# *Integral Calculus

## Proper and Improper Integral

PROPER INTEGRAL

(1) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(2) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(3) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Removal identity

(4) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

(5) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ ; if $f(x)$ is even, $f(-x) = f(x)$

$= 0$ ; if $f(x)$ is odd, $f(-x) = -f(x)$

(6) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ ; if $f(2a-x) = f(x)$

$= 0$ ; if $f(2a-x) = -f(x)$

## *REDUCTION FORMULA

(7) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{(m-1)(m-3)\ldots\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots} \times K$

$K = 1$, if m or n or both are odd

$K = \pi/2$, if m and n are even

Walli's formula

(8) $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \dfrac{(n-1)(n-3)\cdots\cdots}{n(n-2)(n-4)\cdots} \times K$.

$K = 1$, when $n$ is odd

$K = \pi/2$, when $n$ is even

(1) $\int x^n = \dfrac{x^{n+1}}{n+1}$

(2) $\int 1/x = \log|x|$

(3) $\int e^{ax} = \dfrac{e^{ax}}{a}$

(4) $\int \sin x = -\cos x$

(5) $\int \cos x = \sin x$

(6) $\int \sec^2 x = \tan x$

(7) $\int \tan x = \log|\sec x|$

(8) $\int \cot x = \log|\sin x|$

(9) $\int \dfrac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a)$

(10) $\int \dfrac{1}{a^2 + x^2} = \dfrac{1}{a} \tan^{-1}(x/a)$

(11) $\int \dfrac{f'(x)}{f(x)} \, dx = \log f(x)$

(12) $\int \dfrac{f'(x)}{(1 + (f(x))^2)} \, dx = \tan^{-1}(f(x))$

(13) $\int \dfrac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)}$

(14) $\int f^n(x) \cdot f'(x) = \dfrac{f^{n+1}(x)}{n+1}$

(15) $\int u \cdot v = u \int v \, dx - \int \left( \dfrac{du}{dx} \int v \, dx \right) dx$

Choose $u \rightarrow$ LIATE/ILATE

$Q:- \int x^3 e^x dx$

$x^3 e^x - \int 3x^2 e^x$

$x^3 e^x - 3 \cdot \left[\cancel{\phantom{x}}x^2 e^x - \int 2x e^x dx\right]$

$x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x$

$Q:- \int e^x \cdot x^2 dx$

$e^x\left[x^2 \cdot 2x + 2\right]$

$Q:- \int_{\log 2}^{\log 3} \frac{e^x}{(1+e^x)} dx$

$\int_{\log 2}^{\log 3} \left(1 - \frac{1}{1+e^x}\right) dx$ $\quad \underline{\underline{OR}} \quad \frac{f'(x)}{f(x)} = \log f(x)$

$\left[x - \log(1+e^x)\right]_{\log 2}^{\log 3}$

$(\log 3 - \log 2) - \left[\log(1+e^{\log 3}) - \log(1+e^{\log 2})\right]$

$\log\left(\frac{3}{2}\right) - \left[\log \frac{4}{3}\right]$

$\log\left(\frac{3}{2}\right) - \log\left(\frac{4}{3}\right)$

$\log\left(\frac{3 \cdot 3}{2 \cdot 4}\right) = \log\left(\frac{9}{8}\right) = \log\left(\frac{4}{3}\right)$

Q:- $\int_0^1 \frac{1}{e^x+e^{-x}}$

$= \int_0^1 \frac{1}{e^x + 1/e^x}$

$= \int_0^1 \frac{e^x}{1+e^{2x}}$

$= [\tan^{-1} e^x]_0^1$

$= \tan^{-1} e^1 - \tan^{-1} 1 \quad = \tan^{-1} e - \pi/4$

Q:- $\int_2^7 \frac{\sqrt{x}\ dx}{\sqrt{x} + \sqrt{9-x}}$

$\cancel{\int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \times \frac{\sqrt{x}-\sqrt{9-x}}{\sqrt{x}-\sqrt{9-x}}}$

$\cancel{\int_2^7 \frac{x - \sqrt{x}\sqrt{9-x}}{x-(9-x)}} = \cancel{\int_2^7 \frac{x-(x)^{1/2}\sqrt{9-x}}{2x-9}}$

$= \frac{7-2}{2} = \boxed{\frac{5}{2}}$

Q:- $\int_0^2 |1-x|\ dx$

$\int_0^1 1-x + \int_1^2 -1+x$

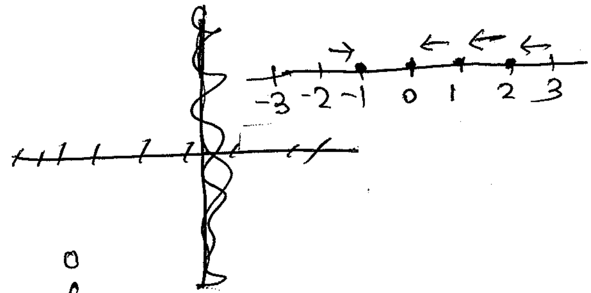
$\left[x - \frac{x^2}{2}\right]_0^1 + \left[-x + \frac{x^2}{2}\right]_1^2 = \left(1-\frac{1}{2}\right) + \left(-\cancel{2}+\cancel{2}+1-\frac{1}{2}\right)$

$= \frac{1}{2} + \frac{1}{2} = 1$

Q:- $\int_{-1.5}^{2} [x]\,dx$

$[x] \rightarrow$ integer function

greatest integer not getting $x$




$\cancel{\int_{0}^{2} [x]\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2}$

$$\int_{-1.5}^{-1} [x]\,dx + \int_{0}^{1} [x]\,dx + \int_{1}^{2} [x]\,dx + \int_{-1}^{0} [x]\,dx$$

**NOTE:-**

$[1.2] = 1$

$[-0.2] = -1$

$[0.1] = 0$

$[1.5] = 1$

Q:- $\int_{-1.5}^{1} [x+1]\,dx$

$$\int_{-1.5}^{-1} [-0.5] + \int_{-1}^{0} [0]\,dx + \int_{0}^{1} [1]\,dx$$

$\cancel{[-0.5x]_{-1.5}^{-1} + 0 + [x]_0^1}$

$(-x)_{-1.5}^{-1} + 0 + (x)_0^1$

$\cancel{+0.5\ (7.5) + 1}$

$\cancel{-7 + 1 = -6}$

$+1 - (+1.5) + 1$

$-0.5 + 1 = \underline{0.5}$

Q:- $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$$\cancel{= 2\int_0^{\pi/4} \frac{x\cdot \sin x}{1+\cos^2 x}\,dx}$$

$$\cancel{\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin x}{1+\cos^2\left(\pi/2 - x\right)}}$$

$$\cancel{= \frac{\pi\cos x}{1+\sin^2 x} - \frac{x\cos x}{1+\sin^2 x}}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}$$

$$I = \int_0^{\pi} \frac{\pi \sin x\,dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$$

$$2I = -\pi\left[\tan^{-1}(\cos x)\right]_0^{\pi}$$

$$2I = -\pi\left(\tan^{-1}(\cos\pi) - \tan^{-1} 1\right)$$

$$2I = -\pi\left(\tan^{-1}(-1) - \pi/4\right)$$

$$2I = -\pi\left(-\pi/4 - \pi/4\right)$$

$$2I = -\pi\left(-\pi/2\right)$$

$$\boxed{I = +\frac{\pi^2}{4}}$$

Q:- $\displaystyle\int_0^{\pi/2} \log(\tan x)\,dx$

$$= \int_0^{\pi/2} \log(\sin x) - \int_0^{\pi/2} \log(\cos x)$$

$$= 0 - \int_0^{\pi/2} \log(\sin x)$$

$$= 0 - 0 = \underline{0}$$

Q:- $\int_{-1}^{1} \frac{1}{1+x^2} dx$

$\left[\frac{1}{1} \tan^{-1} x\right]_{-1}^{1}$

$\pi/4 + \pi/4 = \underline{\pi/2}$

Q:- $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx =$

$= \frac{(4-1)(4-3)(5-1)(5-3)}{(9)(7)(5)(3)(1)}$

$= \frac{\cancel{3}\cdot 1\cdot 4\cdot 2}{9\cdot 7\cdot 5\cdot \cancel{3}} = \frac{8}{35\cdot 9} = \frac{8}{315}$

Q:- Assuming $i = \sqrt{-1}$ and $t$ is a real no.
Find $\int_0^{\pi/3} e^{it} dt$ is

$\int_0^{\pi/3} (\cos t + i \sin t) dt$

$\left[+\sin t - i \cos t\right]_0^{\pi/3}$

$\sin \pi/3 - i\left[\cos \pi/3 - \cos 0\right]$

$\frac{\sqrt{3}}{2} - i\left[1/2 - 1\right] = \frac{\sqrt{3}}{2} + \frac{1}{2} i$

Q:- $\int_0^{\pi/4} \frac{(1-\tan x)}{1+\tan x}$

$\int_0^{\pi/4} \tan(\pi/4 - x)$

$\left[\frac{\log|\sec(\pi/4 - x)|}{-1}\right]_0^{\pi/4}$

$\frac{\log|\sec 0| - \log|\sec \pi/4|}{-1}$

$\frac{\log|1| - \log|\sqrt{2}|}{-1}$

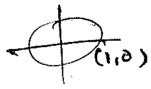
$\frac{0 - \log|\sqrt{2}|}{-1} = +\log\sqrt{2}$

$\int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x}$

$[\log|\cos x + \sin x|]_0^{\pi/4}$

Q:- If $\int_0^{2\pi} |x \cdot \sin x|\, dx = k\pi$. Find integer value of k

$I = \int_0^{\pi} x \sin x\, dx + -\int_{\pi}^{2\pi} (x \sin x) dx$



$= [-x\cos x + \sin x]_{\pi}^{2\pi}$

$= (-2\pi \cdot \cos 2\pi + \pi\cos\pi) - (\pi\cos\pi + \sin\pi)$

$= [-2\pi \cdot 1 + \pi(-1)] - (-\pi(-1))$ $\quad = (-2\pi \cdot 1 + 0) - (-\pi(-1))$

$= -2\pi - \pi$

$= -(-3\pi - \pi) = +4\pi$

$= \underline{4\pi}$

Q:- The value of integral $\int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2+\cos(x-1)}\,dx$ is

Sol$^n$:- $\cancel{\frac{(-x-1)^2\sin(-x-1)}{(-x-1)^2+\cos(-x-1)}}$

$x-1=t$

$\frac{t^2\sin t}{t^2+\cos t}$ odd function

$= \boxed{0}$

Q:- The value of $\int_0^{\pi/6}\cos^4 3\theta\cdot\sin^3 6\theta\,d\theta$

$3\theta = t$

$3d\theta = dt$

$d\theta = \frac{dt}{3}$

$\cancel{\frac{1}{3}\int_0^{\pi/2}\cos^4 t\,\sin^3 t\cdot dt}$

$\cancel{\frac{1}{3}\,\frac{(4-1)(4-3)(3-1)}{7\cdot5\cdot3\cdot1}}$

$\cancel{\frac{3\cdot1\cdot2}{3\cdot7\cdot5\cdot3\cdot1} = \frac{2}{105}}$

$\frac{1}{3}\int_0^{\pi/2}\cos^4 t\cdot\sin^3 2t\,dt$

$=\frac{1}{3}\int_0^{\pi/2}\cos^4 t\cdot(2\sin t\cos t)^3$

$=\frac{8}{3}\,\frac{(7-1)(7-3)(7-5)(3-1)}{10\cdot8\cdot6\cdot4\cdot2}$

$\boxed{I=\frac{1}{15}}$