Chapter 5
Continuity and Differentiability

Miscellaneous Exercise

Q. 1 Differentiate w.r.t. x the function

$(3x^2 – 9x + 5)^9$

Answer:

Let $y = (3x^2 – 9x + 5)^9$

If u = v(w(x))

Then using chain rule $\frac{du}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$

∴ Differentiating y w.r.t. x using chain rule

$\frac{dy}{dx} = \frac{d}{dx}(3x^2 - 9x + 5)^9$

$= 9 (3x^2 – 9x + 5)^8 \times \frac{d}{dx}(3x2 – 9x + 5)$

$= 9(3x^2 – 9x + 5)^8 \times (6x – 9)$

$=9(3x^2 – 9x + 5)^8 \times 3(2x – 3)$

$=27(3x^2 – 9x + 5)^8 (2x – 3)$

∴ $\frac{dy}{dx} = 27 (3x^2 – 9x + 5)^8 (2x – 3)$

Q. 2 Differentiate w.r.t. x the function

$\sin^3 x + \cos^6 x$

Answer:

Let $y = \sin^3 x + \cos^6 x$

Differentiating both sides with respect to x

$\therefore \frac{dy}{dx} = \frac{d}{dx}$ (sin^3 x) + $\frac{d}{dx}$ (cos^6 x) $\because \frac{d}{dx}$ (sin x) = cos x & $\frac{d}{dx}$ (cos x) = -sin x]

= 3 sin2x × $\frac{d}{dx}$ (sin x) + 6 cos5 x × $\frac{d}{dx}$ (cos x)

= 3 sin^2 x × cos x + 6 cos^5 x × (– sin x)

= 3 sin x cos x (sin x– 2 cos^4 x)

$\therefore \frac{dy}{dx}$ = 3 sinx cosx (sinx– 2 cos^4 x)

Q. 3 Differentiate w.r.t. x the function

(5x)$^{3\cos 2x}$

Answer:

Let y = (5x)$^{3\cos 2x}$

Then log y = log (5x)$^{3\cos 2x}$

⇒ log y = 3cox 2x × log 5x

Differentiating both sides with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x \times \frac{d}{dx}(\cos 2x) + \cos 2x \times \frac{d}{dx}(\log 5x)\right]$

$\left[\therefore \frac{d}{dx}(uv) = u \times \frac{dv}{dx} + v \times \frac{du}{dx}\right]$

$= \frac{dy}{dx} = 3y\left[\log 5x(-2\sin 2x) \times \frac{d}{dx}(2x) + \cos 2x \times \frac{1}{5x} \times \frac{d}{dx}(5x)\right]$

$= \frac{dy}{dx} = 3y[-2\sin 2x \log 5x +]\frac{\cos 2x}{x}$

$= \frac{dy}{dx} = y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$

$= \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$

$\because \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$

Q. 4 Differentiate w.r.t. x the function

Sin^{-1} $\left(x\sqrt{x}\right), 0 \leq x \leq 1$

Answer:

Let y = sin$-1\ \left(x\sqrt{x}\right), 0 \leq x \leq 1$

Differentiating both sides with respect to x, we get

Using chain rule we get

$\frac{dy}{dx} = \frac{d}{dx}$ sin^{-1} $\left(x\sqrt{x}\right)$

$= \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}} \times \frac{d}{dx}\left(x\sqrt{x}\right)$

$= \frac{dy}{dx} = \frac{1}{\sqrt{1-x^3}} \times \frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{dy}{dx} = \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$

$= \frac{dy}{dx} = \frac{3}{2}\sqrt{\frac{x}{\sqrt{1-x^3}}}$

$\therefore \frac{dy}{dx} = \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$

Q. 5 Differentiate w.r.t. x the function

$\frac{cos^{-1}}{\sqrt{2x+7}}, -2 < x < 2$

Answer:

Let y = $\frac{cos^{-1}}{\sqrt{2x+7}}, -2 < x < 2$

Differentiating both sides with respect to x, we get

Using Quotient rule

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\frac{d}{dx}\left(cos^{-1}\frac{x}{2}\right)-\left(cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\times\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{2x+7}$$

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\times-\frac{1}{\sqrt{4-x^2}}-\left(cos^{-1}\frac{x}{2}\right)\times\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$\frac{dy}{dx}=-\frac{\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)}-\frac{cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$\therefore\frac{dy}{dx}=-\left[\frac{1}{\sqrt{4-x^2}\times\sqrt{2x+7}}+\frac{cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Q. 6 Differentiate w.r.t. x the function

Cot^{-1} $\left(\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), 0<x<\frac{\pi}{2}$

Answer:

Let y = cot^{-1} $\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right), 0<x<\frac{\pi}{2}$

$$\frac{\sqrt{1+sinx}+\sqrt{1-sinx}}{\sqrt{1+sinx}-\sqrt{1-sinx}}$$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+sinx}-\sqrt{1-sinx}\right)\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)}$$

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)(1-\sin x)}$$

$$=\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\frac{2+2\sqrt{1-sin^2x}}{2\sin x}$$

$= \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \frac{1+\cos x}{\sin x} = \frac{2cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = cot\,\frac{x}{2}$

Substituting the value of $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \cot\frac{x}{2}$ in y.

$\therefore$ y = $\cot^{-1}\left(\cot\frac{x}{2}\right)$

$\Rightarrow$ y = x/2

Differentiating both sides with respect to x, we get

$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x)$

$\therefore \frac{dy}{dx} = \frac{1}{2}$

Q. 7

Differentiate w.r.t. x the function

$(\log x)^{\log x}$, x > 1

Answer:

Let y = $(\log x)^{\log x}$, x > 1

Taking logarithm on both sides

$\Rightarrow$ log y = log (log x) log x = log x × log (log x)

Differentiating both sides with respect to x, we get

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\log x \times \log(\log x)]$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x) \times \frac{d}{dx}(\log x) + \log x \times \frac{d}{dx}[\log(\log x)]$

$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x) \times \frac{1}{x} + \log x \times \frac{1}{\log x} \times \frac{d}{dx}(\log x)\right]$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$

$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$

Q. 8 Differentiate w.r.t. x the function

cos (a cos x + b sin x), for some constant a and b.

Answer:

Let y = cos (a cos x + b sin x)

a and b are some constants

y = cos (a cos x + b sin x)

Differentiating both sides with respect to x, we get

Using chain rule

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$

$\Rightarrow \frac{dy}{dx} = -\sin(\text{acos} + bsin\ x) \times \frac{d}{dx}(a\cos x + b\sin x)$

$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x)\ \times [a\ (-\sin x) + b\cos x]$

$\therefore \frac{dy}{dx} = (a\sin x - b\cos x)\ \times \sin(a\cos x + b\sin x)$

Q. 9 Differentiate w.r.t. x the function

$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer:

Let y = $\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Taking logarithm both sides, we get

log y = log [(sin x – cos x) (sin x – cos x)]

⇒ log y = (sin x – cos x) × log (sin x – cos x)

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[(\sin x - \cos x) \times \log(\sin x - \cos x)]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \times \frac{d}{dx}(\sin x - \cos) + (\sin x - \cos x) \times \frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\sin x - \cos x) \times (\cos x + \sin x) + (\sin x - \cos x) \times \frac{1}{(\sin x - \cos x)} \times \frac{d}{dx}(\sin x - \cos x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\sin x - \cos x) \times (\cos x + \sin x) + (\sin x - \cos x) \times \frac{1}{(\sin x - \cos x)} \times \frac{d}{dx}(\sin x - \cos x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y[(\cos x + \sin x)\log(\sin x - \cos x) + (\cos x + \sin x)]$$

$\therefore \frac{dy}{dx}$ = (sin x – cos x) (sin x – cos x) (cos x + sin x) [1 + log (sin x – cos x)]

Q. 10 Differentiate w.r.t. x the function

$x^x + x^a + a^x + a^a$, for some fixed a > 0 and x > 0

Answer:

Let $y = x^x + x^a + a^x + a^a$, for some fixed a > 0 and x > 0

And let $x^x = u$, $x^a = v$, $a^x = w$ and $a^a = s$

Then y = u + v + w + s

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \ldots \text{(I)}$$

Now,

$u = x^x$

Taking logarithm both sides, we get

log u = log x^x

⇒ log u = x log x

Differentiating both sides w.r.t. x

$\Rightarrow \frac{1}{u}\frac{dy}{dx} = \log x \times \frac{d}{x}(x) + x \times \frac{d}{dx}(\log x)$

$\Rightarrow \frac{dy}{dx} = u\left[\log x + x \times \frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x)$ … (II)

v = x^a

Differentiating both sides with respect to x

$\frac{dy}{dx} = \frac{d}{dx}(x^a)$

$\Rightarrow \frac{dv}{dx} = ax^{a-1}$ ……….. (III)

w = a^x

Taking logarithm both sides

log w = log ax

log w = x log a

Differentiating both sides with respect to x

$\frac{1}{w}\frac{dy}{dx} = \log a \times \frac{d}{dx}(x)$

$\Rightarrow \frac{dw}{dx} = w \log a$

$\Rightarrow \frac{dy}{dx} = a^x \log a$ ……………. (IV)

s = a^a

Differentiating both sides with respect to x

$\frac{ds}{dx} = 0$ ………. (V)

Putting (II), (III), (IV) and (V) in (I)

$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a + 0$

$\therefore \frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a$

Q. 11 Differentiate w.r.t. x the function

$x^{x^2-3} + (x-3)^{x^2}$, for x > 3

Answer:

Let y = $x^{x^2-3} + (x-3)^{x^2}$

And let x^{x^2-3} = u & $(x-3)^{x^2}$ = v

∴ y = u + v

Differentiating both sides w.r.t. x we get

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ …….. (I)

Now,

u = x^{x^2-3}

Taking logarithm both sides

log u = log $x^{x^2 3}$

⇒ log u = (x2 – 3) log x

Differentiating w.r.t. x, we get

$\frac{1}{u}\frac{dy}{dx} = \log x \times \frac{d}{dx}(x^2 - 3) + (x^2 - 3) \times \frac{d}{dx}(\log x)$

$\Rightarrow \frac{dy}{dx} = u \left[\log x \times 2x + (x^2 - 3) \times \frac{1}{x}\right]$

$\Rightarrow \frac{du}{dx} = x^{x^2-3} \left[\frac{x^2-3}{x} + 2x \log x\right]$ …………… (II)

Also,

v = $(x - 3)^{x^2}$

Taking logarithm both sides

log v = log $(x - 3)^{x^2}$

⇒ log v = x^2 log (x – 3)

Differentiating both sides w.r.t. x

$\frac{1}{v}\frac{dv}{dx} = loh\ (x - 3) \times \frac{d}{dx}(x^2) \times \frac{d}{dx}[\log(x - 3)]$

$\Rightarrow \frac{dv}{dx} = v \left[\log(x - 3) \times 2x + x^2 \times \frac{1}{(x-3)} \times \frac{d}{dx}(x - 3)\right]$

$\Rightarrow \frac{dv}{dx} = (x - 3)^{x^2} \left[2x \log(x - 3) + \frac{x^2}{(x-3)} \times 1\right]$

$\Rightarrow \frac{dv}{dx} = (x - 3)^{x^2} \left[\frac{x^2}{(x-3)} + 2x \log(x - 3)\right]$ ……………. (II)

Substituting (II) and (III) in (I)

$\therefore \frac{dy}{dx} = x^{x^2-3} \left[\frac{x^2-3}{x} + 2x \log x\right] + (x - 3)^{x^2} \left[\frac{x^2}{(x-3)} + 2x \log(x - 3)\right]$

Q. 12 Find dy/dx, if y = 12 (1 – cos t), x = 10 (t – sin t),

Answer:

To find $\frac{dy}{dx}$ we need to find out $\frac{dx}{dt}$ and $\frac{dy}{dt}$

So, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given, y = 12 (1 – cos t) and x = 10 (t – sin t) x = 10 (t – sin t)

Differentiating with respect to t.

$\frac{dx}{dt} = \frac{d}{dt}[10\ (t - \sin t)]$

$\Rightarrow \frac{dx}{dt} = 10 \times \frac{dy}{dx}(t - \sin t) = 10\ (1 - \cos t)$

y = 12 (1 – cos t)

Differentiating with respect to t.

$\frac{dy}{dt} = \frac{d}{dx}[12\ (1 - \cos t)]$

$\Rightarrow \frac{dy}{dx} = 12 \times \frac{dy}{dx}(1 - \cos t) = 12\ \times [0 - (-\sin t)] = 12 \sin t$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1-\cos t)} = \frac{12\times 2 \sin\frac{t}{2} \cos\frac{t}{2}}{10\times 2 sin^2\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$

$\therefore \frac{dy}{dx} = \frac{6}{5}\cot\frac{t}{2}$

Q. 13 Find dy/dx, if y = $\sin^{-1}$ x + $\sin^{-1}$ $\sqrt{1-x^2}$, 0 < x < 1

Answer:

Given,

y = $\sin^{-1}$ x + $\sin^{-1}$ $\sqrt{1-x^2}$

Differentiating with respect to x

$\frac{dy}{dx} = \frac{d}{dx}\left[sin^{-1}x + sin^{-1}\sqrt{1-x^2}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(sin^{-1}x) + \frac{d}{dx}\left(sin^{-1}\sqrt{1-x^2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(1-x^2)}} \times \frac{d}{x}\left(\sqrt{1-x^2}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \times \frac{1}{2\sqrt{1-x^2}} \times \frac{d}{dx}(1-x^2)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}} \times (-2x)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dx} = 0$

Q. 14 If x $\sqrt{1+y}$ + y $\sqrt{1+x}$ = 0,for , – 1 < x < 1, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Answer:

Given, x $\sqrt{1+y}$ + y $\sqrt{1+x}$ = 0

x $\sqrt{1+y}$ = -y $\sqrt{1+x} = 0$

Now, squaring both sides, we get

$\Rightarrow \left(x\sqrt{1+y}\right)^2 = (-y\sqrt{1+x})^2$

$\Rightarrow x^2 (1 + y) = y^2 (1+x)$

$\Rightarrow x^2 + x^2y = y^2 + y^2x$

$\Rightarrow x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow (x + y) (x - y) = xy (y - x)$

$\Rightarrow x + y = -xy$

$\Rightarrow y + xy = -x$

$\Rightarrow y (1 + x) = -x$

$\Rightarrow y = -\frac{x}{(1+x)}$

Differentiating both sides with respect to x, we get

$y = -\frac{x}{(1+x)}$

Using Quotient Rule

$y = -\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x^2)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{(1+x)-x}{(1+x)^2} = -\frac{1}{(1+x)^2}$

$\therefore \frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Hence, Proved

Q. 15 If $(x – a)^2 + (y – b)^2 = c^2$, for some c > 0, prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

Answer:

Given, $(x – a)^2 + (y – b)^2 = c^2$

Differentiating with respect to x, we get

$\frac{d}{dx}[(x-a)^2] + \frac{d}{dx}[(y-b)^2] = \frac{d}{dx}(c^2)$

$\Rightarrow 2\,(x – a) \times \frac{d}{dx}(x-a) + 2(y-b) \times \frac{d}{dx}(y-b) = 0$

$\Rightarrow 2\,(x – a) \times 1 + 2\,(y – b) \times \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{(x-a}{y-b}$

Differentiating again with respect to x

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{(x-a)}{y-b}\right]$

Using Quotient Rule

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)\times\frac{d}{dx}(x-a)-(x-a)\times\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)-(x-a)\times\frac{d}{dx}}{(y-b)^2}\right]$$

Substituting the value of dy/dx in the above equation

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)-(x-a)\times\left\{-\frac{(x-a)}{y-b}\right\}}{(y-b)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{c^3}{(y-b)^3}\right]}{\frac{c^2}{(y-b)^3}} = -c$$

$\therefore \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}}$ = - c, which is independent of a and b

Hence, Proved

Q. 16 If cos y = x cos (a + y), with cos a ≠ ± 1, prove that $\frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$

Answer

Given, cos y = x cos (a + y)

Differentiating both sides with respect to x

$$\frac{d}{dy}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\times\frac{d}{dx}(x) + x\times\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x \sin(a+y) - \sin y] \frac{dy}{dx} = \cos(a+y) \text{ .... (I)}$$

Since, cos y = x cos (a + y) ⇒ x = cos y/cos (a + y)

Substituting the value of x in (I)

$$\left[\frac{\cos y}{\cos(a+y)} \times \sin(a+y) - \sin y\right] \frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y \times \sin(a+y) - \sin y \times \cos(a+y)] \frac{dy}{dx} = \cos\,(a+y) \times \cos\,(a+y)$$

$$\Rightarrow \sin(a+y-y) \frac{dy}{dx} = cos^{\,2}(a+y)$$

$$\Rightarrow \sin a \times \frac{dy}{dx} = cos^2(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{cos^2(a+y)}{\sin a}$$

Hence, proved

Q. 17 If x = a (cos t + t sin t) and y = a (sin t – t cos t), find d^2y/dx^2.

Answer:

Given, x = a (cos t + t sin t) and y = (sin t – t cos t)

To find $\frac{dy}{dx}$ we need to find out $\frac{dx}{dt}$ and $\frac{dy}{dt}$

So, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and $\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}}{dt} \times \frac{dt}{dx}$

x = a (cos t + t sin t)

Differentiating with respect to t.

$$\frac{dx}{dt} = \frac{d}{dt}[a\,(\cos t + t \sin t)]$$

$$\Rightarrow \frac{dx}{dt} = a \times \frac{d}{dt}(\cos t + t \sin t) = a[-\sin t + \sin t \ \times \frac{d}{dt}(t) + t \times \frac{d}{dt}(\sin t)]$$

$$\Rightarrow \frac{dx}{dt} = a[-\sin t + \sin t + t \cos t] = at \cos t$$

y = a (sin t – t cos t)

Differentiating with respect to t.

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin t - t\cos t) = a\left[\cos t - \left\{\cos t \times \frac{d}{dt}(t) + t \times \frac{d}{dt}(cost\right\}\right]$$

$$\Rightarrow \frac{dy}{dt} = a[\cos t - \cos t + t \sin t] = a\ t \sin t$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\ t \sin t}{a\ t \cos t} = \tan t$$

Differentiating dy/dx with respect to t

$$\frac{\frac{dy}{dx}}{dt} = \frac{d}{dt}(\tan t) = \ sec^2 t$$

And $\frac{dt}{dx} = \frac{1}{a\ t \cos t} = \frac{\sec t}{a\ t}$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}}{dt} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = sec^2 t \times \frac{\sec t}{a\ t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{sec^3 t}{a\ t}$$

Q. 18 If f (x) = $|x|^3$, show that f″(x) exists for all real x and find it.

Answer:

$$|x| = \begin{cases} x, if\ x \geq 0 \\ -x, if\ x < 0 \end{cases}$$

When, x ≥ 0,

$f(x) = |x|^3 = x^3$

So, $f'(x) = 3x^2$

And $f''(x) = d(f'(x))/dx = 6x$

$\therefore f''(x) = 6x$

When $x < 0$,

$f(x) = |x|^3 = (-x)^3 = -x^3$

$f'(x) = -3x^2$

$f''(x) = -6x$

$\therefore f''(x) = \begin{cases} 6x, x \geq 0 \\ -6x, x < 0 \end{cases}$

Q. 19 Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n.

Answer:

To prove: P(n): $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For n = 1,

LHS = $\frac{d}{dx}(x) = 1$

RHS = $1 \times x1 - 1 = 1$

So, LHS = RHS

∴ P (1) is true.

∴ P(n) is true for n = 1

Let P(k) be true for some positive integer k.

i.e. P(k) = $\frac{dy}{dx}(x^k) = kx^{k-1}$

Now, to prove that P (k + 1) is also true

RHS = (k + 1) x (k + 1) – 1

LHS = $\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \times x^k)$

= $x^k \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(x^k)$

= $x^k \times 1 + x \times k \times x^{k-1}$

= $x^k + kx^k$

= $(k+1) \times x^k$

= $(k+1)x^{(k+1)-1}$

∴ LHS = RHS

Thus, P (k + 1) is true whenever P(k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Q. 20 Using the fact that sin (A + B) = sin A Cos B + cos A Sin B and the differentiation, obtain the sum formula for cosines.

Answer:

sin (A + B) = sin A Cos B + cos A Sin B

Differentiating with respect to x, we get

$\frac{d}{dx}$ [sin (A + B)] = $\frac{d}{dx}$ (sin A cos B) + $\frac{d}{dx}$ (cos A sin B)

$\Rightarrow \cos(A + B)\frac{d}{dx}(A + B) = \cos B\frac{d}{dx}(\sin A) + \sin A\frac{d}{dx}(\cos B) + \sin B\frac{d}{dx}(\cos A) + \cos A\frac{d}{dx}(\sin B)$

$\Rightarrow \cos(A+B)\,\frac{d}{dx}(A+B) = \cos B \cos A\,\frac{dA}{dx} + \sin A\,(-\sin B)\frac{dB}{dx} + \sin B\,(-\sin A)\,\frac{dA}{dx} + \cos A \cos B\,\frac{dB}{dx}$

$\Rightarrow \cos(A+B) \times \left[\frac{dA}{dx} + \frac{dB}{dx}\right] = (\cos A \cos B - \sin A \sin B) \times \left[\frac{dA}{dx} + \frac{dB}{dx}\right]$

∴ cos (A + B) = cos A Cos B – sin A Sin B

Q. 21 Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

Answer:

Considering the function

f(x) = |x| + |x + 1|

The above function f is continuous everywhere, but is not differentiable at x = 0 and x = – 1

$$f(x) = \begin{cases} -x-(x+1), x \le -1 \\ -x+(x+1), -1 < x < 0 \\ x+(x+1), x \ge 0 \end{cases}$$

$$= \begin{cases} -2x-1, x \le -1 \\ 1, -1 < x < 0 \\ 2x+1, x \ge 0 \end{cases}$$

Now, checking continuity

CASE I: At x < – 1

f(x) = – 2x – 1

f(x) is a polynomial

⇒ f(x) is continuous [∵ Every polynomial function is continuous]

CASE II: x > 0

f(x) = 2x + 1

f(x) is a polynomial

⇒ f(x) is continuous [∵ Every polynomial function is continuous]

CASE III: At – 1 < x < 0

f(x) = 1

f(x) is constant

⇒ f(x) is continuous

CASE IV: At x = – 1

f (x) = $\begin{cases} -2x - 1, x \leq -1 \\ 1, -1 < x < 0 \\ 2x + 1, x \geq 0 \end{cases}$

A function will be continuous at x = – 1

If LHL = RHL = f (– 1)

i.e. $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$

LHL = $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -2x - 1$

Putting x = – 1

LHL = – 2 × (– 1) – 1 = 2 – 1 = 1

RHL = $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 1 = 1$

f(x) = – 2x – 1

f (– 1) = – 2 × (– 1) – 1 = 2 – 1 = 1

so, LHL = RHL = f (– 1)

⇒ f is continuous.

CASE V: At x = 0

$$f(x) = \begin{cases} -2x - 1, x \le -1 \\ 1, -1 < x < 0 \\ 2x + 1, x \ge 0 \end{cases}$$

A function will be continuous at x = 0

If LHL = RHL = f (0)

i.e. $\lim\limits_{x\to 0^{-1}} f(x) = \lim\limits_{x\to 0^{+}} f(x) = f(0)$

LHL = $\lim\limits_{x\to 0^{-}} f(x) = \lim\limits_{x\to 0^{-}} 1$ = 1

RHL = $\lim\limits_{x\to 0^{+}} \quad = \lim\limits_{x\to 0^{+}} 2x + 1$

Putting x = 0

RHL = 2 × 0 + 1 = 1

f(x) = 2x + 1

f (0) = 2 × 0 + 1 = 0 + 1 = 1

so, LHL = RHL = f (0)

⇒ f is continuous.

Thus f(x) = |x| + |x + 1| is continuous for all values of x.

Checking differentiability

CASE I: At x < – 1

f(x) = – 2x – 1

f'(x) = – 2

f(x) is polynomial.

⇒ f(x) is differentiable

CASE II: At x > 0

f(x) = 2x + 1

f'(x) = 2

f(x) is polynomial.

⇒ f(x) is differentiable

CASE III: At – 1 < x < 0

f(x) = 1

f(x) is constant.

⇒ f(x) is differentiable

CASE IV: At x = – 1

$$f(x) = \begin{cases} -2x-1, x \le -1 \\ 1, -1 < x < 0 \\ 2x+1, x \ge 0 \end{cases}$$

f is differentiable at x = – 1 if

LHD = RHD = f' (– 1)

i.e. $\lim_{h\to -1^-} \frac{f(-1)-f(-1-h)}{h} = \lim_{h\to -1^+} \frac{f(-1+h)-f(-1)}{h} = f'(-1)$

LHD = $\lim_{h\to -1^-} \frac{f(-1)-f(-1-h)}{h} = \lim_{h\to -1^-} \frac{-2\times(-1)-1-(-2\times(-1-h)-1)}{h} = \lim_{h\to -1^-} \frac{2-1(2+2h-1)}{h}$

LHD = $\lim_{h\to -1^-} \frac{1-2h-1}{h} = \lim_{h\to -1^-} \frac{-2h}{h} = -2$

RHD = $\lim_{h\to -1^+} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to -1^+} \frac{1-(-2\times(-1)-1}{h} = \lim_{h\to -1^+} \frac{1-1}{h} = 0$

Since, LHD ≠ RHD

∴ f is not differentiable at x = – 1

CASE V: At x = 0

f (x) = $\begin{cases} -2x-1, x \le -1 \\ 1, -1 < x < 0 \\ 2x+1, x \ge 0 \end{cases}$

f is differentiable at x = 0 if

LHD = RHD = f' (0)

i.e. $\lim_{h\to 0^-} \frac{f(0)-f(0-h)}{h} = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = f'(0)$

LHD = $\lim_{h\to 0^-} \frac{f(0)-f(0-h)}{h} = \lim_{h\to 0^-} \frac{2\times 0+1-1}{h} = 0$

RHD = $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{2\times(0+h)+1-(2\times 0+1)}{h} = \lim_{h\to -1^+} \frac{2h+1-1}{h} = 0$

Since, LHD ≠ RHD

∴ f is not differentiable at x = 0

So, f is not differentiable at exactly two-point x = 0 and x = 1, but continuous at all points.

Q. 22

If $\begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$

Answer:

Let y = $\begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$

Differentiation of determinant u = $\begin{vmatrix} e & f & g \\ h & i & j \\ k & l & m \end{vmatrix}$ is given b

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(e) & \frac{d}{dx}(f) & \frac{d}{dx}(g) \\ h & i & j \\ k & l & m \end{vmatrix} + \begin{vmatrix} e & f & g \\ \frac{d}{dx}(h) & \frac{d}{dx}(i) & \frac{d}{dx}(j) \\ k & l & m \end{vmatrix} + \begin{vmatrix} e & f & g \\ h & i & j \\ \frac{d}{dx}(k) & \frac{d}{dx}(l) & \frac{d}{dx}(m) \end{vmatrix}$$

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(f(x)) & \frac{d}{dx}(g(x)) & \frac{d}{dx}(h(x)) \\ 1 & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(m) & \frac{d}{dx}(n) \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ \frac{d}{dx}(a) & \frac{d}{dx}(b) & \frac{d}{dx}(c) \end{vmatrix}$$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 1 & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 1 & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

Since, a, b, c and l, m, n are constants so, their differentiation is zero.

Also in a determinant if all the elements of row or column turns to be zero then the value of determinant is zero.

$$\therefore \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} = 0 \text{ and } \begin{vmatrix} f(x) & g(x) & f(x) \\ 1 & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ 1 & m & n \\ a & b & c \end{vmatrix}$$

Hence, proved.

Q. 23 If, Y = $e^{acos^{-1}x}, -1 \leq x \leq 1$ show that (1 – x2) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Answer:

Given, y = $e^{acos^{-1}x}$

Taking logarithm both sides, we get

log y = log $e^{acos^{-1}x}$

⇒ log y = a cos – 1x log e

⇒ log y = a cos – 1x [log e = 1]

Differentiating both sides with respect to x

$\frac{1}{y}\frac{dy}{dx} = a \times -\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = -\frac{ay}{\sqrt{1-x^2}}$

Squaring both sides

$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$

⇒ (1 – x2) $\left(\frac{dy}{dx}\right)^2 = a^2y^2$

Differentiating both sides

$\Rightarrow \left(\frac{dy}{dx}\right)^2 \frac{dy}{dx}(1 - x^2) + (1 - x^2) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}(y^2)$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + (1 - x^2) \times 2 \times \frac{d}{dx} \times \frac{d^2y}{dx^2} = a^2 \times 2y \times \frac{dy}{dx}$

$\Rightarrow -x \times \frac{dy}{dx} + (1 - x^2)\frac{d^2y}{dx^2} = a^2y$

$\therefore (1 - x^2)\frac{d^2y}{dx^2} - x \times \frac{dy}{dx} - a^2y = 0$

Hence, proved