Application of Derivatives — Chapter 20

I. APPLICATION IN MECHANICS & RATE MEASURER

VELOCITY AND ACCELERATION IN RECTILINEAR MOTION

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change of its velocity with respect to time.
Let velocity and acceleration at time t be v and a respectively.

Then, Velocity $(v) = \frac{ds}{dt}$;

Acceleration $(a) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

DERIVATIVE AS THE RATE OF CHANGE

If a variable quantity y is some function of time t *i.e.*, $y = f(t)$, then for a small change in time Δt we have a corresponding change Δy in y.

Thus, the average rate of change $= \frac{\Delta y}{\Delta t}$.

The differential coefficient of y with respect to x, *i.e.*, $\frac{dy}{dx}$ is nothing but the rate of change of y relative to x.

RATE OF CHANGE OF QUANTITY

1. **Chain Rule.** If both x and y are functions of the parameter t, then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$
2. If the rate of change of a variable is positive (negative) then the value of the variable increases (decreases) with the increase in the value of independent variable.

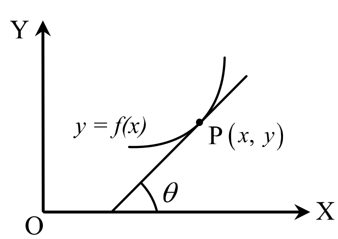
ERROR AND APPROXIMATION

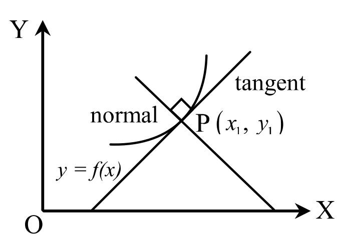
1. **Approximate value** of a function $f(x+h) = f(x) + hf'(x)$
2. **Absolute error :** The error Δx in x is called, the absolute error.
3. **Relative error :** If Δx is error in x, then the ratio $\frac{\Delta x}{x}$ is called relative error.
4. **Percentage error :** If $\frac{\Delta x}{x}$ is relative error, then $\frac{\Delta x}{x} \times 100$ is called percentage error in x.

II. TANGENTS AND NORMALS

1. Geometrical interpretation of $\frac{dy}{dx}$

   If $P(x_1, y_1)$ is a point on the curve $y = f(x)$, then value of $\frac{dy}{dx}$ at P gives the slope of tangent to the curve at P.





2. Equation of tangent to the curve at P is $y - y_1 = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$

3. If $\left[\frac{dy}{dx}\right]_{(x_1, y_1)}$ is zero, then the tangent to the curve $y = f(x)$ at P is $y = y_1$ which is parallel to x-axis.

The equation of normal to the curve at P is given by $x = x_1$ which is parallel to y-axis

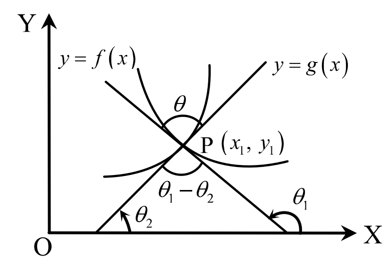
4. If $\left[\frac{dy}{dx}\right]_{(x_1, y_1)} \neq 0,$ then slope of normal at P is $-\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$ and equation of normal is

$$y - y_1 = -\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$$

5. If $\frac{dx}{dy} = 0,$ then the tangent is perpendicular to x-axis and its equation is $x = x_1$ or normal is parallel to x-axis and equation of normal is $y = y_1$

ANGLE OF INTERSECTION OF TWO CURVES

Angle of intersection of two curves is the (acute) angle between the tangents to the two curves at their point of intersection.



If θ is the acute angle between the tangents, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

where $m_1 =$ value of $\frac{dy}{dx}$ at the common point for first curve.

and $m_2 =$ value of $\frac{dy}{dx}$ at the common point for the second curve.

If θ is the required angle of intersection, then, $\theta = \left|(\theta_1 - \theta_2)\right|,$

where θ_1 and θ_2 are the inclination of tangent to the curves $y = f(x)$ and $y = g(x)$ respectively at the point P.

ORTHOGONAL AND TOUCHING CURVES

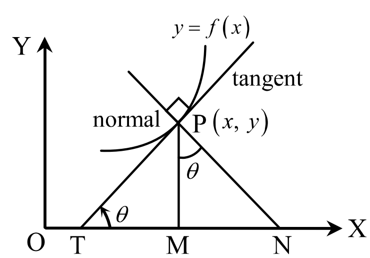
Two curves are said to be orthogonal (or intersect orthogonally) if the angle of intersection of two curves is a right angle. i.e. if $m_1 m_2 = -1$

Two curves touch each other if $m_1 = m_2$

Note : The curve $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ cut each other orthogonally if $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

LENGTH OF TANGENT, NORMAL, SUB-TANGENT AND SUBNORMAL

Let the tangent and normal at the point $P(x, y)$ on the curve meet the axis of x at the points T and N respectively. Let M be the foot of the ordinates at P. Then,



(i) Length of the tangent $= PT = \left| y \operatorname{cosec}\theta \right|$

$$= \left| y\sqrt{1 + \cot^2\theta} \right| = \left| \frac{y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} \right| = \left| y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right|$$

(ii) Length of the normal $= PN = \left| y\sec\theta \right| = \left| y\sqrt{1+\tan^2\theta} \right| = \left| y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \right|$

(iii) Length of subtangent $= TM = \left| y\cot\theta \right| = \left| \frac{y}{\left(\frac{dy}{dx}\right)} \right| = \left| y\,\frac{dx}{dy} \right|$

(iv) Length of subnormal $= MN = \left| y\tan\theta \right| = \left| y\left(\frac{dy}{dx}\right) \right|$.

III. ROLLE'S AND LAGRANGE'S MEAN VALUE THEOREMS

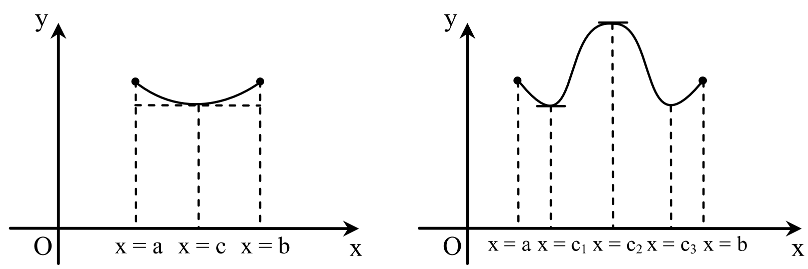
ROLLE'S THEOREM

If a function $f(x)$ defined on $[a, b]$ is

(i) continuous on $[a, b]$,

(ii) differentiable on (a, b) and

(iii) $f(a) = f(b)$, then there exists at least one point c, $a < c < b$ such that $f'(c) = 0$.



(i) (ii)

There is one point c in figure (i) and more than one point c in figure (ii).

Geometrical Interpretation : If a function $f(x)$ satisfies all the above three conditions, then there exists at least one point c between a and b at which tangent to the curve is parallel to x-axis.

Algebraic Interpretation : If $f(x)$ is a polynomial with a, b roots of $f(x) = 0$, i.e. $f(a) = 0 = f(b)$ then $f(x)$ satisfies all the three conditions of Rolle's theorem. Therefore, there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$ i.e. c is a root of $f'(x) = 0$

Thus, we have **Rolle's theorem for polynomials.** If $f(x)$ is a polynomial, then between any two roots of $f(x) = 0$, there always lies a root of $f'(x) = 0$

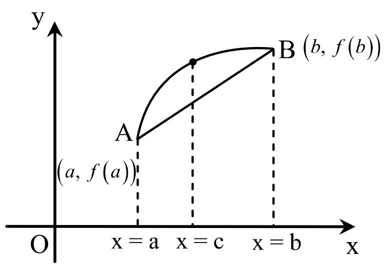
LAGRANGE'S MEAN VALUE THEOREM

If a function $f(x)$ defined on $[a, b]$ is

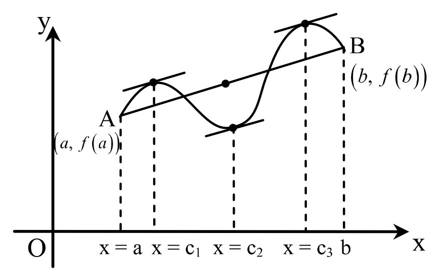
(i) continuous on $[a, b]$,

(ii) differentiable on (a, b), then there exists at least one point c, $a < c < b$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

Geometrical Interpretation : If a function $f(x)$ satisfies the above two conditions, then there exists at least one point c between a and b at which tangent is parallel to the chord joining the point $A(a, f(a))$ and $B(b, f(b))$.



(i)



(ii)

Note :

(i) A polynomial functions is everywhere continuous as well as differentiable.

(ii) An exponential function, sine and cosine functions are everywhere continuous as well as differentiable.

(iii) Logarithmic function is continuous as well as differentiable in its domain.

(iv) $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$ are discontinuous at those points where $\cos x = 0$ i.e. $x = (2n+1)\frac{\pi}{2}$.

(v) $|x|$ is not differentiable at $x = 0$.

(vi) If $f'(x) \to \pm\infty$ as $x \to a$, then $f(x)$ is not differentiable at $x = a$.

For example, if $f(x) = (2x-1)^{1/2}$,

Then $f'(x) = \frac{1}{\sqrt{2x-1}} \to \infty$ as $x \to \left(\frac{1}{2}\right)^+$. So, $f(x)$ is not differentiable at $x = \frac{1}{2}$

and $\cot x = \frac{\cos x}{\sin x}$ and $\operatorname{cosec} x = \frac{1}{\sin x}$

are discontinuous at those points where $\sin x = 0$ i.e. $x = n\pi$.

(vii) The sum, difference, product and quotient of continuous (differentiable) functions is continuous (differentiable) (with Denominator $\neq 0$ in last case).

IV. INCREASING AND DECREASING FUNCTIONS (MONOTONICITY)

1. A function $f(x)$ is said to be increasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in I$. A function $f(x)$ is said to be decreasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$, for all $x_1, x_2 \in I$.

   A function $f(x)$ is said to be strictly increasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, for all $x_1, x_2 \in I$.

   A function $f(x)$ is said to be strictly decreasing in an interval I, if for $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.

   A function $f(x)$ is said to be monotonic on an interval I, if it is either increasing or decreasing on I.

A function $f(x)$ is increasing (decreasing) at a point x_0, if $f(x)$ is increasing (decreasing) on an interval $(x_0-\delta, x_0+\delta)$ for some $\delta>0$. A function $f(x)$ is increasing (decreasing) on $[a, b]$, if it is increasing (decreasing) on (a, b) and it is also increasing (decreasing) at $x=a$ and $x=b$

2. **TEST OF MONOTONICITY**

Let $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f(x)$ is increasing $(\uparrow)$ on $[a, b]$ then $f'(x) \geq 0$ for all $x \in (a, b)$

(b) If $f(x)$ is decreasing $(\downarrow)$ on $[a, b]$ then $f'(x) \leq 0$ for all $x \in (a, b)$

(c) If $f'(x) > 0$ for all $x \in (a, b)$ then $\uparrow^s$

(d) If $f'(x) < 0$ for all $x \in (a, b)$ then $\downarrow_s$

(e) If a function $f(x)$ is defined on (a, b) and $f'(x)>0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x)=0$, then $f(x)$ is strictly increasing $(\uparrow^S)$ on (a, b)

(f) If a function $f(x)$ is defined on (a, b) and $f'(x)<0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x)=0$, then $f(x)$ is strictly decreasing $(\downarrow_S)$ on (a, b)

V. MAXIMA AND MINIMA

1. A real valued function 'f' with domain D_f is said to have absolute maximum at a point $x_0 \in D_f$ iff $f(x_0) \geq f(x)\ \forall x \in D_f$ and $f(x_0)$ is called the absolute maximum value and x_0 is called the point of absolute maxima.

Likewise, 'f' is said to have absolute minimum at a point x_0, if $f(x_0) \leq f(x) \forall x \in D_f$ and x_0 is called the point of absolute minima.

Note : Absolute maximum and Absolute minimum values of a function, if exist are unique and may occur at more than one point of D_f.

2. A real valued function 'f' with domain D_f is said to have local maxima at $x_0 \in D_f$, if for some positive δ, $f(x_0) > f(x)\ \ \forall x \in (x_0-\delta, x_0+\delta)$.$f(x_0)$ is called the local maximum value and $(x_0, f(x_0))$ is called the point of local maxima.

Likewise 'f' is said to have local minimum at $x_0 \in D_f$ if for some positive δ, $f(x_0) < f(x) \forall x \in (x_0-\delta, x_0+\delta)$.$f(x_0)$ is called the local minimum value and $(x_0, f(x_0))$ is called the point of local minima.

3. A point of domain of 'f' is an extreme point of f, if it is either a point of local maxima or local minima. It is also called as turning point.

A point x_0 of domain of 'f' is a critical point, if either f is not differentiable at x_0 or $f'(x_0)=0$.

A point x_0 where $f'(x_0)=0$ is called a stationary point and $f(x_0)$ is called the stationary value at x_0.

Note : A local maxima or minima may not be absolute maxima or absolute minima. A local maximum value at some point may be less than local minimum value of the function at another point.

4. **To find absolute maximum and absolute minimum in [a, b], proceed as follows**

(i) Evaluate $f(x)$ at points where $f'(x)=0$

(ii) Evaluate $f(x)$ at points where $f'(x)$ does not exist

(iii) Find $f(a)$ and $f(b)$

Then, the maximum of these values is the absolute maximum value and minimum of these values is the absolute minimum value of the function f.

But if the given function has domain (a, b) then we will find $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$, but note that $\lim_{x\to a^+} f(x)$ is greatest (or least) then we will say that point of Absolute Maxima (or Absolute Minima) does not exist, similarly for $\lim_{x\to b^-} f(x)$.

5. **To find the local maximum and local minimum.**

First Derivative Test

A point x_0 is a point of local maxima (local minima) if

(i) $f'(x_0)=0$ and

(ii) $f'(x)$ changes sign from positive (negative) to negative (positive) while passing through x_0

Second Derivative Test

A point x_0 is a point of local maxima (local minima) if

(i) $f'(x_0)=0$ and

(ii) $f''(x_0)<0(>0)$

If $f''(x_0)=0,$ then second derivative test fails and find $f'''(x_0)$. If $f'''(x_0)\neq 0$ then it is a point of inflexion. If $f'''(x_0)=0,$ find $f^{(iv)}(x_0)$. If $f^{iv}(x_0)$ is <0 then x_0 is the point of local maxima. If $f^{iv}(x_0)>0$ then x_0 is the point of local minima. If $f^{iv}(x_0)=0$ then we find $f^{v}(x_0)$. If $f^{v}(x_0)\neq 0$ then x_0 given the point of inflexion. If $f^{v}(x_0)=0$ then we find $f^{vi}(x_0)$ is so on in similar way.

6. If $f'(x_0)$ does not exist, but f' exists in a neighbourhood of 'x_0', then

x	slightly $< x_0$	slightly $> x_0$	Nature of the point 'x_0'
$f'(x)$	$+$ve	$-$ve	Maxima
$f'(x)$	$-$ve	$+$ve	Minima

PROPERTIES OF MONOTONIC FUNCTIONS

(i) If $f(x)$ is strictly increasing on $[a, b]$, then f^{-1} exists and is also strictly increasing.

(ii) If $f(x)$ is strictly increasing on $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

(iii) If $f(x)$ and $g(x)$ are monotonically (or strictly) increasing (or decreasing) on $[a, b]$ then $(gof)(x)$ is also monotonically (or strictly) increasing (or decreasing) on $[a, b]$.

(iv) If one of the two functions $f(x)$ and $g(x)$ is strictly (monotonically) increasing and the other is strictly (monotonically) decreasing, then $(gof)(x)$ is strictly (monotonically) decreasing on $[a, b]$.

CONCAVITY. CONVEXITY AND POINT OF INFLEXION

1. **Concavity and Convexity**

Let P be a point on the curve $y=f(x)$, where the tangent PT is not parallel to y-axis. The curve is said to be concave upwards (or convex downwards) at P if in the immediate neighbourhood of P, the curve lies above the tangent PT on both sides (as in figure I).

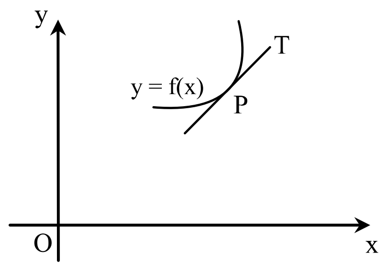


Figure I

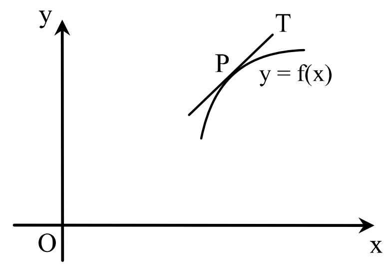


Figure II

The curve is said to be concave downwards (or convex upwards) at P if in the immediate neighbourhood of P, the curve lies below the tangent PT on both sides (as in figure II).

Criteria for concavity and convexity

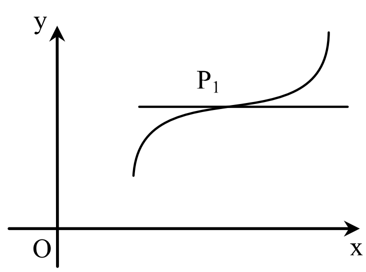
At a point P on the curve $y = f(x)$, the curve is

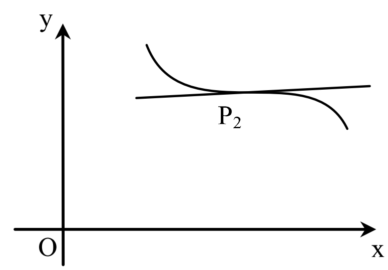
(i) Convex downward if $f''(x) > 0$

(ii) Concave downwards if $f''(x) < 0$.

2. **Point of inflexion**

A point on a curve at which the curve changes from concavity to convexity or vice-versa is called a point of inflexion. At a point of inflexion, the tangent to the curve crosses the curve.





Criteria for a point of inflexion

A point P is a point of inflexion if

(i) $\dfrac{d^2y}{dx^2} = 0$ at this point, and

(ii) $\dfrac{d^2y}{dx^2}$ changes sign in passing through this point.