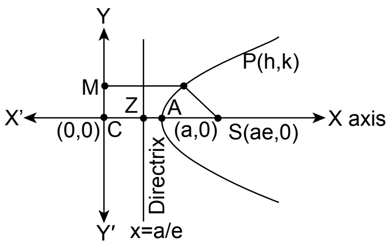
CHAPTER 18

HYPERBOLA

18.1 DEFINITION

It is the locus of a point P whose ratio of distance from a fixed point (S) to a fixed line (Directrix) remains constant (e) is known as the eccentricity of hyperbola (e > 1).



18.1.1 Standard Equation

Given, S(ae, 0) as focus and the line x – (a/e) = 0 as directrix.

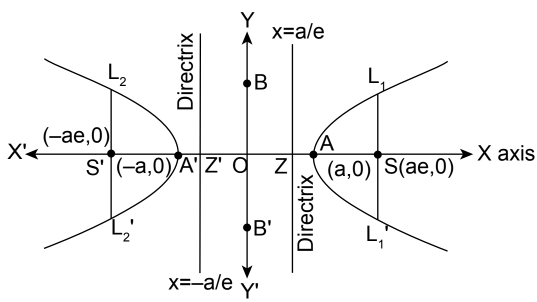
Focal Distance: Focal distance of a point P is given as

$\because$ SP = e.PM = eh – a $\Rightarrow$ $a^2e^2 + h^2 - 2aeh + k^2 = a^2 + e2h^2 - 2aeh$

$\Rightarrow$ $h^2(1 - e^2) + k^2 = a^2(1 - e^2) \Rightarrow \frac{h^2}{a^2} - \frac{k^2}{a^2(e^2-1)} = 1.$ $\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; where $a^2(e^2 - 1) = b^2$.

18.1.2 Tracing of Hyperbola

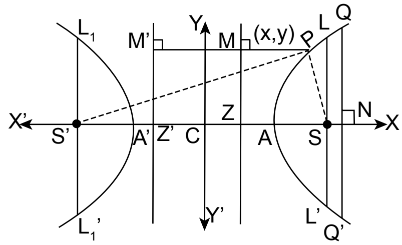
- Equation of hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Eccentricity: $e = \sqrt{1 + \frac{b^2}{a^2}}$
- **Symmetry:** Since equation is even w.r.t. variable x and y so graph is symmetric about both co-ordinate axes. Hence, there should be two foci and two directrix.



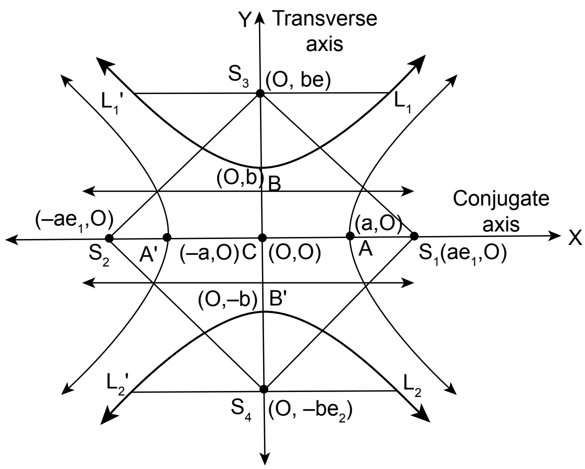
- **Foci:** $S_1(ae, 0)$; $S_2(-ae, 0)$
- Directrices $D_1: x = a/e$; $D_2: x = -a/e$
- Intersection with x-axis: $y = 0 \Rightarrow x = \pm a \Rightarrow A(a, 0),\ A'(-a, 0)$
  AA' is called transverse axis of hyperbola length = 2a, equation y = 0.
- Intersection with y-axis: $x = 0 \Rightarrow y = \pm bi \Rightarrow B(0, b), B'(0, -b)$
  BB' is called conjugate axis length = 2b, equation x = 0.
  The point of intersection of transverse and conjugate is called centre.
- **Normal chord:** Chord normal to transverse axis is called normal chord or double ordinate. If it passes through focus, it is called latus rectum.

  Extremities of Latus rectum: $L_1 = \left(ae, \frac{b^2}{a}\right)$ and $L_1' = \left(ae, -\frac{b^2}{a}\right)$

  Length of $\text{R.R.} = \frac{2b^2}{a}$ equation x = +ae, –ae
- Focal distances $S_1P = ePM = eh - a$; $S_2P = ePM' = eh + a$.
  $|S_2P - S_1P| = 2a$; where 2a is length of transverse axis.
  Case I: If $2a < S_1S_2 = 2ae \Rightarrow$ hyperbola.
  Case II: If $S_1P + S_2P = S_1S_2 \Rightarrow$ union of two rays.
  Case III: If $S_1P + S_2P > S_1S_2 \Rightarrow$ No locus.



Conjugate hyperbola of a hyperbola H = 0 is a hyperbola C = 0 whose transverse axis is conjugate axis of H = 0 and conjugate axis is transverse axis of H = 0 both in the sense of length and equation.

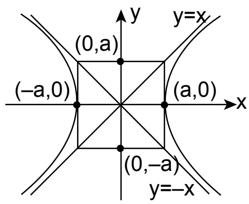


Equation hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

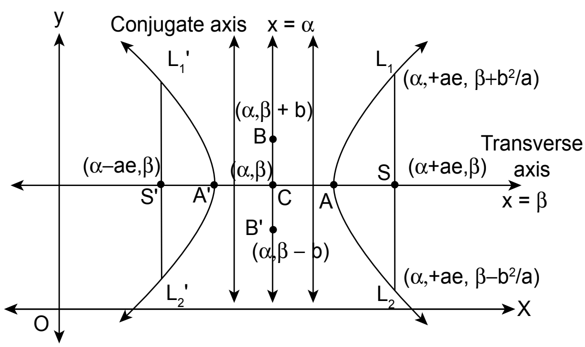
Conjugate hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$

- **Eccentricity:** $e_2 = \sqrt{1 + (a^2/b^2)}$
- Foci: $(0, \pm be_2)$

- **Transverse axis:** $x = 0$, Length $= 2b$
- **Conjugate axis:** $y = 0$, Length $= 2a$
- **Latus Rectum:** $y = \pm be_2$ L′L: $\left(\pm\frac{a^2}{b}, be_2\right)$ and length $= \frac{2a^2}{b}$
- $\frac{1}{e_2^2}+\frac{1}{e_1^2}=\frac{b^2}{a^2+b^2}+\frac{a^2}{a^2+b^2}=1$
- The foci of a hyperbola and its conjugate are con-cylic and form the vertices of a square.
- If $a = b$, hyperbola is said to be equilateral or rectangular and has the equation, $x^2 - y^2 = a^2$. Eccentricity for such a hyperbola is $\sqrt{2}$.

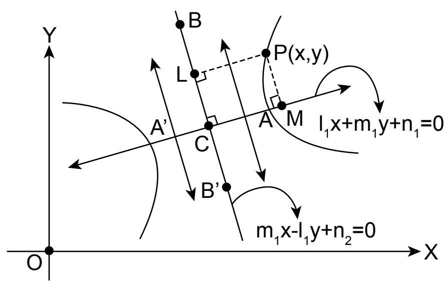


Equation of hyperbola whose centre lies at (α, β) and transverse axis is parallel to x-axis of length $2a$ and conjugate axis of length $2b$. equation: $\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$



- **Transverse axis:** $y = b$, Length $= 2a$
- **Conjugate axis:** $x = a$, Length $= 2b$
- **Foci:** $S_1 = (a + ae, b)$, $S_2 = (a - ae, b)$
- **Directrix:** D_1: $x = a + a/e$, $x = a - a/e$

Equation of Hyperbola Referred to two perpendicular straight lines as their axes, but not parallel to coordinate axes:



$$\frac{\left(\frac{m_1x-l_1y+n_2}{\sqrt{m_1^2+l_1^2}}\right)^2}{a^2}-\frac{\left(\frac{(l_1x+m_1y+n_1)}{\sqrt{l_1^2+m_1^2}}\right)^2}{b^2}=1$$

Centre: C is the point of intersection of line $l_1x + m_1y + n_1 = 0$ and $m_1x - l_1y + n_2 = 0$.

Equations of Directrices: If (x, y) is any point on a directrix, then its ⊥r distance from conjugate axis, i.e., $m_1x - l_1y + n_2 = 0$ is a/e

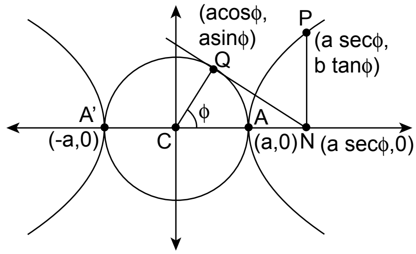
$\therefore$ Equation of directrices are given by $\frac{m_1x-l_1y+n_2}{\sqrt{m_1^2+l_1^2}}=\pm\frac{a}{e}$

Foci: Foci can be obtained by solving the equation $l_1x + m_1y + n_1 = 0$ and the pair of normal chords (Latera Recta) $\frac{m_1x - l_1y + n_2}{\sqrt{m_1^2 + l_1^2}} = \pm ae$.

Length of each Latera Recta $= \frac{2b^2}{a}$. Equations of Latera Recta are given by $\frac{m_1x - l_1y + n_2}{\sqrt{m_1^2 + l_1^2}} = \pm ae$.

18.1.3 Auxiliary Circle of Hyperbola

A circle drawn on transverse axis of the hyperbola as diameter is called auxiliary circle of hyperbola for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ auxiliary circle is given by $x^2 + y^2 = a^2$.

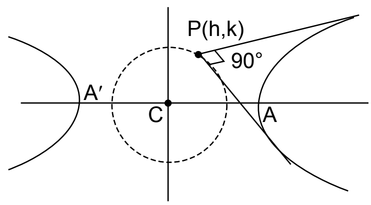


- **Eccentric Angle:** Of any point P on the hyperbola is angle (θ) made by CP′ with positive direction of transverse axis in anticlockwise (where C is centre and P′ is point of contact of tangent drawn from foot of ordinate of P to the Auxiliary circle).
- Parametric Equation $x = a\sec\theta$ and $y = b\tan\theta$; $\theta \in [0, 2\pi) \sim \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and $(a\sec\theta, b\tan\theta)$ is called point θ an the hyperbola.
- The ratio of ordinate of point P on hyperbola and length of tangent from the foot of ordinate (M) to the Auxiliary circle is constant (b/a) $\frac{PM}{P'M} = \frac{b\sin\theta}{a\cos\theta} = \frac{b}{a}$.

18.2 DIRECTOR CIRCLE

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which are perpendicular to each other is called director circle.

The equation of director circle is P(h, k) is $x^2 + y^2 = a^2 - b^2$ $(a > b)$.

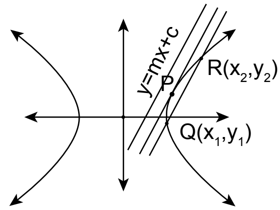


18.2.1 Position of a Point with Respect to Hyperbola

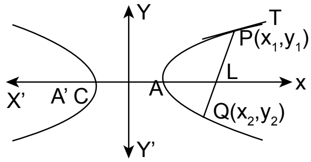
Given hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \Rightarrow y^2 = \frac{b^2}{a^2}(x^2 - a^2)$. A point $P(x_1, y_1)$ lies inside (towards centre)/on/outside (towards focus) of hyperbola as $S_1 < 0/S_1 = 0/S_1 > 0$.

18.2.2 Position of a Line with Respect to Hyperbola $S: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$

The straight line $y = mx + c$ cuts/touches/has no contact with hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as the equation, $b^2x^2 - a^2(mx + c)^2 - a^2b^2 = 0$, has $D > 0/D = 0/D < 0$.



- Condition of tangency $c = \pm\sqrt{a^2m^2 - b^2}$
- Equation of tangent in terms of slope $y = mx \pm \sqrt{a^2m^2 - b^2}$ and point of contact is $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$.
- **Chord of Hyperbola Joining Point θ and ϕ:**
  Equation of chord of hyperbola Joining Point θ and ϕ is $\begin{vmatrix} x & y & 1 \\ a\sec\theta & b\tan\theta & 1 \\ a\sec\phi & b\tan\phi & 1 \end{vmatrix} = 0$ which can also be written as $\frac{x}{a}\cos\frac{\theta-\phi}{2} - \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta+\phi}{2}$.

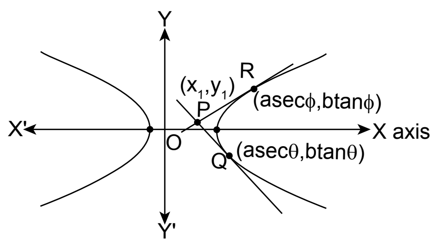


- **Condition for Focal Chord:** Chord becomes focal chord, if it passes through (ae,0) or (–ae, 0). Suppose it passes through (ae,0), then $\Rightarrow \tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\phi}{2}\right) = \frac{1-e}{1+e}$. or $\frac{1+e}{1-e}$ if it passes through (–ae, 0)

18.2.3 Properties of Tangents and Normals

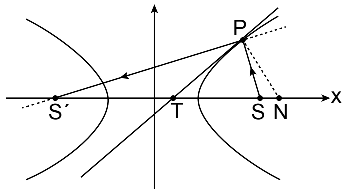
Construction	Slope	Equation
Tangent at (x_1,y_1):	$\frac{b^2x_1}{a^2y_1}$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$
Tangent at θ	$\frac{b}{a}\text{cosec}\theta$	$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$
Normal at (x_1,y_1)	$-\frac{a^2y_1}{b^2x_1}$	$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = \underbrace{a^2+b^2}_{a^2e^2}$
Normal at θ	$-\frac{a}{b}\sin\theta$	$ax\cos\theta + by\cot\theta = \underbrace{a^2+b^2}_{a^2e^2}$

- Point of intersection of tangent at θ and ϕ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $x_1 = a.\frac{\cos\frac{\theta-\phi}{2}}{\cos\frac{\theta+\phi}{2}}$; $y_1 = b\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta+\phi}{2}}$



- Tangent at any point (P) bisects the internal angle and normal at same point bisects the external angle between focal distances of P. This refers to reflection property of the hyperbola which states that rays from one. Focus are reflected such that they appear to be coming from other focus.

- An ellipse and hyperbola if con-focal, always intersect orthogonally.
- **Chord of contact:** $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$



- **Pair of tangents:**

$$SS_1 = T_2 \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

- **Chord with a given middle point:** $T = S_1 \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) \Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

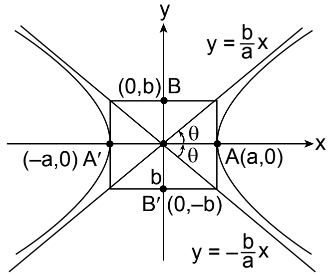
18.2.4 Asymptote Hyperbola

Asymptote to any curve is straight line at finite distance that touches the curve at infinity (μ).

Let $y = mx + c$ be asymptote to hyperbola, then both roots of the equation $(b^2 - a^2m^2)x^2 - 2a^2cmx - a^2(c^2 + b^2) = 0$ approach to μ

$$\Rightarrow \underbrace{b^2 - a^2m^2 = 0}_{\because \text{sum of root infinity}} \quad \text{and} \quad \underbrace{c^2 = a^2m^2 - b^2}_{\text{condition of tangency}} \quad \Rightarrow m = \pm\frac{b}{a} \text{ and } c = 0$$

$$\Rightarrow \begin{cases} y = \dfrac{b}{a}x \\ y = \dfrac{-b}{a}x \end{cases} \text{ are pair of asymptotes.}$$

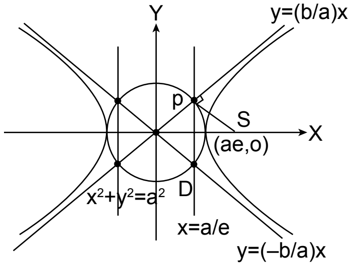


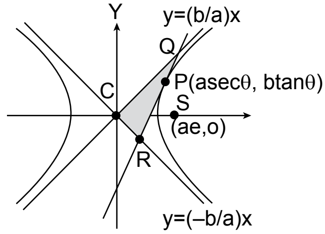
18.2.4.1 Properties of asymptote hyperbola

- Both the asymptotes are pair of tangents to a hyperbola from its centre. Axis of Hyperbola bisects the angle between asymptotes.
- If lines be drawn through A, A′ parallel to C axis and, through B, B′ parallel to T axis, then asymptotes lie along the diagonal of rectangle thus formed.
- Combined equation of asymptotes (A = 0) differs from equation of hyperbola (H = 0) and conjugate hyperbola (C = 0) by same constant, i.e., $A = H + \lambda$ and $A = C - \lambda$.

  As $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ and $A: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.\sqrt{2}$ $C: \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$
- Relation between A, C, H: $A = \frac{C + H}{2}$
- **Angle between Asymptote:** Included angle between two asymptotes is $\tan^{-1}\left(\frac{2ab}{a^2 - b^2}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$ or $2\ \mathrm{Sec}^{-1}(e)$

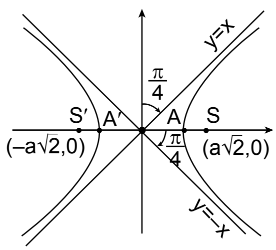
  If the angle asymptotes is 90°, then b = a and hyperbola is called rectangular hyperbola.
- The product of the perpendicular drawn from any point on a hyperbola to its asymptotes is constant.
- The foot of the perpendicular from a focus to an asymptote is a point of intersection of the auxiliary circle and the corresponding directix.





- The portion of any tangent to hyperbola intercepted between asymptote is bisected at the point of contact.
- Any tangent to the hyperbola makes with asymptote a triangle of constant area.

18.3 RECTANGULAR HYPERBOLA



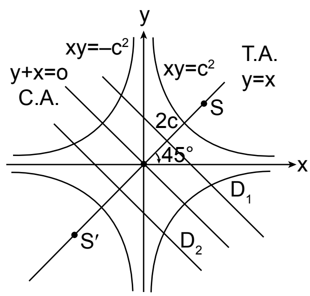
A hyperbola whose asymptotes include a right angle is called rectangular hyperbola; or, if the lengths of transverse and conjugate axes of a hyperbola be equal, it is called rectangular or equilateral hyperbola.

- Equation: $x^2 - y^2 = a^2$.
- TA: y = 0; Length: 2a
- CA: x = 0 , Length: 2a
- Eccentricity: (e) = $\sqrt{2}$
- Foci: $(\pm a\sqrt{2}, 0)$
- Directrix: $x = \pm a/\sqrt{2}$
- Asymptote: y = x and y = x.

18.3.1 Rectangular Hyperbola where Asymptote are Coordinate Axis

Given rectangular hyperbola $x^2 - y^2 = a^2$. If axes rotating by π/4 about the origin.

$$x \to \frac{x+y}{\sqrt{2}} \text{ and } y \to \frac{-x+y}{\sqrt{2}} \text{ the equation transforms to } \frac{(x+y)^2}{2} - \frac{(x-y)^2}{2} = a^2 .$$



- Eccentricity = $\sqrt{2}$
- Transverse axis: Equation: y = x, Length: $2\sqrt{2}c$
- Conjugate axis: Equation: y + x = 0, Length: $2\sqrt{2}c$
- Foci: $S(c\sqrt{2}, c\sqrt{2})$ and $S'(-c\sqrt{2}, -c\sqrt{2})$
- Directrix: $x + y = \pm c\sqrt{2}$
- Parametric equation: x = ct, y = c/t, t ∈ R–{0}.
- Centre: (0, 0)
- Vertex: (c, c) and (– c, –c)

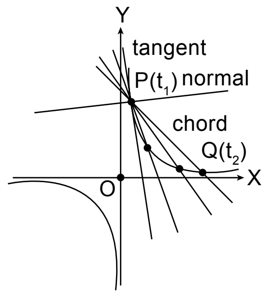
Conjugate Hyperbola of Rectangular Hyperbola $xy = c^2$:

- It is given by: $xy = -c^2$
- Centre: (0, 0)

- Vertex: $(-c, c)$ and $(c, -c)$
- Eccentricity $= \sqrt{2}$
- TA: Equation $y + x = 0$, Length: $2\sqrt{2}c$
- CA: Equation: $y = x$, Length: $2\sqrt{2}c$
- Foci: $S(-c\sqrt{2}, c\sqrt{2})$ and $S'(c\sqrt{2}, -c\sqrt{2})$
- Directrix: $x - y = \pm c\sqrt{2}$
- Parametric equation: $x = ct$, $y = -c/t$, $t \in R - \{0\}$.

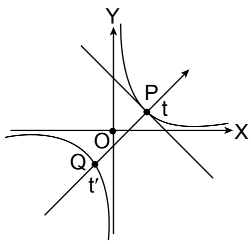
18.3.2 Parametric Equations of Chord, Tangents and Normal

Slope of chord joining the points $P(t_1)$ and $Q(t_2)$: $m = -\frac{1}{t_1 t_2}$.

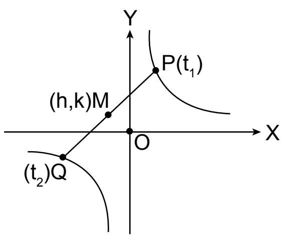


- Equation of chord: $x + t_1 t_2 y = c(t_1 + t_2)$
- Condition for focal chord: $\frac{t_1 + t_2}{1 + t_1 t_2} = \pm\sqrt{2}$
- Equation of the tangent at $P(x_1, y_1)$: $\frac{x}{x_1} + \frac{y}{y_1} = 2$
- Equation of tangent at P(t): $x + yt^2 = 2ct$
- Equation of normal at P (t): $y - \frac{c}{t} = t^2(x - ct)$

  $\Rightarrow$ $xt_3 - yt = c(t_4 - 1)$
- If normal of hyperbola $xy = c^2$ at the point P(T) meet the hyperbola again at T′ the $T^3.T' = -1$.



- Chord with a given middlepoint as (h, k) is $kx + hy = 2hk$.



18.3.3 Co-normal Points

In general, four normals can be drawn on a hyperbola each passing through a common point. The feets of perpendicular of these four normals lying on the hyperbola are called co-normal points.

18.3.3.1 *Properties of co-normal points*

1. In general, four normals can be drawn to a hyperbola from any point and if α, β, γ, δ be the eccentric angles of these four co-nomal points, then α + β + γ + δ is an odd multiple of π.
2. If α, β, γ are the eccentric angles of three points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the normals at which are concurrent, then sin (α + β) + sin (β + γ) + sin (α + γ) = 0.

18.3.3.2 *Diameter of a hyperbola*

The locus of the middle points of a system of parallel chords of a hyperbola is called a diameter.

The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

18.3.3.3 *Conjugate diameters*

Two diameters are said to be conjugate when each bisects all chords parallel to the others.

Two diameters, y = mx and y = kx, are said to be conjugate if their gradients are related as km = b2/a2.

18.3.4 Properties of Conjugate Diameters

- If a pair of diameters are conjugate with respect to a hyperbola, then they are also conjugate with respect to its conjugate hyperbola.
- If a pair of diameters be conjugate with respect to a hyperbola, then one of those diametsrs will meet the hyperbola in real points, while the other diameter will meet the conjugate hyperbola in real points.
- If a pair of conjugate diameters meet the hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right)$ and its conjugate hyperbola $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0\right)$ in P,P′ and D, D′ respective then
  (i) $CP^2 - CD^2 = a^2 - b^2$
  (ii) The parallelogram formed by the tangents at the extremities of conjugate diameters has its vertices lying on the asymptotes and its of constant area.
  (iii) Show that the asymptotes to the hyperbola bisect PD, PD′, P′D and P′D′