x + at^2$
Now, here $a = 2$, so the equation of the tangent at $(2t^2, 4t)$ to the parabola $y^2 = 8x$ is
$ty = x + 2t^2$

Slope of the tangent $= \dfrac{1}{t}$

Step 2:
Now it is given that the angle between the line $y = 3x + 5$ and tangent is 45°

$$\Rightarrow \tan 45° = \left|\frac{3 - \frac{1}{t}}{1 + 3 \times \frac{1}{t}}\right| \qquad \Rightarrow \pm \tan 45° = \frac{3 - \frac{1}{t}}{1 + 3 \times \frac{1}{t}}$$

$$\Rightarrow 3 - \frac{1}{t} = \pm\left(1 + \frac{3}{t}\right) \qquad \Rightarrow t = 2 \text{ or } \frac{-1}{2}$$

Therefore, the points of contact for $t = 2$ and $t = \dfrac{-1}{2}$ are $(8, 8)$ and $\left(\dfrac{1}{2}, -2\right)$, respectively.

MATHEMATICS

PARABOLA

MORE ON PROPERTIES OF TANGENTS AND NORMAL TO A PARABOLA

NOTES

What you already know

- Standard equation of parabola
- Important terms associated with parabola
- Parametric coordinates
- Tangent to a parabola

What you will learn

- More on properties of tangents
- Normal to a parabola
- Equation of normal in point form
- Equation of normal parametric form
- Equation of normal in slope form
- Problems on finding equation of normal

$y = x + 2$ is any tangent to the parabola $y^2 = 8x$. Find point P on the tangent such that the other tangent from it is perpendicular to it.

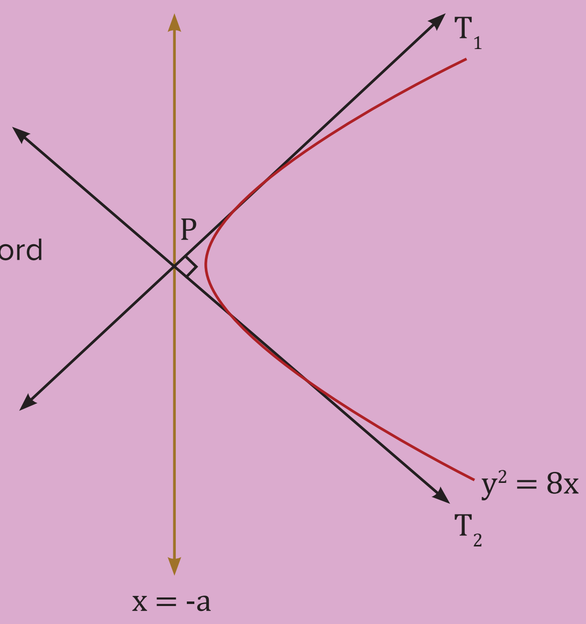
Solution

Step 1 :

Given,

$y^2 = 8x \implies a = 2$

The tangents drawn at the extremities of the focal chord are perpendicular and intersect on the directrix.



Step 2 :

Given,

P lies on the directrix.

$\Rightarrow x = -a = -2$

Satisfying $x = -2$ in the given equation of tangent,

we get the y coordinate of point of intersection,

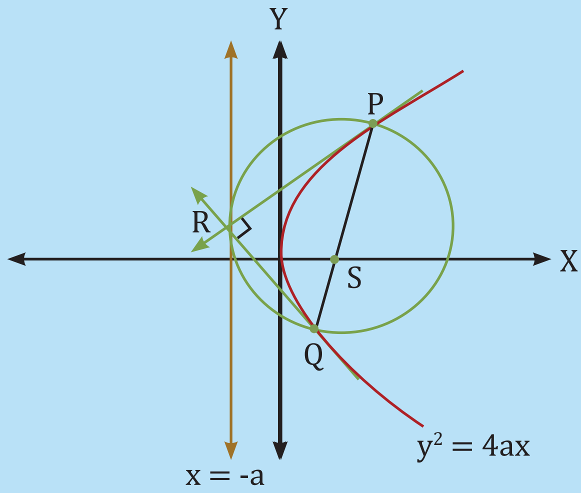
$\Rightarrow\ y = -2 + 2 = 0$

$\therefore P \equiv (-2, 0)$

Note

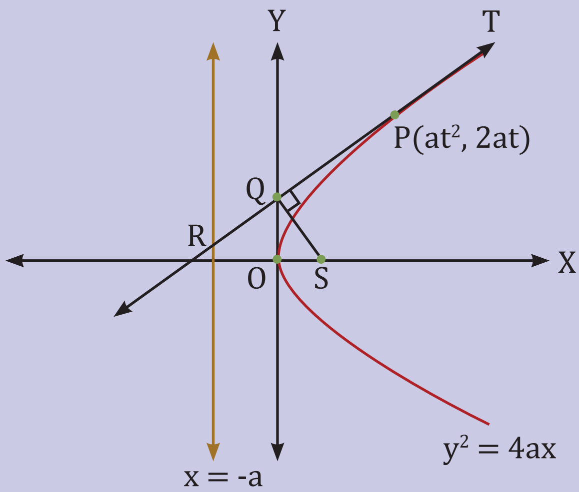
A circle having any focal chord as the diameter touches the directrix.

($\because$ The angle subtended by a diameter on the circumference is 90°.)



Properties of tangents

IV. The foot of the perpendicular drawn from the focus upon any tangent lies on the tangent at the vertex. Hence, a circle described on any focal radii as diameter touches the tangent at the vertex.



Proof

Using the parametric form for $y^2 = 4ax$ at $(at^2, 2at)$

Equation of tangent, $T: ty = x + at^2$..........(i)

Let T meets the y-axis at $Q(0, y_0)$.

By substituting $x = 0$ and $y = y_0$ in (i)
we get the following:

$\Rightarrow ty_0 = 0 + at^2 \Rightarrow y_0 = at$

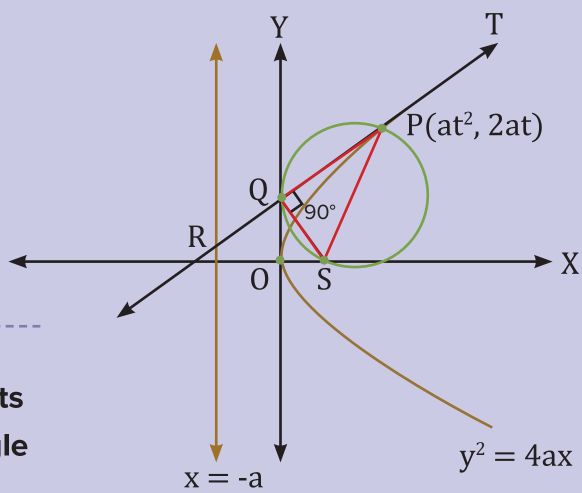
$\therefore Q \equiv (0, at)$

We have $Q \equiv (0, at)$, $S = (a, 0)$ and $P = (at^2, 2at)$

Now to prove $\angle PQS = 90°$ or $m_{PQ} \times m_{QS} = -1$

$$m_{PQ} = \frac{2at - at}{at^2 - 0} = \frac{at}{at^2} = \frac{1}{t} \text{ and } m_{QS} = \frac{0 - at}{a - 0} = -t$$

$$\Rightarrow m_{PQ} = \frac{-1}{m_{QS}} \text{ or } m_{PQ} \times m_{QS} = -1, \text{ i.e., } \angle PQS = 90°$$



V. The area of the triangle formed by three points on the parabola is twice the area of the triangle formed by the tangents at these points.

Proof

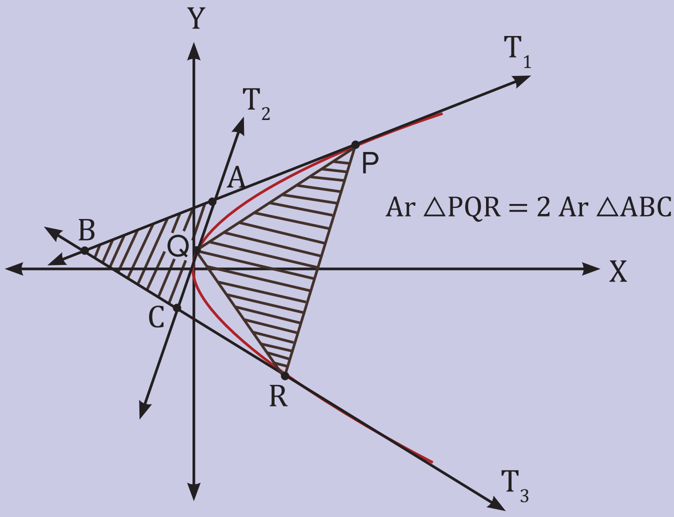
Let the coordinates of the points P, Q, R be $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$ respectively. We can calculate the area of triangle PQR using these coordinates.

Now, using the property 1 of tangents, the coordinates of the points A, B, and C can be obtained.

$A \equiv (at_1t_2, a(t_1 + t_2))$

$B \equiv (at_1t_3, a(t_1 + t_3))$

$C \equiv (at_2t_3, a(t_2 + t_3))$



Now, we have coordinates of vertices of both the triangles. Using them we can find the area of both the triangles and on finding the area we will see that the area of triangle PQR is twice of the area of triangle ABC.

If the focus of a parabola is $(1, 2)$ and the feet of the perpendicular on any two tangents drawn from the focus are $(3, 4)$ and $(4, 5)$, then find the vertex of the parabola.

Solution

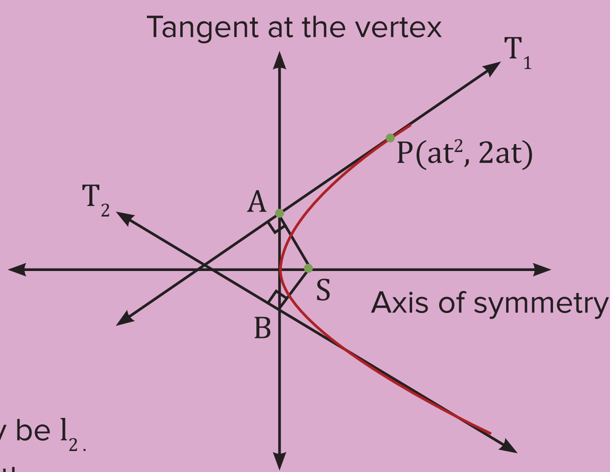
Step 1 :

Let $S \equiv (1, 2)$, $A \equiv (3, 4)$, $B \equiv (4, 5)$

The foot of the perpendicular drawn from the focus upon any tangent lies on the tangent at the vertex.

$SA \perp T_1$ and $SB \perp T_2$

The vertex is the point of intersection of the tangent at the vertex and the axis of symmetry.



Step 2 :

Equation of the tangent at the vertex

$$y - 4 = \frac{5-4}{4-3}(x - 3)$$

$\Rightarrow y = x + 1$, let this be l_1 and the axis of symmetry be l_2.

We know that l_1 and l_2 are perpendicular to each other.

So slope of l_2 will be -1.

$\Rightarrow l_2 : y - 2 = -1\,(x - 1) \Rightarrow y = -x + 3$

Now, the point of intersection of l_1 and l_2 will give the coordinates of the vertex.

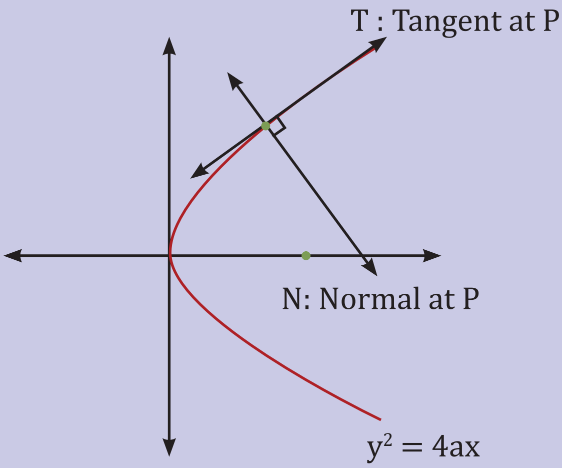
Solving the equations of l_1 and l_2 simultaneously, we get, $V \equiv (1, 2)$.

Concept Check 1

If the tangents are drawn from the parabola, $(x-3)^2+(x+4)^2=\frac{(3x-4y-6)^2}{25}$, at the extremities of chord $2x-3y-18=0$, then find the angle between them.

Normal

The line perpendicular to the tangent at the point of contact is known as the normal to the parabola at that point.



Point form

The equation of the normal to the parabola, $y^2 = 4ax$, at $P(x_1,\ y_1)$ is $y-y_1=\frac{-y_1}{2a}(x-x_1)$

Proof

We have $y^2 = 4ax$

Equation of the tangent at $P(x_1,\ y_1)$ is $T=0$

$\Rightarrow yy_1 = 2a\,(x+x_1)$

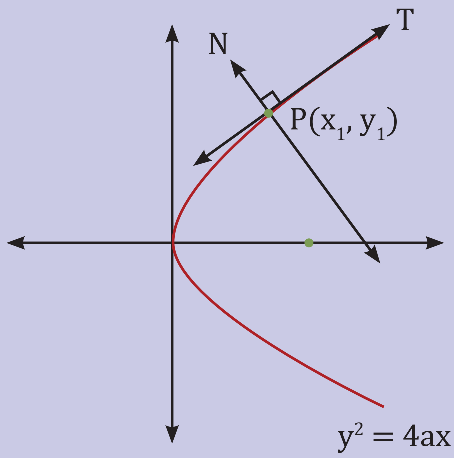
$\Rightarrow$ Slope of the tangent at P $= m_{tangent} = \frac{2a}{y_1}$

$\Rightarrow$ Slope of the normal at P $= \frac{-y_1}{2a}$ ($\because$ Tangent $\perp$ Normal)

$\therefore$ By the point-slope form,

The equation of the normal is as follows:

$y-y_1=\frac{-y_1}{2a}(x-x_1)$

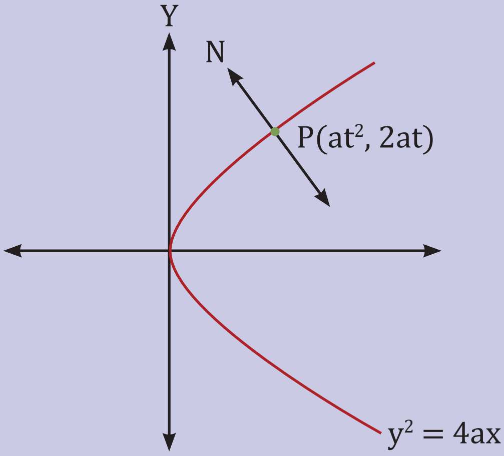


Equations of the normals of standard parabolas at $P(x_1, y_1)$

Equations	**Normals at** (x_1, y_1)
$y^2 = 4ax$	$y - y_1 = \frac{-y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = \frac{-2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

Parametric form

The equation of the normal to the parabola, $y^2 = 4ax$, at $P(t) \equiv P\ (at^2, 2at)$ is given by $y + tx = 2at + at^3$



Proof Here, $(x_1, y_1) \equiv (at^2, 2at)$

Equation of the normal $= y - y_1 = \frac{-y_1}{2a}(x - x_1)$ (Point form)

By substituting $x_1 = at^2$ and $y_1 = 2at$, we get the following:

$\Rightarrow y - 2at = \frac{-2at}{2a}(x - at^2) \quad \Rightarrow \quad y = -tx + 2at + at^3$

Note

Relation between t (parameter) and m (slope of normal) is as follows for $y^2 = 4ax$:

$\because m = \text{Slope of the normal} = \frac{-y_1}{2a}$, where $(x_1, y_1) \equiv (at^2, 2at)$

$\because m = \frac{-2at}{2a} = -t \Rightarrow t = -m$

Equations of parabolas	**Parametric Coordinates**	Relation between t (parameter) and m (slope of normal)
$y^2 = 4ax$	$(at^2, 2at)$	$m = -t$
$y^2 = -4ax$	$(-at^2, 2at)$	$m = t$
$x^2 = 4ay$	$(2at, at^2)$	$m = -\frac{1}{t}$
$x^2 = -4ay$	$(2at, -at^2)$	$m = \frac{1}{t}$

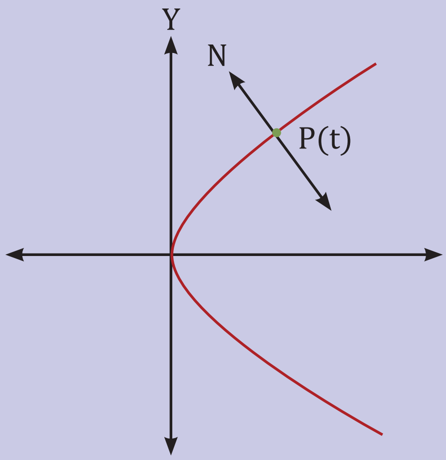
Equations of the normals of standard parabolas at t are as follows:

Equations of parabolas	Parametric Coordinates	Normals at t
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

Slope form

The equation of the normal to parabola $y^2 = 4ax$ of slope m is $y = mx - 2am - am^3$

Proof



We know, $t = -m$ for the parabola $y^2 = 4ax$

Equation of normal in terms of t is $y + tx = 2at + at^3$

By substituting $t = -m$, we get, the following:

$y = mx - 2am - am^3$

Equations of the normals, points of contact, and conditions of normality in terms of slope m are as follows:

Equations of parabolas	Points of contact	Equations of Normals
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$

The equations of the normals at the ends of the latus rectum of parabola $y^2 = 4ax$ are given by which of the following?

(a) $x^2 - y^2 - 6ax + 9a^2 = 0$ **(b)** $x^2 - y^2 - 6ax - 6ay + 9a^2 = 0$

(c) $x^2 - y^2 - 6ay + 9a^2 = 0$ **(d)** **None of the these**

Solution

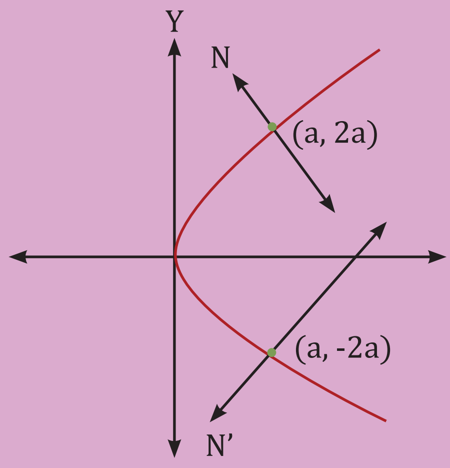
Step 1 :

The coordinates of the ends of the latus rectum of parabola $y^2 = 4ax$ are $(a, 2a)$ and $(a, -2a)$, respectively.

The equation of the normal at $(a, 2a)$ to $y^2 = 4ax$

is $y - 2a = \frac{-2a}{2a}(x - a)$ $\boxed{y - y_1 = \frac{-y_1}{2a}(x - x_1)}$

$\Rightarrow y + x - 3a = 0 \cdots (i)$



Step 2 : Similarly, the equation of the normal at $(a, -2a)$ is $x - y - 3a = 0 \quad \cdots (ii)$

The combined equation of (i) and (ii) is as follows: $x^2 - y^2 - 6ax + 9a^2 = 0$

Hence, option (a) is the correct answer.

Concept Check 2

If $x + y = k$ is a normal to parabola $y^2 = 12x$, then find k.

(a) 3 (b) 9 (c) -9 (d) -3

Summary Sheet

Key Takeaways

- A circle having any focal chord as the diameter touches the directrix.
- The foot of the perpendicular drawn from the focus upon any tangent lies on the tangent at the vertex. Hence, a circle described on any focal radii as the diameter touches the tangent at the vertex.
- The area of the triangle formed by three points on the parabola is twice the area of the triangle formed by the tangents at these points.
- The line perpendicular to the tangent at the point of contact is known as the normal to the parabola at that point.

Key Equations

Equations of the normals of standard parabolas at $P(x_1, y_1)$ (point form) are as follows:

Equations	Normals at (x_1, y_1)
$y^2 = 4ax$	$y - y_1 = \frac{-y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = \frac{-2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

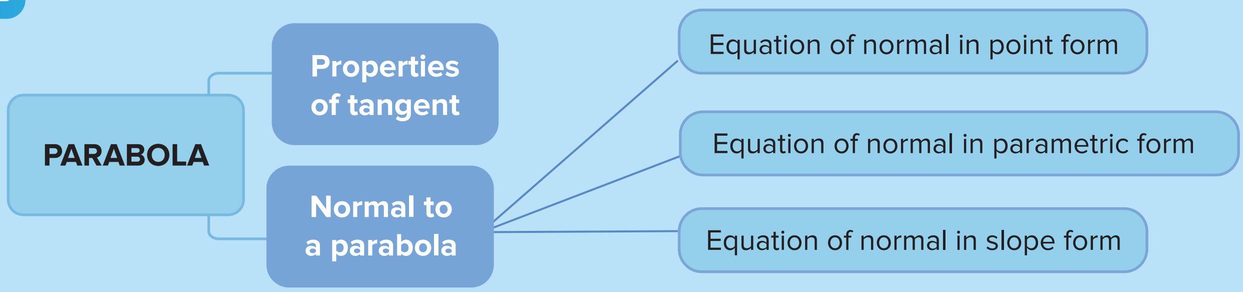
Equations of the normals of standard parabolas at t (parametric form) are as follows:

Equations of parabolas	Parametric Coordinates	Normals at t
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

Equations of the normals, points of contact, and conditions of normality in terms of slope m (slope form) are as follows:

Equations of parabolas	Points of contact	Equations of Normals
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$

Mind map



Self-Assessment

1. If from a point, the two tangents drawn to parabola $y^2 = 4ax$ are normals to the parabola $x^2 = 4by$, then which of the following is correct?

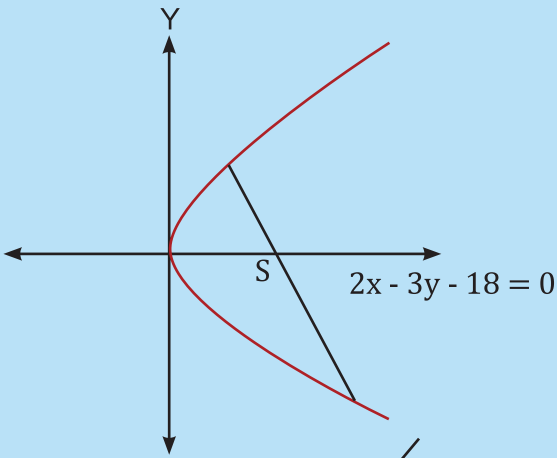
   (a) $a^2 > 8b^2$ (b) $b^2 > 8a^2$ (c) $a^2 < 8b^2$ (d) None of these

2. Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ may be equal to which of the following?

   (a) 3 (b) 5 (c) 2 (d) 4

Answers

Concept Check 1



Step 1 :

$(x-3)^2 + (y+4)^2 = \dfrac{(3x-4y-6)^2}{25}$ → Parabola

$(x-\alpha)^2 + (y-\beta)^2 = \dfrac{(ax+by+c)^2}{(\alpha^2+\beta^2)}$ → General equation

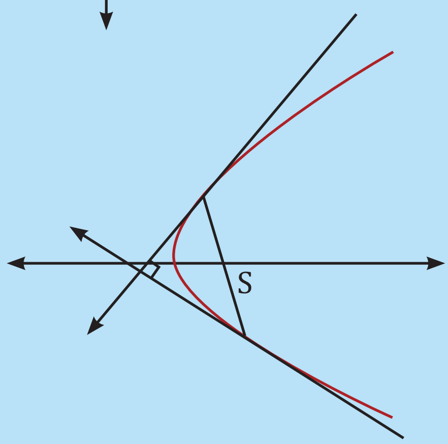
By comparing the given parabola equation with the general equation, we get the following:

Focus, $S \equiv (\alpha, \beta)$ is $(3, -4)$

Step 2 :

Focus, $S \equiv (3, -4)$ satisfies the equation, $2x - 3y - 18 = 0$.

⇒ The given chord is the focal chord.



The tangents drawn at the extremities of the focal chord are perpendicular and intersect on the directrix.

∴ Angle = 90°

Concept Check 2

METHOD 1

Step 1 :

Given,

Equation of the normal, $x + y = k$, and equation of parabola, $y^2 = 12x$

General equation of the normal to the parabola,

$y + tx = 2at + at^3 \implies y + tx = 6t + 3t^3 \;\; (a = 3)$

$$\Rightarrow x + y = k \text{ if } \frac{t}{1} = \frac{1}{1} = \frac{6t+3t^3}{k}$$

$$\therefore\; t = 1 \text{ and } 1 = \frac{6+3}{k} \Rightarrow k = 9$$

METHOD 2

Step 1 :

Equation of the parabola, $y^2 = 12x$

General equation of normal to the parabola,

$y = mx - 2am - am^3 \Rightarrow y = mx - 6m - 3m^3 \; (a = 3)$

$$\Rightarrow\; x + y = k \;\Rightarrow\; y = -x + k \text{ ; if } \frac{1}{1} = \frac{m}{-1} = \frac{-6m-3m^3}{k}$$

$$\therefore\; m = -1 \text{ and } 1 = \frac{6+3}{k}$$

$$\Rightarrow\; k = 9$$

Hence, option (b) is the correct answer.

Self Assessment 1

Step 1 :

The coordinates of any point on the parabola, $x^2 = 4by$, are $(2bt, bt^2)$.

For the parabola, $x^2 = 4by$

Equation of normal is the following: $x + ty = 2bt + bt^2$ .... (1)

Or, $y = \frac{-1}{t}x + 2b + bt^2$

Step 2 :

The normal, $y = \frac{-1}{t}x + 2b + bt^2 \ldots(1)$, will touch the parabola, $y^2 = 4ax$ if :

$$2b + bt^2 = \frac{a}{\frac{-1}{t}} \left(\because c = \frac{a}{m}\right)$$

$bt^2 + at + 2b = 0$

For distinct real roots, discriminant > 0.

$\Rightarrow a^2 - 8b^2 > 0 \ \Rightarrow a^2 > 8b^2$

Hence, option (a) is the correct answer.

Self Assessment 2

Step 1: Equations of the normals to $y^2 = 4ax$ and $x^2 = 4by$ are given by the following:

$$y = mx - 2am - am^3 \text{ and } y = mx + 2b + \frac{b}{m^2}$$

Step 2: For common normals, $-2am - am^3 = 2b + \dfrac{b}{m^2}$

$$am^5 + 2am^3 + 2bm^2 + b = 0$$

Therefore, a maximum of 5 normals are possible.

Hence, option (b) is the correct answer.

PARABOLA

PROPERTIES OF NORMAL AND CONORMAL POINTS

What you already know

- Tangents to parabola and properties of tangents
- Properties of focal chord
- Parametric coordinates of a parabola

What you will learn

- Properties of normal to parabola
- Co-normal points and their properties

Concept Check 1

Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersects the axis of parabola at A and B, respectively. If C is the centre of the circle through the points P, A, B, and $\angle CPB = \theta$, then find $\tan\theta$.

(a) 2 (b) $\frac{4}{3}$ (c) 3 (d) $\frac{1}{2}$

Find the minimum distance between the curves C_1: $y^2 = 4x$ and C_2: $x^2 + y^2 - 12x + 31 = 0$.

Solution

Step 1:

Given, curves C_1: $y^2 = 4x$ and $C_2 : x^2 + y^2 - 12x + 31 = 0$

As we have to find the minimum distance between the given curves,

We can see that parabola C_1 is similar to $y^2 = 4ax$

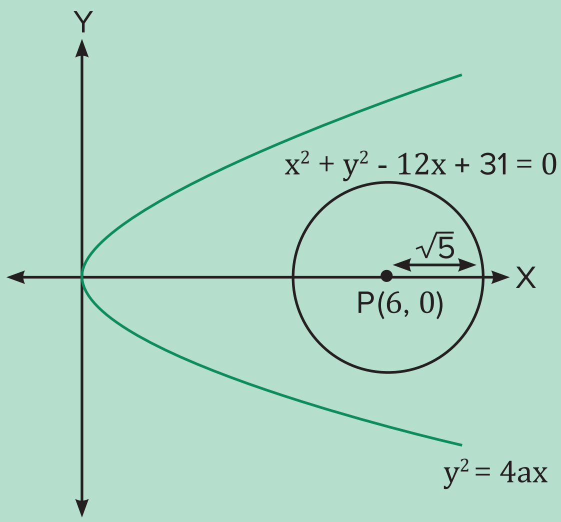
Therefore, a = 1

And in C_2, coefficient of xy = 0, and the coefficient of x^2 is same as the coefficient of y^2, so it will represent a circle.

Therefore, $x^2 + y^2 - 12x + 31 + 36 - 36 = 0$

Or $(x - 6)^2 + (y - 0)^2 = (\sqrt{5})^2$,

Is a circle having centre at (6, 0) and radius = $\sqrt{5}$



Step 2:

As we know that,

The shortest distance between the two curves occurs along the common normal.

N_1: Normal to $y^2 = 4x$ at A and the normal passes through $(6, 0)$

As we know, equation of normal to $y^2 = 4x$,

N_1: $y = -tx + 2t + t^3$ passes through $(6, 0)$.

$\Rightarrow 0 = -6t + 2t + t^3$

$\Rightarrow t^3 - 4t = 0 \Rightarrow t(t - 2)(t + 2) = 0$

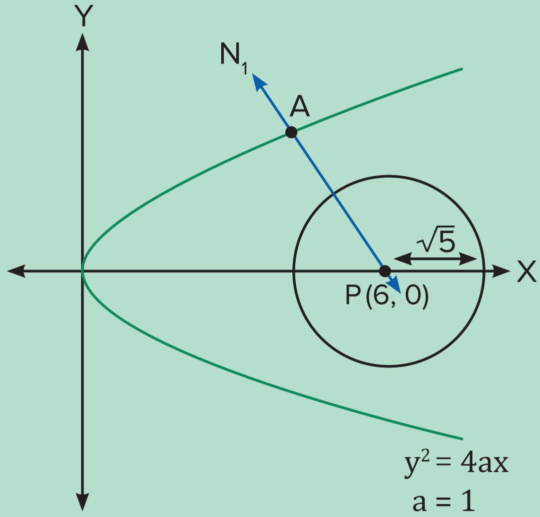
Or $t = 0, 2, -2$

So, the points can be:

A $(t^2, 2t) \equiv (4, 4)$ (when $t = 2$),

B$(t^2, 2t) \equiv (4, -4)$ (when $t = -2$), and

C$(t^2, 2t) \equiv (0, 0)$ (when $t = 0$)



Step 3:

Now, $d_1 = AP - \sqrt{5}$, $d_2 = BP - \sqrt{5}$, and $d_3 = CP - \sqrt{5}$

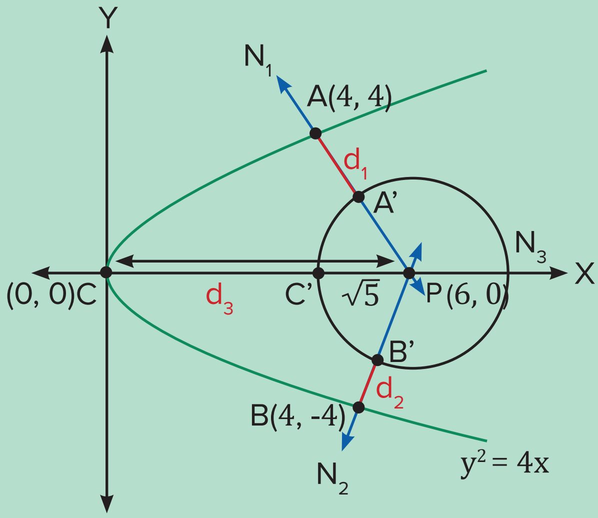
Or

$d_1 = \sqrt{(4 - 6)^2 + (4 - 0)^2} - \sqrt{5}$

$= \sqrt{20} - \sqrt{5} = d_2$ (since BP = AP)

And $d_3 = 6 - \sqrt{5}$

Hence, $\sqrt{20} - \sqrt{5}$ is the shortest distance.



Properties of Normal

1. The normal other than the axis of parabola never passes through the focus.

Proof

Let the normal at P$(am^2, -2am)$,

$y = mx - 2am - am^3$ pass through the focus $(a, 0)$.

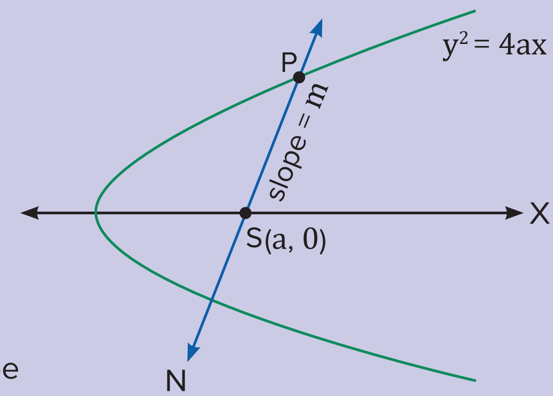
$\Rightarrow 0 = am - 2am - am^3$

$\Rightarrow am^3 + am = 0$

$\Rightarrow m(m^2 + 1) = 0$

Either $m = 0$ or $m^2 + 1 = 0$

As $m^2 + 1 > 0$, therefore, $m = 0$ (which represents the slope of the axis of parabola)



2. If the normal at $P(t_1)$ meets the parabola again at $Q(t_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

Proof

As we know that the relation between the slope of normal m and the parameter t is

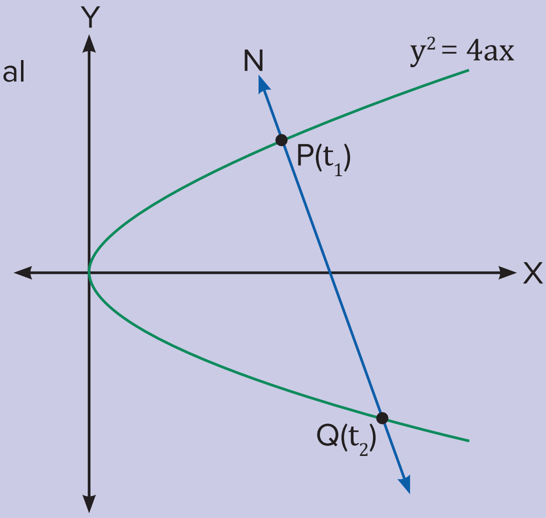
$t = -m$,

$\Rightarrow$ Slope of the normal, $N = m = -t_1$

$$m = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} \text{ or } m = \frac{2}{t_1 + t_2} = -t_1$$

Finally,

$$t_2 = -t_1 - \frac{2}{t_1}$$



3. The point of intersection of normals at $P(t_1)$ and $Q(t_2)$ is $[2a + a(t_1^2 + t_1t_2 + t_2^2), - at_1t_2(t_1 + t_2)]$.

Proof

Let us consider normal (N_1) at $P(t_1)$.

N_1: $y = -t_1x + 2at_1 + at_1^3$ -----(i)

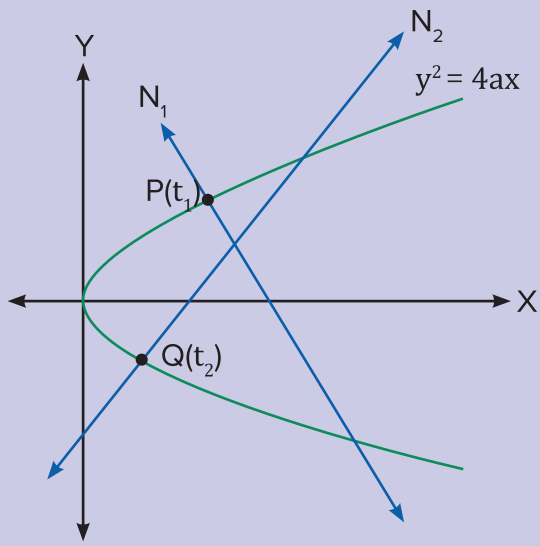
And normal (N_2) at $Q(t_2)$,

N_2: $y = -t_2x + 2at_2 + at_2^3$ -----(ii)

Now, solving equation (i) and (ii) , we get,

$x = 2a + a(t_1^2 + t_1t_2 + t_2^2)$ and

$y = -at_1t_2(t_1 + t_2)$



4. If the normals to the parabola $y^2 = 4ax$ at points $P(t_1)$ and $Q(t_2)$ intersect again on the parabola at the point $R(t_3)$, then $t_1t_2 = 2$ and $t_3 = -(t_1 + t_2)$

Proof

As we know by property 2, $t_3 = -t_1 - \frac{2}{t_1}$ and $t_3 = -t_2 - \frac{2}{t_2}$

Now, equating the two given expressions, we get,

$$\Rightarrow -t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$\Rightarrow t_1 \times t_2 = 2$

Now, as $R(t_3) \equiv (at_3^2, 2at_3)$,

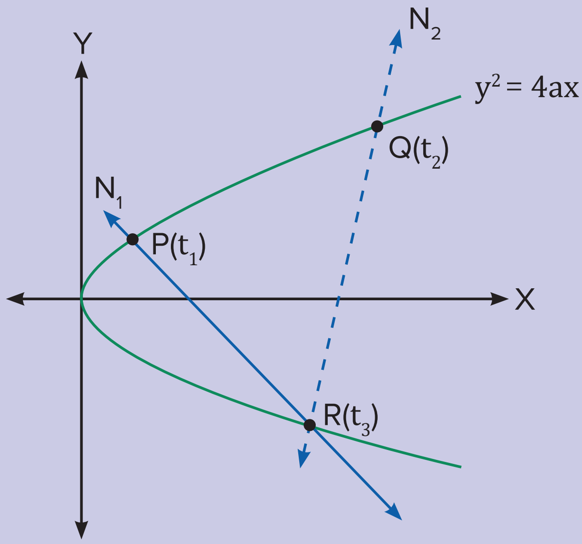
And by property 3, $R(t_3) = [2a + a(t_1^2 + t_1t_2 + t_2^2), - at_1t_2(t_1 + t_2)]$.

Therefore,

$2at_3 = -at_1t_2(t_1 + t_2)$, as $t_1 \times t_2 = 2$

$\Rightarrow 2t_3 = -2(t_2 + t_1)$

Or $t_1 + t_2 + t_3 = 0$



The normal to the parabola $y^2 = 8x$ at the point (2, 4) meets the parabola at which of the following points?

(a) (-18, -12) **(b) (-18, 12)** **(c) (18, 12)** **(d) (18, -12)**

Solution

Step 1:

Given, equation of parabola, $y^2 = 8x \Rightarrow a = 2$

And a point $P(at_1^2, 2at_1) \equiv (2, 4)$

Or $2 \times 2t_1 = 4 \Rightarrow t_1 = 1$

Using

If the normal at $P(t_1)$ meets the parabola again at $Q(t_2)$,

then $t_2 = -t_1 - \dfrac{2}{t_1}$

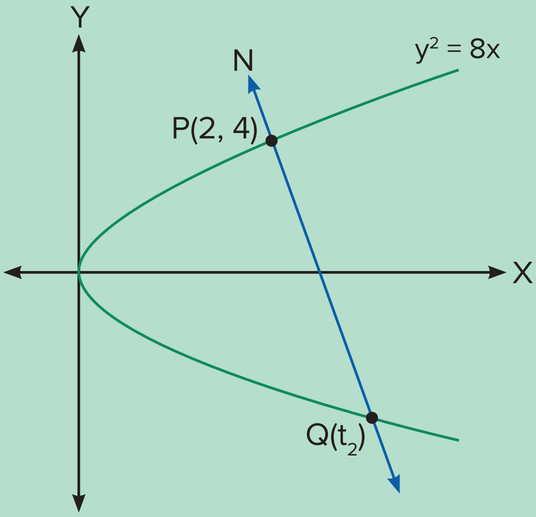
We get, $t_2 = -3$

Step 2:

Now, coordinates of point Q is $(at_2^2, 2at_2)$.

Or $(2 \times 9, -12) = (18, -12)$

Hence, option (d) is the correct answer.

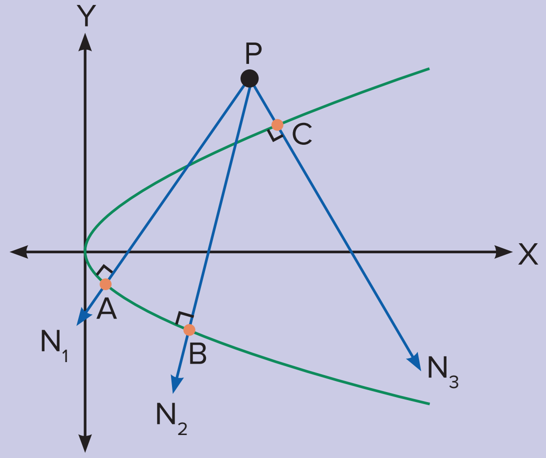


Concept Check 2

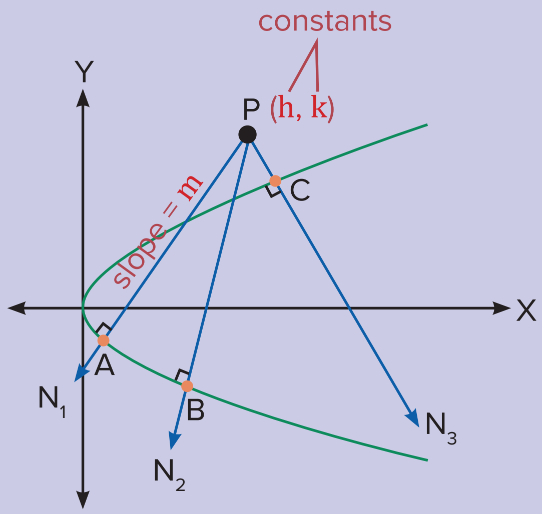
A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at B. If AB subtends a right angle at the vertex of the parabola, then find the slope of AB.

Co-normal Points

The points on the parabola through which the normals drawn are concurrent, are known as co-normal points.
Here, A, B, and C are conormal points.



Let us consider a general point P(h, k).
Now, as we know that the normal equation having slope m will be
$k = mh - 2am - am^3$
Or $am^3 + (2a - h)m + k = 0$
Finally, we have,
$am^3 + 0m^2 + (2a - h)m + k = 0$ is a cubic equation in variable m.
Now, there may be two different cases.



Case 1: All three real roots i.e. we can draw a maximum of three different normals.
Case 2: One real root and two imaginary roots i.e. we can draw only one normal.

Note

A maximum of three normals can be drawn from a point to parabola.
If m_1, m_2, m_3 are the slopes of three normals to the parabola drawn from the point P(h, k), then for cubic equation $am^3 + 0m^2 + (2a - h)m + k = 0$,

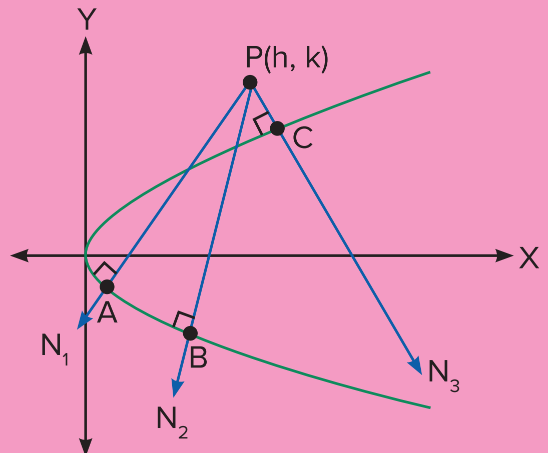
$$m_1 + m_2 + m_3 = 0$$
$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a}$$
$$m_1m_2m_3 = -\frac{k}{a}$$

Properties of Co-normal Points

1. The algebraic sum of ordinates of the feet of three normals (conormal points) drawn to a parabola from a given point is zero.

Proof

Let us consider parametric coordinates for
$A \equiv (at_1^2, 2at_1) = (am_1^2, -2am_1)$,
$B \equiv (at_2^2, 2at_2) = (am_2^2, -2am_2)$, and
$C \equiv (at_3^2, 2at_3) = (am_3^2, -2am_3)$
where m_1, m_2 and m_3 are the slopes of normals N_1, N_2, and N_3, respectively.
Then, as we see,
Sum of ordinates of A, B, and C
$= -2am_1 - 2am_2 - 2am_3$
$= -2a(m_1 + m_2 + m_3) = -2a \times 0 = 0$



Note

As we know that $m_i = -t_i$,
$(m_1 + m_2 + m_3) = -(t_1 + t_2 + t_3) = 0$

2. The centroid of the triangle formed by co-normal points as vertices lies on the axis of the parabola.

Proof

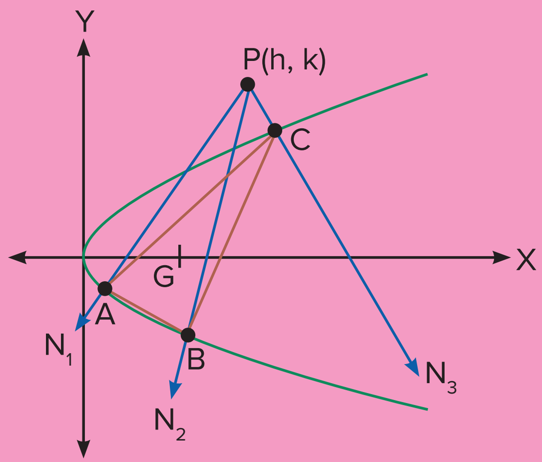
Let A, B, and C be co-normal points.
And
$A \equiv (at_1^2, 2at_1) = (am_1^2, -2am_1)$,
$B \equiv (at_2^2, 2at_2) = (am_2^2, -2am_2)$, and
$C \equiv (at_3^2, 2at_3) = (am_3^2, -2am_3)$

Here, $G \equiv$ Centroid of $\Delta ABC \equiv (x, y)$,

Where $y = \dfrac{2at_1 + 2at_2 + 2at_3}{3} = \dfrac{2a(t_1 + t_2 + t_3)}{3} = 0$

Therefore, G lies on the axis of parabola $y^2 = 4ax$.
Now, in terms of slope,

$y = \dfrac{-2a(m_1 + m_2 + m_3)}{3} = 0 \Rightarrow G_y = 0$



Summary Sheet

Key Takeaways

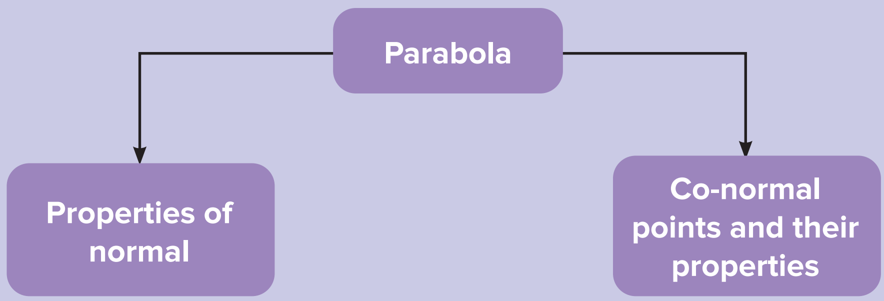
Properties of normal:

- The normal other than the axis of parabola never passes through the focus.
- If the normal at $P(t_1)$ meets the parabola again at $Q(t_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.
- The point of intersection of normals at $P(t_1)$ and $Q(t_2)$ is $[2a + a(t_1^2 + t_1t_2 + t_2^2), - at_1t_2(t_1 + t_2)]$.
- If the normals to the parabola $y^2 = 4ax$ at points $P(t_1)$ and $Q(t_2)$ intersect again on the parabola at the point $R(t_3)$, then $t_1t_2 = 2$ and $t_3 = -(t_1 + t_2)$.

Properties of conormal points:

- The algebraic sum of ordinates of the feet of three normals (conormal points) drawn to a parabola from a given point is zero.
- The centroid of the triangle formed by co-normal points as the vertices lies on the axis of the parabola.

Mind map



Self-Assessment

1. Find the equations of normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum.
2. Find the number of distinct normals that can be drawn from (-2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$

Answers

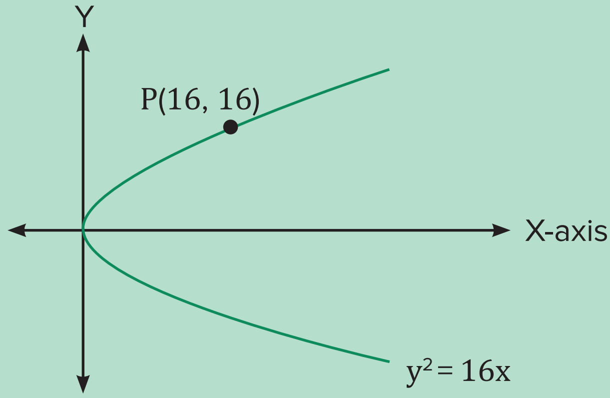
Concept Check 1

Step 1:

Given, parabola $y^2 = 16x \Rightarrow a = 4$

As point P(16, 16) satisfies the parabola, the point lies on the curve.

Now, let us consider that the tangent at point P is T and normal at point P is N.



Step 2:

Therefore, parametric equation will be

T : $ty = x + at^2$ as a = 4

$\Rightarrow$ T : $ty = x + 4t^2$

And N :$y = -tx + 2at + at^3$, as a = 4

$\Rightarrow$ N : $y = -tx + 8t + 4t^3$

Now, as P(16, 16) $\equiv$ P(at^2, 2at),

Put the value of a = 4

Or P(16, 16) $\equiv$ P($4t^2$, 8t)

$\Rightarrow$ 16 = 8t or t = 2

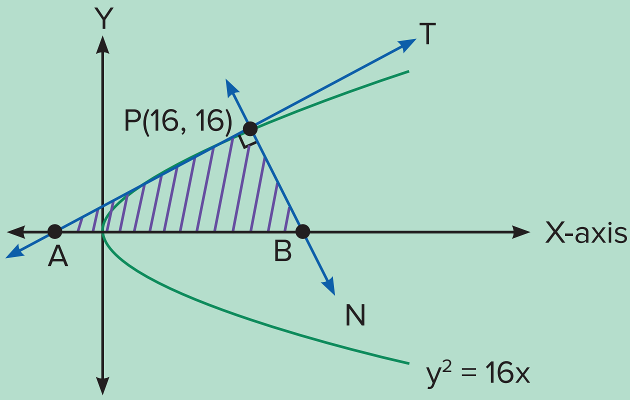
Therefore,

T : $2y = x + 16$

N : $y = -2x + 48$

Now, intersection point A (x_0, 0) and B (x_1, 0) are as follows:

Or A (-16, 0) and B (24, 0)



Step 3:

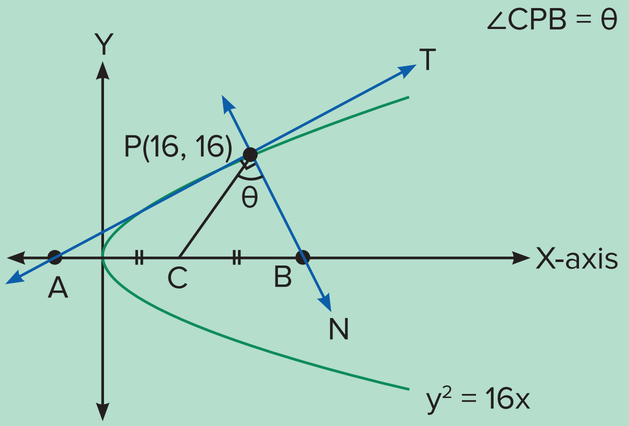
As C is the midpoint of A and B,

$\Rightarrow C\left(\frac{-16 + 24}{2}, 0\right) \equiv C\,(4, 0)$

Now, let us consider $m_{PC} = m_1$ and $m_{PB} = m_2$

So, $m_1 = \frac{16 - 0}{16 - 4} = \frac{4}{3}$ and $m_2 = \frac{16 - 0}{16 - 24} = -2$

As we know, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 \times m_2}\right|$



Or $\tan\theta = \left|\dfrac{\frac{4}{3}+2}{1-\frac{8}{3}}\right|$

Finally, $\tan\theta = 2$

Hence, option (a) is the correct answer.

Concept Check 2

Step 1:

Given, A is a point on the parabola $y^2 = 4ax$, and the normal at A cuts the parabola again at B. AB subtends a right angle at the vertex of the parabola.

Let us consider the slope of normal (N) = m

I.e., also equal to $-t_1$

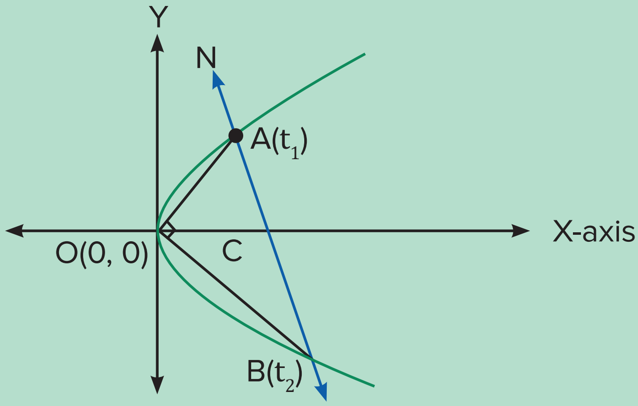
Coordinates of $A(t_1) \equiv (at_1^2, 2at_1)$ and

$B(t_2) \equiv (at_2^2, 2at_2)$

As given,

$m_{OA} \times m_{OB} = -1 \Rightarrow \left(\dfrac{2at_1}{at_1^2}\right)\left(\dfrac{2at_2}{at_2^2}\right) = -1$

Or $t_1t_2 = -4$ ----(i)



Step 2:

Now using,

If the normal at $P(t_1)$ meets the parabola again at $Q(t_2)$, then $t_2 = -t_1 - \dfrac{2}{t_1}$

We get, $t_2 = -t_1 - \dfrac{2}{t_1}$ ----(ii)

Here, solving equation (i) and (ii),

We have, $t_2 = \dfrac{-4}{t_1}$

Now, $\dfrac{-4}{t_1} = -t_1 - \dfrac{2}{t_1}$

Or $t_1 = \dfrac{2}{t_1} \Rightarrow t_1 = \pm\sqrt{2}$

And now, slope $= -t_1 = \mp\sqrt{2}$

Self-Assessment 1

Step 1:

Given, equation of parabola $y^2 = 4ax$ and we have to find the equations of normal at the extreme end of LR.

As we know that the extremities of LR for parabola $y^2 = 4ax$ are $(a, 2a)$ and $(a, -2a)$,

Now, differentiating $y^2 = 4ax$ with respect to x, we have $\dfrac{dy}{dx} = \dfrac{2a}{y}$

Now, slope of normal $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)}$

So, slope of normal $= \dfrac{-y}{2a}$

Step 2:

Hence, slope of normal at point P(a, 2a) = $-\dfrac{2a}{2a} = -1$

Slope of normal at point Q(a, -2a) = $-\dfrac{(-2a)}{2a} = 1$

Now, equations of normal

At P: $(y - 2a) = -1(x - a)$

Or $x + y - 3a = 0$

Now, at Q: $(y + 2a) = 1(x - a)$

Or $x - y - 3a = 0$

Therefore, equations of normals at P and Q are $x + y - 3a = 0$ and $x - y - 3a = 0$

Self-Assessment 2

Step 1:

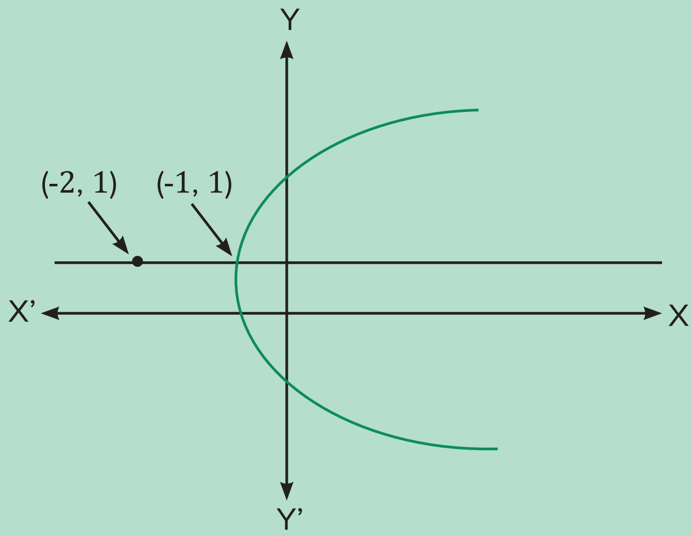
Given, equation of parabola

$y^2 - 4x - 2y - 3 = 0$

$\Rightarrow y^2 - 2y + (-1)^2 - (-1)^2 = 4x + 3$

$\Rightarrow (y - 1)^2 = 4(x + 1)$

So, the axis of parabola is $y - 1 = 0$



Step 2:

Here we can see that point (-2, 1) lies on the axis, and it is exterior to the parabola because $(1)^2 - 4(-2) - 2(1) - 3 = 4 > 0$

Hence, only one normal is possible.

PARABOLA

MORE ON PROPERTIES OF CO-NORMAL POINTS AND CHORD OF CONTACT

NOTES

What you already know

- Parabola and its equation
- Some standard parabolas
- Parametric equation of a parabola
- Focal chord and its properties
- Tangents and normals to the parabola
- Properties of normal and co-normal points

What you will learn

- Properties of co-normal points
- Chord of contact
- Chord bisected at a point

Solve

If two of the three feet of normals drawn from a point to the parabola $y^2 = 4x$ be $(1, 2)$ and $(1, -2)$, then find the third foot.

Solution

Method 1

Step 1

Equation of parabola, $y^2 = 4x$

By expressing points A, B, and C in parametric coordinates, we get the following:

$A \equiv (at_1^2, 2at_1)$

$B \equiv (at_2^2, 2at_2)$

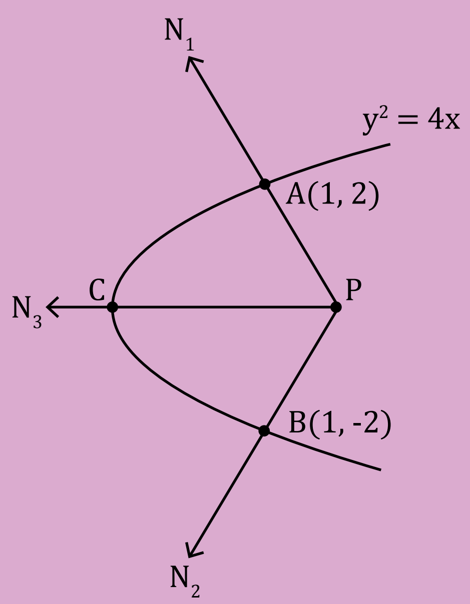
$C \equiv (at_3^2, 2at_3)$

Here, $a = 1$

$A \equiv (t_1^2, 2t_1)$

$B \equiv (t_2^2, 2t_2)$

$C \equiv (t_3^2, 2t_3)$



Step 2

By comparing with the coordinates of A and B, we get the following:

$t_1 = 1$ and $t_2 = -1$

The sum of the parameters of the three co-normal points is 0.

$t_1 + t_2 + t_3 = 0$

$\Rightarrow 1 + (-1) + t_3 = 0$

$\Rightarrow t_3 = 0$

$\Rightarrow C(0, 0)$

Method 2

Let the coordinates of C be (x_1, y_1).

We know that the sum of the y-coordinates of co-normal points is 0.

$2 + (-2) + y_1 = 0$

$\Rightarrow y_1 = 0$

Since, C lies on the parabola, $y^2 = 4x$, it also satisfies its equation as follows:

$y_1^2 = 4x_1$

$\Rightarrow 0 = 4x_1$

$\Rightarrow x_1 = 0$

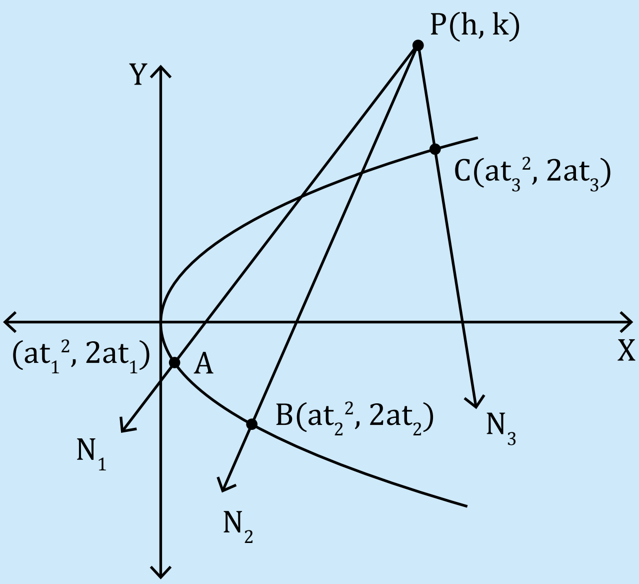
So, the coordinates of C are $(0, 0)$.

Properties of Co-normal Points

1. The algebraic sum of the ordinates of the feet of 3 normals (co-normal points) drawn to a parabola from a given point is 0.

2. The centroid of the Δ formed by the co-normal points lies on the axis of the parabola.

3. If three distinct normals are drawn to any parabola $y^2 = 4ax$ from a given point (h, k), then $h > 2a$.



Proof

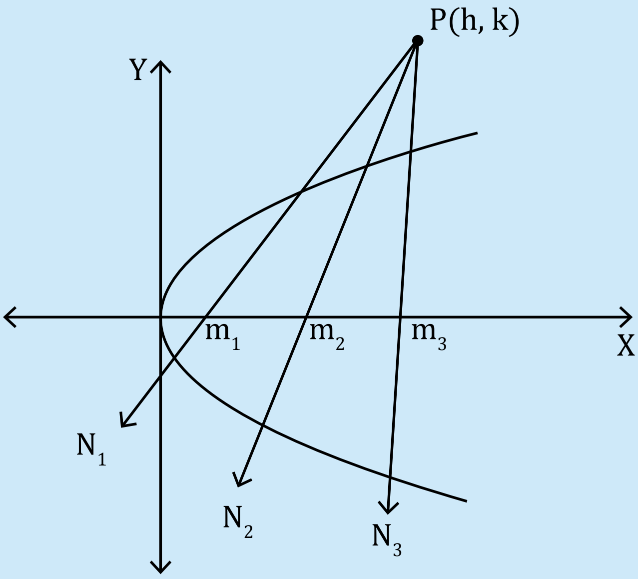
Step 1

Let m_1, m_2, and m_3 be the slopes of N_1, N_2, and N_3, respectively.

$m_1^2 + m_2^2 + m_3^2 \geq 0$ (Sum of 3 non-negative quantities is non-negative)

Case I: $m_1^2 + m_2^2 + m_3^2 = 0$

Case II: $m_1^2 + m_2^2 + m_3^2 > 0$



Step 2

Case I: $m_1^2 + m_2^2 + m_3^2 = 0$

$\Rightarrow m_1^2 + m_2^2 + m_3^2 = 0$

$\Rightarrow m_1 = m_2 = m_3 = 0$

Since, 3 distinct horizontal normals cannot be drawn from a point P, we can ignore this case.

Step 3

Case II: $m_1^2 + m_2^2 + m_3^2 > 0$

$\Rightarrow (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_1m_3) > 0$

We know that the sum of slopes of 3 normals drawn from a point $P(h, k)$ to the parabola is zero.

$m_1 + m_2 + m_3 = 0$

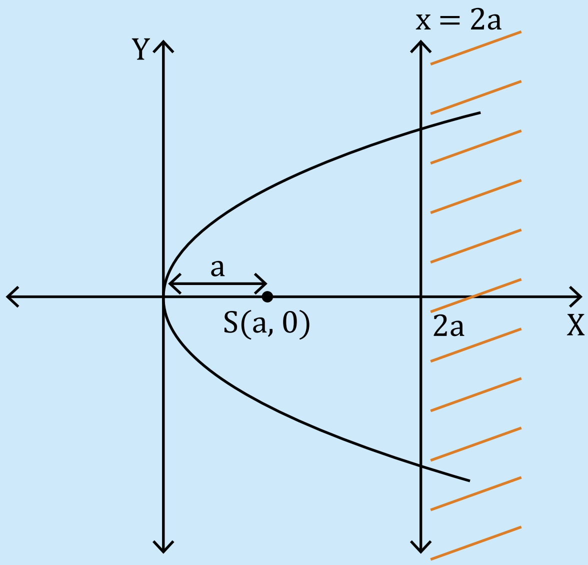
Also, $m_1m_2 + m_2m_3 + m_1m_3 = \frac{2a - h}{a}$

$m_1m_2m_3 = -\frac{k}{a}$

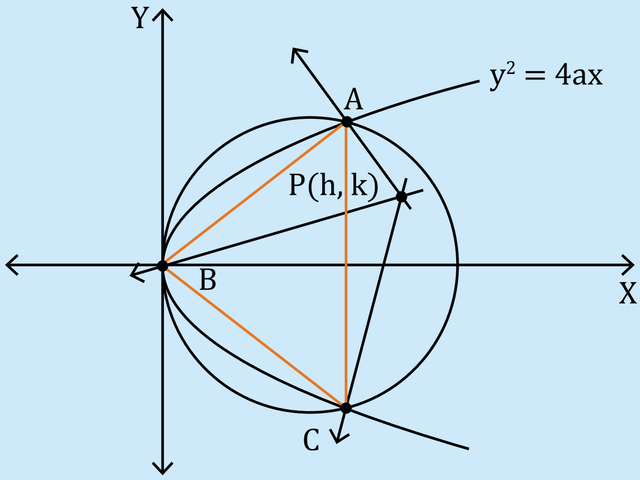
$0^2 - 2\left(\frac{2a - h}{a}\right) > 0$

$\frac{h - 2a}{a} > 0$

Since $a > 0$, the inequality becomes $h > 2a$.



4. A circle circumscribing the triangle formed by three co-normal points passes through the vertex of the parabola and its equation is as follows: $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$.



Chord of Contact

It is a chord joining two points of contact of a pair of tangents drawn from an external point.

Given a parabola $y^2 = 4ax$, the equation of the chord of contact is given by $T = 0$
Equation of parabola, $y^2 - 4ax = 0$
So, $S : y^2 - 4ax$
Using the following transformation formulae:

$x^2 \rightarrow xx_1$

$y^2 \rightarrow yy_1$

$x \rightarrow \frac{x + x_1}{2}$

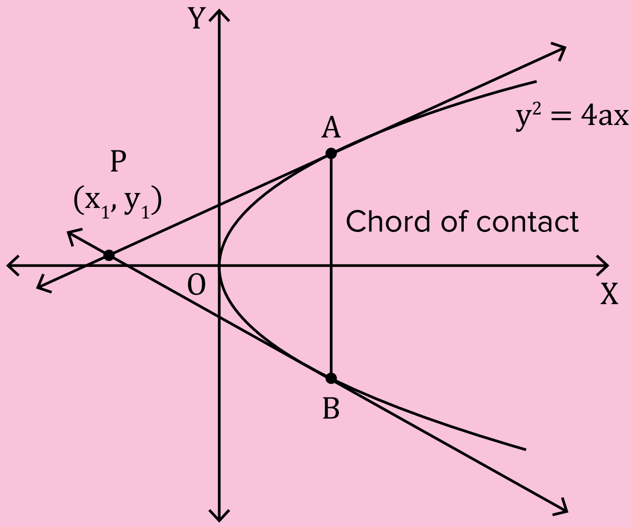
$y \rightarrow \frac{y + y_1}{2}$

$xy \rightarrow \frac{xy_1 + yx_1}{2}$

T becomes $yy_1 - 4a\left(\frac{x + x_1}{2}\right)$

$T = 0$ becomes $yy_1 - 4a\left(\frac{x + x_1}{2}\right) = 0$

The equation of chord of contact is, $yy_1 - 2a(x + x_1) = 0$



Note

1. If $P(x_1, y_1)$ is an external point to the parabola, then $T = 0$ represents the equation of chord of contact with respect to P.
2. If $P(x_1, y_1)$ lies on the parabola, then $T = 0$ represents the equation of tangent through P (point of contact).

Solve

Tangents are drawn to the parabola $y^2 = 16x$ at points where the line $2x + 3y - 1 = 0$ meets the parabola. Find the point of intersection of these tangents.

Solution

Step 1

Let A, B be the points of intersection of the parabola $y^2 = 16x$ and the line $l: 2x + 3y - 1 = 0$

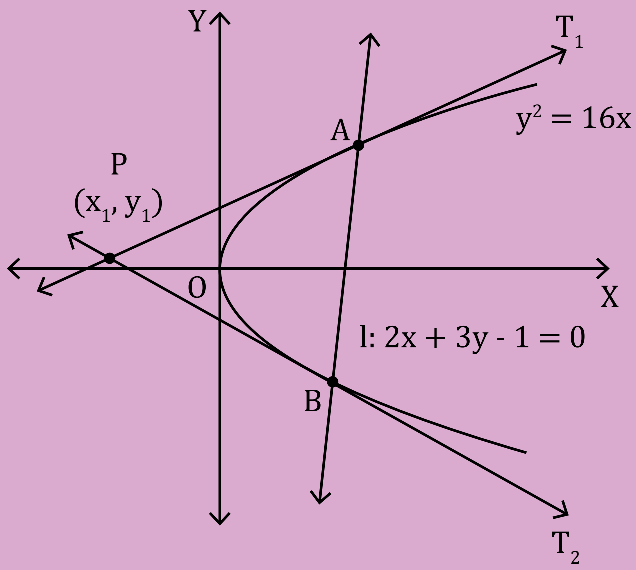
Let T_1: Tangent at A

T_2: Tangent at B

$P \equiv (x_1, y_1)$: Point of intersection of tangents T_1 and T_2.

$S: y^2 - 16x$

$\Rightarrow T = yy_1 - 16\left(\frac{x + x_1}{2}\right)$



Step 2

So, the equation of AB is $T = 0$

$y_1y - 8(x + x_1) = 0$

$8x - y_1y + 8x_1 = 0$ ...(i)

However, the equation of this line is already given as, $2x + 3y - 1 = 0$ ...(ii)

By dividing equation (i) by 4, we get the following:

$2x - \frac{y_1y}{4} + 2x_1 = 0$ ...(iii)

By comparing equations (ii) and (iii), we get the following: $y_1 = -12$ and $x_1 = \frac{-1}{2}$

Hence, the coordinates of P are $(-\frac{1}{2}, -12)$.

Chord Bisected at a Point

The unique chord having $P(x_1, y_1)$ as its midpoint is given by the equation, $T = S_1$.

$S \equiv y^2 - 4ax$

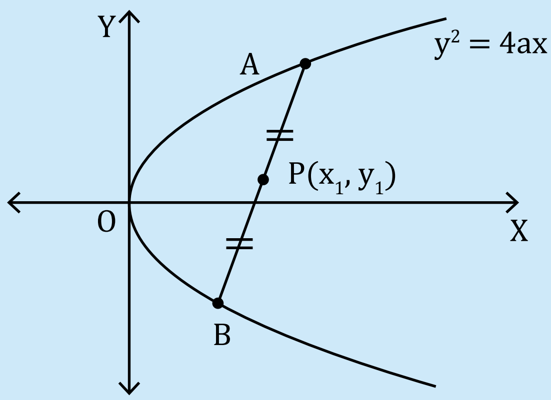
$T = yy_1 - 4a\left(\frac{x + x_1}{2}\right)$

$T = yy_1 - 2a(x + x_1)$

$S_1 = S_{P(x_1, y_1)} = y_1^{\,2} - 4ax_1$

So, $T = S_1$ becomes

$yy_1 - 2a(x + x_1) = y_1^{\,2} - 4ax_1$



Concept Check 1

If a chord, which is not a tangent of the parabola $y^2 = 16x$ has the equation $2x + y = p$ and midpoint (h, k), then which of the following is/are possible values of p, h, and k?

(a) $p = -2, h = 2, k = -4$

(b) $p = -1, h = 1, k = -3$

(c) $p = 2, h = 3, k = -4$

(d) $p = 5, h = 4, k = -3$

Note

If T_1 and T_2 are pair of tangents drawn from an external point $P(x_1, y_1)$, then combined equation of T_1 and T_2 is as follows:

$SS_1 = T^2$

Given, parabola $y^2 = 4ax$

$S = y^2 - 4ax$

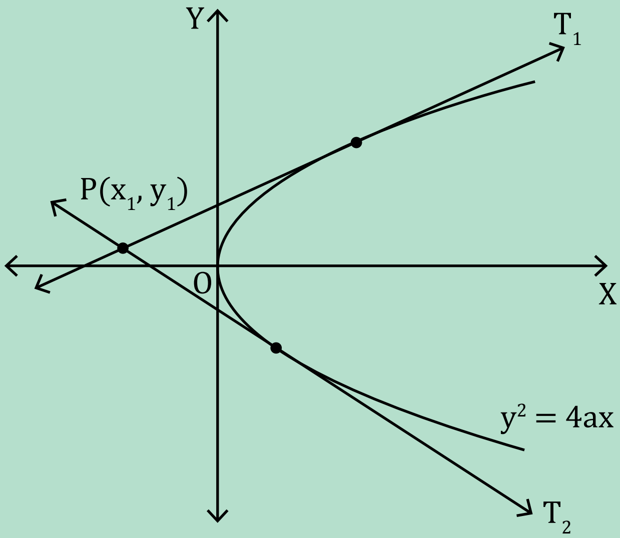
$T = yy_1 - 4a\left(\frac{x + x_1}{2}\right)$

$\Rightarrow T = yy_1 - 2a(x + x_1)$

$S_1 = y_1^2 - 4ax_1$

$SS_1 = T^2$

$\Rightarrow (y^2 - 4ax)(y_1^2 - 4ax_1) = (yy_1 - 2a(x + x_1))^2$



Solve

What is the angle between the tangents drawn from the point $(1, 4)$ to the parabola $y^2 = 4x$?

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

Solution

Step 1

Equation of parabola, $y^2 - 4x = 0$

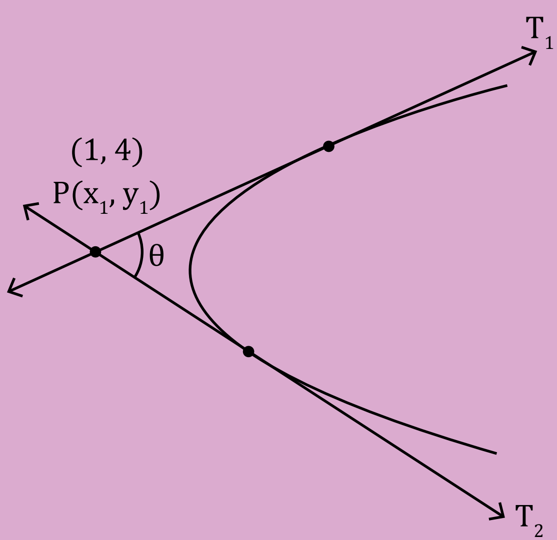
$S = y^2 - 4x$

$S_1 = S_{(1, 4)} = (4)^2 - 4(1) = 12$

The combined equation of pair of intersecting straight lines is given as follows:

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$



Step 2

The combined equation of the pair of tangents drawn from $(1, 4)$ is as follows: $SS_1 = T^2$

$T = yy_1 - 2(x + x_1) = 4y - 2(x + 1)$

$\Rightarrow T = 4y - 2x - 2$

$(y^2 - 4x)12 = (4y - 2x - 2)^2$

$\Rightarrow 12(y^2 - 4x) = 4(2y - x - 1)^2$

$\Rightarrow 3y^2 - 12x = 4y^2 + x^2 + 1 - 4xy - 4y + 2x$

$\Rightarrow x^2 + y^2 - 4xy + 14x - 4y + 1 = 0$

By comparing with the equation of the pair of straight lines, we get the following:

$a = b = 1, h = -2$

$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \left|\frac{2\sqrt{4 - 1}}{2}\right| = \sqrt{3}$

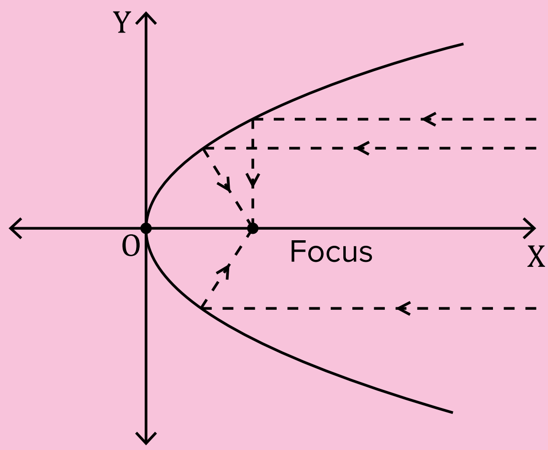
$\Rightarrow \theta = \frac{\pi}{3}$

So, option (b) is the correct answer.

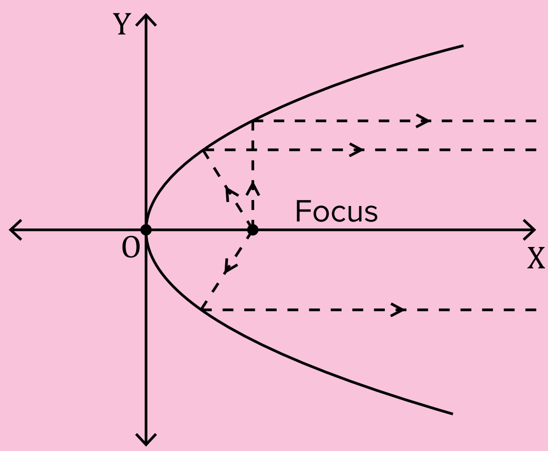
Reflection Property of Parabola

1. Any ray parallel to the axis of symmetry of the parabola will bounce off the parabola and pass through the focus.

   This principle is used extensively in dish antennas. The surface of the antenna is parabolic, so when the electromagnetic waves fall on it, they all converge at a singular point, which is the focus of the parabola. This focus is where the receiver is placed. All the rays collectively give a strong signal.



2. Conversely, any ray (light ray) emanating from the focus will reflect off the parabola in a straight line parallel to the axis.



3. The angle of incidence is equal to the angle of reflection.

Proof

Step 1

The equation of tangent at point $P(at^2, 2at)$ of the parabola $y^2 = 4ax$ is given by

$yt = x + at^2$

The tangent at point P intersects the x-axis at point T. Let the coordinates of T be $(x_0, 0)$.

Since T lies on the tangent, we get the following:

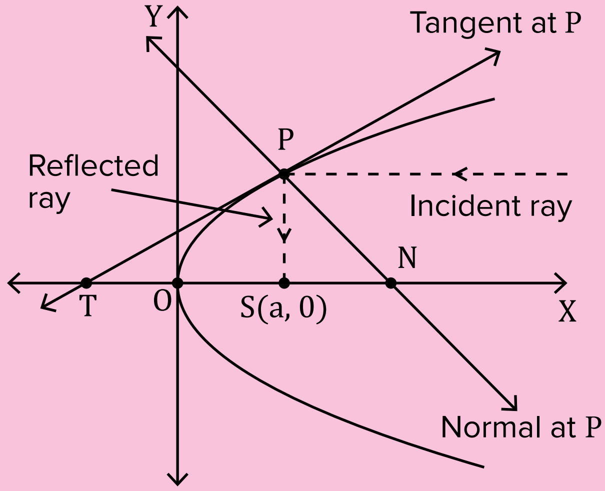
$0 = x_0 + at^2$

$\Rightarrow x_0 = -at^2$

$T \equiv (-at^2, 0)$

$S \equiv (a, 0)$

So, $|ST| = a + at^2$



Step 2

Also, the equation of normal at P(t) is given by,

$y = -tx + 2at + at^3$

The normal at P intersects the x-axis at point N. Let the coordinates of N be $(x_1, 0)$.

$0 = -tx_1 + 2at + at^3$

$\Rightarrow tx_1 = 2at + at^3$

$\Rightarrow x_1 = 2a + at^2$

$N \equiv (2a + at^2, 0)$

$S \equiv (a, 0)$

$|SN| = a + at^2$

Step 3

$P \equiv (at^2, 2at)$

$S \equiv (a, 0)$

$|PS|$= Focal distance of $P(at^2, 2at)$

$|PS| = a + at^2$

So, $|ST| = |SN| = |PS| = a + at^2$

Step 4

With respect to the tangent line

The angle of incidence i and the angle of reflection r with respect to the tangent line are shown in the figure.

In ΔPST, $|PS| = |ST|$, so it is an isosceles triangle.

$\Rightarrow \angle PTS = r$ ...(i)

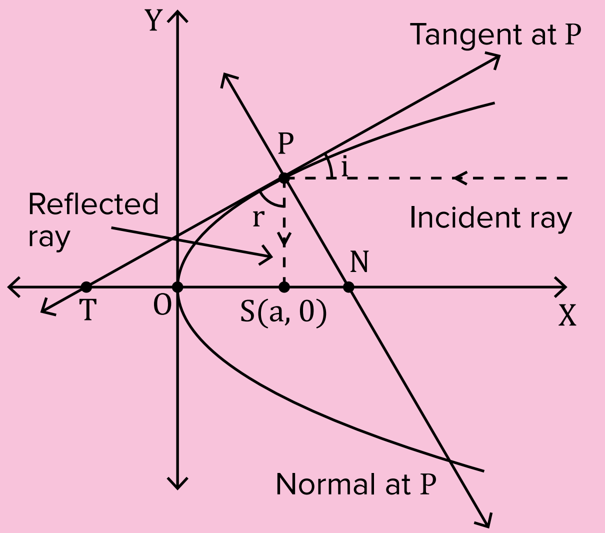
However, the incident ray is parallel to the x-axis and tangent acts as the transversal. The pair of the corresponding angles will be equal.

$\Rightarrow \angle PTS = i$ ...(ii)

By using (i) and (ii), we get the following:

$i = r$

I.e., Angle of incidence = Angle of reflection



Step 5

With respect to the normal line

The angle of incidence i and the angle of reflection r with respect to the normal line are shown in the figure.

In $\triangle PSN$, $|PS| = |SN|$, so it is an isosceles triangle.

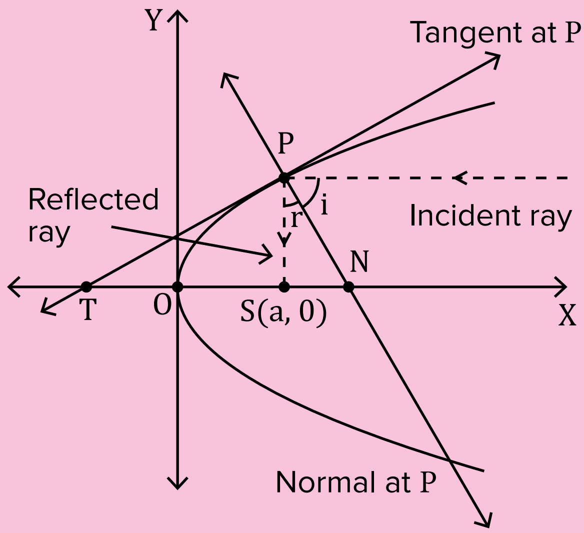
$\angle SNP = r$ ...(i)

The incident ray is parallel to the x-axis and the normal acts as the transversal. The pair of the alternate interior angles will be equal.

$\Rightarrow \angle SNP = i$ ...(ii)

By using (i) and (ii), we get the following: $i = r$

I.e., Angle of incidence = Angle of reflection



Solve

A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $(y - 4)^2 = 8(x + 1)$. After reflection, the ray passes through the point (α, β), find the value of $\alpha + \beta + 10$.

Solution

Upon reflection, all the rays of light parallel to the axis of parabola passes through the focus of the parabola.

The equation of the given parabola is $(y - 4)^2 = 8(x + 1)$

Taking $y' = y - 4$, $x' = x + 1$, we get the following: $(y')^2 = 8x'$

The focus of this parabola is $(x', y') = (2, 0)$

I.e., $(x + 1, y - 4) = (2, 0)$

So, the focus of original parabola is $(x, y) = (1, 4)$

So, $\alpha = 1$ and $\beta = 4$

$\Rightarrow \alpha + \beta + 10 = 15$

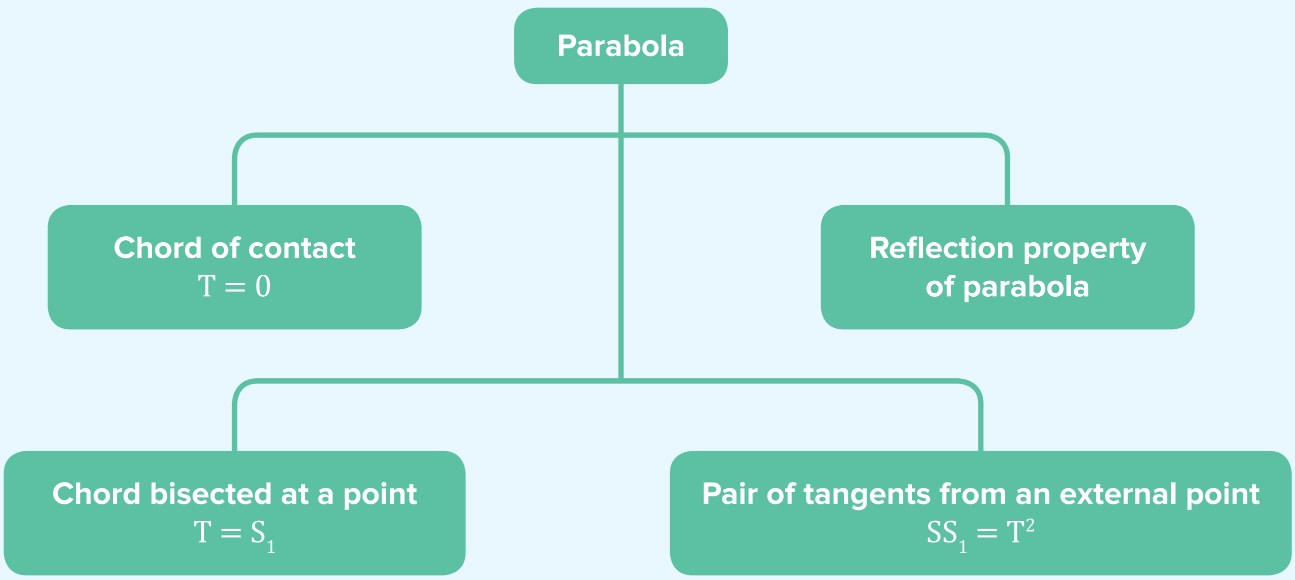
Summary Sheet

Key Takeaways

- The algebraic sum of the ordinates of the feet of 3 normals (co-normal points) drawn to a parabola from a given point is 0.
- The centroid of the $\triangle$ formed by the co-normal points lies on the axis of the parabola.
- If three distinct normals are drawn to any parabola $y^2 = 4ax$ from a given point (h, k), then $h > 2a$.

- Given a parabola $y^2 = 4ax$, the equation of the chord of contact is given by $T = 0$.
- Given a parabola $y^2 = 4ax$, the unique chord having $P(x_1, y_1)$ as its midpoint is given by the equation $T = S_1$.
- If T_1 and T_2 are the pair of tangents drawn from an external point $P(x_1, y_1)$, then the combined equation of T_1 and T_2 is, $SS_1 = T^2$.
- Any ray parallel to the axis of symmetry of the parabola will bounce off the parabola and pass through the focus.
- The angle of incidence is equal to the angle of reflection.

Mind Map



Self-Assessment

1. Find the locus of midpoint of chords of the parabola $y^2 = 4ax$ that passes through the point $(3a, a)$.
2. If the chord of contact of tangents from a point P to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$, then find the locus of P.

Answers

Concept Check 1

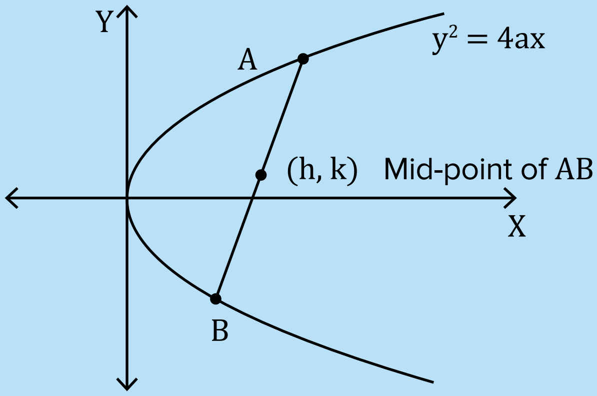
Step 1

Given, equation of parabola, $y^2 = 16x$

So, $S = y^2 - 16x$

Equation of AB, $2x + y = p$ ...(i)

Midpoint $P(h, k)$



Step 2

Equation of AB, i.e., the chord bisected at point P is given by $T = S_1$

$$yy_1 - 16\left(\frac{x + x_1}{2}\right) = y_1^2 - 16x_1$$

$\Rightarrow ky - 8(x + h) = k^2 - 16h$

$\Rightarrow ky - 8x + 8h - k^2 = 0$ ...(ii)

Step 3

$2x + y - p = 0$ ...(i)

$ky - 8x + 8h - k^2 = 0$

$\Rightarrow y - \left(\frac{8}{k}\right)x + \frac{8h - k^2}{k} = 0$ ...(ii)

By comparing (i) and (ii), we get the following:

$k = -4$ and $\frac{8h - k^2}{k} = -p$

$\Rightarrow \frac{8h - (-4)^2}{-4} = -p$

$\Rightarrow 8h - 16 = 4p$

$\Rightarrow 2h - 4 = p$

Only option (c) satisfies both the conditions, I.e., $k = -4$ and $2h - 4 = p$.

Hence, option (c) is the correct answer.

Self-Assessment 1

Step 1

The equation of chord with midpoint $P(h, k)$ is given by $T = S_1$

For $y^2 = 4ax$, S becomes $y^2 - 4ax$

$$yk - 4a\left(\frac{x + h}{2}\right) = k^2 - 4ah$$

$\Rightarrow yk - 2a(x + h) = k^2 - 4ah$

Step 2

Given, $(3a, a)$ satisfies this equation, we get the following:

$\Rightarrow ak - 2a(3a + h) = k^2 - 4ah$

$\Rightarrow ak + 2ah - 6a^2 - k^2 = 0$

Locus of P is given as follows:

$ay + 2ax - y^2 - 6a^2 = 0$

Self-Assessment 2

Step 1

Chord of contact of parabola $y^2 = 4ax$ with respect to $P(x_1, y_1)$ is as follows:

$yy_1 - 2a(x + x_1) = 0$

Given, this line touches $x^2 = 4by$

So, let us solve these equations simultaneously,

$$x^2 = 4b\left(\frac{2a(x + x_1)}{y_1}\right)$$

Step 2

$y_1x^2 - 8ab(x + x_1) = 0$

Since, it is given that these two curves touch each other, it means that they have just one common point. This means that the quadratic in x has one root or discriminant $= 0$.

$(-8ab)^2 - 4y_1(-8abx_1) = 0$

$\Rightarrow 8ab(8ab + 4x_1y_1) = 0$

$\Rightarrow ab = 0$ or $2ab = -x_1y_1$

Locus of P is as follows: $xy = -2ab$