SEQUENCE & SERIES

1. ARITHMETIC PROGRESSION (AP) :

AP is sequence whose terms increase or decrease by a fixed number. This fixed number is called the **common difference**. If 'a' is the first term & 'd' is the common difference, then AP can be written as $a, a + d, a + 2d, ... a + (n - 1) d, ...$

(a) n^{th} term of this AP $\boxed{T_n = a + (n-1)d}$, where $d = T_n - T_{n-1}$

(b) The sum of the first n terms : $\boxed{S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + \ell]}$

where ℓ is the last term.

(c) Also n^{th} term $\boxed{T_n = S_n - S_{n-1}}$

Note :

(i) Sum of first n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. 2A

(ii) n^{th} term of an A.P. is of the form $An + B$ i.e. a linear expression in n, in such case the coefficient of n is the common difference of the A.P. i.e. A

(iii) Three numbers in AP can be taken as $a - d, a, a + d$; four numbers in AP can be taken as $a - 3d, a - d, a + d, a + 3d$ five numbers in AP are $a - 2d, a - d, a, a + d, a + 2d$ & six terms in AP are $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(iv) If for A.P. p^{th} term is q, q^{th} term is p, then r^{th} term is $= p + q - r$ & $(p + q)^{th}$ term is 0.

(v) If a_1, a_2, a_3....... and b_1, b_2, b_3 .... are two A.P.s, then $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3$......... are also in A.P.

(vi) (a) If each term of an A.P. is increased or decreased by the same number, then the resulting sequence is also an A.P. having the same common difference.

(b) If each term of an A.P. is multiplied or divided by the same non zero number (k), then the resulting sequence is also an A.P. whose common difference is kd & d/k respectively, where d is common difference of original A.P.

(vii) Any term of an AP (except the first & last) is equal to half the sum of terms which are equidistant from it.

$$T_r = \frac{T_{r-k} + T_{r+k}}{2},\ k < r$$

2. GEOMETRIC PROGRESSION (GP) :

GP is a sequence of numbers whose first term is non-zero & each of the succeeding terms is equal to the preceeding terms multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4 , ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term $\boxed{T_n = a\,r^{n-1}}$

(b) Sum of the first n terms $\boxed{S_n = \frac{a(r^n - 1)}{r - 1},\ \text{if } r \neq 1}$

(c) Sum of infinite GP when $|r| < 1$ & $n \to \infty,\ r^n \to 0$

$$\boxed{S_\infty = \frac{a}{1-r};\ |r| < 1}$$

(d) Any 3 consecutive terms of a GP can be taken as a/r, a, ar; any 4 consecutive terms of a GP can be taken as a/r^3, a/r, ar, ar^3 & so on.

(e) If a, b, c are in GP $\Rightarrow b^2 = ac \Rightarrow$ loga, logb, logc, are in A.P.

Note :

(i) In an G.P. product of k^{th} term from beginning and k^{th} term from the last is always constant which equal to product of first term and last term.

(ii) Three numbers in **G.P.** : **a/r, a, ar**

Five numbers in **G.P.** : $\mathbf{a/r^2, a/r, a, ar, ar^2}$

Four numbers in **G.P.** : $\mathbf{a/r^3, a/r, ar, ar^3}$

Six numbers in **G.P.** : $\mathbf{a/r^5, a/r^3, a/r, ar, ar^3, ar^5}$

(iii) If each term of a **G.P.** be raised to the same power, then resulting series is also a **G.P.**

(iv) If each term of a G.P. be multiplied or divided by the same non-zero quantity, then the resulting sequence is also a G.P.

(v) If a_1, a_2, a_3 ..... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2 ..... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ...... will also form a G.P. common ratio will be $r_1 r_2$ and $\frac{r_1}{r_2}$ respectively.

(vi) In a positive G.P. every term (except first) is equal to square root of product of its two terms which are equidistant from it.

i.e. $\mathbf{T_r = \sqrt{T_{r-k}T_{r+k}}}$, $k < r$

(vii) If $\mathbf{a_1, a_2, a_3 ..... a_n}$ is a **G.P.** of **non zero, non negative terms,** then $\mathbf{\log a_1, \log a_2, ... \log a_n}$ is an **A.P.** and **vice-versa.**

3. HARMONIC PROGRESSION (HP) :

A sequence is said to HP if the reciprocals of its terms are in AP.

If the sequence a_1, a_2, a_3, ..., a_n is an HP then $1/a_1$, $1/a_2$,.........., $1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots\ldots \frac{1}{a+(n-1)d}$

Note : No term of any H.P. can be zero. If a, b, c are in

$$HP \Rightarrow b = \frac{2ac}{a+c} \text{ or } \frac{a}{c} = \frac{a-b}{b-c}$$

4. MEANS

(a) Arithmetic mean (AM) :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c.

n-arithmetic means between two numbers :

If a,b are any two given numbers & a, A_1, A_2, ..., A_n, b are in AP then $A_1, A_2, ... A_n$ are the n AM's between a & b, then

$A_1 = a + d, A_2 = a + 2d, ..., A_n = a + nd$, where $d = \frac{b-a}{n+1}$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

(b) Geometric mean (GM) :

If a, b, c are in GP, b is the GM between a & c, $b^2 = ac$, therefore $b = \sqrt{ac}$

n-geometric means between two numbers :

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in GP then $G_1, G_2, G_3, ...G_n$ are n GMs between a & b.

$G_1 = ar, G_2 = ar^2, ... G_n = ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to nth power of the single GM between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b

(c) Harmonic mean (HM) :

If a, b, c are in HP, then b is HM between a & c, then $b = \frac{2ac}{a+c}$.

Important note :

(i) If A, G, H, are respectively AM, GM, HM between two positive number a & b then

(a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \geq G \geq H$

(c) $A = G = H \Rightarrow a = b$

(ii) Let $a_1, a_2, \ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \frac{a_1 + a_2 + \ldots + a_n}{n}$

$$G = (a_1 a_2 \ldots a_n)^{1/n} \text{ and } H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots = a_n$

5. ARITHMETICO - GEOMETRIC SERIES :

Sum of First n terms of an Arithmetico-Geometric Series :

Let $S_n = a + (a+d)r + (a+2d)r^2 + \ldots + [a+(n-1)d]r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]\ r^n}{1-r},\ r \neq 1$

Sum to infinity :

If $|r| < 1$ & $n \to \infty$ then $\lim_{n\to\infty} r^n = 0 \Rightarrow S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

6. SIGMA NOTATIONS

Theorems :

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ **(b)** $\sum_{r=1}^{n} k a_r = k\sum_{r=1}^{n} a_r$

(c) $\sum_{r=1}^{n} k = nk$; where k is a constant.

7. RESULTS

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$