UNIT - 16 : FUNCTIONS [JEE – MAIN CRASH COURSE]

Definition

Function can be easily defined with the help of the concept of mapping. Let A and B be any two non-empty sets. "A function from A and B is a rule or correspondence that assigns to each element of set A, one and only one element of set B." Let the correspondence be "f" then mathematically we write $f: A \rightarrow B$ where $y = f(x)$, $x \in A$ and $y \in B$. We say that "y" is the image of "x" under 'f' (or x is the pre-image of y).

- A mapping $f: A \rightarrow B$ is said to be a function if each element in the set A has an image in set B. It is possible that a few elements in the set B are present which are not the images of any element in set A.
- Every element in set A should have one and only one image. That means it is impossible to have more than one image for a specific element in set A. Functions cannot be multi-valued. (A mapping that is multi-valued is called a relation from A and B.)

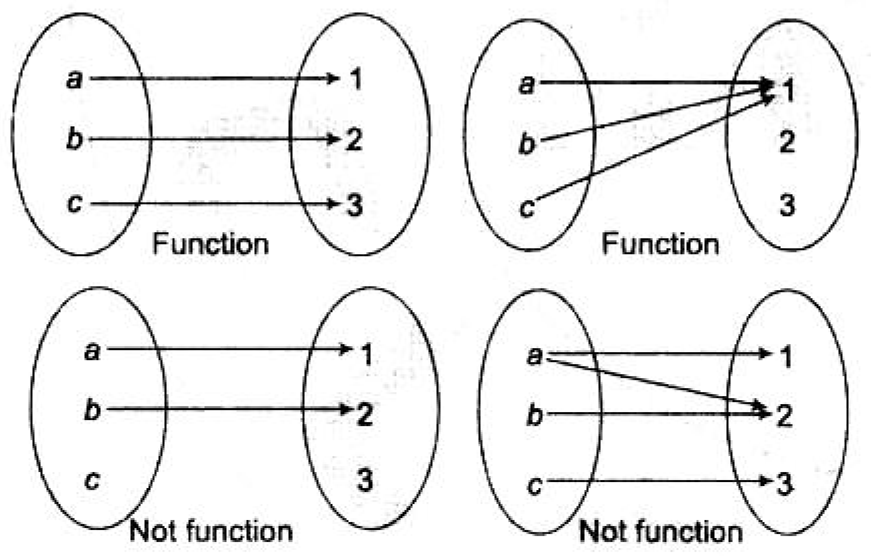


Fig. 1

Let us consider some other examples to make the above-mentioned concepts clear.

1. Let $f: R^+ \rightarrow R$ where $y^2 = x$. This cannot be considered a function as each $x \in R^+$ would have two images namely $\pm\sqrt{x}$. Thus it would be a relation.
2. Let $f: [-2, 2] \rightarrow R$, where $x^2 + y^2 = 4$. Here $y = \pm\sqrt{4 - x^2}$, that means for every $x \in [-2, 2]$ we would have two values of y (except when $x = \pm 2$). Hence it does not represent a function.
3. Let $f: R \rightarrow R$ where $y = x^3$. Here for each $x \in R$ we would have a unique value of y in the set R (as cube of any two distinct real numbers are distinct). Hence it would represent a function.

Domain, Co-Domain, and Range

Let $f: X \rightarrow Y$ be a function. In general sets X and Y could be any arbitrary non-empty sets. But at this level we would confine ourselves only to real-valued functions. That means it would be invariably assumed that X and Y are the subsets of real numbers.

Set "X" is called domain of the function "f".

Set "Y" is called co-domain of the function "f".

Set of images of different elements of set X is called the range of the function "f". It is obvious that range could be a subset of co-domain as we may have few elements in co-domain which are not the images of any element of the set X (of course these elements of co-domain will not be included in the range). Range is also called domain of variation. Domain of function "f" is normally represented as Domain (f). Range is represented as Range (f). Note that sometimes domain of the function is not explicitly defined. In these cases domain would mean the set of values of "x" for which $f(x)$ assumes real values, e.g., if $y = f(x)$ then Domain (f) = $\{x: f(x)$ is a real number$\}$.

Rules for finding the domain of a function

1. Domain $(f(x) + g(x))$ = Domain $f(x) \cap$ Domain $g(x)$.
2. Domain $(f(x) \cdot g(x))$ = Domain $f(x) \cap$ Domain $g(x)$.
3. Domain $\left(\frac{f(x)}{g(x)}\right)$ = Domain $f(x) \cap$ Domain $g(x) \cap$ $\{x : g(x) \neq 0\}$.
4. Domain $\sqrt{f(x)}$ = Domain $f(x) \cap \{x : f(x) \geq 0\}$.
5. Domain (fog) = Domain $(g(x)$, where fog is defined by $fog(x) = f\{g(x)\}$.

Trigonometric functions

Function	Domain	Range
$f(x) = \sin x$	R	$[-1, 1]$
$f(x) = \cos x$	R	$[-1, 1]$
$f(x) = \tan x$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$	R
$f(x) = \cot x$	$R - \{n\pi, n \in Z\}$	R
$f(x) = \sec x$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$	$(-\infty, -1] \cup [1, \infty)$
$f(x) = \text{cosec } x$	$R - \{n\pi, n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$

Important result

$$f(x) = a \cos x + b \sin x$$

$$= \sqrt{a^2 + b^2} \sin(x + \tan^{-1} \frac{a}{b})$$

$$= \sqrt{a^2 + b^2} \cos(x - \tan^{-1} \frac{b}{a})$$

Range of $f(x) = a \cos x + b \sin x$ is

$$[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}]$$

Inverse trigonometric functions

Function	Domain	Range
$f(x) = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$f(x) = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$f(x) = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$f(x) = \cot^{-1} x$	R	$(0, \pi)$
$f(x) = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$
$f(x) = \text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$

Modulus Function

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} = \sqrt{x^2} \qquad y = |x - a| = \begin{cases} x - a, & x \geq a \\ a - x, & x < a \end{cases}$$

$= \max\{x, -x\}$

Domain: R; Range: $[0, \infty)$

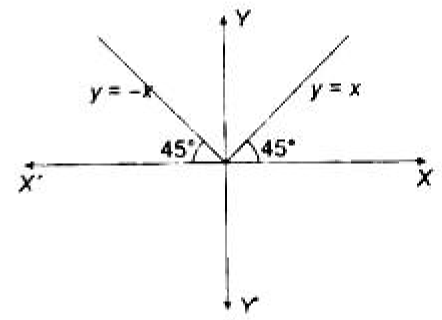


Fig. 2

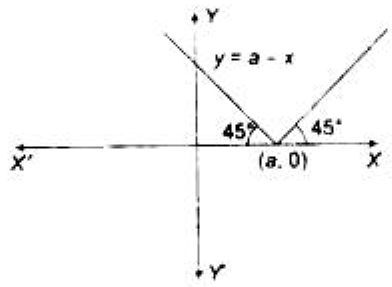


Fig. 3

Properties of modulus function

1. $|x| = a \Rightarrow$ Points on the real number line whose distance from origin is a
   $\Rightarrow \quad x = \pm a$
2. $|x| \leq a \Rightarrow x^2 \leq a^2 \Rightarrow$ Points on the real number line whose distance from origin is a or less than a
   $\Rightarrow \quad -a \leq x \leq a$; $(a \geq 0)$
3. $|x| \geq a \Rightarrow x^2 \geq a^2 \Rightarrow$ Points on the real number line whose distance from origin is a or greater than a.
   $\Rightarrow x \leq -a$ or $x \geq a$; $(a \geq 0)$
4. $a \leq |x| \leq b \quad \Rightarrow a^2 \leq x^2 \leq b^2 \Rightarrow x \in [-b, -a] \cup [a, b]$
5. $|x + y| = |x| + |y| \Leftrightarrow x$ and y have same sign or at least one of x and y is zero or $xy \geq 0$
6. $|x \pm y| \leq |x| + |y|$
7. $|x \pm y| \geq ||x| - |y||$

Signum Function

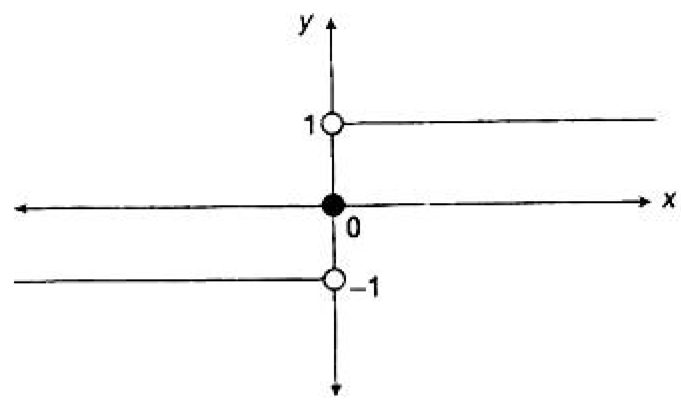


Fig. 4

$y = f(x) = \text{sgn}(x)$

$$\text{sgn}(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

or $\quad \text{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

Domain – R; Range $\rightarrow \{-1, 0, 1\}$; Nature: many one, odd function

In general $\text{sgn}(f(x)) = \begin{cases} \dfrac{|f(x)|}{f(x)}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$

or $\text{sgn}(f(x)) = \begin{cases} -1, & f(x) < 0 \\ 0, & f(x) = 0 \\ 1, & f(x) > 0 \end{cases}$

Exponential and Logarithmic Functions

Exponential function

$y = a^x \,.\, a > 1$

Domain $\rightarrow R$; Range $\rightarrow (0, \infty)$; Nature $\rightarrow$ non-periodic, one-one, neither odd nor even, monotonically increasing, **($a > 1$)**; monotonically decreasing, **($0 < a < 1$)**

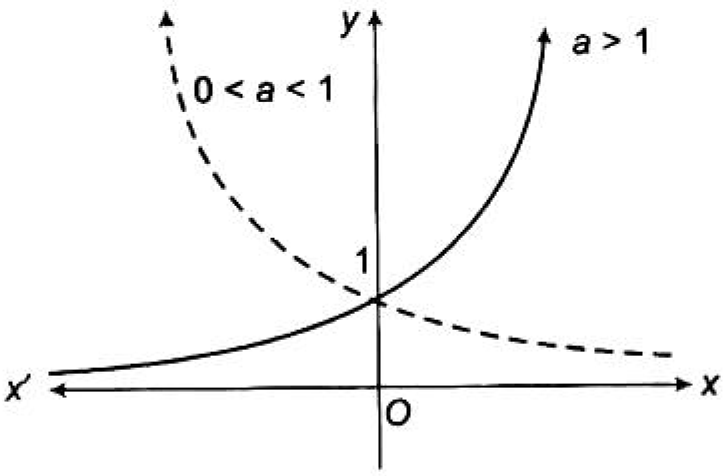


Fig. 5

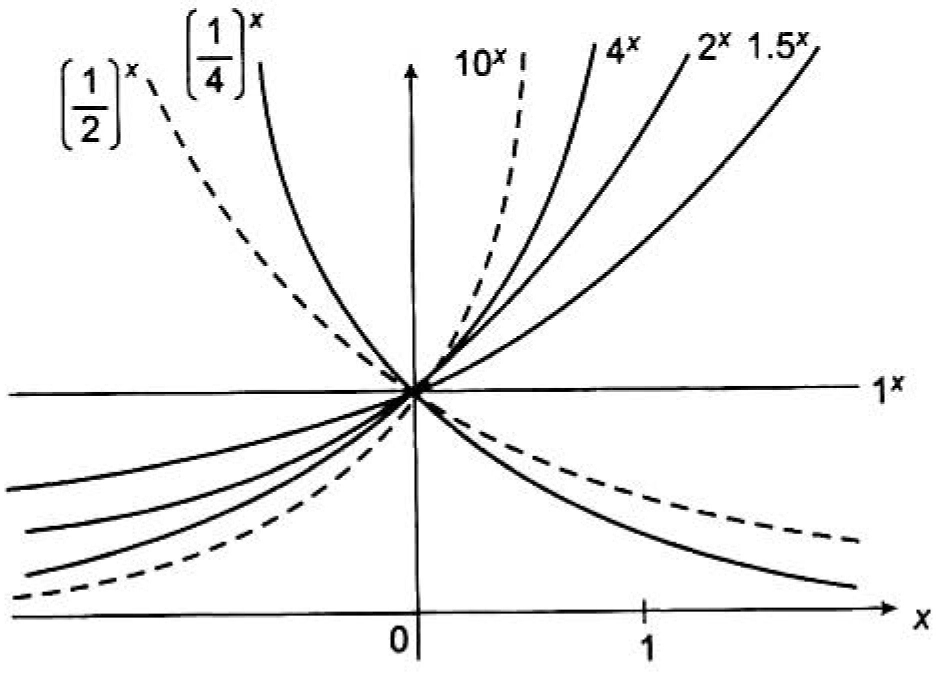


Fig. 6

Logarithmic function

Logarithmic function is inverse of exponential function. Hence domain and range of the logarithmic functions are range and domain respectively of exponential functions. Also graph of a function can be obtained by taking the mirror image of the graph of the exponential function in the line $y = x$.

$y = \log_a x$, $a > 0$ and $\neq 1$

Domain $\rightarrow (0, \infty)$; Range $\rightarrow (-\infty, \infty)$; Period $\rightarrow$ non-periodic; Nature $\rightarrow$ neither odd nor even; Interval which the inverse can be obtained $\rightarrow (0, \infty)$

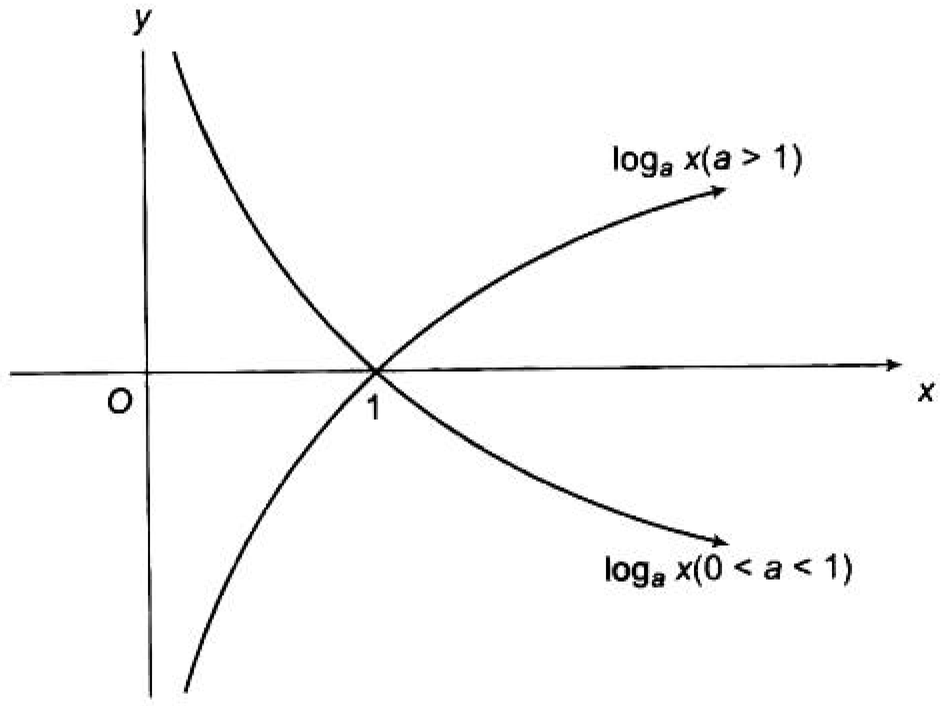


Fig. 7

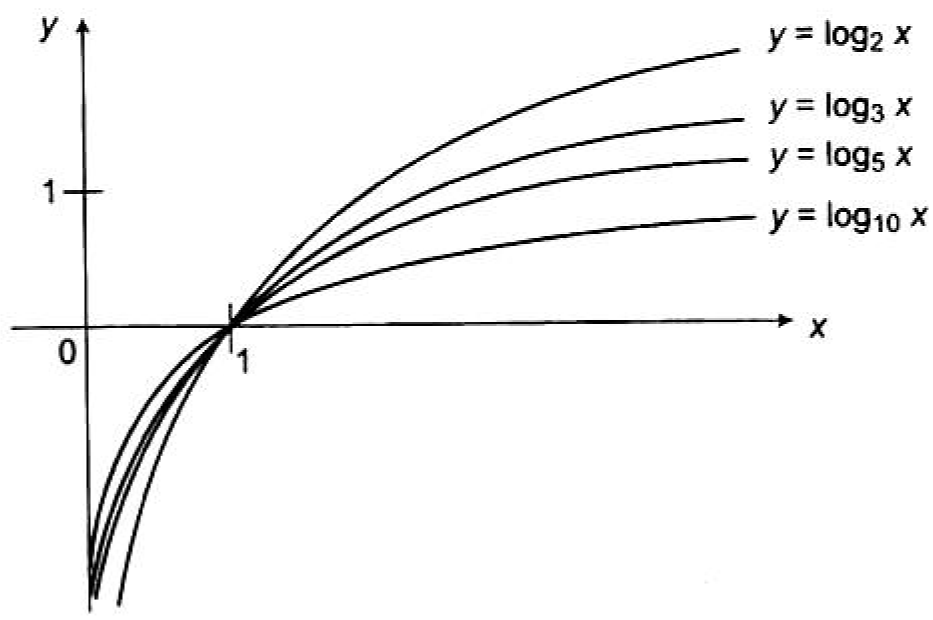


Fig. 8

Properties of logarithmic function

1. $\log_a(x \cdot y) = \log_a x + \log_a y$
2. $\log_a(x/y) = \log_a x - \log_a y$
3. $\log_a(x^b) = b\log_a x$
4. $\log_{x^a} y^b = \dfrac{b}{a}\log_x y$

5. If $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, & \text{if } a > 1 \\ 0 < x < y, & \text{if } 0 < a < 1 \end{cases}$

6. If $\log_a x > y \Rightarrow \begin{cases} x > a^y, & \text{if } a > 1 \\ 0 < x < a^y, & \text{if } 0 < a < 1 \end{cases}$

7. $a^{\log_a x} = x$

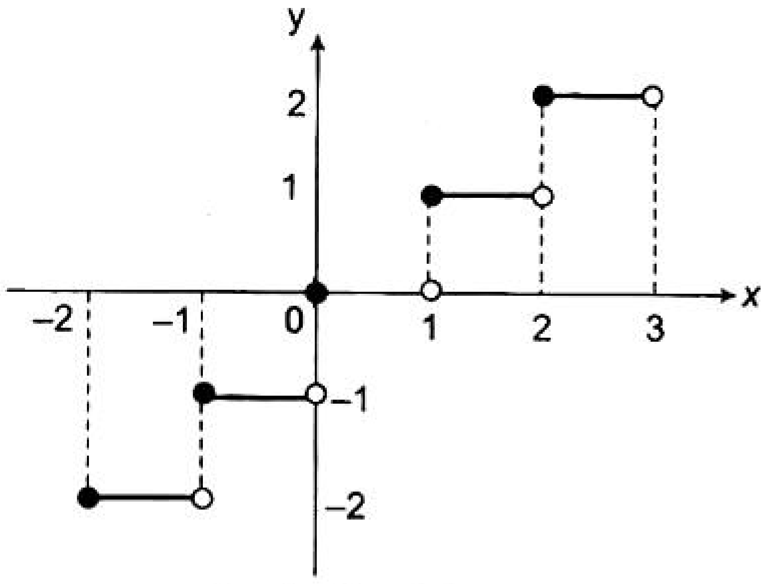
8. $\log_y x = \dfrac{\log_a x}{\log_a y}$

Greatest Integer and Fractional Part Function

Greatest integer function (floor value function)

$y = f(x) = [x]$ (Greatest integer $\le x$)

Domain $\to R$; Range $\to I$;



Graph of $y = [x]$

Fig. 9

Properties of greatest integer function

- $[x] = n \ (n \in I) \Rightarrow x \in [n, n+1)$
- $x - 1 < [x] \le x$
- $[-x] + [x] = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$
- $[x] \ge n \Rightarrow x \ge n, n \in I$
- $[x] \le n \Rightarrow x < n + 1, n \in I$
- $[x] > n \Rightarrow x \ge n + 1, n \in I$

  For example,

  $[x] \ge 2 \Rightarrow x \in [2, \infty)$

  $[x] > 3 \Rightarrow [x] \ge 4 \Rightarrow x \in [4, \infty)$

  $[x] \le 3 \Rightarrow x \in (-\infty, 4)$

Fractional part function

$y = f(x) = \{x\} = x - [x]$

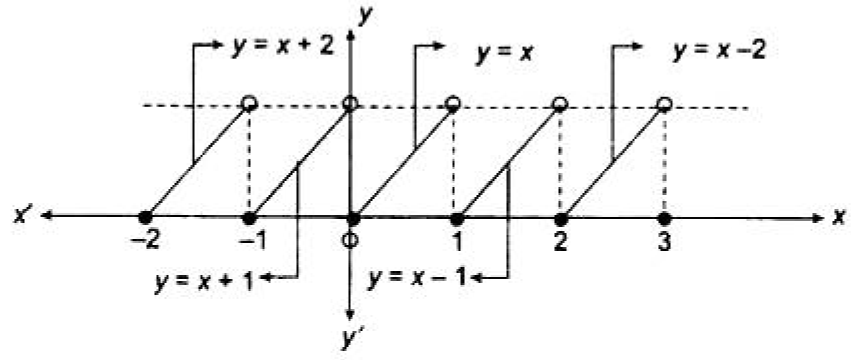
Domain $\to R$; Range $\to [0, 1)$; Period $\to 1$;

$[x + y] = [x] + [y]$, if $0 \le \{x\} + \{y\} < 1$

$[x + y] = [x] + [y] + 1$, $1 \le \{x\} + \{y\} < 2$

$\{x\} + \{-x\} = 0$ if $x \in I$

$\{x\} + \{-x\} = 1$ if $x \notin I$



Graph of $y = \{x\}$

Fig. 10

Different Types of Mappings

One-one and many-one functions

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one-one. One-one functions are also called **injective** functions.

Methods to determine one-one and many-one

1. Let $x_1, x_2 \in$ domain of f and if $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$ for every x_1, x_2 in the domain, then f is one-one else many-one.
2. Conversely if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-one else many-one.
3. If the function is entirely increasing or decreasing in the domain, then f is one-one else many-one.
4. Any continuous function $f(x)$ which has at least one local maxima or local minima is many-one.
5. All even functions are many one.
6. All polynomials of even degree defined in R have at least one local maxima or minima and hence are many one in the domain R. Polynomials of odd degree can be one-one or many-one.
7. If f is a rational function then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain. Hence we can write $f(x_1) - f(x_2) = (x_1 - x_2)\, g(x_1, x_2)$ where $g(x_1, x_2)$ is some function in x_1 and x_2. Now, if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and which lies in the domain, then f is many-one else one-one.
8. Draw the graph of $y = f(x)$ and determine whether $f(x)$ is one-one or many-one.

Onto and into functions

Let $f: X \to Y$ be a function. If each element in the co-domain "Y" has at least one pre-image in the domain X, that is, for every $y \in Y$ three exists at least one element $x \in X$ such that $f(x) = y$, then f is onto. In other words range of $f = Y$, for onto functions.

On the other hand, if there exists at least one element in the co-domain Y which is not an image of any element in the domain X, then f is into.

Onto function is also called **surjective function** and a function which is both one-one and onto is called **bijective function**.

Methods to determine onto or into

1. If range = co-domain, then f is onto. If range is a proper subset of co-domain, then f is into.
2. Solve $f(x) = y$ for x, say $x = g(y)$. Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain of f for all $y \in$ co-domain, then $f(x)$ is onto. If this requirement is not met by at least one value of y in co-domain, then $f(x)$ is into.

Remark:

- An into function can be made onto by redefining the co-domain as the range of the original function.
- Any polynomial function $f: R \to R$ is onto if degree is odd; into if degree of f is even.

Even and odd functions

Even function A function $y = f(x)$ is said to be an even functions if $f(-x) = f(x)\ \forall\ x \in D_f$.

Graph of an even function $y = f(x)$ is symmetrical about the y-axis, i.e., if point (x, y) lies on the graph then $(-x, y)$ also lies on the graph.

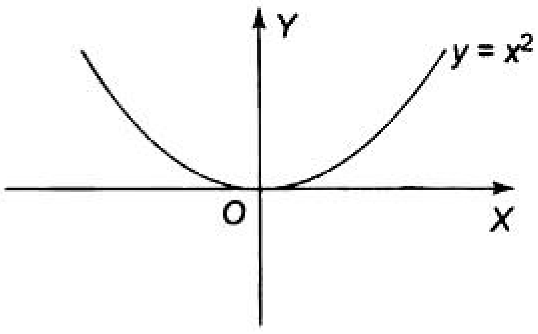


Fig. 11

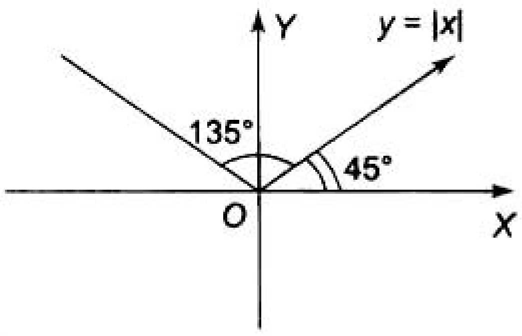


Fig. 12

Odd function A function $y = f(x)$ is said to be an odd function if $f(-x) = -f(x)\ \forall\ x \in D_f$.

Graph of an odd function $y = f(x)$ is symmetrical in opposite quadrants. i.e., if point (x, y) lies on the graph then $(-x, -y)$ also lies on the graph.

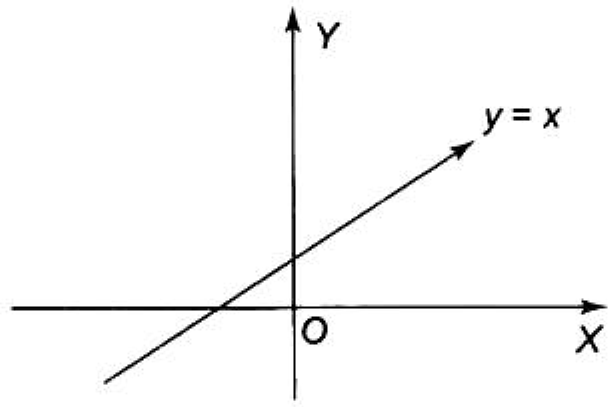


Fig. 13

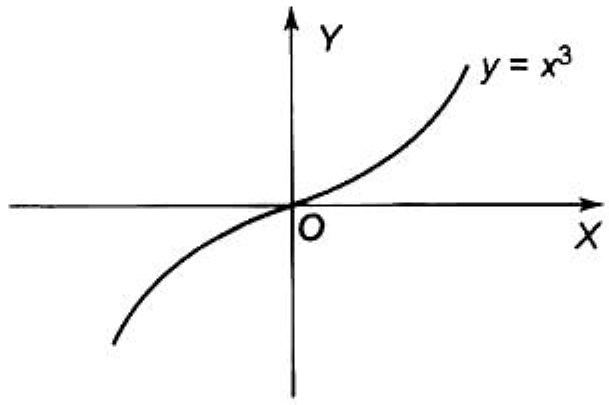


Fig. 14

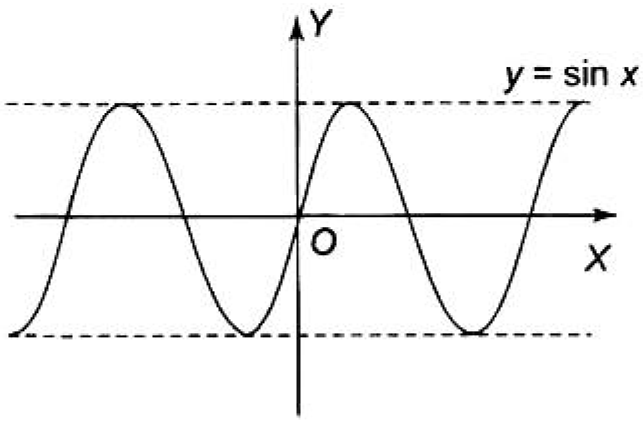


Fig. 15

- The first derivative of an even function is an odd function and vice versa.
- If $x = 0 \in$ domain of f, then for odd function $f(x)$ which is continuous at $x = 0$, $f(0) = 0$, i.e., if for a function, $f(0) \neq 0$, then that function cannot be odd. It follows that for a differentiable even function $f'(0) = 0$, i.e., if for a differentiable function $f'(0) \neq 0$ then the function f cannot be even.
- $f(x) = 0$ is the only function which is defined on the entire number line is even and odd at the same time.
- Every even function $y = f(x)$ are many-one $\forall\ x \in D_f$.

Periodic Function

A function $f: X \to Y$ is said to be a periodic function if there exists a positive real number p such that $f(x + p) = f(x)$, for all $x \in X$. The least of all such positive numbers p is called the principal period or simply period of f. All periodic functions can be analyzed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

1. $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic functions with period 2π. $\tan x$, $\cot x$ are periodic with period π.
2. $f(x) = x - [x]$ is periodic with period 1, where $[\cdot]$ represents greatest integer function.

There are two types of questions asked in the examination. You may be asked to test for periodicity of the function or to find the period of the function. In the former case you just need to show that $f(x + T) = f(x)$ for same $T(> 0)$ independent of x whereas in the latter, you are required to find a least positive number T independent of x for which $f(x + T) = f(x)$ is satisfied.

Notes:

- If $f(x)$ is periodic with period p, then $a f(x + c) + b$ where $a, b, c \in R$ $(a \neq 0)$ is also periodic with period p.
- If $f(x)$ is periodic with period p, then $f(ax) + b$ where $a, b \in R$ $(a \neq 0)$ is also period with period $\frac{p}{|a|}$.
- Let $f(x)$ has period $p = m/n$ ($m, n \in N$ and co-prime) and $g(x)$ has period $q = r/s$ ($r, s \in N$ and co-prime) and let t be the LCM of p and q, i.e.,

$$t = \frac{\text{LCM of } (m, r)}{\text{HCF of } (r, s)}$$

Then t shall be the period of $f + g$ provided there does not exist a positive number k $(< t)$ for which $f(x + k) + g(x + k) = f(x) + g(x)$, else k will be the period. The same rule is applicable for any other algebraic combination of $f(x)$ and $g(x)$.
LCM of p and q always exist if p/q is a rational quantity. If p/q is irrational then algebraic combination of f and g is non-periodic.

- $\sin^n x$, $\cos^n x$, $\operatorname{cosec}^n x$, and $\operatorname{sec}^n x$ have period 2π if n is odd and π if n is even.
- $\tan^n x$ and $\cot^n x$ have period π whether n is odd or even.
- A constant function is periodic but does not have a well-defined period.
- If g is periodic then fog will always be a periodic function. Period of fog may or may not be the period of g.
- If f is periodic and g is strictly monotonic (other than linear) then fog is non-periodic.
- Addition of periodic and non-periodic functions is always non-periodic function.
- Addition of two non-periodic functions may be periodic.

Composite Function

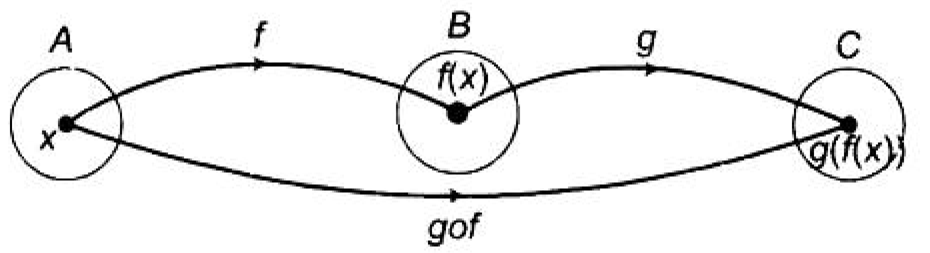


Fig. 16

Let A, B, and C be three non-empty sets.

Let $f: A \to B$ and $g: B \to C$ be two functions then $gof: A \to C$. This function is called composition of f and g, given by $gof(x) = g(f(x))\ \forall\ x \in A$.

Thus the image of every $x \in A$ under the function gof is the g-image of the f-image of x.

The gof is defined only if $\forall\ x \in A$, $f(x)$ is an element of the domain of g so that we can take its g-image.

The range of f must be a subset of the domain of g in gof.

Properties of composite functions

1. The composition of function is not commutative, i.e., $fog \neq gof$.
2. The composition of function is associative, i.e., if $h: A \to B$, $g: B \to C$ and $f: C \to D$ be three functions, then $(fog)oh = fo(goh)$.

3. The composition of any function with the identity function is the function itself. i.e.. $f: A \to B$ then $foI_A = I_Bof = f$ where I_A and I_B are the identity functions of A and B, respectively.

Identical Function

Two functions f and g are said to be identical if

1. The domain of f = the domain of g, i.e., $D_f = D_g$
2. The range of f = the range of g
3. $f(x) = g(x)\ \forall\ x \in D_f$ or $x \in D_g$, e.g.. $f(x) = x$ and $g(x) = \sqrt{x^2}$ are not identical functions as $D_f = D_g$, but $R_f = R$, $R_g = [0, \infty)$