7. TRIGONOMETRIC RATIOS, IDENTITIES AND EQUATIONS

1. INTRODUCTION

The equations involving trigonometric functions of unknown angles are known as Trigonometric equations e.g. $\cos\theta = 0, \cos^2\theta - 4\cos\theta = 1, \sin^2\theta + \sin\theta = 2\cos^2\theta - 4\sin\theta = 1$.

2. TRIGONOMETRIC FUNCTIONS (CIRCULAR FUNCTIONS)

Function	Domain	Range
sin A	R	[–1, 1]
cos A	R	[-1, 1]
tan A	$R - \left[(2n+1)\pi/2, n \in I\right]$	$R = (-\infty, \infty)$
cosec A	$R - \left[n\pi, n \in I\right]$	$(-\infty, -1] \cup [1, \infty)$
sec A	$R - \left\{(2n+1)\pi/2, n \in I\right\}$	$(-\infty, -1] \cup [1, \infty)$
cot A	$R - \left[n\pi, n \in I\right]$	$(-\infty, \infty)$

We find, $|\sin A| \le 1, |\cos A| \le 1, \sec A \ge 1$ or $\sec A \le -1$ and $\operatorname{cosec} A \ge 1$ or $\operatorname{cosec} A \le -1$

2.1 Some Basic Formulae of Trigonometric Functions

(a) $\sin^2 A + \cos^2 A = 1.$

(b) $\sec^2 A - \tan^2 A = 1$

(c) $\operatorname{cosec}^2 A - \cot^2 A = 1$

(d) $\sin A \operatorname{cosec} A = \tan A \cot A = \cos A \sec A = 1$

A system of rectangular coordinate axes divide a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The signs of the trigonometric ratios in the four quadrants are shown in Fig 7.1.

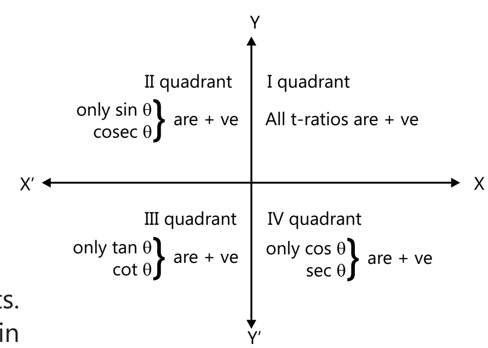

Figure 7.1

PLANCESS CONCEPTS

A crude way to remember the sign is "Add Sugar to Coffee". This implies the 1st letter of each word gives you the trigonometric functions with a +ve sign.

Eg. Add-1st word $\Rightarrow$ 1st quadrant 1st letter=A $\Rightarrow$ All are positive to-3rd word $\Rightarrow$ 3rd quadrant 1st letter-t $\Rightarrow$ tan θ (cot θ) are positive.

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Sine, cosine and tangent of some angles less than 90°:

Trigonometric ratios	0°	15°	18°	30°	36°
sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$
	37°	**45°**	**53°**	**60°**	**90°**
sin	$\approx 3/5$	$\frac{1}{\sqrt{2}}$	$\approx 4/5$	$\frac{\sqrt{3}}{2}$	1
cos	$\approx 4/5$	$\frac{1}{\sqrt{2}}$	$\approx 3/5$	$\frac{1}{2}$	0
tan	$\approx 3/4$	1	$\approx 4/3$	$\sqrt{3}$	Not defined

Illustration 1: Prove the following identities:

(i) $\left(1+\tan^2 A\right)+\left(1+\frac{1}{\tan^2 A}\right)=\frac{1}{\sin^2 A-\sin^4 A}$

(JEE MAIN)

(ii) $\frac{1+\tan^2\theta}{1+\cot^2\theta}=\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2$

Sol: (i) Simply by using Pythagorean and product identities, we can solve these problems.

(i) L.H.S. = $(1 + \tan^2 A) + \left(1+\frac{1}{\tan^2 A}\right) = \sec^2 A + (1 + \cot^2 A)$

$=\sec^2 A + \text{cosec}^2 A = \frac{1}{\cos^2 A}+\frac{1}{\sin^2 A}=\frac{\sin^2 A+\cos^2 A}{\sin^2 A.\cos^2 A}\left[\because \sin^2\theta+\cos^2\theta=1\right]$

$$= \frac{1}{\sin^2 A\left(1-\sin^2 A\right)} = \frac{1}{\sin^2 A - \sin^4 A} = \text{R.H.S.}\left[\because \cos^2\theta = 1-\sin^2\theta\right]$$

Hence proved.

(ii) L.H.S. = $\frac{1+\tan^2\theta}{1+\cot^2\theta} = \frac{\sec^2\theta}{\operatorname{cosec}^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$ ... (i)

Now, R.H.S. = $\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \left(\frac{1-\tan\theta}{1-\frac{1}{\tan\theta}}\right)^2 = \left(\frac{1-\tan\theta}{\frac{\tan\theta-1}{\tan\theta}}\right)^2 = \left(\frac{1-\tan\theta}{-(1-\tan\theta)}.\tan\theta\right)^2 = \tan^2\theta$ ... (ii)

From (i) and (ii), clearly, L.H.S. = R.H.S. **Proved.**

Illustration 2: Prove the following identities: **(JEE MAIN)**

(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A\cos^2 A} - 2 = \sec^2 A\operatorname{cosec}^2 A - 2$

(ii) $\sec^4 A\,(1 - \sin^4 A) - 2\tan^2 A = 1$

Sol: Use algebra and appropriate identities to solve these problems.

(i) $\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{\sin^4 A + \cos^4 A}{\sin^2 A\cos^2 A} = \frac{\left(\sin^2 A\right)^2 + \left(\cos^2 A\right)^2 + 2\sin^2 A\cos^2 A - 2\sin^2 A.\cos^2 A}{\sin^2 A\cos^2 A}$

$$= \frac{\left(\sin^2 A + \cos^2 A\right)^2 - 2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A} = \frac{1-2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}$$

$= \frac{1}{\sin^2 A\cos^2 A} - \frac{2\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A} = \sec^2 A\operatorname{cosec}^2 A - 2$ = R.H.S. **Proved.**

(ii) L.H.S. = $\sec^4 A\,(1 - \sin^4 A) - 2\tan^2 A = \sec^4 A - \frac{\sin^4 A}{\cos^4 A} - 2\tan^2 A = \sec^4 A - \tan^4 A - 2\tan^2 A$

$= \left(1+\tan^2 A\right)^2 - \tan^4 A - 2\tan^2 A = 1 + 2\tan^2 A + \tan^4 A - \tan^4 A - 2\tan^2 A = 1$ = R.H.S. **Proved.**

Illustration 3: Prove the following identities:

(i) $\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \operatorname{cosec}\alpha + \cot\alpha$ (ii) $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \sec\alpha + \tan\alpha$ **(JEE MAIN)**

Sol: By rationalizing L.H.S. we will get required result.

(i) L.H.S. = $\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}\times\frac{1+\cos\alpha}{1+\cos\alpha}} = \sqrt{\frac{(1+\cos\alpha)^2}{1-\cos^2\alpha}}$

$= \sqrt{\frac{(1+\cos\alpha)^2}{\sin^2\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{1}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha} = \operatorname{cosec}\alpha + \cot\alpha = \text{R.H.S.}$ **Proved.**

(ii) L.H.S. = $\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \sqrt{\frac{(1+\sin\alpha)(1+\sin\alpha)}{(1-\sin\alpha)(1+\sin\alpha)}} = \sqrt{\frac{(1+\sin\alpha)^2}{1-\sin^2\alpha}} = \sqrt{\frac{(1+\sin\alpha)^2}{\cos^2\alpha}}$

$= \dfrac{1+\sin\alpha}{\cos\alpha} = \dfrac{1}{\cos\alpha} + \dfrac{\sin\alpha}{\cos\alpha} = \sec\alpha + \tan\alpha = \text{R.H.S.}$ **Proved.**

Illustration 4: In each of the following identities, show that:

(i) $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A.\tan B$ (ii) $\tan^2 A - \tan^2 B = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$ **(JEE ADVANCED)**

Sol: Apply tangent and cotangent identity.

(i) L.H.S. = $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \dfrac{\dfrac{\cos A}{\sin A} + \dfrac{\sin B}{\cos B}}{\dfrac{\cos B}{\sin B} + \dfrac{\sin A}{\cos A}} = \dfrac{\dfrac{\cos A \cos B + \sin A \sin B}{\sin A \cos B}}{\dfrac{\cos A \cos B + \sin A \sin B}{\sin B \cos A}}$

$= \dfrac{\sin B \cos A}{\sin A \cos B} = \left(\dfrac{\cos A}{\sin A}\right)\left(\dfrac{\sin B}{\cos B}\right) = \cot A \tan B = \text{R.H.S.}$ **Proved.**

(ii) L.H.S. = $\tan^2 A - \tan^2 B = \dfrac{\sin^2 A}{\cos^2 A} - \dfrac{\sin^2 B}{\cos^2 B} = \dfrac{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B}{\cos^2 A \cos^2 B}$

$= \dfrac{\sin^2 A\left(1-\sin^2 B\right) - \left(1-\sin^2 A\right)\sin^2 B}{\cos^2 A \cos^2 B} = \dfrac{\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B}{\cos^2 A \cos^2 B} = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B} = \text{R.H.S.}$ **Proved.**

Illustration 5: Prove the following identities: $\dfrac{1}{\operatorname{cosec}\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}$ **(JEE ADVANCED)**

Sol: By rearranging terms we will get $\dfrac{1}{\operatorname{cosec}\theta - \cot\theta} + \dfrac{1}{\operatorname{cosec}\theta + \cot\theta} = \dfrac{2}{\sin\theta}$, and then using Pythagorean identity we can solve this problem.

We have, $\dfrac{1}{\operatorname{cosec}\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}$

$\Rightarrow \dfrac{1}{\operatorname{cosec}\theta - \cot\theta} + \dfrac{1}{\operatorname{cosec}\theta + \cot\theta} = \dfrac{1}{\sin\theta} + \dfrac{1}{\sin\theta} \Rightarrow \dfrac{1}{\operatorname{cosec}\theta - \cot\theta} + \dfrac{1}{\operatorname{cosec}\theta + \cot\theta} = \dfrac{2}{\sin\theta}$

Now, L.H.S. = $\dfrac{1}{\operatorname{cosec}\theta - \cot\theta} + \dfrac{1}{\operatorname{cosec}\theta + \cot\theta} = \dfrac{\operatorname{cosec}\theta + \cot\theta + \operatorname{cosec}\theta - \cot\theta}{(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta)}$

$= \dfrac{2\operatorname{cosec}\theta}{\left(\operatorname{cosec}^2\theta - \cot^2\theta\right)} \quad \left[\because \operatorname{cosec}^2\theta - \cot^2\theta = 1\right]$

$= \dfrac{2\operatorname{cosec}\theta}{1} = \dfrac{2}{\sin\theta} = \text{R.H.S.} \quad \left[\because \operatorname{cosec}\theta = \dfrac{1}{\sin\theta}\right]$ **Proved.**

Alternative Method

R.H.S $= \dfrac{1}{\sin\theta} - \dfrac{1}{\operatorname{cosec}\theta + \cot\theta} = \operatorname{cosec}\theta - \dfrac{(\operatorname{cosec}\theta - \cot\theta)}{\operatorname{cosec}^2\theta - \cot^2\theta}$

$= \operatorname{cosec}\theta - \operatorname{cosec}\theta + \cot\theta$

$= \cot\theta$ **Proved.**

Illustration 6: Prove that:

(i) $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}=\sin^2 A.\cos^2 A$

(ii) $\dfrac{\sin A}{\sec A+\tan A-1}+\dfrac{\cos A}{\operatorname{cosec}A+\cot A-1}=1$ **(JEE ADVANCED)**

Sol: Using algebra and appropriate identities, we can prove this.

(i) L.H.S. = $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$

$$=\frac{\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)}{(\sec A-\operatorname{cosec}A)(\sec^2 A+\sec A\operatorname{cosec}A+\operatorname{cosec}^2 A)}\quad\left[\because a^3-b^3=(a-b)(a^2+ab+b^2)\right]$$

$$=\frac{\frac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A}}{(\sec A-\operatorname{cosec}A)\left(\frac{1}{\cos^2 A}+\frac{1}{\cos A\sin A}+\frac{1}{\sin^2 A}\right)}=\frac{(\sin A\cos A+1)\left(\frac{\sin A}{\sin A\cos A}-\frac{\cos A}{\sin A\cos A}\right)}{(\sec A-\operatorname{cosec}A)\left(\frac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin^2 A\cos^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)(\sec A-\operatorname{cosec}A)}{(\sec A-\operatorname{cosec}A)(1+\sin A\cos A)}\times\sin^2 A\cos^2 A\quad\left[\because \sin^2\theta+\cos^2\theta=1\right]=\sin^2 A\cos^2 A=\text{R.H.S.}$$ **Proved.**

(ii) L.H.S. = $\dfrac{\sin A}{\sec A+\tan A-1}+\dfrac{\cos A}{\operatorname{cosec} A+\cot A-1}$

$$=\frac{\sin A\operatorname{cosec}A+\sin A\cot A-\sin A+\cos A\sec A+\cos A\tan A-\cos A}{(\sec A+\tan A-1)(\operatorname{cosec}A+\cot A-1)}$$

$$=\frac{1+\cos A-\sin A+1+\sin A-\cos A}{\left(\frac{1}{\cos A}+\frac{\sin A}{\cos A}-1\right)\left(\frac{1}{\sin A}+\frac{\cos A}{\sin A}-1\right)}=\frac{2}{\left(\frac{1+\sin A-\cos A}{\cos A}\right)\frac{(1+\cos A-\sin A)}{\sin A}}$$

$$=\frac{2\sin A\cos A}{[1+(\sin A-\cos A)][1-(\sin A-\cos A)]}=\frac{2\sin A\cos A}{1-(\sin A-\cos A)^2}$$

$$=\frac{2\sin A\cos A}{1-(\sin^2 A+\cos^2 A-2\sin A\cos A)}\quad\left[\because (a+b)(a-b)=a^2-b^2\right]$$

$$=\frac{2\sin A\cos A}{1-(1-2\sin A\cos A)}=\frac{2\sin A\cos A}{1-1+2\sin A\cos A}\quad\left[\because \sin^2\theta+\cos^2\theta=1\right]=\frac{2\sin A\cos A}{2\sin A\cos A}=1=\text{R.H.S.}$$ **Proved.**

Illustration 7: Prove that:

$$\left(\frac{1}{\sec^2\theta-\cos^2\theta}+\frac{1}{\operatorname{cosec}^2\theta-\sin^2\theta}\right)\sin^2\theta.\cos^2\theta=\frac{1-\sin^2\theta.\cos^2\theta}{2+\sin^2\theta.\cos^2\theta}$$ **(JEE ADVANCED)**

Sol: Write L.H.S. in terms of cosine and sine functions.

$$\text{L.H.S.} = \left(\frac{1}{\sec^2\theta - \cos^2\theta} + \frac{1}{\operatorname{cosec}^2 - \theta\sec^2\theta}\right)\sin^2\theta.\cos^2\theta$$

$$= \left[\frac{1}{\frac{1}{\cos^2\theta} - \cos^2\theta} + \frac{1}{\frac{1}{\sin^2\theta} - \sin^2\theta}\right]\sin^2\theta\cos^2\theta = \left[\frac{\cos^2\theta}{1-\cos^4\theta} + \frac{\sin^2\theta}{1-\sin^4\theta}\right]\sin^2\theta\cos^2\theta$$

$$= \left[\frac{\cos^2\theta}{(1+\cos^2\theta)(1-\cos^2\theta)} + \frac{\sin^2\theta}{(1-\sin^2\theta)(1+\sin^2\theta)}\right]\sin^2\theta\cos^2\theta \qquad \left[\because a^2 - b^2 = (a-b)(a+b)\right]$$

$$= \left[\frac{\cos^2\theta}{(1+\cos^2\theta)\sin^2\theta} + \frac{\sin^2\theta}{\cos^2\theta(1+\sin^2\theta)}\right]\sin^2\theta\cos^2\theta$$

$$= \frac{\cos^4\theta}{1+\cos^2\theta} + \frac{\sin^4\theta}{1+\sin^2\theta} = \frac{\left[\cos^4\theta(1+\sin^2\theta) + \sin^4\theta(1+\cos^2\theta)\right]}{(1+\cos^2\theta)(1+\sin^2\theta)}$$

$$= \frac{\cos^4\theta + \sin^2\theta\cos^4\theta + \sin^4\theta + \sin^4\theta\cos^2\theta}{(1+\cos^2\theta)(1+\sin^2\theta)} = \frac{\sin^4\theta + \cos^4\theta + \sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)}{(1+\cos^2\theta)(1+\sin^2\theta)}$$

$$= \frac{(\sin^2\theta)^2 + (\cos^2\theta)^2 + 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta}{(1+\cos^2\theta)(1+\sin^2\theta)}$$

$$= \frac{(\sin^2\theta + \cos^2\theta) - \sin^2\theta\cos^2\theta}{1 + \sin^2\theta + \cos^2\theta + \sin^2\theta\cos^2\theta} = \frac{1 - \sin^2\theta\cos^2\theta}{1 + 1 + \sin^2\theta\cos^2\theta} = \frac{1-\sin^2\theta\cos^2\theta}{2 + \sin^2\cos^2\theta} = \text{R.H.S.}$$

Proved.

3. TRANSFORMATIONS

3.1 Compound, Multiple and Sub-Multiple Angles

Circular functions of the algebraic sum of two angles can be expressed as circular functions of separate angles.

$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$; $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A.\tan B}; \qquad \cot(A \pm B) = \frac{\cot A\cot B \mp 1}{\cot B \pm \cot A}$$

Circular functions of multiples of an angle can be expressed as circular functions of the angle.

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A} = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}; \qquad \sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A; \qquad \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Circular functions of half of an angle can be expressed as circular functions of the complete angle.

$$\sin\frac{A}{2}=\sqrt{\frac{1-\cos A}{2}}\ ;\ \cos\frac{A}{2}=\sqrt{\frac{1+\cos A}{2}}\ ;\quad \tan\frac{A}{2}=\sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{1-\cos A}{\sin A}=\frac{\sin A}{1+\cos A}$$

3.2 Complementary and Supplementary Angles

$\sin(-\theta)=-\sin\theta$

$\cos(-\theta)=\cos\theta$

$\tan(-\theta)=-\tan\theta$

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$

$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$

$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$

$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$

$\sin(\pi-\theta)=\sin\theta$

$\cos(\pi-\theta)=-\cos\theta$

$\tan(\pi-\theta)=-\tan\theta$

$\sin(\pi+\theta)=-\sin\theta$

$\cos(\pi+\theta)=-\cos\theta$

$\tan(\pi+\theta)=\tan\theta$

3.3 Product to Sum and Sum to Product

$\sin C+\sin D=2\sin\frac{C+D}{2}.\cos\frac{C-D}{2}$; $\quad \sin C-\sin D=2\cos\frac{C+D}{2}.\sin\frac{C-D}{2}$

$\cos C+\cos D=2\cos\frac{C+D}{2}.\cos\frac{C-D}{2}$; $\quad \cos C-\cos D=2\sin\frac{C+D}{2}.\sin\frac{D-C}{2}$

Note: $\sin C+\cos D=\sin C+\sin\left(\frac{\pi}{2}-D\right)=2\sin\frac{C+\frac{\pi}{2}-D}{2}.\cos\frac{C-\frac{\pi}{2}+D}{2}$

$\tan C+\tan D=\frac{\sin C}{\cos C}+\frac{\sin D}{\cos D}=\frac{\sin(C+D)}{\cos C.\cos D}$; $\quad \sin A.\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$\sin A.\sin B=\frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$; $\quad \cos A.\cos B=\frac{1}{2}\{\cos(A-B)+\cos(A+B)\}$

$\sin(A+B).\sin(A-B)=\sin^2 A-\sin^2 B$; $\quad \cos(A+B).\cos(A-B)=\cos^2 A-\sin^2 B$

3.4 Power Reduction

$\sin^2 A=\frac{1}{2}(1-\cos 2A)\qquad \cos^2 A=\frac{1}{2}(1+\cos 2A)$

$\tan^2 A=\frac{1-\cos 2A}{1+\cos 2A}$; $\quad \sin^3 A=\frac{3\sin A-\sin 3A}{4}$; $\quad \cos^3 A=\frac{3\cos A+\cos 3A}{4}$

PLANCESS CONCEPTS

- $\cos A.\cos 2A.\cos 2^2.A\cos 2^3 A......\cos 2^{n-1} A = \begin{cases} \dfrac{\sin 2^n A}{2^n \sin A} & \text{if} \quad A \neq n\pi \\ 1 & \text{if} \quad A = 2n\pi \\ -1 & \text{if} \quad A = (2n+1)\pi \end{cases}$
- $\sin\left(A_1 + A_2 + ... + A_n\right) = \cos A_1 \cos A_2 .... \cos A_n \left(S_1 - S_3 + S_5 - S_7 + ...\right)$
- $\cos\left(A_1 + A_2 + ... + A_n\right) = \cos A_1 \cos A_2 .... \cos A_n \left(1 - S_2 + S_4 - S_6 ....\right)$
- $\tan\left(A_1 + A_2 + ... + A_n\right) = \dfrac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}$

Where,

$S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = The sum of the tangents of the separate angles.

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + ...$ = The sum of the tangents taken two at a time.

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ...$ = Sum of tangents three at a time, and so on.

If $A_1 = A_2 = ... = A_n = A$, then $S_1 = n \tan A$, $S_2 = {}^nC_2 \tan^2 A$. $S_3 = {}^nC_3 \tan^3 A$, ...

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4. TRIGONOMETRIC IDENTITY

A trigonometric equation is said to be an identity if it is true for all values of the angle or angles involved. A given identity may be established by (i) Reducing either side to the other one, or (ii) Reducing each side to the same expression, or (iii) Any convenient, modification of the methods given in (i) and (ii).

4.1 Conditional Identity

When the angles, A, B and C satisfy a given relation, we can establish many interesting identities connecting the trigonometric functions of these angles. To prove these identities, we require the properties of complementary and supplementary angles. For example, if A + B + C = π, then

1. $\sin\left(B + C\right) = \sin A, \cos B = -\cos\left(C + A\right)$
2. $\cos\left(A + B\right) = -\cos C, \sin C = \sin\left(A + B\right)$
3. $\tan\left(C + A\right) = -\tan B, \cot A = -\cot\left(B + C\right)$
4. $\cos\dfrac{A+B}{2} = \sin\dfrac{C}{2}, \cos\dfrac{C}{2} = \sin\dfrac{A+B}{2}$
5. $\sin\dfrac{C+A}{2} = \cos\dfrac{B}{2}, \sin\dfrac{A}{2} = \cos\dfrac{B+C}{2}$
6. $\tan\dfrac{B+C}{2} = \cot\dfrac{A}{2}, \tan\dfrac{B}{2} = \cot\dfrac{C+A}{2}$

Some Important Identities: If $A + B + C = \pi$, then

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$
3. $\tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} + \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1$
4. $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

5. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

6. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

8. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

9. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustration 8: Show that:

(i) $\sin(40^\circ+\theta)\cos(10^\circ+\theta) - \cos(40^\circ+\theta)\sin(10^\circ+\theta) = \frac{1}{2}$

(ii) $\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\phi\right) - \sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}-\phi\right) = \sin(\theta+\phi)$ **(JEE MAIN)**

Sol: Use sum and difference formulae of sine and cosine functions.

(i) L.H.S. = $\sin(40^\circ+\theta)\cos(10^\circ+\theta) - \cos(40^\circ+\theta)\sin(10^\circ+\theta)$

$= \sin\{(40^\circ+\theta)-(10^\circ+\theta)\}$ $[\because \sin(A-B) = \sin A\cos B - \cos A\sin B]$ $= \sin 30^\circ = \frac{1}{2} = $ R.H.S. **Proved.**

(ii) L.H.S. = $\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\phi\right) - \sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}-\phi\right)$

$= \cos\left\{\left(\frac{\pi}{4}-\theta\right)+\left(\frac{\pi}{4}-\phi\right)\right\}$ $[\because \cos(A+B) = \cos A\cos B - \sin A\sin B]$ $= \cos\left\{\frac{\pi}{2}-(\theta+\phi)\right\}$

$= \sin(\theta+\phi) = $ R.H.S. $\left[\because \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\right]$ **Proved**

Illustration 9: Find the value of $\tan(\alpha+\beta)$, given that $\cot\alpha = \frac{1}{2}, \alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = -\frac{5}{3}, \beta \in \left(\frac{\pi}{2}, \pi\right)$. **(JEE MAIN)**

Sol: As we know, $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha.\tan\beta}$, therefore by using product and Pythagorean identities we can obtain the values of $\tan\alpha$ and $\tan\beta$.

Given, $\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$

Also, $\sec\beta = -\frac{5}{3}$. Then $\tan\beta = \sqrt{\sec^2\beta - 1} = \pm\sqrt{\frac{25}{9}-1} = \pm\frac{4}{3}$

But $\beta \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \tan\beta = -\frac{4}{3}$ $[\because \tan\beta \text{ is } -\text{ve in II quadrant}]$

Substituting $\tan\alpha = 2$ and $\tan\beta = -\frac{4}{3}$ in (1), we get $\tan(\alpha+\beta) = \frac{2+\left(-\frac{4}{3}\right)}{1-(2)\left(-\frac{4}{3}\right)} = \frac{+\frac{2}{3}}{\frac{11}{3}} = +\frac{2}{11}$

Illustration 10: Prove that: $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$ **(JEE MAIN)**

Sol: Here we can write $\tan 3A$ as $\tan(2A+A)$, and then by using $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ we can solve this problem.

We have: 3A = 2A + A $\Rightarrow \tan 3A = \tan(2A+A) \Rightarrow \tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$\Rightarrow \tan 3A(1-\tan 2A \tan A) = \tan 2A + \tan A \ \Rightarrow \tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$

$\Rightarrow \tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$ **Proved.**

Illustration 11: Prove that: $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$ **(JEE MAIN)**

Sol: Use $1+\cos 2\theta = 2\cos^2\theta$, to solve this problem.

L.H.S. = $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}}$ $[\because 1+\cos 2\theta = 2\cos^2\theta]$

$= \sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}} = \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} = \sqrt{2+2\cos 2\theta}$

$= \sqrt{2(1+\cos 2\theta)} = \sqrt{2.2\cos^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$ = R.H.S. **Proved.**

Illustration 12: If $\tan A = \frac{m}{m-1}$ and $\tan B = \frac{1}{2m-1}$, prove that $A - B = \frac{\pi}{4}$ **(JEE ADVANCED)**

Sol: Simply using $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A.\tan B}$, we can prove above equation.

We have, $\tan A = \frac{m}{m-1}$ and $\tan B = \frac{1}{2m-1}$

Now, $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A.\tan B}$ ... (i)

Substituting the values of tan A and tan B in (i), we get

$$\tan(A-B) = \frac{\frac{m}{m-1} - \frac{1}{2m-1}}{1+\left(\frac{m}{m-1}\right)\left(\frac{1}{2m-1}\right)} = \frac{2m^2-m-m+1}{(m-1)(2m-1)} \times \frac{(m-1)(2m-1)}{2m^2-3m+1+m} = 1$$

$\Rightarrow \tan(A-B) = \tan\frac{\pi}{4}$ $\left[\because \tan\frac{\pi}{4} = 1\right]$ $\Rightarrow A - B = \frac{\pi}{4}$ **Proved.**

Illustration 13: If $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$; prove that $\tan(\alpha-\beta) = (1-n)\tan\alpha$ **(JEE ADVANCED)**

Sol: Same as above problem $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$, therefore by substituting

$\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, we can prove given equation.

L.H.S. = $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$ ... (i)

Substituting $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$ in (i), we get L.H.S. = $\frac{\frac{\sin\alpha}{\cos\alpha} - \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\frac{\sin\alpha}{\cos\alpha}.\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}$ $\left[\because \tan\alpha = \frac{\sin\alpha}{\cos\alpha}\right]$

$$= \frac{\sin\alpha\left(1-n\sin^2\alpha\right)-n\sin\alpha\cos^2\alpha}{\cos\alpha\left(1-n\sin^2\alpha\right)+n\sin^2\alpha\cos\alpha} = \frac{\sin\alpha-n\sin^3\alpha-n\sin\alpha\cos^2\alpha}{\cos\alpha-n\sin^2\alpha\cos\alpha+n\sin^2\alpha\cos\alpha}$$

$$= \frac{\sin\alpha-n\sin\alpha\left(\sin^2\alpha+\cos^2\alpha\right)}{\cos\alpha} = \frac{\sin\alpha-n\sin\alpha}{\cos\alpha}$$

$\left[\because \sin^2\alpha+\cos^2\alpha=1\right]$ $= \frac{(1-n)\sin\alpha}{\cos\alpha} = (1-n)\tan\alpha = \text{R.H.S.}$ **Proved.**

Illustration 14: If $\theta+\phi=\alpha$ and $\sin\theta=k\sin\phi$, prove that $\tan\theta=\frac{k\sin\alpha}{1+k\cos\alpha}, \tan\phi=\frac{\sin\alpha}{k+\cos\alpha}$ **(JEE ADVANCED)**

Sol: Here $\phi=\alpha-\theta$, substitute this in $\sin\theta=k\sin\phi$ and then use compound angle formula to obtain required result.

We have, $\theta+\phi=\alpha \Rightarrow \phi=\alpha-\theta$ ... (i)

and $\sin\theta=k\sin\phi$ ... (ii)

$\Rightarrow \sin\theta=k\sin(\alpha-\theta)$ [Using (i)] $=k\left[\sin\alpha\cos\theta-\cos\alpha\sin\theta\right]$

$\Rightarrow \sin\theta=k\sin\alpha\cos\theta-k\cos\alpha\sin\theta$ ... (iii)

Dividing both sides of (iii) by $\cos\theta$, we get $\tan\theta=k\sin\alpha-k\cos\alpha.\tan\theta$

$\Rightarrow \tan\theta+k\cos\alpha.\tan\theta=k\sin\alpha \Rightarrow \tan\theta(1+k\cos\alpha)=k\sin\alpha \Rightarrow \tan\theta=\frac{k\sin\alpha}{1+k\cos\alpha}$ **Proved.**

Again, $\sin\theta=k\sin\phi$ $\Rightarrow \sin(\alpha-\phi)=k\sin\phi$ $\left[\theta+\phi=\alpha\Rightarrow\theta=\alpha-\phi\right]$

$\Rightarrow \sin\alpha\cos\phi-\cos\alpha\sin\phi=k\sin\phi$ ... (iv)

Dividing both side of (iv) by $\cos\phi$, we get

$\Rightarrow \sin\alpha-\cos\alpha\tan\phi=k\tan\phi \Rightarrow (k+\cos\alpha)\tan\phi=\sin\alpha \Rightarrow \tan\phi=\frac{\sin\alpha}{k+\cos\alpha}$ **Proved.**

Illustration 15: Prove that: $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$ **(JEE ADVANCED)**

Sol: Use $\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, to solve this problem.

L.H.S = $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$ = $(\cos\alpha+\cos\beta)+\left[\cos\gamma+\cos(\alpha+\beta+\gamma)\right]$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right).\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+2\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\} = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right)$ = R.H.S. **Proved.**

Illustration 16: If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, then show that xy + yz + zx = 0. **(JEE ADVANCED)**

Sol: Consider $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$, obtain the value of x , y and z in terms of k, and solve L.H.S. of given equation.

Let $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$ ... (i)

$\Rightarrow \frac{1}{x} = \frac{\cos\theta}{k}, \frac{1}{y} = \frac{\cos\left(\theta+\frac{2\pi}{3}\right)}{k}, \frac{1}{z} = \frac{\cos\left(\theta+\frac{4\pi}{3}\right)}{k}$

Now, L.H.S. = xy + yz + zx = $\frac{xyz}{z}+\frac{xyz}{x}+\frac{xyz}{y} = xyz\left(\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right)$

$= xyz\left[\frac{\cos\left(\theta+\frac{4\pi}{3}\right)}{k}+\frac{\cos\theta}{k}+\frac{\cos\left(\theta+\frac{2\pi}{3}\right)}{k}\right]$ [Using (i)] $= \frac{xyz}{k}\left[\cos\left(\theta+\frac{4\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\theta\right]$

$= \frac{xyz}{k}\left[2\cos\frac{2\theta+2\pi}{2}\cos\frac{\pi}{3}+\cos\theta\right] = \frac{xyz}{k}\left[2\cos(\pi+\theta)\cos\frac{\pi}{3}+\cos\theta\right] = \frac{xyz}{k}\left[-2\cos\theta.\left(\frac{1}{2}\right)+\cos\theta\right]$

$= \frac{xyz}{k}\left[-\cos\theta+\cos\theta\right] = \frac{xyz}{k}[0] = 0 \Rightarrow xy+zy+zx = 0$ **Proved.**

Illustration 17: Prove that: $\cos\theta\cos2\theta\cos4\theta......\cos2^{n-1}\theta = \frac{\sin(2^n\theta)}{2^n(\sin\theta)}$ **(JEE ADVANCED)**

Sol: Multiply and divide L.H.S. by $2\sin\theta$ and apply $\sin(2\theta) = 2\sin\theta\cos\theta$.

Here, we observe that each angle in L.H.S. is double of the preceding angle.

L.H.S. = $\cos\theta\cos2\theta\cos4\theta......\cos2^{n-1}\theta$

$= \frac{1}{2\sin\theta}(2\sin\theta.\cos\theta)\cos2\theta.\cos4\theta.....\cos2^{n-1}\theta = \frac{1}{2^2\sin\theta}(2\sin2\theta.\cos2\theta)(\cos4\theta......\cos2^{n-1}\theta)$

$= \frac{1}{2^3\sin\theta}(2\sin4\theta.\cos4\theta)\left[\cos8\theta\cos16\theta......\cos2^{n-1}\theta\right]\left[\because \sin2n\theta = 2\sin n\theta\cos n\theta\right]$

$= \frac{1}{2^4\sin\theta}(2\sin8\theta.\cos8\theta)\left[\cos16\theta......\cos2^{n-1}\theta\right] = \frac{1}{2^n\sin\theta}\left[2\sin2^{n-1}\theta\cos2^{n-1}\theta\right] = \frac{\sin(2^n\theta)}{2^n\sin\theta}$ = R.H.S. **Proved.**

Illustration 18: If $\cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$, prove that $\tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$ **(JEE ADVANCED)**

Sol: Substitute $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ and $\cos\phi = \frac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}$ in given equation i.e. $\cos\theta = \frac{a\cos\phi+b}{a+b\cos\phi}$.

Now, $\cos\theta = \dfrac{a\cos\phi + b}{a + b\cos\phi}$ $\Rightarrow \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \dfrac{a.\left(\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)+b}{a+b\left(\dfrac{1-\tan^2\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}}\right)}$ [Using (i)]

$$\Rightarrow \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{a\left[1-\tan^2\frac{\phi}{2}\right]+b\left[1+\tan^2\frac{\phi}{2}\right]}{a\left[1+\tan^2\frac{\phi}{2}\right]+b\left[1-\tan^2\frac{\phi}{2}\right]} = \frac{a-a\tan^2\frac{\phi}{2}+b+b\tan^2\frac{\phi}{2}}{a+a\tan^2\frac{\phi}{2}+b-b\tan^2\frac{\phi}{2}}$$

Applying componendo and dividendo, we get

$$\frac{2\tan^2\frac{\theta}{2}}{2} = \frac{2a\tan^2\left(\frac{\phi}{2}\right)-2b\tan^2\left(\frac{\phi}{2}\right)}{2a+2b} = \frac{(a-b)\tan^2\frac{\phi}{2}}{a+b} \Rightarrow \tan\frac{\theta}{2} = \sqrt{\frac{a-b}{a+b}}\tan\frac{\phi}{2}$$

Proved

5. SOLUTION OF TRIGONOMETRIC EQUATION

A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.

Eg.: if $\sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}, ...$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature). These solutions can be classified as:

(i) Principal solution (ii) General solution

5.1 Principal Solution

The solutions of a trigonometric equation which lie in the interval $[-\pi, \pi)$ are called principal solutions.

Methods for Finding Principal Value

Suppose we have to find the principal value of θ satisfying the equation $\sin\theta = -\dfrac{1}{2}$. Since $\sin\theta$ is negative, θ will be in 3rd or 4th quadrant. We can approach the 3rd and the 4th quadrant from two directions. Following the anticlockwise direction, the numerical value of the angle will be greater than π. The clockwise approach would result in the angles being numerically less than π. To find the principal value, we have to take the angle which is numerically smallest.

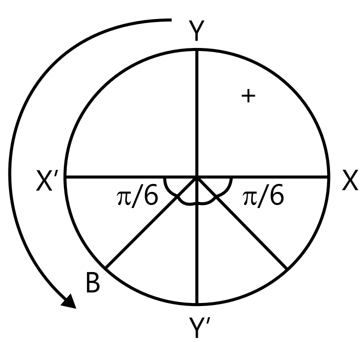


Figure 7.2

For Principal Value

(a) If the angle is in the 1st or 2nd quadrant, we must select the anticlockwise direction and if the angles are in the 3rd or 4th quadrant, we must select the clockwise direction.

(b) Principal value is never numerically greater than π.

(c) Principal value always lies in the first circle (i.e. in first rotation)

On the above criteria, θ will be $-\frac{\pi}{6}$ or $-\frac{5\pi}{6}$. Among these two $-\frac{\pi}{6}$ has the least numerical value. Hence $-\frac{\pi}{6}$ is the principal value of θ satisfying the equation $\sin\theta = -\frac{1}{2}$

From the above discussion, the method for finding principal value can be summed up as follows:

(a) First identify the quadrants in which the angle lies.

(b) Select the anticlockwise direction for the 1st and 2nd quadrants and select clockwise direction for the 3rd and 4th quadrants.

(c) Find the angle in the first rotation.

(d) Select the numerically least value among these two values. The angle thus found will be the principal value.

(e) In case, two angles, one with a positive sign and the other with a negative sign have the same numerical value, then it is the convention to select the angle with the positive sign as the principal value.

5.2 General Solution

The expression which gives all solutions of a trigonometric equation is called a General Solution.

General Solution of Trigonometric Equations

In this section we shall obtain the general solutions of trigonometric equations

$\sin\theta = 0, \cos\theta = 0, \tan\theta = 0$ and $\cot\theta = 0$.

General Solution of $\sin\theta = 0$

By Graphical approach:

The graph clearly shows that $\sin\theta = 0$ at

$\theta = 0, \pi, 2\pi, \ldots\ldots, -\pi, -2\pi \ldots\ldots$

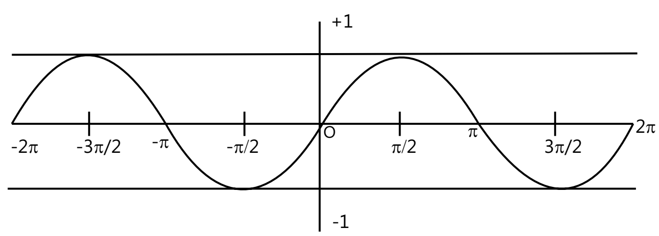


Figure 7.3

So the general solution of $\sin\theta = 0$ is $\theta = n\pi : n \in I$ where n = 0, ± 1, ± 2 ....

Note: Trigonometric functions are periodic functions. Therefore, solutions of trigonometric equations can be generalized with the help of periodicity of trigonometric functions.

PLANCESS CONCEPTS

A trigonometric identity is satisfied by any value of an unknown angle while a trigonometric equation is satisfied by certain values of the unknown.

Vaibhav Krishnan (JEE 2009, AIR 22)

Method for Finding Principal Value

(a) First note the quadrants in which the angle lies.

(b) For the 1st and 2nd quadrants, consider the anticlockwise direction. For the 3rd and 4th quadrants, consider the clockwise direction.

(c) Find the angles in the 1st rotation.

(d) Select the numerically least value among these two values. The angle found will be the principal value.

Illustration 19: Principal value of $\tan\theta = -1$ is **(JEE MAIN)**

Sol: Solve it by using above mentioned method.

$\because \tan\theta$ is negative

$\therefore \theta$ will lie in 2nd or 4th quadrant

For the 2nd quadrant, we will choose the anticlockwise direction and for the 4th quadrant, we will select the clockwise direction.

In the first circle, two values $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are obtained.

Among these two, $-\frac{\pi}{4}$ is numerically least angle.

Hence, the principal value is $-\frac{\pi}{4}$

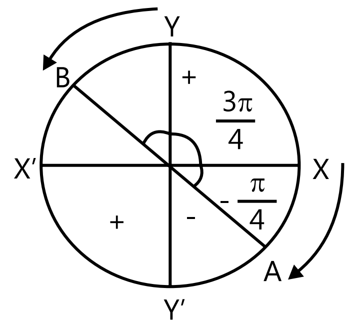


Figure 7.4

Illustration 20: Principal value of cos $\theta = \frac{1}{2}$ is: **(JEE MAIN)**

Sol: Here $\cos\theta$ is $(+)$ve hence θ will lie in 1st or 4th quadrant.

$\because \cos\theta$ is $(+)$ve $\therefore \theta$ will lie in the 1st or the 4th quadrant.

For the 1st quadrant, we will select the anticlockwise direction and for the 4th quadrant, we will select the clockwise direction.

As a result, in the first circle, two values $\frac{\pi}{3}$ and $\frac{-\pi}{3}$ are found.

Both $\frac{\pi}{3}$ and $-\frac{\pi}{3}$ have the same numerical value.

In such a case, $\frac{\pi}{3}$ will be selected as the principal value, as it has a positive sign.

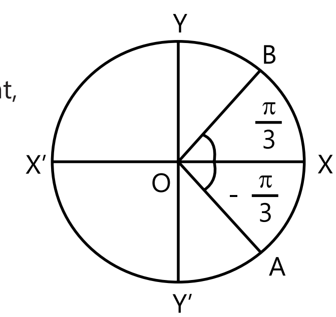


Figure 7.5

Illustration 21: Find the general solutions of the following equations:

(i) $\sin 2\theta = 0$ (ii) $\cos\left(\frac{3}{2}\theta\right) = 0$ (iii) $\tan^2 2\theta = 0$ **(JEE MAIN)**

Sol: By using above mentioned method of finding general solution we can solve these equation.

(i) We have, $\sin 2\theta = 0 \Rightarrow 2\theta = n\pi \Rightarrow \theta = \frac{n\pi}{2}$ where, $n = 0, \pm 1, \pm 2, \pm 3$......

Hence, the general solution of $\sin 2\theta = 0$ is $\theta = \frac{n\pi}{2}, n \in Z$

(ii) We know that, the general solution of the equation $\cos\theta = 0$ is $\theta = (2n+1)\frac{\pi}{2}, n \in Z$

Therefore, $\cos\left(\frac{3\theta}{2}\right) = 0 \Rightarrow \frac{3\theta}{2} = (2n+1)\frac{\pi}{2} \Rightarrow \theta = (2n+1)\frac{\pi}{3}$, where n = 0, ± 1, ± 2 ....

Which is the general solution of $\cos\left(\frac{3\theta}{2}\right) = 0$

(iii) We know that the general solution of the equation $\tan\theta = 0$ is $\theta = n\pi$, $n \in Z$

Therefore, $\tan^2 2\theta = 0 \Rightarrow \tan 2\theta = 0 \Rightarrow 2\theta = n\pi \Rightarrow \theta = \frac{n\pi}{2}$, where n = 0, ± 1, ± 2 ....

Which is the required solution.

6. PERIODIC FUNCTION

A function $f(x)$ is said to be periodic if there exists T > 0 such that $f(x + T) = f(x)$ for all x in the domain of definition of $f(x)$. If T is the smallest positive real number such that $f(x + T) = f(x)$, then it is called the period of $f(x)$.

We know that, $\sin(2n\pi + x) = \sin x, \cos(2n\pi + x) = \cos x$, $\tan(n\pi + x) = \tan x$ for all $n \in Z$

Therefore, sinx, cosx and tanx are periodic functions. The period of sinx and cosx is 2π and the period of tanx is π.

Function	Period
$\sin(ax+b), \cos(ax+b), \sec(ax+b), \operatorname{cosec}(ax+b)$	$2\pi / a$
$\tan(ax+b), \cot(ax+b)$	π / a
$\lvert\sin(ax+b)\rvert, \lvert\cos(ax+b)\rvert, \lvert\sec(ax+b)\rvert, \lvert\operatorname{cosec}(ax+b)\rvert$	π / a
$\lvert\tan(ax+b)\rvert, \lvert\cot(ax+b)\rvert$	$\pi / 2a$

(a) Trigonometric equations can be solved by different methods. The form of solutions obtained in different methods may be different. From these different forms of solutions, it is wrong to assume that the answer obtained by one method is wrong and those obtained by another method are correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

To test the equivalence of two solutions obtained from two different methods, the simplest way is to put values of n = ... – 2, – 1, 0, 1, 2, 3, ... etc. and then to find the angles in [0, 2π]. If all the angles in both the solutions are same, the solutions are equivalent.

(b) While manipulating the trigonometric equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For example, suppose we have the equation tan x = 2 sin x. Here by dividing both sides by sin x, we get cos x = 1/2.

(c) While equating one of the factors to zero, we must take care to see that the other factor does not become infinite. For example, if we have the equation sin x = 0, which can be written as cos x tan x = 0. Here we cannot put cos x = 0, since for cos x = 0, tan x = sin x / cos x is infinite.

(d) Avoid squaring: When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions that do not satisfy the given equation.

For example: Consider the equation, $\sin\theta + \cos\theta = 1$ .... (i)

Squaring, we get $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ .... (ii)

This gives $\theta = 0, \pi/2, \pi, 3\pi/2$ ....

Verification shows that π and $3\pi/2$ do not satisfy the equation as $\sin\pi + \cos\pi = -1, \neq 1$ and $\sin 3\pi/2 + \cos 3\pi/2 = -1, \neq 1$.

The reason for this is simple.

The equation (ii) is not equivalent to (i) and (ii) contains two equations: $\sin\theta + \cos\theta = 1$ and $\sin\theta + \cos\theta = -1$. Therefore, we get extra solutions.

Thus if squaring is a must, verify each of the solutions.

Some Necessary Restriction: If the equation involves tan x, sec x, take cos x $\neq$ 0. If cot x or cosec x appear, take sin x $\neq$ 0. If log appears in the equation, then number > 0 and base of log > 0, $\neq$ 1.

Also note that $\sqrt{[f(\theta)]}$ is always positive. For example, $\sqrt{\sin^2\theta} = \lvert\sin\theta\rvert$, not ± sin θ.

Verification: Students are advised to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.

7. SOME TRIGONOMETRIC EQUATIONS WITH THEIR GENERAL SOLUTIONS

Trigonometric equation	General solution
If $\sin\theta = 0$	$\theta = n\pi$
If $\cos\theta = 0$	$\theta = (n\pi + \pi/2) = (2n+1)\pi/2$
If $\tan\theta = 0$	$\theta = n\pi$
If $\sin\theta = 1$	$\theta = 2n\pi + \pi/2 = (4n+1)\pi/2$
If $\cos\theta = 1$	$\theta = 2n\pi$
If $\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha$ where $\alpha \in [-\pi/2, \pi/2]$
If $\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$
If $\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha$ where $\alpha \in [-\pi/2, \pi/2]$
If $\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$
If $\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
If $\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$
If $\sin\theta = \sin\alpha$ $\cos\theta = \cos\alpha$	$\theta = 2n\pi + \alpha$
If $\sin\theta = \sin\alpha$ $\tan\theta = \tan\alpha$	$\theta = 2n\pi + \alpha$
If $\tan\theta = \tan\alpha$ $\cos\theta = \cos\alpha$	$\theta = 2n\pi + \alpha$

Note: Everywhere in this chapter, "n" is taken as an integer.

Illustration 22: Solve: $\sin m\theta + \sin n\theta = 0$ **(JEE MAIN)**

Sol: By using $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, we can solve this problem.

We have, $\sin m\theta + \sin n\theta = 0$

$$\Rightarrow \sin\left(\frac{m+n}{2}\right)\theta.\cos\left(\frac{m-n}{2}\right)\theta = 0 \Rightarrow \sin\left(\frac{m+n}{2}\right)\theta = 0 \quad \text{or} \quad \cos\left(\frac{m-n}{2}\right)\theta = 0$$

Now, $\sin\left(\frac{m+n}{2}\right)\theta = 0 \Rightarrow \left(\frac{m+n}{2}\right)\theta = r\pi, r \in Z$ ... (i)

And $\cos\left(\frac{m-n}{2}\right)\theta = 0 \quad \Rightarrow \cos\left(\frac{m-n}{2}\right)\theta = \cos\frac{\pi}{2}$

$\Rightarrow \left(\frac{m-n}{2}\right)\theta = (2p+1)\frac{\pi}{2},\ p \in Z \ \Rightarrow \theta = \left(\frac{2p+1}{m-n}\right)\pi,$ ... (ii)

From (i) and (ii), we have $\theta = \frac{2r\pi}{m+n}$ or $\theta = \left(\frac{2p+1}{m-1}\right)\pi$ where, $m,n \in Z$

Illustration 23: Solve: $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$ **(JEE ADVANCED)**

Sol: Simply using algebra and method of finding general equation, we can solve above equation.

We have, $4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$

$\Rightarrow 2\sin x(2\cos x + 1) + 1(2\cos x + 1) = 0 \quad \Rightarrow (2\sin x + 1)(2\cos x + 1) = 0$

$\Rightarrow 2\sin x + 1 = 0$ or $2\cos x + 1 = 0 \quad \Rightarrow \sin x = -\frac{1}{2}$ or $\cos x = -\frac{1}{2}$

$\sin x = -\frac{1}{2} \Rightarrow \sin x = \sin\left(-\frac{\pi}{6}\right) \quad \Rightarrow x = -\frac{\pi}{6}$ The general solution of this is

$x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right) = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right) \quad \Rightarrow x = \pi\left[n + \frac{(-1)^{n+1}}{6}\right]$ ... (i)

and $\cos x = -\frac{1}{2} \quad \Rightarrow \cos x = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \quad \Rightarrow x = \frac{2\pi}{3}$

The general solution of this is $x = 2n\pi \pm \frac{2\pi}{3}$ i.e. $x = 2\pi\left(n \pm \frac{1}{3}\right)$ ... (ii)

From (1) and (2), we have $\pi\left[n + \frac{(-1)^{n+1}}{6}\right]$ and $2\pi\left(n \pm \frac{1}{3}\right)$ are the required solutions

8. METHODS OF SOLVING TRIGONOMETRIC EQUATIONS

8.1 Factorization

Trigonometric equations can be solved by use of factorization.

Illustration 24: Solve: $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ **(JEE MAIN)**

Sol: Use factorization method to solve this illustration.

$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x \ \Rightarrow \quad (2\sin x - \cos x)(1 + \cos x) - \sin^2 x = 0$

$(2\sin x - \cos x)(1 + \cos x) - (1 - \cos x)(1 + \cos x) = 0$; $(1 + \cos x)(2\sin x - 1) = 0$

$1 + \cos x = 0$ or $2\sin x - 1 = 0$

$\cos x = -1$ or $\sin x = \frac{1}{2}$

$\cos x = \cos\pi$ or $\sin x = \sin \pi/6$

$\Rightarrow x = (2n+1)\pi, n \in I$ or $x = n\pi + (-1)^n \pi/6, n \in I$

$\therefore$ The solution of given equation is $(2n+1)\pi, n \in I$ or $n\pi + (-1)^n \pi/6,\ n \in I$

8.2 Sum to Product

Trigonometric equations can be solved by transforming a sum or difference of trigonometric ratios into their product.

Illustration 25: If sin 5x + sin 3x + sin x = 0 and $0 \le x \le \pi/2$, then x is equal to. **(JEE MAIN)**

Sol: By using sum to product formula i.e. $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

sin 5x + sinx = – sin 3x $\Rightarrow 2\sin3x\cos2x + \sin3x = 0$ $\Rightarrow \sin3x\left(2\cos2x+1\right) = 0$

$\Rightarrow \sin3x = 0, \cos2x = -1/2$ $\Rightarrow x = n\pi, x = n\pi \pm \left(\pi/3\right)$

Illustration 26: Solve $\cos3x + \sin2x - \sin4x = 0$ **(JEE MAIN)**

Sol: Same as above illustration, by using formula

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ We can solve this illustration.

cos 3x + sin 2x – sin 4x = 0 $\Rightarrow$ $\cos3x + 2\cos3x.\sin\left(-x\right) = 0$

$\Rightarrow \cos3x - 2\cos3x.\sin x = 0$ $\Rightarrow$ $\cos3x\left(1 - 2\sin x\right) = 0$

$\Rightarrow \cos3x = 0$ or 1 – 2 sin x = 0 $\Rightarrow 3x = \left(2n+1\right)\frac{\pi}{2}, n \in I$ or $\sin x = \frac{1}{2}$

$\Rightarrow x = \left(2n+1\right)\frac{\pi}{6}, n \in I$ or $x = n\pi + \left(-1\right)^n\frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is $\left(2n+1\right)\frac{\pi}{6}, n \in I$ or $n\pi + \left(-1\right)^n\frac{\pi}{6}, n \in I$

8.3 Product to Sum

Trigonometric equations can also be solved by transforming product into a sum or difference of trigonometric ratios.

Illustration 27: The number of solutions of the equation $\sin5x\cos3x = \sin6x\cos2x$, in the interval $\left[0, \pi\right]$, is: **(JEE MAIN)**

Sol: Simply by using product to sum method.

The given equation can be written as $\frac{1}{2}\left(\sin8x + \sin2x\right) = \frac{1}{2}\left(\sin8x + \sin4x\right)$

$\Rightarrow$ sin 2x – sin 4x = 0 $\Rightarrow$ – 2 sin x cos 3x = 0

Hence sin x = 0 or cos 3x = 0. That is, x = $n\pi\left(n \in I\right)$, or 3x = $k\pi + \frac{\pi}{2}\left(k \in I\right)$.

Therefore, since $x \in \left[0, \pi\right]$, the given equation is satisfied if x = 0, $\pi, \frac{\pi}{6}, \frac{\pi}{2}$ or $\frac{5\pi}{6}$.

Hence, no. of solutions is 5.

8.4 Parametric Methods

General solution of trigonometric equation $a\cos\theta + b\sin\theta = c$

To solve the equation $a\cos\theta + b\sin\theta = c$, put $a = r\cos\phi, b = r\sin\phi$ such that $r = \sqrt{a^2+b^2}, \phi = \tan^{-1}\frac{b}{a}$

Substituting these values in the equation, we have, $r\cos\phi\cos\theta + r\sin\phi\sin\theta = c$

$\cos(\theta-\phi) = \frac{c}{r}$ $\quad\Rightarrow\quad \cos(\theta-\phi) = \frac{c}{\sqrt{a^2+b^2}}$

If $|c| > \sqrt{a^2+b^2}$, then the equation $a\cos\theta + b\sin\theta = c$ has no solution.

If $|c| \le \sqrt{a^2+b^2}$, then put $\frac{|c|}{\sqrt{a^2+b^2}} = \cos\alpha$, so that $\cos(\theta-\phi) = \cos\alpha$

$\Rightarrow (\theta-\phi) = 2n\pi \pm \alpha \quad\Rightarrow \theta = 2n\pi \pm \alpha + \phi$

Illustration 28: Solve: $\sin x + \sqrt{3}\cos x = \sqrt{2}$ **(JEE MAIN)**

Sol: Solve by using above mentioned parametric method.

Given, $\sqrt{3}\cos x + \sin x = \sqrt{2}$, dividing both sides by $\sqrt{a^2+b^2}$

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \cos\left(x - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4} \quad \Rightarrow x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6} \Rightarrow x = 2n\pi + \frac{5\pi}{12}, 2n\pi - \frac{\pi}{12} \text{ where } n \in I$$

Note: Trigonometric equations of the form a sin x + b cos x = c can also be solved by changing sin x and cos x into their corresponding tangent of half the angle. i.e t=tan x/2. The following example gives you insight.

Illustration 29: Solve: 3 cos x + 4 sin x = 5 **(JEE MAIN)**

Sol: As we know, $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$. Therefore by substituting these values and solving we will be get the result.

3 cos x + 4 sin x = 5 ... (i)

$\because \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ $\therefore$ Equation (i) becomes

$$\Rightarrow 3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = 5 \quad \text{... (ii)}$$

Let $\tan\frac{x}{2} = t$ $\quad\therefore$ Equation (ii) becomes

$$3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 5 \Rightarrow 4t^2 - 4t + 1 = 0 \Rightarrow (2t-1)^2 = 0 \Rightarrow t = 1/2 \because t = \tan x/2$$

$\Rightarrow \tan\frac{x}{2} = \frac{1}{2} \Rightarrow \tan\frac{x}{2} = \tan\alpha$, where $\tan\alpha = \frac{1}{2}$ $\Rightarrow \frac{x}{2} = n\pi + \alpha$ $\Rightarrow x = 2n\pi + 2\alpha$ where, $\alpha = \tan^{-1}\left(\frac{1}{2}\right), n \in I$

8.5 Functions of sin x and cos x

Trigonometric equations of the form P (sin x ± cos x, sin x cos x) = 0, where P (y, z) is a polynomial, can be solved by using the substitution sin x ± cos x = C.

Illustration 30: Solve: sin x + cos x = 1 + sin x. cos x **(JEE MAIN)**

Sol: Consider sin x + cos x = t, and solve it by using parametric method.

∵ sin x + cos x = 1 + sin x. cos x ... (i)

Let sin x + cos x = t

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$ $\Rightarrow \sin x.\cos x = \frac{t^2 - 1}{2}$

Now, put sin x + cos x = t and sin x. cos x = $\frac{t^2-1}{2}$ in (i), we get t = 1 + $\frac{t^2-1}{2}$

$\Rightarrow t^2 - 2t + 1 = 0$ $\Rightarrow t = 1$ ∵ t = sin x + cos x $\Rightarrow \sin x + \cos x = 1$ ... (ii)

Dividing both sides of equation (ii) by $\sqrt{2}$, we get:

$\Rightarrow \sin x\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$ $\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

If we take the positive sign, we get x = $2n\pi + \frac{\pi}{2}$, $n \in I$

If we take the negative sign, we get x = $2n\pi$, $n \in I$

8.6 Using Boundaries of sin x and cos x

Trigonometric equations can be solved by the use of boundness of the trigonometric ratios sinx and cos x.

PLANCESS CONCEPTS

(i) The answer should not contain such values of angles which make any of the terms undefined or infinite.

(ii) Never cancel terms containing unknown terms on the two sides, which are in product. It may cause loss of the general solution.

Suppose the equation is sin x = (tan x)/2. Now, cancelling sinx on both the sides, we get only $\cos x = \frac{1}{2}$, sin x = 0 is not counted.

(iii) Check that the denominator is not zero at any stage while solving equations.

(iv) While solving a trigonometric equation, squaring the equation at any step must be avoided if possible. If squaring is necessary, check the solution for extraneous values.

Suppose the equation is sin x = – sin x. We know that the only solution of this is sin x = 0 but on squaring, we get $(\sin x)^2 = (\sin x)^2$ which is always true.

(v) Domain should not change, it if changes, necessary corrections must be made.

Shivam Agarwal (JEE 2009, AIR 27)

Illustration 31: Solve: sin 3x + cos 2x = – 2 **(JEE ADVANCED)**

Sol: By using boundary condition of sin x and cos x.

Since $\sin 3x \ge -1$ and $\cos 2x \ge -1$, we have, $\sin 3x + \cos 2x \ge -2$

Thus, the equality holds true if and only if $\sin 3x = -1$ and $\cos 2x = -1$

$\Rightarrow 3x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$ and $2x = 2n\pi \pm \pi$ i.e. $x = \frac{n\pi}{3} + (-1)^n\left(-\frac{\pi}{6}\right)$ and $x = n\pi \pm \frac{\pi}{2}$, $n \in I$

$\therefore$ Solution set is, $\left\{x \mid x = \frac{n\pi}{3} + (-1)^n\left(-\frac{\pi}{6}\right)\right\} \cap \left\{x \mid x = n\pi \pm \left(\frac{\pi}{2}\right)\right\}$

Note: Here, unlike all other problems, the solution set consists of the intersection of two solution sets and not the union of the solution sets.

Illustration 32: $\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right)(\cos x) = 0$. Find the general solution. **(JEE ADVANCED)**

Sol: Open all brackets of given equation and then by using sum to product formula and method of finding general solution we will get the result.

$$\sin x\cos\frac{x}{4} - 2\sin^2 x + \cos x + \sin\frac{x}{4}\cos x - 2\cos^2 x = 0$$

$$\sin\left(x + \frac{x}{4}\right) + \cos x = 2 \Rightarrow \sin\frac{5x}{4} + \cos x = 2 \qquad \Rightarrow \sin\frac{5x}{4} = 1 \text{ and } \cos x = 1$$

$$\sin\frac{5x}{4} = 1 \Rightarrow \frac{5x}{4} = 2n\pi + \frac{\pi}{2} \qquad \Rightarrow x = 2(4n+1)\frac{\pi}{5}\ ;\ \cos x = 1 \Rightarrow x = 2m\pi$$

$$\Rightarrow x = 2\pi, 10\pi, 18\pi \ \ldots\ldots\text{AP} \qquad \Rightarrow x = 2\pi + (m-1)8\pi$$

$$\Rightarrow x = 2\pi(4m-3) \qquad m \in I$$

Illustration 33: Find the general solution of $2\sin\left(3x + \frac{\pi}{4}\right) = \sqrt{1 + 8\sin 2x\cos^2 2x}$ **(JEE ADVANCED)**

Sol: First square on both side and then using sum and difference formula we can solve this illustration.

$$4\sin^2\left(3x + \frac{\pi}{4}\right)^2 = 1 + 8\sin 2x\cos^2 2x \Rightarrow 4\left(\frac{\sin 3x}{\sqrt{2}} + \frac{\cos 3x}{\sqrt{2}}\right)^2 = 1 + 8\sin 2x\cos^2 2x$$

$$\Rightarrow \frac{4\sin^2 3x}{2} + \frac{4\cos^2 3x}{2} + 4\sin 3x\cos 3x = 1 + 8\sin 2x\cos^2 2x$$

$$\Rightarrow 2\sin^2 3x + 2\cos^2 3x + 2\sin 6x = 1 + 8\sin 2x\cos^2 2x$$

$\Rightarrow$ 1 + 2 sin 6x = 8 sin 2x $\cos^2$ 2x $\qquad \Rightarrow$ 1 + 2 sin 6x = 4 sin 4x cos 2x

$\Rightarrow$ 1 + 2 sin 6x = 2 (sin 6x + sin 2x) $\qquad \Rightarrow$ 1 = 2 sin 2x $\Rightarrow$ sin 2x = ½

$$\Rightarrow x = \frac{\pi}{12} + 2n\pi \qquad x = \frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5} \qquad n \in I$$

9. SIMULTANEOUS EQUATIONS

Two equations are given and we have to find the value of variable θ which may satisfy both the given equations, like $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$

So, the common solution is $\theta = 2n\pi + \alpha, n \in I$

Similarly, $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$

So, the common solution is $\theta = 2n\pi + \alpha,\ n \in I$

Illustration 34: The most general value of θ satisfying the equations $\cos\theta = \frac{1}{\sqrt{2}}$ and $\tan\theta = -1$ is: **(JEE MAIN)**

Sol: As above mentioned method we can find out the general value of θ.

$\cos\theta = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$

$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{4};\quad n \in I$ Put n = 1 $\theta = \frac{9\pi}{4}, \frac{7\pi}{4}$

$\tan\theta = -1 = \tan\left(\frac{-\pi}{4}\right) \quad \Rightarrow \theta = n\pi - \pi/4,\quad n \in I \quad$ Put $n = 1, \theta = \frac{3\pi}{4}$; Put $n = 2, \theta = \frac{7\pi}{4}$

The common value which satisfies both these equation is $\left(\frac{7\pi}{4}\right)$.

Hence, the general value is $2n\pi + \frac{7\pi}{4}$.

Illustration 35: The most general value of θ satisfying equations $\sin\theta = -\frac{1}{2}$ and $\tan\theta = 1/\sqrt{3}$ are: **(JEE MAIN)**

Sol: Similar to above illustration.

We shall first consider values of θ between 0 and 2π

$\sin\theta = -\frac{1}{2} = -\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right)$ or $\sin(2\pi - \pi/6)$

$\therefore \theta = 7\pi/6, 11\pi/6$; $\quad \tan\theta = 1/\sqrt{3} = \tan(\pi/6) = \tan(\pi + \pi/6)$

$\therefore \theta = \pi/6, 7\pi/6$

Thus, the value of θ between 0 and 2π which satisfies both the equations is $7\pi/6$.

Hence, the general value of θ is $2n\pi + 7\pi/6$ where $n \in I$

PROBLEM SOLVING TACTICS

(a) Any formula that gives the value of $\sin\frac{A}{2}$ in terms of sin A shall also give the value of sin $\frac{n\pi + (-1)^n A}{2}$.

(b) Any formula that gives the value of $\cos\frac{A}{2}$ in terms of cos A shall also give the value of cos $\frac{2n\pi \pm A}{2}$.

(c) Any formula that gives the value of $\tan\frac{A}{2}$ in terms of tan A shall also give the value of tan $\frac{n\pi \pm A}{2}$.

(d) If α is the least positive value of θ which satisfies two given trigonometric equations, then the general value of θ will be $2n\pi+\alpha$. For example, $\sin\theta=\sin\alpha$ and $\cos\theta=\cos\alpha$, then, $\theta=2n\pi+\alpha, n\in I$

(i) $\sin(n\pi+\theta)=(-1)^n\sin\theta,\ n\in I$

(ii) $\cos(n\pi+\theta)=(-1)^n\cos\theta,\ n\in I$

(iii) $\sin(n\pi-\theta)=(-1)^{n-1}\sin\theta,\ n\in I$

FORMULAE SHEET

Tangent and cotangent Identities	$\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$
Product Identities	$\sin\theta\times\operatorname{cosec}\theta=1$, $\cos\theta\times\sec\theta=1$, $\tan\theta\times\cot\theta=1$
Pythagorean Identities	$\sin^2\theta+\cos^2\theta=1$, $\tan^2\theta+1=\sec^2\theta$, $1+\cot^2\theta=\csc^2\theta$
Even/Odd Formulas	$\sin(-\theta)=-\sin\theta$, $\cos(-\theta)=\cos\theta$, $\tan(-\theta)=-\tan\theta$, $\cot(-\theta)=-\cot\theta$, $\sec(-\theta)=\sec\theta$, $\operatorname{cosec}(-\theta)=-\operatorname{cosec}\theta$
Periodic Formulas (If n is an integer)	$\sin(2n\pi+\theta)=\sin\theta$, $\cos(2n\pi+\theta)=\cos\theta$, $\tan(n\pi+\theta)=\tan\theta$, $\cot(n\pi+\theta)=\cot\theta$, $\sec(2n\pi+\theta)=\sec\theta$, $\operatorname{cosec}(2n\pi+\theta)=\operatorname{cosec}\theta$
Double and Triple Angle Formulas	$\sin(2\theta)=2\sin\theta\cos\theta$, $\sin3\theta=3\sin\theta-4\sin^3\theta$, $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ $\cos3\theta=4\cos^3\theta-3\cos\theta$, $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$, $\tan3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$
Complementary angles	$\sin\left(\dfrac{\pi}{2}\pm\theta\right)=\cos\theta$, $\cos\left(\dfrac{\pi}{2}\pm\theta\right)=\mp\sin\theta$, $\tan\left(\dfrac{\pi}{2}\pm\theta\right)=\mp\cot\theta$, $\cot\left(\dfrac{\pi}{2}-\theta\right)=\tan\theta$, $\sec\left(\dfrac{\pi}{2}-\theta\right)=\operatorname{cosec}\theta$, $\operatorname{cosec}\left(\dfrac{\pi}{2}-\theta\right)=\sec\theta$
Half Angle	$\sin^2\theta=\dfrac{1}{2}\left[1-\cos(2\theta)\right]$, $\cos^2\theta=\dfrac{1}{2}\left[1+\cos(2\theta)\right]$, $\tan^2\theta=\dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Sum and Difference	$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$, $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$, $\tan(\alpha\pm\beta)=\dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$,

Product to Sum	$\sin\alpha\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$, $\sin\alpha\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$, $\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$, $\cos\alpha\sin\beta=\frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$,
Sum to Product	$\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$, $\sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Solved Examples

JEE Main/Boards

Example 1: Solve: $2\cos 2\theta+\sqrt{2\sin\theta}=2$

Sol: Solve this example by using $\sin^2\theta=\frac{1}{2}\left[1-\cos(2\theta)\right]$.

$\sqrt{2\sin\theta}=2(1-\cos 2\theta)=4\sin^2\theta$

$\therefore$ 2 sin $\theta=16\sin^4\theta:\sin\theta\geq 4$

$\sin\theta=0$ or $\sin^3\theta=\frac{1}{8}$ $\therefore\sin\theta=0$ or

$\frac{1}{2}$, $\theta=m\pi:m\in I$ or $\theta=n\pi+(-1)^n\frac{\pi}{6}:n\in I$

Example 2: Solve: $8\tan^2\frac{x}{2}=1+\sec x$

Sol: As we know that

$\tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}$, substitute this to solve above example.

$8\tan^2\frac{x}{2}=1+\sec x$ ... (i)

$\Rightarrow 8\left(\frac{1-\cos x}{1+\cos x}\right)=\frac{\cos x+1}{\cos x}:\cos x\neq 0,-1$

or (8 – 8 cosx)cosx = $(\cos x+1)^2$

or 8 cosx – 8 $\cos^2 x$ = $\cos^2 x$ + 2cosx + 1

or $9\cos^2 x$ – 6 cosx + 1 = 0

or $(3\cos x-1)^2=0$

or cosx = $\frac{1}{3}$ = $\cos\beta,(\text{say})$, $\beta=\cos^{-1}\left(\frac{1}{3}\right)$

$\therefore x=2n\pi\pm\beta:n\in I$

Example 3: Solve: $\sin x+\cos x-2\sqrt{2}\sin x\cos x=0$

Sol: We can write given equation a sinx + cosx = $\sqrt{2}\sin 2x$, multiplying and dividing L.H.S. by $\sqrt{2}$, we will get the result.

or $\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)=\sqrt{2}\sin 2x$

or $\sin\left(x+\frac{\pi}{4}\right)=\sin 2x$

$\Rightarrow 2x=n\pi+(-1)^n\left(x+\frac{\pi}{4}\right):n\in I$

Example 4: Find the general value of θ which satisfies both the equations $\cos\theta=-\frac{\sqrt{3}}{2}$ and $\sin\theta=\frac{1}{2}$.

Sol: Use the method for simultaneous equations.

$\cos\theta=-\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{5\pi}{6},\frac{7\pi}{6}....$

$\sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6},\frac{5\pi}{6}.....$

Hence, the general solution is given by

$\theta=2n\pi+\frac{5\pi}{6}, n\in I$

Example 5: Show that the equation

$2\cos^2\left(\frac{x}{2}\right)\sin^2 x=x^2+x^{-2}$ for $0<x\le\frac{\pi}{2}$

has no real solution.

Sol: Here $2\cos^2\frac{x}{2}\sin^2 x=x^2+\frac{1}{x^2}$ holds only If $x^2=1$, hence by substituting x =1 in above equation we can conclude that given equation has real solution or not.

$x^2+\frac{1}{x^2}\ge 2\forall x$ with equality for

$x^2=1$ alone. Since $2\cos^2\frac{x}{2}\sin^2 x\le 2$,

$\therefore 2\cos^2\frac{x}{2}\sin^2 x=x^2+\frac{1}{x^2}$ holds only

If $x^2=1$ $\therefore x=1$ and $\cos\frac{x}{2}\sin x=\pm 1$

i.e. $\cos\left(\frac{1^C}{2}\right)\sin 1^C=\pm 1$, which is not true.

Hence, the given equation has no solution.

Example 6: Determine 'a' for which the equation $a^2-2a+\sec^2(\pi(a+x))=0$ has solutions and find the solutions.

Sol: By using algebra and tangent of angle we can get the result.

$a^2-2a+1+\tan^2(\pi(a+x))=0$

or $(a-1)^2+[\tan\pi(a+x)]^2=0$

$\Rightarrow a-1=0$ and $\tan\pi(a+x)=0$

$\Rightarrow \tan(1+x)\pi=0 \Rightarrow (1+x)\pi=n\pi : n\in I$

$\therefore x=n-1:n\in I:a=1$

Example 7: Solve the equation $\cos^7 x+\sin^4 x=1$

Sol: Here $\cos^7 x\le\cos^2 x$ and $\sin^4 x\le\sin^2 x$, hence by solving this we will get the result.

$\cos^7 x\le\cos^2 x$ and $\sin^4 x\le\sin^2 x$

$\therefore 1=\cos^7 x+\sin^4 x\le\cos^2 x+\sin^2 x=1$

$\therefore \cos^7 x=\cos^2 x$ and $\sin^4 x=\sin^2 x$

$\cos^7 x=\cos^2 x\Rightarrow\cos^2 x(\cos^5 x-1)=0$

$\therefore \cos x=0$ or $\cos x=1$

$\therefore x=(2n+1)\frac{\pi}{2};\quad n\in I$

or $x=2m\pi;\quad m\in I$ ...(i)

$\sin^2 x=\sin^4 x\Rightarrow\sin^2 x\cos^2 x=0$

$\Rightarrow \cos x=0$ or $\sin x=0$ ... (ii)

Since (i) satisfies the system (ii),

$\therefore$ Solution set is given by (i)

Example 8: Solve for x and y:

$12\sin x+5\cos x=2y^2-8y+21$

Sol: Multiply and divide L.H.S. by 13 and solve to get the result.

$12\sin x+5\cos x=2y^2-8y+21$

$\therefore \sqrt{12^2+5^2}\left(\frac{12}{13}\sin x+\frac{5}{13}\cos x\right)=2(y^2-4y+4)+13$

or $13\cos(x-\alpha)=2(y-2)^2+13$

: $\cos\alpha=\frac{5}{13}$ and $\sin\alpha=\frac{12}{13}$

Thus, $\cos(x-\alpha)=1$ and y = 2 or

$x-\alpha=2n\pi:n\in I$ and y = 2

$\therefore x=2n\pi+\alpha:n\in I$ and y = 2

JEE Advanced/Boards

Example 1: Solve for x, y: $x\cos^3 y + 3x\cos y\sin^2 y = 14$
$x\sin^3 y + 3x\cos^2 y\sin y = 13$

Sol: Divide equation 1 by equation 2 and then by applying componendo and dividendo we can solve this problem.

We note that, "x = 0; sin y = 0 or cos y = 0" do not yield a solution to given system.

$$\frac{\cos^3 y + 3\cos y\sin^2 y}{\sin^3 y + 3\cos^2 y\sin y} = \frac{14}{13}$$

By componendo and dividendo,

$$\frac{\cos^3 y + 3\cos y\sin^2 y + 3\cos^2 y\sin y + \sin^3 y}{\cos^3 y + 3\cos y\sin^2 y - 3\cos^2 y\sin y - \sin^3 y}$$

$$= \frac{14+13}{14-13} \text{ or } \left(\frac{\cos y + \sin y}{\cos y - \sin y}\right)^3 = 27 = (3)^3$$

$$\text{or } \frac{\cos y + \sin y}{\cos y - \sin y} = 3 \text{ or } \frac{1+\tan y}{1-\tan y} = \frac{3}{1}$$

$$\tan y = \frac{1}{2} = \tan\alpha; y = n\pi + \alpha : n \in I$$

Since sin y and cos y have signs, we have the following cases:

(i) $\sin y = \frac{1}{\sqrt{5}}$ and cos y = $\frac{2}{\sqrt{5}}$; then

$$x\left[\frac{8}{5\sqrt{5}} + 3\frac{2}{\sqrt{5}}.\frac{1}{5}\right] = 14 \Rightarrow x = 5\sqrt{5}$$

(ii) $\sin y = -\frac{1}{\sqrt{5}}$ and cos y = $-\frac{2}{\sqrt{5}}$; then

$$x\left[\frac{-8}{5\sqrt{5}} + 3\left(\frac{-2}{\sqrt{5}}\right)\frac{1}{5}\right] = 14 \Rightarrow x = -5\sqrt{5}$$

Example 2: Solve: $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x\cos x$

Sol: By substituting $2\sin x\cos x$ = t and solving we will be get the result.

$$\sin^4 x + \cos^4 x = \frac{7}{4}\sin 2x; \therefore \sin 2x > 0$$

$$\left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x\cos^2 x = \frac{7}{4}\sin 2x$$

$$\therefore 1 - \frac{t^2}{2} = \frac{7t}{4} \text{ or } 2t^2 + 7t - 4 = 0$$

$$\text{or } (2t-1)(t+4) = 0 \ \therefore \sin 2x = t = \frac{1}{2} = \sin\frac{\pi}{6}$$

$\Rightarrow 2x = n\pi + (-1)^n\frac{\pi}{6} : n \in I$ General solution is

$$x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}, n \in I$$

Example 3: Solve: $3\tan(\theta - 15^\circ) = \tan(\theta + 15^\circ)$

Sol: We can write given equation as $\frac{\tan(\theta+15^\circ)}{\tan(\theta-15^\circ)} = \frac{3}{1}$, hence by applying componendo and dividendo we will get the result.

Given, $3\tan(\theta - 15^\circ) = \tan(\theta + 15^\circ)$

$$\text{or } \frac{\tan(\theta+15^\circ)}{\tan(\theta-15^\circ)} = \frac{3}{1}$$

$$\text{or } \frac{\tan(\theta+15^\circ) + \tan(\theta-15^\circ)}{\tan(\theta+15^\circ) - \tan(\theta-15^\circ)} = \frac{4}{2}$$

$$\text{or } \frac{\sin(\theta+15^\circ+\theta-15^\circ)}{\sin(\theta+15^\circ-\theta+15^\circ)} = 2 \text{ or } \sin 2\theta = 1 = \sin\frac{\pi}{2}$$

$$\Rightarrow 2\theta = n\pi + (-1)^n\frac{\pi}{2} : n \in I \therefore \theta = \frac{n\pi}{2} + (-1)^n\frac{\pi}{4} : n \in I$$

Example 4: Find value of θ for $\sin 2\theta = \cos 3\theta$, where $0 \le \theta \le 2\pi$; Use the above equation to find the value of sin 18°.

Sol: Here as we know $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$, hence we can write given equation as $\cos 3\theta = \cos\{(\pi/2) - 2\theta\}$.

Therefore by comparing their angle we will get the result.

The given equation is $\sin 2\theta = \cos 3\theta$ or, $\cos 3\theta = \sin 2\theta$

or, $\cos 3\theta = \cos\{(\pi/2) - 2\theta\}$

or, $3\theta = 2n\pi \pm \{(\pi/2) - 2\theta\}$ where $n \in I$

Taking + sign, $3\theta = 2n\pi + \{\pi/2 - 2\theta\}$

or $5\theta = (4n+1)(\pi/2)$

or, $\theta = (4n+1)(\pi/10)$, where $n \in I$ ... (i)

Again taking – sign, $3\theta - 2n\pi - \{(\pi/2) - 2\theta\}$

or $\theta = (4n+1)(\pi/2)$ ... (ii)

Putting n = 0, 1, 2, 3, ...in (i) the values of θ in the interval $0 \le \theta \le 2\pi$ are given by

$\theta = \pi/10, 5\pi/10, 9\pi/10, 13\pi/10, 17\pi/10$ or 18°, 90°, 162°, 234°, 346°.

Again putting $n=0,\pm1,\pm2,......$, in (ii) the value of θ in the interval $0\le\theta\le2\pi$ is $3\pi/2$ i.e. 270° only.

Hence the required values of θ in $0\le\theta\le2\pi$ are 18°, 90°, 162°, 234°, 270°, 306°.

Example 5: Solve the equation:

$$\cos\left(\pi3^x\right)-2\cos^2\left(\pi3^x\right)+2\cos\left(4\pi3^x\right)-\cos\left(7\pi3^x\right)$$
$$=\sin\left(\pi3^x\right)+2\sin^2\left(\pi3^x\right)-2\sin\left(4\pi3^x\right)$$
$$+2\sin\left(\pi3^{x+1}\right)-\sin\left(7\pi3^x\right)$$

Sol: Substitute $\pi3^x=y$, and then by using sum to product formula we can solve this equation.

Denote $\pi3^x$ by y to get

$$\cos y-2\cos^2 y+2\cos 4y-\cos 7y$$
$$=\sin y+2\sin^2 y-2\sin 4y+2\sin 3y-\sin 7y \qquad ...(i)$$

Transposing all terms to the left side,

we have, $(\cos y-\cos 7y)+(\sin 7y-\sin y)$

$+2(\cos 4y+\sin 4y)-2(\sin 3y+1)=0$

or, $2\sin 4y\sin 3y+2\cos 4y\sin 3y$

$+2(\cos 4y+\sin 4y)-2(\sin 3y+1)=0$

[Use C & D formulae]

or, $2\sin 3y(\sin 4y+\cos 4y)$

$+2(\cos 4y+\sin 4y)-2(\sin 3y+1)=0$

or, $(\sin 3y+1)(\sin 4y+\cos 4y-1)=0$

This enables us to write down three groups of solutions:

$$y_1=-\frac{\pi}{6}+\frac{2k\pi}{3},\ y_2=\frac{n\pi}{2}, y_3=\frac{\pi}{8}+\frac{m\pi}{2}$$

where k, n and m are arbitrary integers. Recalling that $y=\pi3^x$, we obtain an infinity of equations for determining the roots of the original equations:

$$3^x=-\frac{1}{6}+\frac{2k}{3},\ k=0,\pm1,\pm2,......\ 3^x=\frac{n}{2}, n=0,\pm2,$$
$$...=\frac{1}{8}+\frac{m}{2}, m=0,\pm1,\pm2,....$$

The equation $3^x=a$ has a (unique) root only for positive a and it is given by the formula $x=\log_3 a$. Therefore, the equation (i) has a solution only for those (integral) values of k, n, m for which the corresponding right members of the relations (i) are positive.

It is easy to see that of the first equation of (i) that is positive for integer k > 0, the right side of second equation of (i) is positive for integral n > 0; and the right side of the third equation of (i) is positive for $m\ge0$. Thus, we have to solve (i) only for the indicated values of k, m, n. The resulting values of x are then the roots of the original equation:

$$x=\log_3\left(-\frac{1}{6}+\frac{2k}{6}\right); k=1,2,......$$

$$x=\log_3\left(\frac{n}{2}\right), n=1,2,......=\log_3\left(\frac{1}{8}+\frac{1}{m}\right)m=0,1,2$$

Example 6: Solve the equation:

$$\sqrt{17\sec^2 x+16\left(\frac{1}{2}\tan x\sec x-1\right)}=2\tan x(1+4\sin x)$$

Sol: Solve it like algebra by using product and Pythagorean identity.

The given equation is

$$\sqrt{17\sec^2 x+16\left(\frac{1}{2}\tan x\sec x-1\right)}$$
$$=2\tan x(1+4\sin x) \qquad ...(i)$$

$$\Rightarrow\sqrt{17\sec^2 x+8\tan x\sec x-16\times1}=2\tan x(1+4\sin x)$$

$$\Rightarrow\sqrt{17\sec^2 x+8\tan x\sec x-16\times1}=2\tan x(1+4\sin x)$$

$$\Rightarrow\sqrt{17\sec^2 x+8\tan x\sec x-16\left(\sec^2 x-\tan^2 x\right)}=2\tan x(1+4\sin x)$$

$$\Rightarrow\sqrt{16\tan^2 x+8\tan x\sec x+\sec^2 x}$$
$$=2\tan x(1+4\sin x)$$

$$\sqrt{(4\tan x)^2+2\times4\tan x.\sec x+\sec^2 x}$$
$$=2\tan x(1+4\sin x)$$

$$4\tan x+\sec x=2\tan x(1+4\sin x)$$

$\Rightarrow 8\sin x.\tan x - 2\tan x - \sec x = 0$

$\Rightarrow 8\frac{\sin^2 x}{\cos x} - 2\frac{\sin x}{\cos x} - \frac{1}{\cos x} = 0$

$\Rightarrow 8\sin^2 x - 2\sin x - 1 = 0$

$\therefore \sin x = \frac{1}{2}$ and $\sin x = -\frac{1}{4}$

$\sin x = \sin \pi / 6 \Rightarrow x = \sin^{-1}\left(-\frac{1}{4}\right)$

$\therefore$ Solution of (i) is given by

$x = n\pi(-1)^n \frac{\pi}{6}$ and $x = n\pi + (-1)^{n+1} \sin^{-1}\left(\frac{1}{4}\right)$,

where $n \in I$

Example 7: Solve the equation:

$\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$

Sol: we can represent given equation as

$\left(\frac{2\sin^2 x}{2}\right)^5 + \left(\frac{2\cos^2 x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$, then use half angle formula to solve this problem.

Given, $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$

$\left(\frac{2\sin^2 x}{2}\right)^5 + \left(\frac{2\cos^2 x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$

$\left(\frac{1-\cos 2x}{2}\right)^5 + \left(\frac{1+\cos 2x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$

$\frac{(1-\cos 2x)^2.(1-\cos 2x)^3 + (1+\cos 2x)^2.(1+\cos 2x)^3}{32}$

$= \frac{29}{16}\cos^4 2x$

$\Rightarrow 10\cos^4 2x + 20\cos^2 2x + 2 = 58\cos^4 2x$

$\Rightarrow 48\cos^4 2x - 20\cos^2 2x - 2 = 0$

$\Rightarrow 24\cos^4 2x - 10\cos^2 2x - 1 = 0$

$\Rightarrow (2\cos^2 2x - 1)(12\cos^2 2x + 1) = 0$

$\therefore 2\cos^2 2x - 1 = 0 \quad \left[12\cos^2 2x + 1 \neq 0\right]$

$\Rightarrow \cos 4x = 0$

$\Rightarrow 4x = n\pi + \frac{\pi}{2}$

$\Rightarrow x = \frac{n\pi}{4} + \frac{\pi}{8}, n \in I$

Example 8: Consider the system of linear equations in x, y and z:

$(\sin 3\theta)x - y + z = 0$ ... (i)

$(\cos 2\theta)x + 4y + 3z = 0$ ... (ii)

$\frac{\pi}{4}$ 2x + 7y + 7z = 0 ... (iii)

Find the value of θ for which the system has a non-trivial solution.

Sol: Here we can write given linear equation in matrix form, and as we know for the system having non-trivial solution $|A|$ must be 0.

We can write the given linear equation in the form of AX=O

$A = \begin{bmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

As the system has a non-trivial solution,

$|A|$ must be 0

$\therefore \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

$\Rightarrow 7\sin 3\theta + 14\cos 2\theta - 14 = 0$

$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$,

$(3\sin\theta - 4\sin^3\theta) + 2(1 - 2\sin^2\theta) - 2 = 0$ or

$4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\sin\theta(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta = 1/2$ or $\sin\theta = -3/2$

But $\sin\theta = 0$ $\sin\theta = \frac{1}{2}$ is possible

And $\sin\theta = \frac{-3}{2}$ is not possible.

Now, $\sin\theta = 0 \therefore \theta = n\pi$; $n \in I$

and $\sin\theta = 1/2 = \sin(\pi/6)$,

$\therefore \theta = n\pi + (-1)^n \pi/6; n \in I$

Therefore the required values of θ are

$\theta = n\pi$ and $n\pi + (-1)^n(\pi/6)$, where $n \in I$

Example 9: Find the value of x,

$$2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+\frac{1}{2}}\le 1$$

Sol: Re-write the expression inside the square root and then by using algebra we can find out the value of x.

Given

$$2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+\frac{1}{2}}\le 1 \qquad \text{...(i)}$$

$$2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+\frac{1}{4}+\frac{1}{4}}\le 1$$

$$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\le 1$$

Minimum value of $2^{\frac{1}{\cos^2 x}}=2$

Minimum value of $\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{2}$

$\Rightarrow$ Minimum value of (i) is $2\times\frac{1}{2}$ = 1.

$\therefore\ 2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1 \Rightarrow \cos^2 x=1$ and $y=\frac{1}{2}$

$\Rightarrow \cos^2 2n\pi=1$

$\therefore\ x=2n\pi$

JEE Main/Boards

Exercise 1

Q.1 Solve the following trigonometric equations:

(i) $\sin 2\theta=\frac{1}{2}$ (ii) $\cos 5\theta=-\frac{1}{2}$

Q.2 Solve $7\cos^2\theta+3\sin^2\theta=4$

Q.3 Solve: $\tan x+\tan 2x+\sqrt{3}\tan x\tan 2x=\sqrt{3}$

Q.4 Solve: $3\tan(\theta-15°)=\tan(\theta+15°)$

Q.5 Solve the equations

$\sin(x-y)=\frac{\sqrt{3}}{2}$ and $\cos(x+y)=\frac{1}{2}$

Q.6 Solve the equation sin x = tanx

Q.7 Solve the equation $2\tan\theta-\cot\theta+1=0$

Q.8 Solve the equations $\sin mx+\sin nx=0$

Q.9 Solve the equation $\sec^2 2x=1-\tan 2x$

Q.10 Solve the equation:

$4\sin x\cos x+2\sin x+2\cos x+1=0$

Q.11 Solve the equation: $\tan x+\sec x=2\cos x$

Q.12 Solve: $2\sin^2 x-5\sin x\cos x-8\cos^2 x=-2$

Q.13 Solve: $4\sin x\sin 2x\sin 4x=\sin 3x$

Q.14 Solve the equation

$(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$

Q.15 Solve $\sin x+\sqrt{3}\cos x=\sqrt{2}$

Q.16 Find the general solution of the following trigonometric equations:

(i) $\tan 3\theta=-1$ (ii) $\cos 5x=-\frac{1}{\sqrt{2}}$

Q.17 Solve the following trigonometric equations:

(i) $3\cos^2\theta+7\sin^2\theta=4$

(ii) $\tan x+\tan 2x+\tan 3x=\tan 2x\tan 3x$

(iii) $\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=4$

Q.18 Solve the equation $\tan x+\cot x=2$

Q.19 Find the general solution of the trigonometric equation: $\tan^3 x - 3\tan x = 0$

Q.20 Solve the following trigonometric equations:

(i) $\cos x + \sin x = 1$ (ii) $\sec x - \tan x = \sqrt{3}$

(iii) $\sin x + \cos x = \frac{1}{\sqrt{2}}$ (iv) $\cos x + \sqrt{3}\sin x = 1$

Q.21 Find the degree measures corresponding to the following radian measures.

(i) $\left(\frac{\pi}{6}\right)^C$ (ii) $\left(\frac{4\pi}{5}\right)^C$ (iii) $(1.2)^C$

Q.22 The angles in a triangle are in A.P. and the ratio of the smallest angle in degree to the greatest angle in radians is $60:\pi$. Find the angles of the triangle in degrees and radians.

Q.23 Assuming the distance of the earth from the moon to be 38400 km and the angle subtended by the moon at the eye of a person on the earth to be 31′, find the diameter of the moon.

Q.24 Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find the height of the letters that he can read at a distance of 12 meters.

Q.25 Solve the equation $4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$

Q.26 Solve the equation:

$5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0, -\pi < \theta < \pi$

Q.27 Solve the equation: $4\sin^4 x + \cos^4 x = 1$

Q.28 Solve the equation: $\tan 2\theta \tan\theta = 1$

Q.29 Show that the equation: $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution.

Q.30 Does the equation $\sin^4\theta - 2\sin^2\theta - 1 = 0$ has a solution?

Exercise 2

Single Correct Choice Type

Q.1 If in a triangle ABC, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$, then a, c, b are:

(A) In A.P. (B) In G.P. (C) In H.P. (D) None

Q.2 Given $a^2 + 2a + \operatorname{cosec}^2\left(\frac{\pi}{2}(a+x)\right) = 0$

then, which of the following holds good?

(A) a = 1; $\frac{x}{2} \in I$

(B) a = –1; $\frac{x}{2} \in I$

(C) $a \in R; x \in \phi$

(D) a, x are finite but not possible to find

Q.3 In any triangle ABC, $(a+b)^2\sin^2\frac{C}{2} + (a-b)^2\cos^2\frac{C}{2} =$

(A) c (a + b) (B) b (c + a) (C) a (b + c) (D) c^2

Q.4 If in a $\Delta ABC, \sin^3 A + \sin^3 B + \sin^3 C$

$= 3\sin A \cdot \sin B \cdot \sin C$ then

(A) ΔABC may be a scalene triangle

(B) ΔABC is a right triangle

(C) ΔABC is an obtuse angled triangle

(D) ΔABC is an equilateral triangle

Q.5 $\sin 3\theta = 4\sin\theta\sin 2\theta\sin 4\theta$ in $0 \le \theta \le \pi$ has:

(A) 2 real solutions (B) 4 real solutions

(C) 6 real solutions (D) 8 real solutions

Q.6 With usual notations, in a triangle ABC, $a\cos(B-C) + b\cos(C-A) + c\cos(A-B)$ is equal to

(A) $\frac{abc}{R^2}$ (B) $\frac{abc}{4R^2}$ (C) $\frac{4abc}{R^2}$ (D) $\frac{abc}{2R^2}$

Q.7 If $\cos\alpha = \frac{2\cos\beta - 1}{2 - \cos\beta}$ then $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$ has the value equal to, where $(0 < \alpha < \pi \text{ and } 0 < \beta < \pi)$

(A) 2 (B) $\sqrt{2}$ (C) 3 (D) $\sqrt{3}$

Q.8 If $x\sin\theta = y\sin\left(\theta+\frac{2\pi}{3}\right) = z\sin\left(\theta+\frac{4\pi}{3}\right)$ then

(A) $x+y+z=0$ (B) $xy+yz+zx=0$

(C) $xyz+x+y+z=1$ (D) None of these

Q.9 If $a\cos^2\alpha + 3a\cos\alpha\sin^2\alpha = m$ and $a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n$. Then $(m+n)^{2/3}+(m-n)^{2/3}$ is equal to:

(A) $2a^2$ (B) $2a^{1/3}$ (C) $2a^{2/3}$ (D) $2a^3$

Q.10 The number of solutions of $\tan(5\pi\cos\theta) = \cot(5\pi\sin\theta)$ for θ in $(0,2\pi)$ is

(A) 28 (B) 14 (C) 4 (D) 2

Q.11 In a ΔABC if B + C = 3A then $\cot\frac{B}{2}.\cot\frac{C}{2}$ has the value equal to

(A) 4 (B) 3 (C) 2 (D) 1

Q.12 The set of value of 'a' for which the equation, $\cos 2x + a\sin x = 2a-7$ possess a solution is-

(A) $(-\infty,2)$ (B) [2, 6] (C) $(6,\infty)$ (D) $(-\infty,\infty)$

Q.13 In ΔABC, the minimum value of $\dfrac{\sum\cot^2\frac{A}{2}.\cot^2\frac{B}{2}}{\prod\cot^2\frac{A}{2}}$ is

(A) 1 (B) 2

(C) 3 (D) Non-existent

Q.14 The general solution of $\sin x + \sin 5x = \sin 2x + \sin 4x$ is:

(A) $2n\pi$ (B) $n\pi$

(C) $n\pi/3$ (D) $2n\pi/3$ Where $n\in I$

Q.15 Number of values of $\theta\in[0,2\pi]$ satisfying the equation cotx – cosx = 1 – cotx. cosx

(A) 1 (B) 2 (C) 3 (D) 4

Q.16 The exact value of $\cos^2 73° + \cos^2 47° + (\cos 73°.\cos 47°)$ is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1

Q.17 The maximum value of $(7\cos\theta + 24\sin\theta)\times(7\sin\theta - 24\cos\theta)$ for every $\theta\in R$.

(A) 25 (B) 625 (C) $\frac{625}{2}$ (D) $\frac{625}{4}$

Q.18 If $x=\frac{n\pi}{2}$, satisfies the equation $\sin\frac{x}{2}-\cos\frac{x}{2}=1-\sin x$ & the inequality $\left|\frac{x}{2}-\frac{\pi}{2}\right|\le\frac{3\pi}{4}$, then :

(A) n = –1, 0, 3, 5 (B) n = 1, 2, 4, 5

(C) n = 0, 2, 4 (D) n = – 1, 1, 3, 5

Q.19 The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2\cos 2x + a_3\sin^2 x = 0$ for all x is

(A) 0 (B) 1 (C) 3 (D) Infinite

Q.20 If A and B are complimentary angles, then:

(A) $\left(1+\tan\frac{A}{2}\right)\left(1+\tan\frac{B}{2}\right)=2$

(B) $\left(1+\cot\frac{A}{2}\right)\left(1+\cot\frac{B}{2}\right)=2$

(C) $\left(1+\sec\frac{A}{2}\right)\left(1+\operatorname{cosec}\frac{B}{2}\right)=2$

(D) A set containing two values

Previous Years' Questions

Q.1 The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}$, $0 < x \le \frac{\pi}{2}$ has ***(1980)***

(A) No real solution

(B) One real solution

(C) More than one real solution

(D) None of above

Q.2 The smallest positive root of the equation, tan x – x = 0 lies in ***(1987)***

(A) $\left(0,\frac{\pi}{2}\right)$ (B) $\left(\frac{\pi}{2},\pi\right)$ (C) $\left(\pi,\frac{3\pi}{2}\right)$ (D) $\left(\frac{3\pi}{2},2\pi\right)$

Q.3 The number of solution of the equation $\sin(e^x) = 5^x + 5^{-x}$ is ***(1991)***

(A) 0 (B) 1 (C) 2 (D) Infinitely many

Q.4 The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution, is ***(2002)***

(A) 4 (B) 8 (C) 10 (D) 12

Q.5 The set of values of θ satisfying the in equation $2\sin^2\theta - 5\sin\theta + 2 > 0$, where $0 < \theta < 2\pi$, is ***(2006)***

(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (B) $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$

(C) $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, 2\pi\right]$ (D) None of these

Q.6 The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\cos\theta = 0$ in the interval $[0, 2\pi]$ is ***(2007)***

(A) 0 (B) 1 (C) 2 (D) 4

Q.7 Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then ***(2011)***

(A) $P \subset Q$ and $Q - P \neq \varnothing$

(B) $Q \not\subset P$

(C) $P \not\subset Q$

(D) P = Q

Q.8 If $\alpha + \beta + \gamma = 2\pi$, then ***(1979)***

(A) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(B) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

(C) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(D) None of these

Q.9 Given A = $\sin^2\theta + \cos^4\theta$, then for all real values of θ ***(1980)***

(A) $1 \le A \le 2$ (B) $\frac{3}{4} \le A \le 1$

(C) $\frac{13}{16} \le A \le 1$ (D) $\frac{3}{4} \le A \le \frac{13}{16}$

Q.10 The expression ***(1986)***

$$3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right]$$

$$-2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$$ is equal to

(A) 0 (B) 1

(C) 3 (D) $\sin 4\alpha + \cos 6\alpha$

Q.11 If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan\alpha$, equals ***(2001)***

(A) $2(\tan\beta + \tan\gamma)$ (B) $\tan\beta + \tan\gamma$

(C) $\tan\beta + 2\tan\gamma$ (D) $2\tan\beta + \tan\gamma$

Q.12 Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{\cot\theta}$

$t_3 = (\cos\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then ***(2006)***

(A) $t_1 > t_2 > t_3 > t_4$ (B) $t_4 > t_3 > t_1 > t_2$

(C) $t_3 > t_1 > t_2 > t_4$ (D) $t_2 > t_3 > t_1 > t_4$

Q.13 The expression $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$ can be written as ***(2013)***

(A) secA cosecA + 1 (B) tanA + cotA

(C) secA + cosecA (D) sinA cosA + 1

Q.14 If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation cosx + cos2x + cos3x + cos4x = 0 is: ***(2016)***

(A) 5 (B) 7 (C) 9 (D) 3

JEE Advanced/Boards

Exercise 1

Q.1 Solve the equation: $\sin 5x = 16\sin^5 x$

Q.2 Find all the solutions, of $4\cos^2 x\sin x - 2\sin^2 x = 3\sin x$

Q.3 Find the number of solutions of the equation $1+\cos x+\cos 2x+\sin x+\sin 2x+\sin 3x = 0$.
Which satisfy the condition $\frac{\pi}{2} < \left|3x-\frac{\pi}{2}\right| \le \pi$.

Q.4 Solve for x, $(-\pi \le x \le \pi)$ the equation; $2(\cos x+\cos 2x)+\sin 2x(1+2\cos x) = 2\sin x$

Q.5 Find the general solution of the following equation:

$$2(\sin x-\cos 2x)-\sin 2x\ (1+2\sin x)+2\cos x = 0$$

Q.6 Find the values of x, between 0 & 2π. Satisfying the equation $\cos 3x+\cos 2x = \sin\frac{3x}{2}+\sin\frac{x}{2}$.

Q.7 Solve: $\tan^2 2x+\cot^2 2x+2\tan 2x+2\cot 2x = 6$

Q.8 Solve the equation: $1 + 2\ \text{cosec}x = -\frac{\sec^2\frac{x}{2}}{2}$ and the inequality $\frac{2\cos 7x}{\cos 3+\sin 3} > 2^{\cos 2x}$

Q.9 Solve $\sin\left(\frac{\sqrt{x}}{2}\right)+\cos\left(\frac{\sqrt{x}}{2}\right) = \sqrt{2}\sin\sqrt{x}$

Q.10 Find all values of 'a' for which every root of the equation, $a\cos 2x + |a|\cos 4x + \cos 6x = 1$ is also a root of the equation, $\sin x\cos 2x = \sin 2x\cos 3x - \frac{1}{2}\sin 5x$, and conversely, every root of the second equation is also a root of the first equation.

Q.11 Solve for x, the equation $\sqrt{13-18\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.

Q.12 Determine the smallest positive value of x which satisfy the equation $\sqrt{1+\sin 2x}-\sqrt{2}\cos 3x = 0$

Q.13 $2\sin\left(3x+\frac{\pi}{4}\right) = \sqrt{1+8\sin 2x.\cos^2 2x}$

Q.14 Find the number of principal solutions of the equation.
$\sin x-\sin 3x+\sin 5x = \cos x-\cos 3x+\cos 5x$

Q.15 Find the general solution of the trigonometric equation

$$3^{\left(\frac{1}{2}+\log_3(\cos x+\sin x)\right)}-2^{\log_2(\cos x-\sin x)} = \sqrt{2}$$

Q.16 Find all values of θ between 0° & 180° satisfying the equation; $\cos 6\theta+\cos 4\theta+\cos 2\theta+1 = 0$

Q.17 Find the solution set of the equation,

$$\log_{\frac{-x^2-6x}{10}}(\sin 3x+\sin x) = \log_{\frac{-x^2-6x}{10}}(\sin 2x)$$

Q.18 Find the value of θ, which satisfy
$3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$.

Q.19 Find the sum of the roots of the equation $\cos 4x+6 = 7\cos 2x$ on the interval $[0,314]$.

Q.20 Find the least positive angle measured in degrees satisfying the equation

$$\sin^3 x+\sin^3 2x+\sin^3 3x = (\sin x+\sin 2x+\sin 3x)^3$$

Q.21 Find the number of solution of the equation

$$\sin(\pi-6x)+\sqrt{3}\sin\left(\frac{\pi}{2}+6x\right) = \sqrt{3} \text{ in } [0,2\pi]$$

Q.22 Find the general values of θ for which the quadratic function
$(\sin\theta)x^2+(2\cos\theta)x+\frac{\cos\theta+\sin\theta}{2}$
is the square of a linear function.

Q.23 Prove that the equations

(a) $\sin x\cdot\sin 2x\cdot\sin 3x = 1$

(b) $\sin x\cdot\cos 4x\cdot\sin 5x = -1/2$

(c) $\sin x \cos x \cos 2x + 1/2 = 0$

(d) $4\sin 2x + \cos x = 5$

(e) $\sin 3x - \cos x = 2$

Have no solution

Q.24 Let $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k. Determine

(a) All real number k for which f(x) is constant for all values of x.

(b) All real numbers k for which there exists a real number 'c' such that f(c) = 0

(c) If k = – 0.7, determine all solutions to the equation f(x) = 0

Q.25 If α and β are the roots of the equation, $a\cos\theta + b\sin\theta = c$ then match the entries of column I with the entries of column II.

Column I	Column II
(A) $\sin\alpha + \sin\beta$	(p) $\frac{2b}{a+c}$
(B) $\sin\alpha.\sin\beta$	(q) $\frac{c-a}{c+a}$
(C) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}$	(r) $\frac{2bc}{a^2+b^2}$
(D) $\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}$	(s) $\frac{c^2-a^2}{a^2+b^2}$

Q.26 Solve the equations for 'x' given in column I and match the entries of column II.

Column I	Column II
(A) $\cos 3x.\cos^3 x + \sin 3x.\sin^3 x = 0$	(p) $n\pi \pm \frac{\pi}{3}$
(B) $\sin 3\alpha = 4\sin\alpha \sin(x+\alpha)$ $\sin(x-\alpha)$ Where α is a constant $\neq n\pi$	(q) $n\pi + \frac{\pi}{4}, n \in I$
(C) $\lvert 2\tan x - 1\rvert + \lvert 2\cot x - 1\rvert = 2$	(r) $\frac{n\pi}{4} + \frac{\pi}{8}, n \in I$
(D) $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$	(s) $\frac{n\pi}{2} \pm \frac{\pi}{4}$

Q.27

Column I	Column II
(A) The general solution of the equation $\sin^2 x + \cos^2 3x = 1$ is equal to	(p) $n\pi$(where n $\in$ I)
(B) The general solution of the equation $e^{\cot^2\theta} + \sin^2\theta - 2\cos^2 2\theta + 4 = 4\sin\theta$, is	(q) $\frac{n\pi}{4}$
(C) For all real values of a, the general solution of the equation $a^2 \sin x - a\sin 2x + \sin x = 0$, is equal to	(r) $n\pi + \frac{\pi}{4}$
(D) The general solution of the equation $\sqrt[3]{2\tan\theta - 1} + \sqrt[3]{\tan\theta - 1} = 1$, is	(s) $(4n+1)\frac{\pi}{2}$

Exercise 2

Single Correct Choice Type

Q.1 If in a ΔABC, cos A. cos B + sin A sinB sin 2C =1 then, the statement which is correct?

(A) ΔABC is isosceles and right angled

(B) ΔABC is acute angled

(C) ΔABC is not right angled

(D) Least angle of the triangle is $\frac{\pi}{3}$

Q.2 The set of values of x satisfying the equation,

$$2^{\tan\left(x-\frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x-\frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0 \text{ is}$$

(A) An empty set

(B) A singleton

(C) A set containing two elements

(D) An infinite set

Q.3 The number of solutions of the equation, $\sum_{r=1}^{5}\cos(rx) = 0$ lying in (0, π) is:

(A) 2 (B) 3 (C) 5 (D) More than 5

Q.4 The value of $\cot 7\frac{1^\circ}{2}+\tan 67\frac{1^\circ}{2}-\cot 67\frac{1^\circ}{2}-\tan 7\frac{1^\circ}{2}$ is

(A) A rational number (B) Irrational number

(C) $2\left(3+2\sqrt{3}\right)$ (D) $2\left(3-\sqrt{3}\right)$

Q.5 If A = 580° then which one of the following is true?

(A) $2\sin\left(\frac{A}{2}\right)=\sqrt{1+\sin A}-\sqrt{1-\sin A}$

(B) $2\sin\left(\frac{A}{2}\right)=-\sqrt{1+\sin A}+\sqrt{1-\sin A}$

(C) $2\sin\left(\frac{A}{2}\right)=-\sqrt{1+\sin A}-\sqrt{1-\sin A}$

(D) $2\sin\left(\frac{A}{2}\right)=\sqrt{1+\sin A}+\sqrt{1-\sin A}$

Q.6 If $\tan\alpha=\frac{x^2-x}{x^2-x+1}$ and

$\tan\beta=\frac{1}{2x^2-2x+1}\left(x\neq 0,1\right)$, where

$0<\alpha,\beta<\frac{\pi}{2}$, then $\tan\left(\alpha+\beta\right)$ has the value equal to:

(A) 1 (B) –1 (C) 2 (D) $\frac{3}{4}$

Q.7 Minimum value of $8\cos^2 x+18\sec^2 x\ \forall\ x\in R$ wherever it is defined, is:

(A) 24 (B) 25 (C) 26 (D) 18

Q.8 If ϕ is eliminated from the equations x=a $\cos(\phi-\alpha)$ and y=b $\cos(\phi-\beta)$ then $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos(\alpha-\beta)$ is equal to

(A) $\cos^2(\alpha-\beta)$ (B) $\sin^2(\alpha-\beta)$

(C) $\sec^2(\alpha-\beta)$ (D) $\mathrm{cosec}^2(\alpha-\beta)$

Q.9 The general solution of the trigonometric equation $\tan x+\tan 2x+\tan 3x=\tan x.\tan 2x.\tan 3x$ is

(A) $x=n\pi$ (B) $n\pi\pm\frac{\pi}{3}$

(C) $x=2n\pi$ (D) $x=\frac{n\pi}{3}$ Where $n\in I$

Q.10 Number of principal solutions of the equation tan3x-tan2x-tanx=0 , is

(A) 3 (B) 5 (C) 7 (D) More than 7

Q.11 The value of x that satisfies the relation $x=1-x+x^2-x^3+x^4-x^5+\ldots\ldots\ldots\ldots\infty$

(A) $2\cos 36^\circ$ (B) $2\cos 144^\circ$

(C) $2\sin 18^\circ$ (D) None of these

Multiple Correct Choice Type

Q.12 An extreme value of $1+4\sin\theta+3\cos\theta$ is:

(A) -3 (B) -4 (C) 5 (D) 6

Q.13 It is known that $\sin\beta=\frac{4}{5}\ \&\ 0<\beta<\pi$ then the value of

$$\frac{\sqrt{3}\sin(\alpha+\beta)-\frac{2}{\cos\frac{\pi}{6}}\cos(\alpha+\beta)}{\sin\alpha}$$ is:

(A) Independent of α for all β in (0, $\pi/2$)

(B) $\frac{5}{\sqrt{3}}$ for tan $\beta>0$

(C) $\frac{\sqrt{3}(7+24\cot\alpha)}{15}$ for tan $\beta<0$

(D) None of these

Q.14 If sint +cost= $\frac{1}{5}$ then tan $\frac{1}{5}$ is equal to:

(A) -1 (B) -1/3 (C) 2 (D) -1/6

Previous Years' Questions

Q.1 Show that the equation $e^{\sin x}-e^{-\sin x}-4=0$ has no real solution. ***(1982)***

Q.2 Find the values of $x\left(-\pi,\pi\right)$ which satisfy the equation $2^{1+|\cos x|+|\cos^2 x|+\ldots\ldots}=4$ ***(1984)***

Q.3 If exp $\left\{\left(\sin^2 x+\sin^4 x+\sin^6 x+\ldots\ldots+\infty\right)\log_e 2\right\}$, satisfies the equation $x^2-9x+8=0$, find the value of $\frac{\cos x}{\cos x+\sin x},0<x<\frac{\pi}{2}$. ***(1991)***

Q.4 Determine the smallest positive value of x (in degree) for which tan (x + 100°) = tan (x + 50°) tan (x) tan (x – 50°) ***(1993)***

Q.5 Find the smallest positive number p for which the equation cos (p sin x) –sin (pcos x)=0 has a solution $x \in [0, 2\pi]$ ***(1995)***

Q.6 Find all value of θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfying the equation

$(1-\tan\theta)(1+\tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$ ***(1996)***

Q.7 The values of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$ ***(1988)***

(A) $\frac{7\pi}{24}$ (B) $\frac{5\pi}{24}$ (C) $\frac{11\pi}{24}$ (D) $\frac{\pi}{24}$

Q.8 For $0 < \phi < \pi/2$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then ***(1993)***

(A) $xyz = xy + y$ (B) $xyz = xy + z$

(C) $xyz = x + y + z$ (D) $xyz = yz + x$

Q.9 $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if ***(1996)***

(A) $x = y \neq 0$ (B) $x = y, x \neq 0$

(C) $x = y$ (D) $x \neq 0, y \neq 0$

Q.10 If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then ***(2009)***

(A) $\tan^2 x = \frac{2}{3}$ (B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$ (D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

Q.11 For $0 < \theta < \frac{\pi}{2}$, the solution(s) of

$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is/are ***(2009)***

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

Q.12 The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for n = 0, ± 1, ± 2 and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is .... ***(2010)***

Q.13 The positive value of n > 3 satisfying the equation

$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is ...... ***(2011)***

Q.14 The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$(y+z)\cos 3\theta = (xyz)\sin 3\theta$ $x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$

And $(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is.... ***(2010)***

Q.15 The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$ is ***(2010)***

Q.16 The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is ***(2010)***

Q.17 Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$

and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then ***(2011)***

(A) $P \subset Q$ and $Q - P \neq \varnothing$ (B) $Q \not\subset P$

(C) $P \not\subset Q$ (D) P = Q

Q.18 The positive integer value of n > 3 satisfying the equation

$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is ***(2011)***

Q.19 Let $\theta, \phi \in [0, 2\pi]$ be such that

$$2\cos\theta(1-\sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)\cos\phi - 1,$$

$\tan(2\pi-\theta)>0$ and $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$

Then ϕ cannot satisfy ***(2012)***

(A) $0 < \phi < \frac{\pi}{2}$ (B) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$

(C) $\frac{4\pi}{3} < \phi < \frac{3\pi}{2}$ (D) $\frac{3\pi}{2} < \phi < 2\pi$

Q.20 For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has ***(2014)***

(A) Infinitely many solutions

(B) Three solutions

(C) One solution

(D) No solution

Q.21 The number of distinct solution of the equation

$$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$$

In the interval $[0, 2\pi]$ is ***(2015)***

Q.22 Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$.

The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to ***(2016)***

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

PlancEssential Questions

JEE Main/Boards

Exercise 1

Q.11 Q.17 Q.24
Q.29

Exercise 2

Q.6 Q.9 Q.14
Q.17 Q.19

Previous Years' Questions

Q.1 Q.4 Q.5
Q.8 Q.10 Q.13

JEE Advanced/Boards

Exercise 1

Q.7 Q.12 Q.17
Q.21 Q.24 Q.26

Exercise 2

Q.4 Q.7 Q.8
Q.11

Previous Years' Questions

Q.4 Q.5 Q.6
Q.8 Q.10 Q.12

Answer Key

JEE Main/Boards

Exercise 1

Q.1 (i) $\frac{n\pi}{2}+\left(-1\right)^n\frac{\pi}{12},\ n\in I$ (ii) $\frac{2n\pi}{5}\pm\frac{2\pi}{15},\ n\in I$

Q.2 $2n\pi\pm\frac{\pi}{3},2n\pi\pm\frac{2\pi}{3},\ n\in I$

Q.3 $\frac{n\pi}{3}+\frac{\pi}{9},n\in I$

Q.4 $\frac{n\pi}{2}+\left(-1\right)^n\frac{\pi}{4},n\in I$

Q.5 $x=\frac{1}{2}\left[n\pi+\left(-1\right)^n\frac{\pi}{3}+2m\pi\pm\frac{\pi}{3}\right],n,m\in I,$

$y=\frac{1}{2}\left[2m\pi\pm\frac{\pi}{3}-n\pi-\left(-1\right)^n\frac{\pi}{3}\right],n,m\in I$

Q.6 $x=n\pi,n\in I$

Q.7 $\tan\alpha=\frac{1}{2}$

Q.8 $x=\frac{2a\pi}{m+n}$ or $x=\frac{\left(2b+1\right)\pi}{m-n},a,b\in I$

Q.9 $x=\frac{n\pi}{2},\frac{m\pi}{2}-\frac{\pi}{8},n\in I$

Q.10 $x=n\pi-\left(-1\right)^n\frac{\pi}{6},2n\pi\pm\frac{2\pi}{3},n\in I$

Q.11 $x=n\pi+\left(-1\right)^n\frac{\pi}{6}$ or $(2n+1)\frac{\pi}{2},n\in I$

Q.12 $x=n\pi+\tan^{-1}(-\frac{3}{4})$ or $x=n\pi+\tan^{-1}(2)$

Q.13 $n\pi$ or $\frac{n\pi}{3}\pm\frac{\pi}{9}$, where $n\in I$

Q.14 $n\pi,n\pi-\frac{\pi}{4}$, where $n\in I$

Q.15 $2n\pi+\frac{5\pi}{12},2n\pi-\frac{\pi}{12},n\in I$

Q.16 (i) $\frac{n\pi}{3}-\frac{\pi}{12},n\in I$ (ii) $\frac{2n\pi}{5}\pm\frac{3\pi}{20},n\in I$

Q.17 (i) $n\pi+\left(-1\right)^n\left(\pm\frac{\pi}{6}\right)$ (ii) $x=\frac{n\pi}{3},n\in I$

(iii) $n\pi,n\pi\pm\frac{n\pi}{3},n\in I$

Q.18 $n\pi+\frac{\pi}{4},n\in I$

Q.19 $n\pi,n\pi\pm\frac{\pi}{3},n\in I$

Q.20(i) $2n\pi\pm\frac{\pi}{4}+\frac{\pi}{4},n\in I$ i.e.

$2n\pi$ or $2n\pi+\frac{\pi}{2},n\in I$ (ii) $2n\pi\pm\frac{\pi}{3}+\frac{\pi}{6},n\in I$

(iii) $2n\pi+\frac{7\pi}{12},n\in I$ (iv) $2n\pi,2n\pi+\frac{2\pi}{3},n\in I$

Q.21 (i) 30° (ii) 144° (iii) 68° 43′ 37.8″

Q.22 30°, 60°, 90° and $\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$

Q.23 3466.36 km

Q.24 17.4 cm

Q.25 $x=n\pi,n\pi+\left(-1\right)^{n+1}\frac{3\pi}{10},n\pi+\left(-1\right)^n\frac{\pi}{10}$

where $n=0,\pm1,\pm2,\ldots\ldots$

Q.26 $\theta=\frac{\pi}{3},\frac{-\pi}{3},\pi-\cos^{-1}\left(\frac{3}{5}\right)$

Q.27 $x=n\pi,x=n\pi\pm\ \alpha$ where $\sin\alpha=\frac{\sqrt{2}}{5}$

Q.28 $\theta=n\pi\pm\frac{\pi}{6}$

Q.30 NO real Solution

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** B	**Q.3** D	**Q.4** D	**Q.5** D	**Q.6** A
Q.7 D	**Q.8** B	**Q.9** C	**Q.10** A	**Q.11** C	**Q.12** B
Q.13 A	**Q.14** C	**Q.15** B	**Q.16** C	**Q.17** C	**Q.18** B
Q.19 D	**Q.20** A				

Previous Years' Questions

Q.1 A	**Q.2** C	**Q.3** A	**Q.4** B	**Q.5** A	**Q.6** C
Q.7 D	**Q.8** A	**Q.9** B	**Q.10** B	**Q.11** C	**Q.12** B
Q.13 A	**Q.14** B				

JEE Advanced/Boards

Exercise 1

Q.1 $x = n\pi$ or $x = n\pi \pm \frac{\pi}{6}$

Q.2 $n\pi; n\pi + (-1)^n \frac{\pi}{10}$ or $n\pi + (-1)^n \left(\frac{3\pi}{10}\right)$

Q.3 2

Q.4 $\frac{\pm\pi}{3}, \frac{-\pi}{2}, \pm\pi$

Q.5 $x = 2n\pi$ or $x = n\pi + (-1)^n \left(-\frac{\pi}{2}\right)$ or $x = n\pi + (-1)^n \frac{\pi}{6}$

Q.6 $\frac{\pi}{7}, \frac{5\pi}{7}, \pi \frac{9\pi}{7}, \frac{13\pi}{7}$

Q.7 $x = \frac{n\pi}{2} + \frac{\pi}{8}, \frac{n\pi}{2} - \frac{\pi}{24}, \frac{n\pi}{2} - \frac{5\pi}{24}$

Q.8 $x = 2n\pi - \frac{\pi}{2}$

Q.9 $x = \left(4n\pi - \frac{\pi}{2}\right)^2$ or $x = \left(\frac{4m\pi}{3} + \frac{\pi}{2}\right)^2$

where $m, n \in W$

Q.10 $x = 0$ or $a < -1$

Q.11 $\alpha - 2\pi; \alpha - \pi, \alpha + \pi$, where $\tan\alpha = \frac{2}{3}$

Q.12 $x = \pi/16$

Q.13 $x = n\pi + \frac{\pi}{12}; n \in I$

Q.14 10 solutions

Q.15 $x = 2n\pi + \frac{\pi}{12}$

Q.16 30°, 45°, 90°, 135°, 150°

Q.17 No Solution

Q.18 $\theta = 2n\pi$ or $2n\pi + \frac{\pi}{2}; n \in I$

Q.19 4950 π

Q.20 72°

Q.21 13

Q.23 $2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \tan^{-1} 2; n \in I$

Q.24 (a) $-\frac{3}{2}$; (b) $k \in \left[-1, -\frac{1}{2}\right]$; (c) $x = \frac{n\pi}{2} \pm \frac{\pi}{6}$

Q.25 A $\rightarrow$ r; B $\rightarrow$ s; C $\rightarrow$ p; D $\rightarrow$ q

Q.26 A $\rightarrow$ s; B $\rightarrow$ p; C $\rightarrow$ q; D $\rightarrow$ r

Q.27 A $\rightarrow$ q; B $\rightarrow$ s; C $\rightarrow$ p; D $\rightarrow$ r

Exercise 2

Single Correct Choice Type

Q.1 A	**Q.2** A	**Q.3** A	**Q.4** B	**Q.5** C	**Q.6** A
Q.7 C	**Q.8** B	**Q.9** D	**Q.10** C	**Q.11** C	

Multiple Correct Choice Type

Q.12 B, D	**Q.13** A, B, C	**Q.14** B, C

Previous Years' Questions

Q.2 $\left\{\pm\frac{\pi}{3},\pm\frac{2\pi}{3}\right\}$	**Q.3** $\frac{\sqrt{3}-1}{2}$	**Q.4** 30°	**Q.5** $\frac{\pi}{2\sqrt{2}}$	**Q.6** $\theta=\pm\frac{\pi}{3}$	**Q.7** A C
Q.8 B, C	**Q.9** A, B	**Q.10** A, B	**Q.11** D	**Q.12** 3	**Q.13** 7
Q.14 3	**Q.15** 3	**Q.16** 2	**Q.17** D	**Q.18** 7	**Q.19** A, C, D
Q.20 D	**Q.21** 8	**Q.22** C			

Solutions

JEE Main/Boards

Exercise 1

Sol 1: (i) $\sin 2\theta = \frac{1}{2}$

General solution of $\sin x = \frac{1}{2}$ is

$x = n\pi + (-1)^n\frac{\pi}{6}$, $n \in I$

$\therefore 2\theta = n\pi + (-1)^n\frac{\pi}{6}$

$\Rightarrow \theta = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12}$, $n \in I$

(ii) $\cos 5\theta = -\frac{1}{2}$

General solution of $\cos x = -\frac{1}{2}$ is

$x = 2n\pi \pm \frac{2\pi}{3}$, $n \in I$

$\therefore 5\theta = 2n\pi \pm \frac{2\pi}{3}$, $n \in I$

$\Rightarrow \theta = \frac{2n\pi}{5} \pm \frac{2\pi}{15}$, $n \in I$

Sol 2: $7\cos^2\theta + 3\sin^2\theta = 4$

Since $\sin^2\theta + \cos^2\theta = 1$

$\therefore 4\cos^2\theta + 3(\cos^2\theta + \sin^2\theta) = 4$

$\Rightarrow 4\cos^2\theta + 3 = 4$

$\therefore \cos^2\theta = \frac{1}{4}$ or $\cos\theta = \frac{1}{2}, -\frac{1}{2}$

$\therefore \theta = 2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \frac{2\pi}{3}$, $n \in I$

Sol 3: tanx + tan2x + $\sqrt{3}$ tanx tan2x = $\sqrt{3}$

$\Rightarrow$ tanx + tan2x = $\sqrt{3}$ (1 – tanx tan2x)

$\therefore \dfrac{\tan x + \tan 2x}{1 - \tan x \tan 2x} = \sqrt{3}$

$\because \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ = tan(A + B)

Applying the above formula

tan (x + 2x) = $\sqrt{3}$ $\Rightarrow$ tan3x = $\sqrt{3}$

General solution of tanθ = $\sqrt{3}$ is

$\theta = n\pi + \dfrac{\pi}{3}, n \in I$

$\therefore 3x = n\pi + \dfrac{\pi}{3} \Rightarrow x = \dfrac{n\pi}{3} + \dfrac{\pi}{9}$

Sol 4: 3tan(θ – 15°) = tan(θ + 15°)

We can write it as

$$\frac{\tan(\theta + 15°)}{\tan(\theta - 15°)} = \frac{\tan\left(\theta + \frac{\pi}{12}\right)}{\tan\left(\theta - \frac{\pi}{12}\right)} = 3$$

Applying $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\Rightarrow \frac{\sin\left(\theta + \frac{\pi}{12}\right)\cos\left(\theta - \frac{\pi}{12}\right)}{\sin\left(\theta - \frac{\pi}{12}\right)\cos\left(\theta + \frac{\pi}{12}\right)} = 3$$

Using sin A cos B = $\dfrac{1}{2}$ [sin(A + B) + sin(A – B)]

Above expression can be written as

$$\frac{\frac{1}{2}\left[\sin\left(\theta + \frac{\pi}{12} + \theta - \frac{\pi}{12}\right) + \sin\left(\theta + \frac{\pi}{12} - \theta + \frac{\pi}{12}\right)\right]}{\frac{1}{2}\left[\sin\left(\theta + \frac{\pi}{12} + \theta - \frac{\pi}{12}\right) + \sin\left(\theta - \frac{\pi}{12} - \theta - \frac{\pi}{12}\right)\right]} = 3$$

$$\Rightarrow \frac{\sin 2\theta + \sin\frac{\pi}{6}}{\sin 2\theta - \sin\frac{\pi}{6}} = 3$$

$\Rightarrow$ 2sin2θ = 4sin$\dfrac{\pi}{6}$ $\Rightarrow$ sin2θ = 2 × sin $\dfrac{\pi}{6}$ = 1

$\therefore$ sin2θ = 1

General solution of sin x = 1

is x = nπ + $(-1)^n \dfrac{\pi}{2}$, n ∈ I

$\therefore \theta = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{4}, n \in I$

Sol 5: sin(x – y) = $\dfrac{\sqrt{3}}{2}$, cos(x + y) = $\dfrac{1}{2}$

$\therefore$ x – y = nπ + $(-1)^n \dfrac{\pi}{3}$ n ∈ I. ... (i)

and x + y = 2mπ ± $\dfrac{\pi}{3}$ n ∈ I ... (ii)

$\therefore$ 2x = nπ + $(-1)^n \dfrac{\pi}{3}$ + 2mπ ± $\dfrac{\pi}{3}$

[Adding (i) and (ii)]

and x = $\dfrac{1}{2}\left[n\pi + (-1)^n \dfrac{\pi}{3} + 2m\pi \pm \dfrac{\pi}{3}\right]$

Similarly,

$$y = \frac{1}{2}\left[2n\pi \pm \frac{\pi}{3} - \left(n\pi + (-1)^3 \frac{\pi}{3}\right)\right], m, n \in I$$

[Subtracting (i) from (ii)]

Sol 6: sinx = tanx

$\because$ tanx = $\dfrac{\sin x}{\cos x}$ $\Rightarrow$ sinx = $\dfrac{\sin x}{\cos x}$

$\Rightarrow$ sin x cos x=sin x $\Rightarrow$ sin x(cos x–1) = 0

$\therefore$ sin x = 0 or cos x = 1 for sin x = 0

x = nπ, n ∈ I

and for cos x = 1, x = 2mπ, m ∈ I

As the equation is valid for sin x = 0 or cos x = 1, is the solution will be union of both.

$\therefore$ x = nπ, n ∈ I

Sol 7: 2tanθ – cotθ + 1 = 0

cotθ = $\dfrac{1}{\tan\theta}$

$\therefore$ 2tanθ – $\dfrac{1}{\tan\theta}$ + 1 = 0

$$\Rightarrow \frac{2\tan^2\theta + \tan\theta - 1}{\tan\theta} = 0$$

$$\Rightarrow \frac{(2\tan\theta - 1)(\tan\theta + 1)}{\tan\theta} = 0$$

$\Rightarrow$ tanθ ≠ 0 and tanθ = $\dfrac{1}{2}$ or tanθ = –1

$\therefore$ From tanθ ≠ 0, θ ≠ nπ and

tanθ = –1, θ = nπ – $\dfrac{\pi}{4}$ n ∈ I

∴ Solution of equation is $\tan\theta = \frac{1}{2}, -1$

i.e. $\theta = n\pi + \tan^{-1}\frac{1}{2}, n\pi - \frac{\pi}{4} \; n \in I$

Sol 8: sin mx + sin nx = 0

$\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\therefore\ 2\sin\left((m+n)\frac{x}{2}\right)\cos\left((m-n)\frac{x}{2}\right) = 0$

$\therefore \sin(m+n)\frac{x}{2} = 0$ or $\cos\left((m-n)\frac{x}{2}\right) = 0$

$\Rightarrow (m+n)\frac{x}{2} = a\pi$ or $(m-n)\frac{x}{2} = (2b+1)\frac{\pi}{2}\ \ a,b \in I$

$\therefore x = \frac{2a}{m+n}\pi$ or $\frac{(2b+1)}{m-n}\pi\ \ a,b \in I$

Sol 9: $\sec^2 2x = 1 - \tan 2x$

$[1 + \tan^2\theta = \sec^2\theta]$

$\therefore 1 + \tan^2 2x = 1 - \tan 2x$

$\Rightarrow \tan^2 2x + \tan 2x = 0$

$\Rightarrow \tan 2x(1 + \tan 2x) = 0$

$\tan 2x = 0$ or $\tan 2x = -1$

$2x = n\pi$ or $2x = m\pi - \frac{\pi}{4}\ \ n, m \in I$

$\therefore x = \frac{n\pi}{2}$ or $\frac{m\pi}{2} - \frac{\pi}{8}\ \ n, m \in I$

Sol 10: 4sinx cosx + 2sinx + 2cosx + 1 = 0

$\Rightarrow (2\sin x + 1) \times (2\cos x + 1) = 0$

$\Rightarrow \sin x = -\frac{1}{2}$ or $\cos x = -\frac{1}{2}$

$\therefore x = n\pi - (-1)^n \frac{\pi}{6}$ (when $\sin x = -\frac{1}{2}$)

or $x = 2n\pi \pm \frac{2\pi}{3}$ (when $\cos x = -\frac{1}{2}$)

Sol 11: tanx + secx = 2cosx

$\tan x = \frac{\sin x}{\cos x}, \sec x = \frac{1}{\cos x}$

$\therefore \frac{1+\sin x}{\cos x} = 2\cos x$

$\Rightarrow 1 + \sin x = 2\cos^2 x$

$[\because \sin^2 x + \cos^2 x = 1]$

$\therefore 1 + \sin x = 2(1 - \sin^2 x) = 2(1 - \sin x)(1 + \sin x)$

$\Rightarrow [2(1 - \sin x) - 1](1 + \sin x) = 0$

$\Rightarrow [(1 - 2\sin x)(1 + \sin x)] = 0$

$\therefore \sin x = \frac{1}{2}$ or -1

For $\sin x = \frac{1}{2}$ $\quad x = n\pi + (-1)^n\frac{\pi}{6}\ \ n \in I$

For $\sin x = -1$ $\quad x = (2n+1)\frac{\pi}{2}\ \ n \in I$

$\therefore x = n\pi + (-1)^n\frac{\pi}{6}$ or $(2n+1)\frac{\pi}{2}\ n \in I$

Sol 12: $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x + 2 = 0$

We can write 2 as $2(\cos^2 x + \sin^2 x)$

$\because \cos^2 x + \sin^2 x = 1$

$\therefore 2\sin^2 x - 5\sin x\cos x - 8\cos^2 x + 2\cos^2 x + 2\sin^2 x = 0$

$\Rightarrow 4\sin^2 x - 5\sin x\cos x - 6\cos^2 x = 0$

$4\sin^2 x - 8\sin x\cos x + 3\sin x\cos x - 6\cos^2 x = 0$

$4\sin x(\sin x - 2\cos x) + 3\cos x(\sin x - 2\cos x) = 0$

$\therefore (4\sin x + 3\cos x)(\sin x - 2\cos x) = 0$

$\Rightarrow \sin x = -\frac{3}{4}\cos x$ or $\sin x = 2\cos x$

or $\tan x = -\frac{3}{4}$ or $\tan x = 2$

$x = n\pi + \tan^{-1}\left(-\frac{3}{4}\right)$ or $x = n\pi + \tan^{-1}(2)$

Sol 13: 4sinx sin2x sin4x = sin3x

2sinA sinB = cos(A – B) – cos(A + B)

2sinA cosB = sin(A + B) + sin(A – B)

$\therefore 2\sin x[2\sin 2x \sin 4x] = 2\sin x[\cos 2x - \cos 6x]$

$\Rightarrow 2\sin x\cos 2x - 2\sin x\cos 6x = \sin 3x$

$\Rightarrow \sin 3x + \sin(-x) - [\sin 7x + \sin(-5x)] = \sin 3x$

$\Rightarrow -\sin x = \sin 7x - \sin 5x$

$-\sin x = 2\cos 6x \sin x$

$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\Rightarrow \sin x(2\cos 6x + 1) = 0$

$\sin x = 0$ or $\cos 6x = -\frac{1}{2}$

$x = n\pi$ or $6x = 2n\pi \pm \frac{2\pi}{3} \Rightarrow x = n\pi, \frac{n\pi}{3} \pm \frac{\pi}{9}$

Sol 14: $(1 - \tan\theta)(1 + \sin 2\theta) = (\tan\theta + 1)$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\therefore \left(\dfrac{\cos\theta - \sin\theta}{\cos\theta}\right)(1 + \sin 2\theta) = \left(\dfrac{\cos\theta + \sin\theta}{\cos\theta}\right)$

$\Rightarrow (1 + \sin 2\theta) = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

$= \dfrac{(\cos\theta + \sin\theta)^2}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$

$\therefore (1 + \sin 2\theta) = \dfrac{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$

$(1 + \sin 2\theta) = \dfrac{1 + \sin 2\theta}{\cos 2\theta}$

$\Rightarrow \cos 2\theta (1 + \sin 2\theta) - (1 + \sin 2\theta) = 0$

$(\cos 2\theta - 1)(\sin 2\theta + 1) = 0$

$\sin 2\theta = -1$ or $\cos 2\theta = 1$

$\therefore 2\theta = 2n\pi - \dfrac{\pi}{2}$ or $2\theta = 2n\pi, n \in I$

$\therefore \theta = n\pi - \dfrac{\pi}{4}$ or $n\pi$ $n \in I$

Sol 15: $\sin x + \sqrt{3}\cos x = \sqrt{2}$

$\Rightarrow \dfrac{1}{2}\sin x + \dfrac{\sqrt{3}}{2}\cos x = \dfrac{\sqrt{2}}{2}$

$\Rightarrow \sin\dfrac{\pi}{6}\sin x + \cos\dfrac{\pi}{6}\cos x = \cos\dfrac{\pi}{4}$

$\Rightarrow \cos\left(x - \dfrac{\pi}{6}\right) = \cos\dfrac{\pi}{4}$

$x - \dfrac{\pi}{6} = 2n\pi \pm \dfrac{\pi}{4}$

$\therefore x = 2n\pi + \dfrac{5\pi}{12}, 2n\pi - \dfrac{\pi}{12}$ $n \in I$

Sol 16: (i) $\tan 3\theta = -1$

General solution of $\tan x = -1$ is

$x = n\pi - \dfrac{\pi}{4}, n \in I$

$\therefore 3\theta = n\pi - \dfrac{\pi}{4} \Rightarrow \theta = \dfrac{n\pi}{3} - \dfrac{\pi}{12}, n \in I$

(ii) $\cos 5x = -\dfrac{1}{\sqrt{2}}$

General solution for $\cos\theta = -\dfrac{1}{\sqrt{2}}$ is

$\theta = 2n\pi \pm \dfrac{3\pi}{4}$ $n \in I$

$\therefore 5x = 2n\pi \pm \dfrac{3\pi}{4} \Rightarrow x = \dfrac{2n\pi}{5} \pm \dfrac{3\pi}{20}, n \in I$

Sol 17: (i) $3\cos^2\theta + 7\sin^2\theta = 4$

$3(\cos^2\theta + \sin^2\theta) + 4\sin^2\theta = 4$

$\Rightarrow 4\sin^2\theta = 4 - 3 = 1$

$\therefore \sin\theta = \dfrac{1}{2}, -\dfrac{1}{2} \Rightarrow \theta = n\pi + (-1)^n\left(\pm\dfrac{\pi}{6}\right)$

(ii) $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$

$\Rightarrow \tan x + \tan 2x + \tan 3x(1 - \tan x \tan 2x) = 0$

$\Rightarrow \dfrac{\tan x + \tan 2x}{1 - \tan x \tan 2x} = -\tan 3x$

$\because \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A + B)$

$\therefore \tan(x + 2x) = -\tan 3x$

$2\tan 3x = 0$

i.e. $\tan 3x = 0$

$\because 3x = n\pi, n \in I$ or $x = \dfrac{n\pi}{3}, n \in I$

(iii) $\tan\left(\dfrac{\pi}{4} + \theta\right) + \tan\left(\dfrac{\pi}{4} - \theta\right) = 4$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\therefore \tan\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$

and $\tan\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

$\therefore \dfrac{1 + \tan\theta}{1 - \tan\theta} + \dfrac{1 - \tan\theta}{1 + \tan\theta} = 4$

$\Rightarrow \dfrac{(1 + \tan\theta)^2 + (1 - \tan\theta)^2}{(1 - \tan^2\theta)} = 4$

$\therefore (1 + \tan^2\theta) \times 2 = 4(1 - \tan^2\theta) \Rightarrow 3\tan^2\theta = 1$

$\tan\theta = \pm\dfrac{1}{\sqrt{3}} \Rightarrow \theta = n\pi \pm \dfrac{\pi}{6}, n \in I$

Sol 18: $\tan x + \cot x = 2$

$\cot x = \dfrac{1}{\tan x} \therefore \tan x + \dfrac{1}{\tan x} = 2$

$\tan^2 x - 2\tan x + 1 = 0$

$\Rightarrow (\tan x - 1)^2 = 0 \Rightarrow \tan x = 1$

$\therefore x = n\pi + \frac{\pi}{4}, n \in I$

Sol 19: $\tan^3 x - 3\tan x = 0,$

$\tan x (\tan^2 x - 3) = 0$

$\tan x = 0$ or $\tan x = \pm\sqrt{3}$

$x = n\pi$ or $x = n\pi \pm \frac{\pi}{3}, n \in I$

Sol 20: (i) $\cos x + \sin x = 1$

Multiply whole equation by $\frac{1}{\sqrt{2}}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x = \cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\therefore x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\therefore x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4} = 2n\pi$ or $2n\pi + \frac{\pi}{2}, n \in I$

(ii) $\sec x - \tan x = \sqrt{3}$

$\Rightarrow \frac{1}{\sqrt{3}}\sec x - \frac{\tan x}{\sqrt{3}} = 1 \Rightarrow \frac{1 - \sin x}{\sqrt{3}\cos x} = 1$

$\Rightarrow \sqrt{3}\cos x + \sin x = 1$

Divide the equation by 2

$\Rightarrow \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{1}{2}$

$\cos\left(x - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$

$\therefore x = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{6}, n \in I$

$x = 2n\pi - \frac{\pi}{6}$ or $2n\pi + \frac{\pi}{2}, n \in I$

(iii) $\sin x + \cos x = \frac{1}{\sqrt{2}}$

Multiply the equation by $\frac{1}{\sqrt{2}}$

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{2}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)$

$\Rightarrow x = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{4}$

$2n\pi + \frac{7x}{12}$ or $2n\pi - \frac{\pi}{12}$

(iv) $\cos x + \sqrt{3}\sin x = 1$

Divide the equation by 2

$\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = \frac{1}{2}$

$\Rightarrow \cos\left(x - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$

$\therefore x = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{3}$

i.e. $x = 2n\pi$ or $2n\pi + \frac{2\pi}{3}, n \in I$

Sol 21: (i) $\left(\frac{\pi}{6}\right)^C$

$180° = \pi$ radian

$\therefore 1$ radian $\rightarrow \frac{180^0}{\pi}$

$\frac{\pi}{6} \rightarrow \frac{180}{\pi} \times \frac{\pi}{6} = 30°$

(ii) $\left(\frac{4\pi}{5}\right)^C$

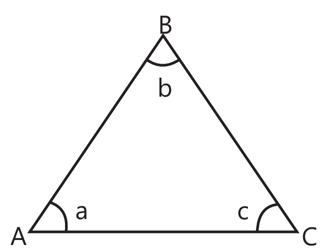
$\frac{4\pi}{5}$ radian $= \frac{180}{\pi} \times \frac{4\pi}{5} = 144°$

(iii) $(1.2)^c$

1.2 radian $= \frac{180}{\pi} \times 1.2 = 68°43'37.8''$

Note: $1° = 60'$, $1' = 60''$

Sol 22: $a + b + c = 180°$



Given angles are in A. P.

Let common difference = d

$\therefore b = a + d, c = a + 2d$

$\Rightarrow a + (a + d) + (a + 2d) = 3(a + d) = 180°$

$a + d = 60°$ ... (i)

Also given $\frac{a}{c} = \frac{a}{a+2d} = \frac{60°}{\pi} = \frac{\frac{\pi}{3}}{\pi} = \frac{1}{3}$

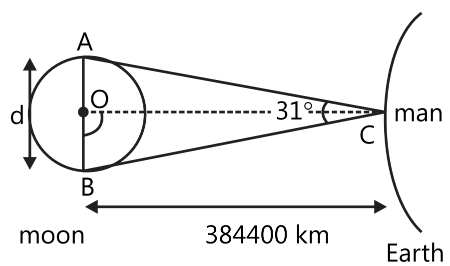
$\Rightarrow 3a = a + 2d \Rightarrow a = d$ ... (ii)

From (i) and (ii)

$a = d = 30°$

$\therefore a = 30°, b = 60°, c = 90°$ Ans.

Sol 23:



Line OC divides AB into two equal parts

In ΔOBC

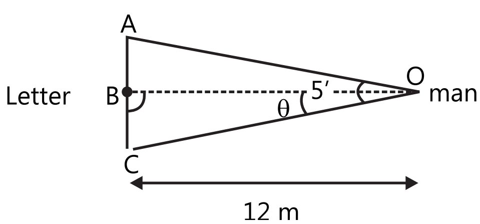
$\tan\left(\frac{31'}{2}\right) = \frac{OB}{OC} \therefore OB = 384400 \times \tan(15.5°)$

$= 384400 \times \tan\frac{15.5^0}{60} = 173318$ Km

$AB = 2(OB) = 346636$ Km

$\therefore$ Diameter of moon = 346636 Km.

Sol 24:



Assuming letter to be symmetrically placed

$\tan\theta = \frac{BC}{OB}$

$\tan(2.5') = \frac{BC}{12}$

$\therefore BC = 12\tan(2.5') = 0.0873$ m

$\therefore$ Total length of letter = 2BC = 0. 174 m = 17.4 m.

Sol 25: $4\cos^2x \sin x - 2\sin^2x = 3\sin x$

One of the obvious solution is $\sin x = 0$

i.e. $x = n\pi$

Now if $x \neq n\pi$ i.e. $\sin x \neq 0 \Rightarrow 4\cos^2x = 3 + 2\sin x$

$\therefore 4 - 4\sin^2x = 3 + 2\sin x \Rightarrow 4\sin^2x + 2\sin x - 1 = 0$

$\sin x = \frac{-2 \pm \sqrt{(2)^2 - 4(4)(-1)}}{2 \times 4} = \frac{-2 \pm \sqrt{20}}{8}$

$\Rightarrow \sin x = \frac{-1 \pm \sqrt{5}}{4}$

$\therefore$ Possible solution is

$\sin x = \frac{\sqrt{5}-1}{4} = \sin\left(\frac{\pi}{10}\right) \Rightarrow x = n\pi + (-1)^n \frac{\pi}{10}$

and $\sin x = \frac{-1-\sqrt{5}}{4} = \sin\left(-\frac{3\pi}{10}\right) = \sin\left(\pi + \frac{3\pi}{10}\right)$

$\therefore x = n\pi + (-1)^n\left(-\frac{3\pi}{10}\right) = n\pi + (-1)^{n+1}\frac{3\pi}{10}$

$\therefore x = n\pi, n\pi + (-1)^n\frac{\pi}{10}, n\pi + (-1)^{n+1}\frac{3\pi}{10}$

Sol 26: $5\cos2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0 \quad -\pi < \theta < n$

$\cos2\theta = 2\cos^2\theta - 1$

$\therefore 2\cos^2\frac{\theta}{2} = \cos\theta + 1$

and $\cos2\theta = 2\cos^2\theta + 1$

Putting both these in given equation

$\therefore 5(2\cos^2\theta - 1) + \cos\theta + 1 + 1 = 0$

$10\cos^2\theta + \cos\theta - 3 = 0$

$10\cos^2\theta - 5\cos\theta + 6\cos\theta - 3 = 0$

$(5\cos\theta + 3)(2\cos\theta - 1) = 0$

$\Rightarrow \cos\theta = -\frac{3}{5}$ or $\cos\theta = \frac{1}{2}$

$\therefore \theta = \cos^{-1}\left(-\frac{3}{5}\right)$ or $\theta = \pm\frac{\pi}{3} = \pi - \cos^{-1}\frac{3}{5}$

$\therefore \theta = \frac{\pi}{3}, -\frac{\pi}{3}, \pi - \cos^{-1}\frac{3}{5}$ (As $-\pi < \theta < \pi$)

Sol 27: $4\sin^4x + \cos^4x = 1$

$4\sin^4x = 1 - \cos^4x = (1 - \cos^2x)(1 + \cos^2x)$

$4\sin^4x = \sin^2x(1 + \cos^2x)$

One of obvious solution is $\sin x = 0$ i.e. $x = n\pi$

If $\sin x \neq 0$

$\Rightarrow 4\sin^2 x = 1 + \cos^2 x = 2 - \sin^2 x$

$\Rightarrow \sin^2 x = \frac{2}{5}$

$\sin x = \pm\sqrt{\frac{2}{5}}$

$\therefore x = n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$

$\therefore x = n\pi, n\pi \pm \alpha, \sin\alpha = \sqrt{\frac{2}{5}}, n \in I$

Sol 28: $\tan 2\theta \tan\theta = 1$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$,

Substitute this in given equation, we get

$\therefore \frac{2\tan^2\theta}{1-\tan^2\theta} = 1 \Rightarrow 3\tan^2\theta = 1$

$\tan\theta = \pm\frac{1}{\sqrt{3}} \quad \therefore \theta = n\pi \pm \frac{\pi}{6}$

Sol 29: $e^{\sin x} - e^{-\sin x} - 4 = 0$

$e^{\sin x} - \frac{1}{e^{\sin x}} = 4$

Max. Value of L. H. S. can be attained only when $e^{\sin x}$ is max and $\frac{1}{e^{\sin x}}$ is min.

As max. Value of sinx is 1

$\therefore e^{\sin x} \le e^1$ and $\frac{1}{e^{\sin x}} \ge \frac{1}{e}$

$\therefore$ Max. Value of LHS $= e - \frac{1}{e} \approx 2\ 35$

So there is no real value of x for which LHS = 4

Sol 30: $\sin^4\theta - 2\sin^2\theta - 1 = 0$

Let $\sin^2\theta = t \Rightarrow t^2 - 2t - 1 = 0$

$\therefore t = \frac{2 \pm \sqrt{(-2)^2 + 4\times 1}}{2} = \frac{2\pm\sqrt{8}}{2} = 1 \pm 2\sqrt{2}$

$\therefore \sin^2\theta = 1 + \sqrt{2}$ or $1 - \sqrt{2}$

Since $-1 < \sin\theta < 1$ and $0 < \sin^2\theta < 1$

$\therefore$ No real solution.

Exercise 2

Single Correct Choice Type

Sol 1: (A) $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3}{2}c$

$b\frac{(\cos A + 1)}{2} + a\frac{(\cos B + 1)}{2} = \frac{3}{2}c$

$b\cos A + b + a\cos B + a = 3c$

$\Rightarrow a + b + (a\cos B + b\cos A) = 3c$

$\Rightarrow a\cos B + b\cos A = c$

$\therefore a + b = 2c \Rightarrow a, b, c$ are in A.P.

Sol 2: (B) $a^2 + 2a + \operatorname{cosec}^2\left(\frac{\pi}{2}(a+x)\right) = 0$

$[\operatorname{cosec}^2\theta = 1 + \cot^2\theta]$

$\Rightarrow a^2 + 2a + 1 + \cot^2\left(\frac{\pi}{2}(a+x)\right) = 0$

$\Rightarrow (a+1)^2 + \cot^2\left(\frac{\pi}{2}(a+x)\right) = 0$

For the above equation to be valid

$a + 1 = 0$ and $\cot\left(\frac{\pi}{2}(a+x)\right) = 0$

$\Rightarrow a = -1$ and $\frac{\pi}{2}(a+x) = (2n+1)\frac{\pi}{2}$

$\Rightarrow a + x = 2n + 1 \Rightarrow x = 2n + 2$

$\therefore a = -1$ and $\frac{x}{2} \in I$

Sol 3: (D) $(a+b)^2\sin^2\frac{C}{2} + (a-b)^2\cos^2\frac{C}{2}$

$= (a^2 + b^2 + 2ab)\sin^2\frac{C}{2} + (a^2 + b^2 - 2ab)\cos^2\frac{C}{2}$

$= (a^2 + b^2)\frac{\pi}{3} + 2ab\left(\sin^2\frac{c}{2} - \cos^2\frac{c}{2}\right)$

$= a^2 + b^2 + 2ab(-\cos C)$

$= a^2 + b^2 - 2ab\frac{a^2 + b^2 - c^2}{2ab} = c^2$

Sol 4: (D) $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A \sin B \sin C$

$\because$ In triangle $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

i.e. $\sin A \propto a$, $\sin B \propto b$, $\sin C \propto c$

$\therefore a^3 + b^3 + c^3 = 3abc$

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$

$\therefore a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0 \Rightarrow a = b = c$

$\therefore$ Triangle should be equilateral

Sol 5: (D) $\sin3\theta = 4\sin\theta \sin2\theta \sin4\theta$ $\theta \in [0, \pi]$

$\sin3\theta = 2\sin\theta [\cos2\theta - \cos6\theta]$

$\sin3\theta = \sin3\theta + \sin(-\theta) - [\sin7\theta + \sin(-5\theta)]$

$\Rightarrow \sin7\theta - \sin5\theta = -\sin\theta$

$\therefore -\sin\theta = 2\cos6\theta \sin\theta$

$\sin\theta [2\cos6\theta + 1] = 0 \Rightarrow \sin\theta = 0$ or $\cos6\theta = \frac{-1}{2}$

$\theta = n\pi$ or $6\theta = 2n\pi \pm \frac{2\pi}{3} \Rightarrow \theta = \frac{n\pi}{3} \pm \frac{\pi}{9}$

$\therefore \theta = 0, \pi, \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{2\pi}{9}, \frac{5\pi}{9}, \frac{8\pi}{9}$

Sol 6: (A) $a\cos(B-C) + b\cos(C-A) + c\cos(A-B)$

$= a(\cos B \cos C + \sin B \sin C) + b(\cos C \cos A + \sin C \sin A)$

$+ c(\cos A \cos B + \sin A \sin B)$

$= \cos C[a\cos B + b\cos A] + c \cos A \cos B$

$+ a \sin B \sin C + b \sin C \sin A + c \sin A \sin B$

$a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$

and $a\cos B + b\cos A = c$

$= c \cos C + c \cos A \cos B + \frac{abc}{(2R)^2} + \frac{abc}{(2R)^2} + \frac{abc}{(2R)^2}$

$= c[\cos(\pi - (A + B)) + \cos A\cos B] + \frac{3abc}{4R^2}$

$= c[-\cos A\cos B + \sin A\sin B + \cos A\cos B] + \frac{3abc}{4R^2}$

$= c \sin A\sin B + \frac{3abc}{4R^2} = \frac{abc}{4R^2} + \frac{3abc}{4R^2} = \frac{abc}{R^2}$

Sol 7: (D) $\cos\alpha = \frac{2\cos\beta - 1}{2 - \cos\beta}$

$$\Rightarrow = \frac{2\dfrac{\left(1-\tan^2\frac{\beta}{2}\right)}{1+\tan^2\frac{\beta}{2}} - 1}{2 - \dfrac{\left(1-\tan^2\frac{\beta}{2}\right)}{1+\tan^2\frac{\beta}{2}}}$$

$$\Rightarrow = \frac{2\left(1-\tan^2\frac{\beta}{2}\right) - \left(1-\tan^2\frac{\beta}{2}\right)}{2(1+\tan^2\frac{\beta}{2}) - \left(1+\tan^2\frac{\beta}{2}\right)}$$

$$\Rightarrow = \frac{2 - 2\tan^2\frac{\beta}{2} - 1 - \tan^2\frac{\beta}{2}}{2 + 2\tan^2\frac{\beta}{2} - 1 + 2\tan^2\frac{\beta}{2}}$$

$$\Rightarrow \frac{1 - 3\tan^2\beta/2}{1 + 3\tan^2\beta/2} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

$$\Rightarrow \left(1 - 3\tan^2\frac{\beta}{2}\right)\left(1 + \tan^2\frac{\beta}{2}\right) = \left(1 - \tan^2\frac{\beta}{2}\right)\left(1 + 3\tan^2\frac{\beta}{2}\right)$$

$$\Rightarrow 1 - 3\tan^2\frac{\beta}{2} + \tan^2\frac{\alpha}{2} - 3\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$$

$$= 1 + 3\tan^2\frac{\beta}{2} - \tan^2\frac{\alpha}{2} = -3\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$$

$$\Rightarrow 2\tan^2\frac{\alpha}{2} = 6\tan^2\frac{\beta}{2}$$

$$\Rightarrow \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2}$$

$$\Rightarrow \tan^2\frac{\alpha}{2} = \sqrt{3}\tan^2\frac{\beta}{2}$$

$$\Rightarrow \tan^2\frac{\alpha}{2} \cdot \cot\frac{\beta}{2} = \sqrt{3}$$

Sol 8: (B) $x \sin\theta = y \sin\left(\theta + \frac{2\pi}{3}\right) = z \sin\left(\theta + \frac{4\pi}{3}\right)$

$$\Rightarrow \frac{x}{y} = \frac{\sin\theta\left(-\frac{1}{2}\right) + \cos\theta\left(+\frac{\sqrt{3}}{2}\right)}{\sin\theta} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\cot\theta$$

$$\frac{x}{z} = \frac{\sin\theta\left(-\frac{1}{2}\right) + \cos\theta\left(-\frac{\sqrt{3}}{2}\right)}{\sin\theta} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\cot\theta$$

$\therefore \frac{x}{y} + \frac{x}{z} = -1 \quad \Rightarrow xz + xy + yz = 0$

Sol 9: (C) $m = a\cos^3\alpha + 3a\cos\alpha \sin^2 a$

$n = a\sin^3\alpha + 3a\cos^2\alpha \sin a$

$(m+n)=a(\sin^3\alpha+\cos^3\alpha+3\cos\alpha\ \sin\alpha(\cos\alpha+\sin\alpha))$

$= a(\sin\alpha + \cos\alpha)^3$

$(m-n)=a(\cos^3\alpha-\sin^3\alpha+3\cos\alpha\ \sin\alpha\ (\sin\alpha - \cos\alpha))$

$= a(\cos\alpha - \sin\alpha)^3$

$\therefore (m + n)^{2/3} + (m - n)^{2/3}$

$= a^{2/3}(\sin\alpha + \cos\alpha)^2 + a^{2/3}(\cos\alpha - \sin\alpha)^2 = 2a^{2/3}$

Sol 10: (A) $\tan(5\pi\cos\theta) = \cot(5\pi\sin\theta)$

$$\frac{\sin(5\pi\cos\theta)}{\cos(5\pi\cos\theta)} = \frac{\cos(5\pi\sin\theta)}{\sin(5\pi\sin\theta)}$$

$\Rightarrow \cos(5\pi\cos\theta)\cos(5\pi\sin\theta) - \sin(5\pi\cos\theta)\sin(5\pi\sin\theta) = 0$

$\Rightarrow \cos(5\pi\cos\theta + 5\pi\sin\theta) = 0$

$\Rightarrow 5\pi(\cos\theta + \sin\theta) = 2n\pi \pm \frac{-2\pm\sqrt{4+4\times4}}{2\times4} = -\frac{1\pm\sqrt{5}}{4}$

$\Rightarrow \cos\theta + \sin\theta = \left(\frac{2n}{5} \pm \frac{1}{10}\right)$

$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}\left(\frac{2n}{5} \pm \frac{1}{10}\right)$

$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\frac{2n}{5} \pm \frac{1}{10}\right) n \in I$

For this relation to satisfy

$\frac{1}{\sqrt{2}}\left(\frac{2n}{5} \pm \frac{1}{10}\right) \in [-1,1]$

$\therefore -\sqrt{2} \le \frac{2n}{5} + \frac{1}{10} \le \sqrt{2}$

and $-\sqrt{2} \le \frac{2n}{5} - \frac{1}{10} \le \sqrt{2}$

From following both condition

$\left(-\sqrt{2} - \frac{1}{10}\right)\frac{5}{2} \le n \le \left(\sqrt{2} - \frac{1}{10}\right)\frac{5}{2}$

and $\left(\frac{1}{10} - \sqrt{2}\right)\frac{5}{2} \le n \le \left(\sqrt{2} + \frac{1}{10}\right)\frac{5}{2}$

Considering values of n and $\theta \in [0, 2\pi]$ total of 28 values are possible.

Sol 11: (C) $\tan\frac{B}{2}\tan\frac{C}{2} = \frac{s-a}{s}$

$2s = a + b + c$

Given $b + c = 3a$

$\therefore \cot\frac{B}{2}\cot\frac{C}{2} = \frac{s}{s-a}$

$$= \frac{\frac{(a+b+c)}{2}}{\frac{(a+b+c)}{2} - a} \text{ or } \frac{\frac{(a+3a)}{2}}{\left(\frac{a+3a}{2}\right) - a} = \frac{2a}{2a-a}$$

$\therefore \cot\frac{B}{2}\cot\frac{C}{2} = 2$

Sol 12: (B) $\cos2x + a\sin x = 2a - 7$

$1 - 2\sin^2x + a\sin x = 2a - 7$

$2\sin^2x - a\sin x + 2a - 8 = 0$

$\sin x = \frac{a \pm \sqrt{a^2 - 4(2a-8)\times2}}{2\times2}$

$\therefore \sin x = \frac{a \pm \sqrt{(a-8)^2}}{4} = \frac{a \pm |a-8|}{4}$

For $a > 8$

$\sin x = \frac{a \pm |a-8|}{4} \Rightarrow \sin x = \frac{2a-8}{4}$ or 2

Since $\sin x \le 1$ no value of $a > 8$ satisfies the equation

For $a < 8$

$\sin x = \frac{a \pm (-a+8)}{4}$

$\sin x = 2, \frac{2a-8}{4}$

$\therefore -1 \le \frac{2a-8}{4} \le 1$

$2 \le a \le 6$

$\therefore$ The solution is $a \in [2, 6]$

Sol 13: (A) $\frac{\sum\cot^2\frac{A}{2}\cot^2\frac{B}{2}}{\prod\cot^2\frac{A}{2}} = \frac{1}{\cot^2\frac{C}{2}} + \frac{1}{\cot^2\frac{B}{2}} + \frac{1}{\cot^2\frac{C}{2}}$

$= \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}$

$= \left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)^2 - 2\sum\tan\frac{A}{2} + \tan\frac{B}{2}$

For triangle $\sum\tan\frac{A}{2} + \tan\frac{B}{2} = 1$

$= \left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right)^2 - 2$

$\therefore$ For LHS to be minimum $\left(\sum \tan\frac{A}{2}\right)^2$ should take its minimum value which is only possible for A = B = C = $\frac{\pi}{3}$

$\therefore \left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2 = \left(\frac{1}{\sqrt{3}}\times 3\right)^2 = 3$

$\therefore$ Min. Value of given function = 3 – 2 = 1

Sol 14: (C) sinx + sin5x = sin2x + sin4x

2sin3x cos2x = 2sin3x cosx

2sin3x(cos2x – cosx) = 0

$\therefore$ 3x = nπ or cos2x = cosx

$x = \frac{n\pi}{3}$ or $2x = 2n\pi \pm x \Rightarrow x = \frac{2n\pi}{3}$, 2np

$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}.........$

$\therefore$General solution is $\frac{n\pi}{3}$

Sol 15: (B) cotx – cosx = 1 – cotx cosx

$\cos x\left(\frac{1-\sin x}{\sin x}\right) = \frac{\sin x-\cos^2 x}{\sin x}$

For sinx ≠ 0 i.e. x ≠ np

cosx – cosx sinx = sinx – $\cos^2$x

(cosx – sinx) + cosx(cosx – sinx) = 0

⇒(1 + cosx) (cosx – sinx) = 0

cosx = –1 or cosx = sinx

$\therefore x = \pi, \frac{x}{4}, \frac{3\pi}{4}$

Θn ≠ π as sinx ≠ 0

$\therefore x = \frac{x}{4}, \frac{3\pi}{4}$

$\therefore$ Two solutions

Sol 16: (C) $\cos^2 73 + \cos^2 47$ + (cos73 cos47)

= $(\cos 73 + \cos 47)^2$ – cos47 cos73

$\Rightarrow \left(2\cos\frac{120}{2}\cos\frac{26}{2}\right)^2 - \left(\frac{\cos 120+\cos 26}{2}\right)$

$= \cos^2 13 - \frac{1}{2}\left(-\frac{1}{2}+2\cos^2 13-1\right)$

$= \cos^2 13 + \frac{1}{4} - \cos^2 13 + \frac{1}{2} = \frac{1}{4}+\frac{1}{2}=\frac{3}{4}$

Sol 17: (C) (7cosθ + 24sinθ) × (7sinθ – 24cosθ)

$= 25\left(\frac{7\cos\theta+24\sin\theta}{25}\right)\times 25\left(\frac{7\sin\theta-24\cos\theta}{25}\right)$

= –625 sin(α + θ) cos(α + θ)

Where $\sin\alpha = \frac{7}{25}$, $\cos\alpha = \frac{24}{24}$

$= -\frac{625}{2}\sin 2(\alpha+\theta)$

$\therefore$ Maximum value occurs when

sin2(α + θ) = –1 $\therefore$ Maximum value = $\frac{625}{2}$

Sol 18: (B) $\sin\frac{x}{2} - \cos\frac{x}{2} = 1 - \sin x$

We know that $1 - \sin x = \left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2$

$\Rightarrow \sin\frac{x}{2} - \cos\frac{x}{2} = \left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2$

$\Rightarrow \sin\frac{x}{2} = \cos\frac{x}{2}$ or $\sin\left(\frac{x}{2}-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{x}{2} = m\pi+\frac{\pi}{4}$ or $\frac{x}{2}-\frac{\pi}{4} = m\pi+(-1)^m\frac{\pi}{4}$

$\Rightarrow x = (4m+1)\frac{\pi}{2}$ or $x = (4m+1)\frac{\pi}{2}+(-1)^m\frac{\pi}{2}$

Also $\left|\frac{x}{2}-\frac{\pi}{2}\right| \le \frac{3\pi}{4}$

$\Rightarrow -\frac{3\pi}{4} \le \frac{x}{2}-\frac{\pi}{2} \le \frac{3\pi}{4} \Rightarrow -\frac{3\pi}{4}+\frac{\pi}{2} \le \frac{x}{2} \le \frac{3\pi}{4}+\frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{4} \le \frac{x}{2} \le \frac{5\pi}{4} \Rightarrow -\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

$\therefore x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{4\pi}{2}, \frac{5\pi}{2}$

i.e. n = 1, 2, 4, 5

Sol 19: (D) $a_1 + a_2\cos 2x + a_3\sin^2 x = 0$

$a_1 + a_2(1 - 2\sin^2 x) + a_3\sin^2 x = 0$

$\Rightarrow a_1 + a_2 + (a_3 - a_2)\sin^2 x = 0$

For all x of this function has to be zero

then $a_1 + a_2 = 0$, $a_3 - 2a_2 = 0$

As there are three Variables and two equations so infinite solution are possible.

Sol 20: (A) Given A + B = $\frac{\pi}{2}$

$$\left(1+\tan\frac{A}{2}\right)\left(1+\tan\frac{B}{2}\right)=\left(1+\tan\frac{A}{2}\right)\left(1+\tan\frac{\left(\frac{\pi}{2}-A\right)}{2}\right)$$

$$=\left(1+\tan\frac{A}{2}\right)\left(1+\frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}\right)=\left(1+\tan\frac{A}{2}\right)\frac{(2)}{1+\tan\frac{A}{2}}=2$$

Previous Years' Questions

Sol 1: (A) Given equation is

$2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+x^{-2},\ x\le\frac{\pi}{9}$

LHS = $2\cos^2\left(\frac{x}{2}\right)\sin^2 x<2$ and RHS = $x^2+\frac{1}{x^2}\ge 2$

$\therefore$ The equation has no real solution.

Sol 2: (C) Let $f(x)=\tan x-x$

We know, for $0<x<\frac{\pi}{2}$

$\Rightarrow \tan x>x$

$\therefore$ f(x) = tan x – x has no root in $(0,\pi/2)$

For $\pi/2<x<\pi$, tan x is negative

$\therefore$ f (x) = tan x – x < 0

So, f(x) = 0 has no root in $\left(\frac{\pi}{2},\pi\right)$

For $\frac{3\pi}{2}<x<2\pi, \tan x$ is negative

$\therefore f(x)=\tan x-x<0$

So, $f(x)=0$ has no root in $\left(\frac{3\pi}{2},2\pi\right)$

We have, $f(\pi)=0-\pi<0$ and $f\left(\frac{3\pi}{2}\right)=\tan\frac{3\pi}{2}-\frac{3\pi}{2}>0$

$\therefore f(x)=0$ has at least one root between π and $\frac{3\pi}{2}$

Sol 3: (A) Given equation is $\sin\left(e^x\right)=5^x+5^{-x}$ is

LHS = sin $\left(e^x\right)\le 1$, for all $x\in R$ and RHS = $5^x+5^{-x}\ge 2$

$\therefore$ $\sin\left(e^x\right)=5^x+5^{-x}$ has no solution.

Sol 4: (B) We know

$-\sqrt{a^2+b^2}\le a\sin x+b\cos x\le\sqrt{\alpha^2+b^2}$

$\therefore -\sqrt{74}\le 7\cos x+5\sin x\le\sqrt{74}$

i.e. $\therefore -\sqrt{74}\le 2k+1\le\sqrt{74}$

Since, k is integer, $-9<2k+1<9$

$\Rightarrow$ $-10<2k<8$ $\Rightarrow -5<k<4$

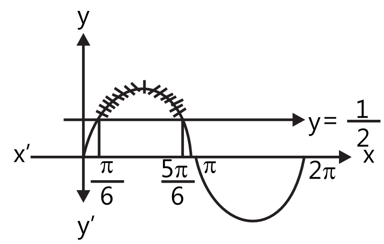
$\Rightarrow$ Number of possible integer values of k is equal to 8.

Sol 5: (A) Since, $2\sin^2\theta-5\sin\theta+2>0$

$\Rightarrow(2\sin\theta-1)(\sin\theta-2)>0$

[Where, $(\sin\theta-2)<0$ for all $\theta\in R$]

$\Rightarrow(2\sin\theta-1)<0$



$\Rightarrow \sin\theta<\frac{1}{2}$

$\therefore$ From the graph, $\theta\in\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},2\pi\right)$

Sol 6: (C) $2\sin^2\theta-\cos2\theta=0\ \Rightarrow \sin^2\theta=\frac{1}{4}$

Also, $2\cos^2\theta=3\sin\theta\ \Rightarrow \sin\theta=\frac{1}{2}$

$\Rightarrow$ Two solutions exist in the interval $[0,2\pi]$.

Sol 7: (D) $P=\left\{\theta:\sin\theta-\cos\theta=\sqrt{2}\cos\theta\right\}$

$\Rightarrow \cos\theta\left(\sqrt{2}+1\right)=\sin\theta$

$\Rightarrow \tan\theta=\sqrt{2}+1$ ...... (i)

$Q=\left\{\theta:\sin\theta+\cos\theta=\sqrt{2}\sin\theta\right\}\ \Rightarrow \sin\theta\left(\sqrt{2}-1\right)=\cos\theta$

$\tan\theta=\frac{1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}=\left(\sqrt{2}+1\right)$ ..... (ii)

$\therefore P=Q$

Sol 8: (A) $\alpha+\beta+\gamma=2\pi$

$\tan(\alpha+\beta+\gamma)=0$

$\Rightarrow \dfrac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}=0$

$\Rightarrow \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma=0$

$\Rightarrow \tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$

Sol 9: (B) $A\sin^2\theta+\cos^4\theta$

$=\sin^2\theta+\left(1-\sin^2\theta\right)^2$

$=\sin^2\theta+\sin^4\theta+1-2\sin^2\theta$

$=1+\sin^4\theta-\sin^2\theta$

$=\left(\sin^2\theta-\dfrac{1}{2}\right)^2+1-\dfrac{1}{4}$

$=\left(\sin^2\theta-\dfrac{1}{2}\right)^2+\dfrac{3}{4}$

For minimum, $\sin\theta=\dfrac{1}{2}$

$A=\dfrac{3}{4}$

For maximum, $\sin\theta=0$

$A=1$

$\Rightarrow \dfrac{3}{4}\le A\le 1$

Sol 10: (B) $3\left[\sin^4\left(\dfrac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$
$-2\left[\sin^6\left(\dfrac{\pi}{2}+2\right)+\sin^6(5\pi-2)\right]$

$=3\left[\cos^2\alpha+\sin^4\alpha\right]-2\left[\cos^6\alpha+\sin^6\alpha\right]$

$=3\cos^4\alpha+3\sin^4\alpha-2\cos^4$
$\alpha\left(1-\sin^2\alpha\right)-2\sin^4\alpha\left(1-\cos^2\alpha\right)$

$=\sin^4\alpha+\cos^4\alpha+2\cos^4\alpha\sin^2\alpha+2\sin^4\alpha\cos^2\alpha$

$=\left(\sin^2\alpha+\cos^2\alpha\right)^2-2\sin^2\alpha\cos^2\alpha$
$+2\cos^2\alpha\sin^2\alpha\left(\cos^2\alpha+\sin^2\alpha\right)$

$=1-2\sin^2\alpha\cos^2\alpha+2\cos^2\alpha\sin^2\alpha$

$=1$

Sol 11: (C) $\alpha+\beta=\dfrac{\pi}{2}\Rightarrow\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=0$

$\gamma=\alpha-\beta$

$\Rightarrow \tan\alpha\tan\beta=1$

$\gamma=\alpha-\beta$

$\tan\gamma=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$\Rightarrow \tan\gamma=\dfrac{\tan\alpha-\tan\beta}{2}$

$\Rightarrow 2\tan\gamma=\tan\alpha-\tan\beta$

$\Rightarrow \tan\alpha=2\tan\gamma+\tan\beta$

Sol 12: (B) $\theta\leftarrow\left(0,\dfrac{\pi}{4}\right)$

$\Rightarrow \tan\theta\in(1,0)$

$\Rightarrow \cos\theta\in\left(\dfrac{1}{\sqrt{2}},1\right)$

$\Rightarrow \cot\theta\in(1,0)$

As $\theta\in\left(0,\dfrac{\pi}{4}\right)$

$\cot\theta>\tan\theta$

$\Rightarrow (\cot\theta)^{\cot\theta}>(\tan\theta)^{\cot\theta}$

$\Rightarrow t_4>t_2$

Sol 13: (A) $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}$

$=\dfrac{\tan A}{1-\dfrac{1}{\tan A}}+\dfrac{1/\tan A}{1-\tan A}$

$=\dfrac{\tan^2 A}{1-\tan A}+\dfrac{1}{\tan A\,(1-\tan A)}$

$=\dfrac{-\tan^3 A+1}{(1-\tan A)\tan A}$

$=\dfrac{(1-\tan A)\left(1+\tan^2 A+\tan A\right)}{\tan A(1-\tan A)}$

$= \dfrac{\sec^2 A + \tan A}{\tan A}$

$= \dfrac{\sec^2 A}{\tan A} + 1$

= sec A cosec A + 1

Sol 14: (B) cos x + cos 2x + cos 3x + cos 4x = 0

$\Rightarrow \cos x + \cos 3x + \cos 2x + \cos 4x = 0$

$\Rightarrow 2\cos 2x \cos x + 2\cos 3x \cos x = 0$

$\Rightarrow 2\cos x(\cos 2x + \cos 3x) = 0$

$\Rightarrow 2\cos x\left[2\cos\frac{5x}{2}\cos\frac{x}{2}\right] = 0$

$\Rightarrow \cos\frac{x}{2} = 0$ or $\cos x = 0$ or $\cos\frac{5x}{2} = 0$

$\Rightarrow \frac{x}{2}(2p+1)\frac{\pi}{2}$ or $x = (2q+1)\frac{\pi}{2}$ or $\frac{5x}{2} = (2r+1)\frac{\pi}{2}$

$\Rightarrow x = (2p+1)\pi$ or $x = (2q+1)\frac{\pi}{2}$ or $x = (2r+1)\frac{\pi}{5}$

$\Rightarrow x = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Total 7 solutions

JEE Advanced/Boards

Exercise 1

Sol 1: $\sin 5x = 16\sin^5 x$

$\sin(3x + 2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$

$= (3\sin x - 4\sin^3 x)(1 - 2\sin^2 x)$

$\quad + (-3\cos x + 4\cos^3 x)2\sin x \cos x$

$= \sin x[(3 - 4\sin^2 x)(1 - 2\sin^2 x)$

$\quad + 2(3 - 4(1 - \sin^2 x))(1 - \sin^2 x)] = 16\sin^5 x$

One of the obvious solution is

$\sin x = 0 \Rightarrow x = n\pi$

If $\sin x \neq 0$

$\Rightarrow 3 - 10\sin^2 x + 8\sin^4 x$

$\quad + 2(-1 + 4\sin^2 x)(1 - \sin^2 x) = 16\sin^4 x$

$\Rightarrow 3 - 10\sin^2 x + 8\sin^4 x + 10\sin^2 x - 8\sin^4 x - 2 = 16\sin^4 x$

$\Rightarrow 1 = 16\sin^4 x$

$\Rightarrow \sin^4 x = \frac{2\pi}{3} \Rightarrow \sin x = \pm\frac{\pi}{3}$

$\therefore x = n\pi \pm \frac{\pi}{6}, n\pi$

Sol 2: $4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$

One above solution is $\sin x = 0$ i.e. $x = n\pi$

If $\sin x \neq 0$

$4\cos^2 x = 3 + 2\sin x$

$4(1 - \sin^2 x) = 3 + 2\sin x$

$4\sin^2 x + 2\sin x - 1 = 0$

$\sin x = = \dfrac{-2 \pm \sqrt{4 + 4 \times 4}}{2 \times 4} = \dfrac{1 \pm \sqrt{5}}{4}$

$x = n\pi + (-1)^{n+1}\frac{3\pi}{10}$ or $n\pi + (-1)^n\frac{\pi}{10}$

Sol 3: 1 + cosx+cos2x+sinx+sin2x+sin3x = 0

Given condition $\frac{\pi}{2} < \left|3x - \frac{\pi}{2}\right| \le \pi$

$1 + \cos x + 2\cos^2 x - 1 + (\sin x + \sin 3x) + \sin 2x = 0$

$\cos x(2\cos x + 1) + 2\sin 2x \cos x + \sin 2x = 0$

$(\cos x + \sin 2x)(2\cos x + 1) = 0$

$(2\sin x + 1)(2\cos x + 1)\cos x = 0$

$\sin x = -\frac{1}{2}$ or $\cos x = -\frac{1}{2}$ or $\cos x = 0$

According to given condition

$\frac{\pi}{2} < \left|3x - \frac{\pi}{2}\right| \le \pi$

$\Rightarrow \frac{\pi}{2} < 3x - \frac{\pi}{2} \le \pi$ and $-\pi \le 3x - \frac{\pi}{2} < -\frac{\pi}{2}$

$\Rightarrow \pi < 3x \le \frac{3\pi}{2}$ and $-\frac{\pi}{2} \le 3x < 0$

$\therefore$ common condition is $x \in \left[-\frac{\pi}{6}, 0\right) \cup \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$

Only two solutions are possible

$\sin x = -\frac{1}{2}$ and $\cos x = 0$

Sol 4: $2(\cos x + \cos 2x) + \sin 2x(1 + 2\cos x) = 2\sin x$

$2(\cos x + 2\cos^2 x - 1) + \sin 2x(1 + 2\cos x) = 2\sin x$

$2\cos x(2\cos x + 1) - 2 + \sin 2x(1 + 2\cos x) = 2\sin x$

$2(\cos x + \sin 2x)(2\cos x + 1) - 2(1 + \sin x) = 0$

$2\cos x(1 + \sin x)(2\cos x + 1) - 2(1 + \sin x) = 0$

$(1 + \sin x)[\cos x(2\cos x + 1) - 1] = 0$

$\Rightarrow \sin x + 1 = 0$ or $2\cos^2 x + \cos x - 1 = 0$

i.e. $(2\cos x - 1)(\cos x + 1) = 0$

$\therefore \sin x = -1$ or $\cos x = \frac{1}{2}$ or $\cos x = -1$

$x = (4n-1)\frac{\pi}{2}$

$x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$

$\because x \in [-\pi, \pi]$

$\therefore x = -\frac{\pi}{2}, -\frac{\pi}{3}, \frac{\pi}{3}; -\pi, +\pi$

Sol 5: $2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$

$2(\sin x + 2\sin^2 x) - 2 - \sin 2x(1 + 2\sin x) + 2\cos x = 0$

$(2\sin x - \sin 2x)(1 + 2\sin x) - 2(1 - \cos x) = 0$

$2\sin x(1 - \cos x)(1 + 2\sin x) - 2(1 - \cos x) = 0$

$2(1 - \cos x)[\sin x(1 + 2\sin x) - 1] = 0$

$\Rightarrow \cos x = 1$ or $2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0 \Rightarrow \sin x = \frac{1}{2}$ or -1

$x = 2n\pi$ or $x = n\pi + (-1)^n\frac{\pi}{6}$ or $x = (4n-1)\frac{\pi}{2}$

$\Rightarrow x = 2n\pi, n\pi + (-1)^{n+1}\frac{\pi}{2}, n\pi + (-1)^n\frac{\pi}{6}$

Sol 6: $\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}$

$\Rightarrow 2\cos\frac{5x}{2}\cos\frac{x}{2} = 2\sin x\cos\frac{x}{2}$

$\Rightarrow \cos\frac{x}{2}\left(\cos\frac{5x}{2} - \sin x\right) = 0$

$\cos\frac{x}{2} = 0$ or $\cos\frac{5x}{2} = \sin x = \cos\left(\frac{\pi}{2} - x\right)$

$\Rightarrow \frac{x}{2} = 2n\pi \pm \frac{\pi}{2}$ or $\frac{5x}{2} = 2n\pi \pm \left(\frac{\pi}{2} - x\right)$

$\Rightarrow x = 4n\pi \pm \pi$ or $x = 4n\frac{\pi}{7} + \frac{\pi}{7}$ or $4n\frac{\pi}{3} - \frac{\pi}{3}$ $n \in I$

$\because x \in (0, 2\pi)$

$\therefore x = \pi, \frac{\pi}{7}, \frac{5\pi}{7}, \frac{9\pi}{7}, \frac{13\pi}{7}$

Sol 7: $\tan^2 2x + \cot^2 2x + 2\tan 2x + 2\cot 2x = 6$

Let $\tan 2x = t$

$\Rightarrow t^2 + \frac{1}{t^2} + 2t + \frac{2}{t} = 6$

$\Rightarrow t^4 + 1 + 2(t^2 + 1)t = 6t^2$

$\Rightarrow t^4 + 2t^3 - 6t^2 + 2t + 1 = 0$

$\Rightarrow (t-1)(t-1)(t^2 + 4t + 1) = 0$

$\Rightarrow \tan 2x = 1$ or $\frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$

$\therefore 2x = n\pi + \frac{\pi}{4}$ or $2x = n\pi - \frac{\pi}{12}$ or $n\pi - \frac{5\pi}{12}$

$\Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}, \frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} - \frac{5\pi}{12}$

Sol 8: $1 + 2\operatorname{cosec} x = \frac{-\sec^2\frac{x}{2}}{2}$

$1 + \frac{2}{\sin x} = -\frac{1}{2} \times \frac{1}{\cos^2\frac{x}{2}}$

$\Rightarrow \frac{2 + \sin x}{2\sin\frac{x}{2}\cos\frac{x}{2}} = -\frac{1}{2\cos^2\frac{x}{2}}$

$\cos\frac{x}{2} \neq 0 \Rightarrow \frac{x}{2} \neq 2n\pi \pm \frac{x}{2} \Rightarrow x \neq 4n\pi \pm p$

$\Rightarrow 2 + \sin x = -\tan\frac{x}{2}$

$\Rightarrow 2 + \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = -\tan\frac{x}{2}$ (Let $\tan\frac{x}{2} = t$)

$\Rightarrow 2 + \frac{2t}{1+t^2} = -t \Rightarrow 2 + 2t^2 + 2t = -t - t^3$

$\Rightarrow t^3 + 2t^2 + 3t + 2 = 0$

$\Rightarrow (t+1)(t^2 + t + 2) = 0$

$\Rightarrow t = -1$ or $t^2 + t + 2 = 0$

$\therefore \tan\frac{x}{2} = -1$

$\frac{x}{2} = n\pi - \frac{\pi}{4} \Rightarrow x = 2n\pi - \frac{\pi}{2}$

Sol 9: $\sin\left(\frac{\sqrt{x}}{2}\right) + \cos\left(\frac{\sqrt{x}}{2}\right) = \sqrt{2}\sin\sqrt{x}$

$\frac{1}{\sqrt{2}}\sin\left(\frac{\sqrt{x}}{2}\right) + \frac{1}{\sqrt{2}}\cos\left(\frac{\sqrt{x}}{2}\right) = \cos\left(\frac{\pi}{2} - \sqrt{x}\right)$

$\Rightarrow \cos\left(\frac{\sqrt{x}}{2}-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{2}-\sqrt{x}\right)$

$\therefore \frac{\sqrt{x}}{2}-\frac{\pi}{4}=2n\pi\pm\left(\frac{\pi}{2}-\sqrt{x}\right)$

$\Rightarrow \sqrt{x}=\frac{4n\pi}{3}+\frac{\pi}{2}$, - $4n\pi+\frac{\pi}{2}$

$\therefore x=\left(\frac{4n\pi}{3}+\frac{\pi}{2}\right)^2, \left(4n\pi-\frac{\pi}{2}\right)^2 \; n\in I$

Sol 10: As the roots are same both equations should be same.

Let us solve the second equation

$\sin x \cos 2x = \sin 2x \cos 3x - \frac{1}{2}\sin 5x$

$\Rightarrow 2\sin x \cos 2x = 2\sin 2x \cos 3x - \sin 5x$

$\Rightarrow \sin 3x - \sin x = \sin 5x - \sin x - \sin 5x$

$\Rightarrow \sin 3x = 0 \therefore x = n\pi$

Put $x = n\pi$ is given equation

$a\cos 2x + |a|\cos 4x + \cos 6x = 1$

$a\cos 2n\pi + |a|\cos 4n\pi + \cos 6n\pi = 1$

$\Rightarrow a + |a| + 1 = 1$

$\therefore a \le 0$

Sol 11: $\sqrt{13-18\tan x} = 6\tan x - 3, -2\pi, x < 2\pi$

$\Rightarrow 13 - 18\tan x = (6\tan x - 3)^2$

$13 - 18\tan x = 36\tan^2 x + 9 - 36\tan x$

$18\tan^2 x - 9\tan x - 2 = 0$

$\Rightarrow (6\tan x + 1)(3\tan x - 2) = 0$

Also $6\tan x - 3 > 0 \Rightarrow \tan x > \frac{1}{2}$

$\therefore \tan x = \frac{2}{3}$

$x = a - 2\pi, a - \pi, a, a + \pi$, where $a = \tan^{-1}\frac{2}{3}$

Sol 12: $\left(\sqrt{1+\sin 2x}-\sqrt{2}\cos 3x=0\right)$

$1 + \sin 2x = 2\cos^2 3x = 2(1 - \sin^2 3x)$

$\Rightarrow 2\sin^2 3x + \sin 2x = 1$

$\Rightarrow 1 - \cos 6x + \sin 2x = 1$

$\Rightarrow \sin 2x = \cos 6x$

$\Rightarrow \cos\left(\frac{\pi}{2}-2x\right) = \cos 6x$

$\Rightarrow \frac{\pi}{2}-2x = 2n\pi \pm 6x$

$\Rightarrow x = \frac{n\pi}{4}+\frac{\pi}{16}, \frac{n\pi}{2}-\frac{\pi}{8}$

$\therefore$ Smallest positive value $= \frac{\pi}{16}$

Sol 13: $2\sin\left(3x+\frac{\pi}{4}\right) = \sqrt{1+8\sin 2x\cos^2 2x}$

$\Rightarrow 2\left(\frac{1}{\sqrt{2}}\sin 3x+\frac{1}{\sqrt{2}}\cos 3x\right)$

$= \sqrt{1+4(2\sin 2x\cos 2x)\cos 2x}$

$\Rightarrow [\sqrt{2}(\sin 3x + \cos 3x)]^2 = 1 + 4\sin 4x \cos 2x$

$\Rightarrow 2(\sin^2 3x + \cos^2 3x + 2\sin 3x \cos 3x)$

$= 1 + 4\sin 4x \cos 2x$

$\Rightarrow 2(1 + 2\sin 3x \cos 3x) = 1 + 4\sin 4x \cos 2x$

$\Rightarrow 1 = 2(2\sin 4x \cos 2x) - 2(2\sin 3x \cos 3x)$

$= 2(\sin 6x + \sin 2x) - 2\sin 6x$

$\Rightarrow \sin 2x = \frac{1}{2}$

$\therefore \left(2x-\frac{\pi}{2}\right) = 2n\pi \pm \frac{\pi}{3}$

$x = n\pi + \frac{5\pi}{12}, n\pi + \frac{\pi}{12}, n \in I$

If $x = \frac{5\pi}{12} \Rightarrow \sin\left(3x+\frac{\pi}{4}\right) = -1$

Which is not possible

$\therefore x = n\pi + \frac{\pi}{12}$

Sol 14: $\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x$

$\frac{1}{\sqrt{2}}(\sin x - \cos x) - \frac{1}{\sqrt{2}}(\sin 3x - \cos 3x)$

$+ \frac{1}{\sqrt{2}}(\sin 5x - \cos 5x) = 0$

$\sin\left(x-\frac{\pi}{4}\right) - \sin\left(3x-\frac{\pi}{4}\right) + \sin\left(5x-\frac{\pi}{4}\right) = 0$

$\Rightarrow 2\sin\left(\frac{6x-\frac{\pi}{2}}{2}\right)\cos\left(\frac{x-5x}{2}\right) - \sin\left(3x-\frac{\pi}{4}\right) = 0$

$\Rightarrow \sin\left(3x - \frac{\pi}{4}\right)$ [2cos2x – 1] = 0

$\therefore \sin\left(3x - \frac{\pi}{4}\right) = 0$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow 3x = n\pi + \frac{\pi}{4}$ or $2x = 2n\pi \pm \frac{\pi}{3}$

$\therefore x = \frac{n\pi}{3} + \frac{\pi}{12}, n\pi \pm \frac{\pi}{6}$

$\therefore$ Principle solution are $\frac{\pi}{12}, \frac{\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12},$

$\frac{17\pi}{12}, \frac{21\pi}{12}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

i.e. 10 solutions.

Sol 15: $3^{\left(\frac{1}{2}+\log_3(\cos x+\sin x)\right)} - 2^{\log_2(\cos x - \sin x)} = \sqrt{2}$

$3^{\frac{1}{2}} 3^{\log_3(\cos x+\sin x)} - 2^{\log_2(\cos x-\sin x)} = \sqrt{2}$

$\sqrt{3}$ (cosx + sinx) – (cosx – sinx) = $\sqrt{2}$

$\Rightarrow \frac{\sqrt{3}-1}{2\sqrt{2}}\cos x + \frac{\sqrt{3}+1}{2\sqrt{2}}\sin x = \frac{1}{2}$

$\cos\left(x - \frac{5\pi}{12}\right) = 2n\pi \pm \frac{\pi}{3}$

cosx – sinx and cosx + sinx > 0

$\therefore x = 2n\pi + \frac{\pi}{12}, 2n\pi + \frac{3\pi}{4}$

$\because$ For $x = 2n\pi + \frac{3\pi}{4}$, cosx + sinx < 0, which is not a solution.

$\therefore x = 2n\pi + \frac{\pi}{12}$

Sol 16: cos6θ + cos4θ + cos2θ + 1 = 0

2cos4θ cos2θ + cos4θ + 1 = 0

2cos4θ cos2θ + $2\cos^2 2\theta$ = 0

2cos2θ (cos4θ + cos2θ)

2cos2θ × 2cos3θ cosθ = 0

$\Rightarrow$ cosθ = 0 or cos2θ = 0 or cos3θ = 0

$\theta = 2n\pi \pm \frac{\pi}{2}$ or $\theta = n\pi + \frac{\pi}{4}$ or $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{4}$

= 30°, 45°, 90°, 150°, 135°

Sol 17: $\log_{\frac{-x^2-6x}{10}}(\sin 3x + \sin x) = \log_{\frac{-x^2-6x}{10}}(\sin 2x)$

$\Rightarrow -x^2 - 6x > 0$ and $-x^2 - 6x \neq 10$

$\Rightarrow x(x + 6) < 0$ and $x^2 + 6x + 10 \neq 0$

$\therefore x \in (-6, 0)$

But if x is negative then sin2x will be negative.

Hence no solution is possible.

Sol 18: 3 – 2cosθ – 4sinθ – cos2θ + sin2θ=0

3 – 2cosθ – 4sinθ – (1 – $2\sin^2\theta$) + 2sinθ cosθ = 0

$\Rightarrow$ 2cosθ(sinθ–1)+2sinθ (sinθ–1)–2(sinθ – 1) = 0

$\Rightarrow$ (sinθ – 1) (2cosθ + 2sinθ – 2) = 0

$\therefore$ sinθ = 1 or sinθ + cosθ = 1

$\Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{2}$ or $\cos\left(\theta - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$

$\because \theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}$

$\therefore \theta = 2n\pi, 2n\pi + \frac{\pi}{2}$

Sol 19: cos4x + 6 = 7cos2x

cos4x – cos2x = 6(cos2x – 1)

$\Rightarrow$ 2sin3x sin(–x) = 6($-2\sin^2 x$)

$\Rightarrow$ sin3x sinx = $6\sin^2 x$

sin3x sinx = $6\sin^2 x$

sinx(sin3x – 6sinx) = 0

sinx(3sinx – $4\sin^3 x$ – 6sinx) = 0

$\Rightarrow \sin^2 x(-3 - 4\sin^2 x) = 0$

$\Rightarrow \sin x = 0$ or $\sin^2 x = \frac{-3}{4}$

Which is not possible

$\therefore x = n\pi$

$\therefore$ Possible solutions are

0, π, 2π, 3π, ... 99π

$\because$ 100π = 314.159 > 314

$\therefore \text{Sum} = \frac{99 \times 100\pi}{2} = 4950\pi$

Sol 20: $\sin^3 x + \sin^3 2x + \sin^3 3x$

= $(\sin x + \sin 2x + \sin 3x)^3$

$a^3 + b^3 + c^3 = (a + b + c)^3 - 3(a + b)$

(b + c) (c + a)

$\Rightarrow$ (sinx + sin2x) (sin2x + sin3x) (sin3x + sinx) =0

$\left(2\sin\frac{3x}{2}\cos\frac{x}{2}\right)\left(2\sin\frac{5x}{2}\cos\frac{x}{2}\right)$ (2sin2xcosx)=0

$\Rightarrow \sin\frac{3x}{2} = 0$ or $\sin\frac{5x}{2} = 0$ or sin2x = 0 or

$\cos\frac{x}{2} = 0$ or cosx = 0

$\therefore x = \frac{2n\pi}{3}, \frac{2n\pi}{5}, n\pi, 4n\pi \pm \pi, 2n\pi \pm \frac{\pi}{2}$

$\therefore$ Least positive angle would be $\frac{2\pi}{5} = 72°$.

Sol 21: $\sin(\pi - 6x) + \sqrt{3}\sin\left(\frac{\pi}{2}+6x\right) = \sqrt{3}$ in $[0, 2\pi]$

$\frac{1}{2}\sin 6x + \frac{\sqrt{3}}{2}\cos 6x = \frac{\sqrt{3}}{2}$

$\cos\left(6x-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$

$6x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{6} \Rightarrow x = \frac{n\pi}{3}, \frac{n\pi}{3} + \frac{\pi}{18}$

$\therefore x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi, \frac{\pi}{18},$

$\frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$

$\therefore$ Total no. of solutions = 13

Sol 22: $(\sin\theta)x^2 + (2\cos\theta)x + \frac{\cos\theta+\sin\theta}{2}$

For perfect square of linear equation D = 0

$\Rightarrow b^2 - 4ac = 0$

$4\cos^2\theta - 4\left(\frac{\cos\theta+\sin\theta}{2}\right)\sin\theta = 0$

$2\cos^2\theta = (\cos\theta + \sin\theta)\sin\theta$

$\Rightarrow 2\cot\theta = 1 + \tan\theta$

$\Rightarrow \tan^2\theta + \tan\theta - 2 = 0$

$(\tan\theta + 2)(\tan\theta - 1) = 0$

$\tan\theta = -2, 1$

$n\pi+\frac{\pi}{4}, n\pi - \tan^{-1}2 \qquad n \in I$

Sol 23: (a) sinx sin2x sin3x = 1

$\frac{1}{\tan x}(\cos(-2x) - \cos 4x)\sin 2x = 1$

2cos2x sin2x – 2cos4x sin2x = 4

$\Rightarrow$ sin4x + sin2x – sin6x = 4

As maximum value of $\sin\theta = 1$ so if sin4x, sin2x takes maximum value 1 and sin6x takes the value –1 even then LHS will be less than RHS so no solution possible.

(b) $\sin x \cos 4x \sin 5x = -\frac{1}{2}$

(cos4x – cos6x) cos4x = –1

$2\cos^2 4x - 2\cos 6x\cos 4x = -2$

$\Rightarrow$ 1+cos8x–cos10x–cos2x=– 2

$\Rightarrow$ cos8x – cos10x – cos2x = –3

For LHS = RHS

cos10x = cos2x = 1 and cos8x = 1

$\therefore$ 8x = 2nπ and 10x = (2m + 1) π and

2x = (2m + 1)p

$\Rightarrow x = \frac{n\pi}{4}, x = (2m+1)\frac{\pi}{10}, x = (2m+1)\frac{\pi}{2}$

$\frac{n\pi}{4} = (2m+1)\frac{\pi}{10}$ and $\frac{n\pi}{4} = (2m+1)\frac{\pi}{2}$

There is no integer value of n and m for which above results hold. So no solution.

(c) 2sinx cosx cos2x = –1

2sinx[cos3x + cosx] = –2

$\Rightarrow$ sin4x + sin(–2x) + sin2x=– 2

$\Rightarrow$ sin4x = –2

$\therefore$ No solution

(d)4sin2x + cosx = 5

For this result to hold

sin2x = 1 and cosx = 1

$\therefore 2x = n\pi + (-1)^n\frac{\pi}{2}$ and $x = 2m\pi$

$\therefore x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{2}$ and $x = 2m\pi$

There exist no m, n for which the above relation as valid. No solution

(e) sin3x – cosx = 2

For this

sin3x = 1 and cosx = –1

$3x = n\pi + (-1)^n\frac{\pi}{2}$ and $x = (2m+1)\pi$

$x = \frac{n\pi}{3} + (-1)^n\frac{\pi}{6}$ and $x = (2m+1)\pi$

Both the values cannot be same for any integral value of m and n. So no. solution

Sol 24: (a) $f(x) = \sin^6x + \cos^6x + k(\sin^4x + \cos^4x)$

$= (\sin^2x + \cos^2x)^3 - 3\sin^4x\cos^2x$

$- 3\sin^2x\cos^4x + k(1 - 2\sin^2x\cos^2x)$

$= 1 + k - 2k\sin^2x\cos^2x - 3\sin^2x\cos^2x$

$= (1 + k) - \dfrac{(2k+3)}{4}\sin^2 2x$

For f(x) = constant

$\dfrac{2k+3}{4} = 0 \Rightarrow k = -\dfrac{3}{2}$

(b) $(1 + k) - \dfrac{(2k+3)}{4}\sin^2 2x = 0$

$\Rightarrow \sin^2 2x = \dfrac{(1+k)^4}{(2k+3)}$

$\therefore 0 \le \dfrac{4(1+k)}{2k+3} \le 1 \Rightarrow k \in \left[-1, \dfrac{-1}{2}\right]$

(c) If k = –0. 7

$\Rightarrow (1-0.7) - \dfrac{2(-0.7)+3}{4}\sin^2 2x = 0$

$\Rightarrow \sin^2 2x = \dfrac{0.3\times 4}{1.6} = \dfrac{3}{4}$

$\therefore \sin 2x = \pm\dfrac{\sqrt{3}}{2}$

$\Rightarrow 2x = n\pi \pm \dfrac{\pi}{3} \Rightarrow x = \dfrac{n\pi}{2} + \dfrac{\pi}{6}$

Sol 25: a & b

$a\cos\theta + b\sin\theta = c$

$\Rightarrow a\cos\theta = c - b\sin\theta$

$\Rightarrow a^2\cos^2\theta = c^2 - b^2\sin^2\theta - 2bc\sin\theta$

$\Rightarrow a^2\left(1-\sin^2\theta\right) = c^2 + b^2\sin^2\theta - 2bc\sin\theta$

$\Rightarrow \left(a^2+b^2\right)\sin^2\theta - 2bc\sin\theta + c^2 - a^2 = 0$

If α & β are roots, then

$\sin\alpha + \sin\beta = \dfrac{2bc}{a^2+b^2}$

$\sin\alpha . \sin\beta = \dfrac{c^2-a^2}{a^2+b^2}$

(c) $\tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2}$

$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$,

$\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Let $\tan\dfrac{\theta}{2} = t$

$\therefore a\cos\theta + b\sin\theta = c$

$a(1 - t^2) + b(2t) = c(1 + t^2)$

$= (a + c)t^2 - 2bt + c - a = 0$

$\therefore \tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2} = \dfrac{2b}{a+c}$ (P) (sum of roots)

$\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = \dfrac{c-a}{c+a}$

(Q) (Product of roots)

Sol 26:

(A) $\cos3x\cos^3x + \sin3x\sin^3x = 0$

$\Rightarrow (4\cos^3x - 3\cos x)\cos^3x + (3\sin x - 4\sin^3x)\sin^3x = 0$

$\Rightarrow 4\cos^6x - 4\sin^6x + 3\sin^4x - 3\cos^4x = 0$

$\Rightarrow 4[(\cos^2x)^3 - (\sin^2x)^3]$

$+3(\sin^2x - \cos^2x)(\sin^2x + \cos^2x) = 0$

$\Rightarrow 4(\cos^2x - \sin^2x)(1 + \cos^2x\sin^2x) - 3\cos2x = 0$

$\Rightarrow 4\cos2x(1 + \cos^2x\sin^2x) = 3\cos2x$

$\Rightarrow \cos2x + 4\cos2x(\sin2x)^2 = 0$

$\Rightarrow \cos2x(1 + \sin^2 2x) = 0$

$\sin^2 2x = -1$

Not possible (No real solution)

$\cos2x = 0 \Rightarrow 2x = 2n\pi \pm \dfrac{\pi}{2} \Rightarrow x = \dfrac{n\pi}{2} \pm \dfrac{\pi}{4}$

(B) $\sin3\alpha = 4\sin\alpha\sin(x + \alpha)\sin(x - \alpha)$

$\sin3\alpha = 2\sin\alpha[\cos2\alpha - \cos2x]$

$\sin3\alpha = \sin3\alpha + \sin(-\alpha) - 2\sin\alpha\cos2x$

$\Rightarrow \sin\alpha(1 + 2\cos2x) = 0$

$\cos2x = -\dfrac{1}{2}$

$2x = 2n\pi \pm \dfrac{2\pi}{3}$

$x = n\pi \pm \dfrac{\pi}{3}$

(C) $|2\tan x - 1| + |2\cot x - 1| = 2$

For $\tan x > 2$

$2\tan x - 1 + 1 - 2\cot x = 2$

$\tan x - \frac{1}{\tan x} = 1$

$\tan^2 x - \tan x - 1 = 0$

$\tan x = \frac{1 \pm \sqrt{5}}{2} < 2$

$\therefore$ No. solution.

for $\tan x \in \left[\frac{1}{2}, 2\right]$

$2\tan x - 1 + 2\cot x - 1 = 2$

$2\tan x + \frac{2}{\tan x} = 4$

$\tan^2 x - 2\tan x + 1 = 0$

$\tan x = 1 \in \left[\frac{1}{2}, 2\right]$

$\therefore x = n\pi + \frac{\pi}{4}$ is one solution

For $x < \frac{1}{2}$

$1 - 2\tan x + \frac{2}{\tan x} - 1 = 2$

$\tan^2 x + \tan x - 1 = 0$

$\tan x = \frac{-1 \pm \sqrt{5}}{2}$

$\tan x = \frac{-1 - \sqrt{5}}{2}$ is acceptable

$\therefore$ From option (Q)

(D) $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$

$$\left(\frac{1 - \cos 2x}{2}\right)^5 + \left(\frac{1 + \cos 2x}{5}\right)^5 = \frac{29}{16}\cos^4 2x$$

Let $\cos 2x = t$

$$\left(\frac{1 - t}{2}\right)^5 + \left(\frac{1 + t}{2}\right)^5 = \frac{29}{16}t^4$$

$24t^4 - 10t^2 - 1 = 0$ or $(2t^2 - 1)(12t^2 + 1) = 0$

$\Rightarrow t = \pm\frac{1}{2}$

$\therefore \cos^2 x = \frac{1}{2}$

$\Rightarrow 2\cos^2 2x - 1 = 0$

$\Rightarrow \cos 4x = 0$

$4x = n\pi + \frac{\pi}{2}$ or $x = \frac{n\pi}{4} + \frac{\pi}{8}$ $n \in I$

Sol 27: (A) $\sin^2 x + \cos^2 3x = 1$

$\cos^2 3x = \cos^2 x$

$\Rightarrow \cos 3x = \pm\cos x$

$\therefore 3x = 2n\pi \pm x$ or $3x = 2n\pi \pm (\pi - x)$

$x = n\pi, \frac{n\pi}{2}$ or $x = \frac{n\pi}{2} + \frac{\pi}{4}, n\pi - \frac{\pi}{2}$

From options one can say that all the options which satisfying the equation are p, q, r, s

(B) $e^{\cot^2\theta} + \sin^2\theta - 2\cos^2 2\theta + 4 = 4\sin\theta$

$e^{\cot^2\theta}$ is not possible at $\theta = n\pi, n \in I$

So at $\theta = \frac{\pi}{4}$

$\Rightarrow e^1 + \frac{1}{2} - 2(0) + 4, \frac{4}{\sqrt{2}} = 2\sqrt{2}$

L. H. S. $\neq$ R. H. S.

at $\theta = \frac{\pi}{2}$

$\Rightarrow e^0 + 1^2 - 2(1)^2 + 4 = 4(1)$

$1 + 1 - x + 4 = 4$

L. H. S. = R. H. S.

So Ans (s) $(4n + 1)\frac{\pi}{2}$

(C) $a^2\sin x - a\sin 2x + \sin x = 0$

$(a^2 + 1)\sin x = 2a\sin x\cos x$

$\sin x[(a^2 + 1) - 2a\cos x] = 0$

$\sin x = 0, \cos x = \frac{a^2 + 1}{2a} \geq 1$

From options Ans. (p)

(D) $\sqrt[3]{2\tan\theta - 1} + \sqrt[3]{\tan\theta - 1} = 1$

From all the given options we can directly reject P, Q, S as they are not satisfying the given equation and $\frac{\pi}{2}$ is not in domain of $\tan\theta$.

Exercise 2

Single Correct Choice Type

Sol 1: (A) If B = C then ∠A = 90° ∠B = 45° ∠C= 45°

Sol 2: (A) $2^{\tan\left(x-\frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x-\frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0$

$$\frac{\sin^2\left(x-\frac{\pi}{4}\right)}{\cos 2x} = \frac{1}{2}\frac{\left(1-\cos 2\left(x-\frac{\pi}{4}\right)\right)}{\cos 2x}$$

$$= \frac{1}{2}\frac{(1-\sin 2x)}{\cos 2x} = \frac{1}{2}\tan\left(\frac{\pi}{4}-x\right)$$

∴ Take $\tan\left(x-\frac{\pi}{4}\right) = t$

Then expression would be

$$2^t - 2\left(\frac{1}{4}\right)^{t/2} + 1 = 0$$

$$\Rightarrow 2^t - 2\left(\frac{1}{2}\right)^t + 1 = 0$$

$(2^t)^2 + (2)^t - 2 = 0$

$(2^t + 2)(2^t - 1) = 0$

$\Rightarrow 2^t = -2$ (Not possible) or $2^t = 1$

$\Rightarrow t = 0$

∴ $\tan\left(x-\frac{\pi}{4}\right) = 0$; $x = n\pi + \frac{\pi}{4}$

But in equation $\frac{1}{\cos 2x}$ does not exist at

$x = n\pi + \frac{\pi}{4}$, therefore no value of x exists.

Sol 3: (A) $\sum_{r=1}^{5}\cos rx = 0$

cos x + cos 2x + cos 3x + cos 4x + cos 5x = 0

$\Rightarrow \cos x + \cos 5x + \cos 2x + \cos 4x + \cos 3x = 0$

$\Rightarrow 2\cos 3x\cos 2x + 2\cos 3x\cos x + \cos 3x = 0$

$\Rightarrow \cos 3x\left[2\cos 2x + 2\cos x + 1\right] = 0$

$\Rightarrow \cos 3x\left[2\times 2\cos\frac{3x}{2}\cos\frac{x}{2} + 1\right] = 0$

$\Rightarrow \cos 3x\left[4\cos\frac{3x}{2}\cos\frac{x}{2} + 1\right] = 0$

$\Rightarrow \cos 3x = 0 = \cos(2n+1)\frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}$

⇒ Two solutions

Sol 4: (B) $\cot 7\frac{1}{2}° + \tan 67\frac{1}{2}° - \cot 67\frac{1}{2}° - \tan 7\frac{1}{2}°$

$= \cot\frac{A}{2} + \tan\frac{B}{2} - \cot\frac{B}{2} - \tan\frac{A}{2}$

A = 15°, B = 135°

$$= \frac{1-\tan^2\frac{A}{2}}{\tan\frac{A}{2}} + \frac{\tan^2\frac{B}{2}-1}{\tan\frac{B}{2}}$$

= 2cotA – 2cotB = 2(cot15° – cot135°)

$= 2(2 + \sqrt{3} + 1) = 2(3 + \sqrt{3})$

Which is an irrational number.

Sol 5: (C) $A = 580° = 3\pi + \frac{2\pi}{9}$

$$\sqrt{1+\sin A} = \sqrt{\sin^2\frac{A}{2} + \cos^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \sqrt{\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2} = \left|\sin\frac{A}{2}+\cos\frac{A}{2}\right|$$

∵ for $A = 3\pi + \frac{2\pi}{9}$

$$\sqrt{1+\sin A} = -\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$$

$$\sqrt{1-\sin A} = \sqrt{\sin^2\frac{A}{2} + \cos^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \left|\sin\frac{A}{2}-\cos\frac{A}{2}\right|$$

For $A = 3\pi + \frac{2\pi}{9}$

$$\sqrt{1-\sin A} = -\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$$

$$\therefore 2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$$

Sol 6: (A) $\tan\alpha = \dfrac{x^2 - x}{x^2 - x + 1}$ and $\tan\beta = \dfrac{1}{2x^2 - 2x + 1}$

$\Rightarrow 2(x^2 - x) + 1 = \dfrac{1}{\tan\beta}$

$x^2 - x = \left(\dfrac{1-\tan\beta}{\tan\beta}\right)\dfrac{1}{2}$

$\therefore \tan\alpha = \dfrac{\frac{1}{2}\left(\frac{1-\tan\beta}{\tan\beta}\right)}{\frac{1}{2}\left(\frac{1-\tan\beta}{\tan\beta}\right)+1} = \dfrac{\frac{1}{2}\left(\frac{1-\tan\beta}{\tan\beta}\right)}{\frac{1}{2}\left(\frac{1+\tan\beta}{\tan\beta}\right)}$

$\Rightarrow \tan\alpha = \dfrac{1-\tan\beta}{1+\tan\beta}$

$\Rightarrow \tan\alpha + \tan\alpha\tan\beta = 1 - \tan b$

$\Rightarrow \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \tan(\alpha+\beta) = 1$

Sol 7: (C) $8\cos^2 x + 18\sec^2 x$

$f(X) = 8X^2 + \dfrac{18}{X^2}$ let $\cos^2 x = X$

$f'(X) = 16X - \dfrac{36}{X^3} = 0$

$X = \left(\dfrac{36}{16}\right)^{1/4} = (2\ 25)^{1/4} > 1$

$\therefore$ Min. Value of this function will be

When $\cos x = 1$

$\therefore$ Min. Value = 8 + 18 = 26

Sol 8: (B) $x = a\cos(\phi - \alpha)$

$y = b\cos(\phi - \beta)$

$\Rightarrow \cos^{-1}\left(\dfrac{x}{a}\right) = \phi - \alpha$ .....(i)

$\Rightarrow \cos^{-1}\left(\dfrac{y}{b}\right) = \phi - \beta$ .....(ii)

(ii) – (i)

$\Rightarrow \alpha - \beta = \cos^{-1}\left(\dfrac{x}{a}\right) - \cos^{-1}\left(\dfrac{y}{b}\right)$

$\Rightarrow \alpha - \beta = \cos^{-1}\left[\dfrac{xy}{ab} + \sqrt{1-\dfrac{x^2}{a^2}}\sqrt{1-\dfrac{y^2}{b^2}}\right]$

$\Rightarrow \cos(\alpha-\beta) = \dfrac{xy}{ab} + \sqrt{1-\dfrac{x^2}{a^2}}\cdot\sqrt{1-\dfrac{y^2}{b^2}}$

$\Rightarrow \cos(\alpha-\beta) = \dfrac{xy}{ab} + \sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{x^2y^2}{a^2b^2}}$

$\Rightarrow \left(\cos(\alpha-\beta) - \dfrac{xy}{ab}\right)^2 = \left(\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{x^2y^2}{a^2b^2}}\right)^2$

$\Rightarrow \cos^2(\alpha-\beta) + \dfrac{x^2y^2}{a^2b^2} - 2\cos(\alpha-\beta)\times\dfrac{xy}{ab}$

$= 1 - \left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right) + \dfrac{x^2y^2}{a^2b^2}$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{2xy}{ab}\cos(\alpha-\beta) = 1 - \cos^2(\alpha-\beta)$

$= \sin^2(\alpha-\beta)$

Sol 9: (D) $\tan x + \tan 2x + \tan 3x = \tan x.\ \tan 2x.\ \tan 3x$

$\dfrac{\tan x + \tan 2x}{1 - \tan x\tan 2x} = -\tan 3x$

$\Rightarrow \tan 3x = -\tan 3x \Rightarrow \tan 3x = 0$

$\Rightarrow 3x = n\pi$

$\Rightarrow x = \dfrac{n\pi}{3}$

Sol 10: (C) $\tan 3x - \tan 2x - \tan x = 0$

$\tan(3x–2x)[1+\tan 3x\tan 2x] - \tan x = 0$

$\Rightarrow \tan x\tan 2x\tan 3x = 0$

$\therefore x = n\pi, \dfrac{n\pi}{2}, \dfrac{n\pi}{3}$

$\therefore x = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi. \dfrac{4\pi}{3}, \dfrac{5\pi}{3}, 2\pi$

i.e. 7 solutions

$\because x \neq \dfrac{\pi}{2}$

As at $x = \dfrac{\pi}{2}$ tan x, is not defined

Sol 11: (C) $x = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ \infty$

$x = 1 - x(1 - x + x^2 - x^3 + x^4 \ldots\ldots\ \infty)$

$x = 1 - x.\ x$

$\Rightarrow x^2 + x - 1 = 0$

$x = \dfrac{-1 \pm \sqrt{5}}{2} = 2\sin 18°$

Multiple Correct Choice Type

Sol 12: (B, D) $1 + 4\sin\theta + 3\cos\theta = 1 + 5\left(\frac{4}{5}\sin\theta + \frac{3}{5}\cos\theta\right)$

$= 1 + 5\sin(\theta + \alpha)\cos\alpha = \frac{4}{5}$

$\therefore$ Maximum value is $1 + 5 = 6$

When $\sin(\theta + \alpha) = 1$

Minimum value is $1 - 5 = -4$

When $\sin(\theta + \alpha) = -1$

Sol 13: (A,B, C) $\dfrac{\sqrt{3}\sin(\alpha+\beta) - \dfrac{2}{\cos\frac{\pi}{6}}\cos(\alpha+\beta)}{\sin\alpha}$

$$= \left(\frac{\sqrt{3}\times\frac{\sqrt{3}}{2}\sin(\alpha+\beta) - 2\cos(\alpha+\beta)}{\sin\alpha}\right)\frac{2}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\frac{3}{2}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) - 2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}{\sin\alpha}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{3}{2}\cos\beta + \frac{3}{2}\cot\alpha\sin\beta - 2\cot\alpha\cos\beta + 2\sin\beta\right]$$

It $\sin\beta = \frac{4}{5}$ and $\cos\beta = \frac{3}{5}$

i.e. $\tan\beta > 0$ i.e. $\beta \in \left(0, \frac{\pi}{2}\right)$

R. H. S.

$$= \frac{2}{\sqrt{3}}\left[\frac{3}{2}\times\frac{3}{5} + \frac{3}{2}\cot\alpha\times\frac{4}{5} - 2\cot\alpha\times\frac{3}{5} + 2\times\frac{4}{5}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{9}{10} + \frac{8}{5}\right] = \frac{5}{\sqrt{3}}$$

If $\cos\beta = -\frac{3}{5}$ i.e. $\tan\beta < 0$

$\Rightarrow$ R. H. S.

$$= \frac{2}{\sqrt{3}}\left[\frac{3}{2}\left(-\frac{3}{5}\right) + \frac{3}{2}\cot\alpha\times\frac{4}{5} + 2\cot\alpha\times\frac{3}{5} + 2\times\frac{4}{5}\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{12}{5}\cot\alpha + \frac{7}{10}\right]$$

$$= \frac{[24\cot\alpha + 7]}{15}\sqrt{3}$$

Sol 14: (B, C) $\sin t + \cos t = \frac{1}{5}$

$$\Rightarrow \frac{2\tan\frac{t}{2}}{1+\tan^2\frac{t}{2}} + \frac{1-\tan^2\frac{t}{2}}{1+\tan^2\frac{t}{2}} = \frac{1}{5}$$

Let $\tan\frac{t}{2} = a$

$5(2a + 1 - a^2) = 1 + a^2$

$\Rightarrow 6a^2 - 10a - 4 = 0 \Rightarrow 3a^2 - 5a - 2 = 0$

$\Rightarrow (3a + 1)(a - 2) = 0$

$\therefore a = -\frac{1}{3}, 2$

$\tan\frac{t}{2} = \frac{-1}{3}, 2$

Previous Years' Questions

Sol 1: Given, $e^{\sin x} - \dfrac{1}{e^{\sin x}} = 4$

$\Rightarrow \left(e^{\sin x}\right)^2 - 4\left(e^{\sin x}\right) - 1 = 0$

$\Rightarrow e^{\sin x} = \dfrac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$

But since, $e \sim 2.72$ and we know, $0 < e^{\sin x} < e$

$\therefore e^{\sin x} = 2 \pm \sqrt{5}$ is not possible.

Hence, no solution.

Sol 2: Given, $2^{1+|\cos x|+|\cos^2 x|+|\cos^3 x|+.....} = 2^2$

$\Rightarrow 2^{\frac{1}{1-|\cos x|}} = 2^2 \qquad \Rightarrow \dfrac{1}{1-|\cos x|} = 2$

$\Rightarrow \cos x = \pm\frac{1}{2}$

$(\because x \in (-\pi, \pi))$

Thus, the solution set is

$\left\{\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}\right\}$

Sol 3: Exp $\left\{\left(\sin^2 x+\sin^4 x+\sin^6 x+.....+\infty\right)\log_e 2\right\}$

$= e^{\frac{\sin^2 x}{1-\sin^2 x}\log_e 2} = e^{\log_e 2\frac{\sin^2 x}{\cos^2 x}}$

$\Rightarrow 2^{\tan^2 x}$ satisfys $x^2-9x+8=0$

$\Rightarrow x = 1, 8$

$\therefore\ 2^{\tan^2 x}=1$ and $2^{\tan^2 x}=8$

$\Rightarrow\ \tan^2 x=0$ and $\tan^2 x=3$

$\Rightarrow\ x=n\pi$ and $\tan^2 x=\left(\tan\frac{\pi}{3}\right)^2$

$\Rightarrow\ x=n\pi$ and $x=n\pi\pm\frac{\pi}{3}$

Neglecting $x = n\pi$ as $0 < x < \frac{\pi}{2}$

$\Rightarrow x=\frac{\pi}{3}\in\left(0,\frac{\pi}{2}\right)$

$\therefore\ \frac{\cos x}{\cos x+\sin x}=\frac{\frac{1}{2}}{\frac{1}{2}+\frac{\sqrt{3}}{2}}=\frac{1}{1+\sqrt{3}}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$

$\Rightarrow\ \frac{\cos x}{\cos x+\sin x}=\frac{\sqrt{3}-1}{2}$

Sol 4: $\tan\left(x+100^\circ\right)=\tan\left(x+50^\circ\right)\tan x\tan\left(x-50^\circ\right)$

$\Rightarrow\frac{\tan\left(x+100^\circ\right)}{\tan x}=\tan\left(x+50^\circ\right)\tan\left(x-50^\circ\right)$

$\Rightarrow\frac{\sin\left(x+100^\circ\right)}{\cos\left(x+100^\circ\right)}\cdot\frac{\cos x}{\sin x}=\frac{\sin\left(x+50^\circ\right)\sin\left(x-50^\circ\right)}{\cos\left(x+50^\circ\right)\cos\left(x-50^\circ\right)}$

$\Rightarrow\left[\sin\left(2x+100^\circ\right)+\sin100^\circ\right]\left[\cos100^\circ+\cos2x\right]$

$=\left[\cos100^\circ-\cos2x\right]\times\left[\sin\left(2x+100^\circ\right)-\sin100^\circ\right]$

$\Rightarrow\sin\left(2x+100^\circ\right)\cdot\cos100^\circ$

$+\sin\left(2x+100^\circ\right)\cdot\cos2x\ +\sin100^\circ\cos100^\circ$

$+\sin100^\circ\cos2x$

$=\ \cos100^\circ\sin\left(2x+100^\circ\right)$

$-\cos100^\circ\sin100^\circ-\cos2x\sin$

$\left(2x+100^\circ\right)+\cos2x\sin100^\circ$

$\Rightarrow 2\sin\left(2x+100^\circ\right)\cos2x\ +2\sin100^\circ\cos100^\circ=0$

$\Rightarrow\sin\left(4x+100^\circ\right)+\sin100^\circ+\sin200^\circ=0$

$\Rightarrow\sin\left(4x+100^\circ\right)+2\sin150^\circ\cos50^\circ=0$

$\Rightarrow\sin\left(4x+100^\circ\right)+2.\frac{1}{2}\sin\left(90^\circ-50^\circ\right)=0$

$\Rightarrow\sin\left(4x+100^\circ\right)+\sin40^\circ=0$

$\Rightarrow\sin\left(4x+100^\circ\right)+\sin\left(-40^\circ\right)$

$\Rightarrow 4x=n\left(180^\circ\right)+\left(-1\right)^n\left(-40^\circ\right)-100^\circ$

$\Rightarrow x=\frac{1}{4}\left[n\left(180^\circ\right)+\left(-1\right)^n\left(-40^\circ\right)-100^\circ\right]$

The smallest positive value of x is obtained when n = 1.

Therefore, $x=\frac{1}{4}\left(180^\circ+40^\circ-100^\circ\right)$

$\Rightarrow x=\frac{1}{4}\left(120^\circ\right)=30^\circ$

Sol 5: Given,

$\cos\left(p\sin x\right)=\sin\left(p\cos x\right),\ \forall x\in\left[0,2\pi\right]$

$\Rightarrow\cos\left(p\sin x\right)=\cos\left(\frac{\pi}{2}-p\cos x\right)$

$\Rightarrow p\sin x=2n\pi\pm\left(\frac{\pi}{2}-p\cos x\right),n\in I$

$\left(\because\cos\theta=\cos\alpha\Rightarrow\theta=2n\pi\pm\alpha,n\in I\right)$

$\Rightarrow p\sin x+p\cos x=2n\pi+\pi/2$

or $p\sin x-p\cos x=2n\pi-\pi/2,n\in I$

$\Rightarrow p\left(\sin x+\cos x\right)2n\pi+\pi/2$

or $p\left(\sin x-\cos x\right)=2n\pi-\pi/2,\ n\in I$

$\Rightarrow p\sqrt{2}\left(\cos\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}\cos x\right)=2n\pi-\frac{\pi}{2}$

or $p\sqrt{2}\left(\cos\frac{\pi}{4}\sin x-\sin\frac{\pi}{4}\cos x\right)=2n\pi-\frac{\pi}{2},n\in I$

$\Rightarrow p\sqrt{2}\left[\sin\left(x+\pi/4\right)\right]=\frac{\left(4n+1\right)\pi}{2}$

or $p\sqrt{2}\left[\sin\left(x-\pi/4\right)\right]=\left(4n-1\right)\frac{\pi}{2},\ n\in I$

Now, $-1\le\sin\left(x\pm\pi/4\right)\le1$

$\Rightarrow -p\sqrt{2}\le p\sqrt{2}\sin\left(x\pm\pi/4\right)\le p\sqrt{2}$

$\Rightarrow -p\sqrt{2}\le\frac{\left(4n+1\right)\pi}{2}\le p\sqrt{2},\ n\in I$

Second inequality is always a subset of first, therefore, we have to consider only first.

It is sufficient to consider $n \le 0$, because for n > 0, the solution will be same for $n \ge 0$.

If $n \ge 0, -\sqrt{2}p \le (4n+1)\pi/2$

$\Rightarrow (4n+1)\pi/2 \le \sqrt{2}p$

For p to be least, n should be least.

$\Rightarrow$ n = 0

$\Rightarrow \sqrt{2}p \ge \pi/2$

$\Rightarrow p \ge \frac{\pi}{2\sqrt{2}}$

Therefore, least value of p = $\frac{\pi}{2\sqrt{2}}$

Sol 6: Given,

$(1-\tan\theta)(1+\tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$

$\Rightarrow (1-\tan^2\theta)\cdot(1+\tan^2\theta) + 2^{\tan^2\theta} = 0$

$\Rightarrow 1-\tan^4\theta + 2^{\tan^2\theta} = 0$

Substitute $\tan^2\theta = x$

$\therefore 1 - x^2 + 2^x = 0$

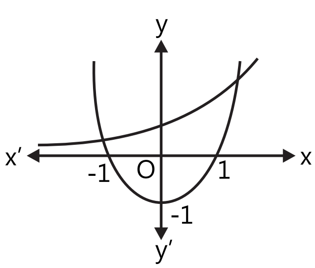
$\Rightarrow x^2 - 1 = 2^x$

Note: 2^x and x^2-1 are incompatible functions, therefore, we have to consider range of both functions.

Curvesy = $x^2 - 1$

Andy = 2^x

It is clear from the graph that two curves interest at one point at x = 3, y = 8.



Therefore, $\tan^2\theta = 3$

$\Rightarrow \tan\theta = \pm\sqrt{3}$

$\Rightarrow \theta = \pm\frac{\pi}{3}$

Sol 7: (A, C) Given,

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Applying

$R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$ we get

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2$

$$\Rightarrow \begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2 + 4\sin 4\theta = 0$

$\Rightarrow \sin 4\theta = \frac{-1}{2}$

$\Rightarrow 4\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\Rightarrow 4\theta = \frac{n\pi}{4} + (-1)^{n+1}\left(\frac{\pi}{24}\right)$

Clearly, $\theta = \frac{7\pi}{24}, \frac{11\pi}{24}$ are two values of θ lying between 0^o and $\frac{\pi}{2}$

Sol 8: (B, C) For $0 < \phi < /\pi/2$ we have

$x = \sum_{n=0}^{\infty}\cos^{2n}\phi = 1 + \cos^2\phi$

$+\cos^4\phi + \cos^6\phi + ...$

It is clearly a GP with common ratio of $\cos^2\phi$ which is ≤ 1.

Hence, x = $\frac{1}{1-\cos^2\phi} = \frac{1}{\sin^2\phi}$

$\left[\because S_\infty = \frac{a}{1-r}, -1 < r < 1\right]$

Similarly, $y = \frac{1}{\cos^2\phi}$

And $z = \frac{1}{1-\sin^2\phi\cos^2\phi}$

Now, $x+y=\frac{1}{\sin^2\phi}+\frac{1}{\cos^2\phi}$

$=\frac{\cos^2\phi+\sin^2\phi}{\cos^2\phi\sin^2\phi}=\frac{1}{\cos^2\phi\sin^2\phi}$

Again, $\frac{1}{x}=1-\sin^2\phi\cos^2\phi=1-\frac{1}{xy}$

$\Rightarrow \frac{1}{x}=\frac{xy-1}{xy}\Rightarrow xy=xyz-z$

$\Rightarrow xy+z=xyz$

Therefore, (b) is the answer from eq. (i) (putting the value of xy)

$\Rightarrow xyz=x+y+z$

Sol 9: (A, B) We know that, $\sec^2\theta\geq 1$

$\Rightarrow \frac{4xy}{(x+y)}\geq 1$

$\Rightarrow 4xy\geq (x+y)^2$

$\Rightarrow (x+y)^2-4xy\leq 0$

$\Rightarrow (x-y)^2\leq 0$

$\Rightarrow x-y=0$

$\Rightarrow x=y$

Therefore , x + y = 2x (add x both sides)

But x + y $\neq$ 0 since it lies in the denominator,

$\Rightarrow 2x\neq 0$

$\Rightarrow x\neq 0$

Hence, x = y, x $\neq$ 0 is the answer.

Sol 10: (A, B) $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$

$\Rightarrow \frac{3\sin^4 x+2\cos^4 x}{6}=\frac{1}{5}$

$\Rightarrow \sin^4 x+2\left(\sin^4 x+\cos^4 x\right)=\frac{6}{5}$

$\Rightarrow \sin^4 x+2\left[1-2\sin^2 x\left(\cos^2 x\right)\right]=\frac{6}{5}$

$\Rightarrow \sin^4 x+2-4\sin^2 x\left(1-\sin^2 x\right)=\frac{6}{5}$

$\Rightarrow \sin^4 x+2-4\sin^2 x+4\sin^4 x=\frac{6}{5}$

$\Rightarrow 5\sin^4 x-4\sin^2 x+\frac{4}{5}=0$

$\Rightarrow \sin^2 x=\frac{4\pm\sqrt{16-4\times 5\times\frac{4}{5}}}{10}$

$=\frac{4\pm\sqrt{16-16}}{10}$

$\Rightarrow \sin^2 x=\frac{2}{5}$

$\Rightarrow \sin x=\pm\sqrt{\frac{2}{5}}$

$\sin^2 x=\frac{2}{5}$

$\Rightarrow \cos^2 x=\frac{3}{5}$

$\Rightarrow \sec^2 x=\frac{5}{3}$

$\Rightarrow \tan^2 x=\sec^2 x-1$

$=\frac{5}{3}-1=\frac{2}{3}$

Now, $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}$

$=\frac{\left(\frac{2}{5}\right)^4}{8}+\frac{\left(\frac{3}{5}\right)^4}{27}=\frac{2}{5^4}+\frac{3}{5^4}=\frac{1}{125}$

Sol 11: (D)

$$\sum_{m=1}^{6}\operatorname{cosec}\left[\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right]\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2}$$

$$\sum_{m=1}^{6}\sqrt{2}\times\frac{\sin\left[\left(\theta+\frac{m}{\pi}\right)-\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)\right]}{\sin\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)}=4\sqrt{2}$$

$$\sum_{m=1}^{6}\frac{\sin\left(\theta+\frac{m\pi}{4}\right)\cos\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)-\cos\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)}{\sin\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)}$$

$$\sum_{m=1}^{6}\sqrt{2}\ x\left[\cot\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)\right]=4\sqrt{2}$$

$$\sqrt{2}\left[\cot\theta-\cot\left(\theta+\frac{3\pi}{2}\right)\right]=4\sqrt{2}$$

$$\Rightarrow \cot\theta-\cot\left(\frac{3\pi}{2}+\theta\right)=4$$

$\Rightarrow \cot\theta - \tan\theta = 4$

$\Rightarrow \tan^2\theta - 4\tan\theta + 1 = 0$

$\Rightarrow \tan\theta = \dfrac{4 \pm \sqrt{16-4}}{2}$

$= \dfrac{4 \pm \sqrt{12}}{2}$

$= \dfrac{4 \pm 2\sqrt{3}}{2}$

$\Rightarrow \tan\theta = \pm 2\sqrt{3}$

$\Rightarrow \theta = \dfrac{5\pi}{12}$

Sol 12: Given, $\tan\theta = \cot 5\theta$

$\Rightarrow \tan\theta = \tan\left(\dfrac{\pi}{2} - 5\theta\right)$

$\Rightarrow \dfrac{\pi}{2} - 5\theta = n\pi + \theta$

$\Rightarrow 6\theta = \dfrac{\pi}{2} - n\pi$

$\Rightarrow \theta = \dfrac{\pi}{12} - \dfrac{n\pi}{6}$

Also, $\cos 4\theta = \sin 2\theta = \cos\left(\dfrac{\pi}{2} - 2\theta\right)$

$\Rightarrow 4\theta = 2n\pi \pm \left(\dfrac{\pi}{2} - 2\theta\right)$

Taking positive Δ sign

$\Rightarrow 6\theta = 2n\pi + \dfrac{\pi}{2}$

$\Rightarrow \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{12}$

Taking negative sign

$\Rightarrow 2\sin\dfrac{2\pi}{n}.\cos\dfrac{2\pi}{n} = \sin\dfrac{3\pi}{n}$

$\Rightarrow \sin\dfrac{4\pi}{n} = \sin\dfrac{3\pi}{n}$

$\Rightarrow 2\theta = 2n\pi - \dfrac{\pi}{2}$

$\Rightarrow \theta = n\pi - \dfrac{\pi}{4}$

Above values of θ suggest that there are only 3 common solutions.

Sol 13: Given, $n > 3 \in$ Integer

and $\dfrac{1}{\sin\left(\dfrac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\dfrac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\dfrac{3\pi}{n}\right)}$

$\Rightarrow \dfrac{1}{\sin\dfrac{\pi}{n}} - \dfrac{1}{\sin\dfrac{3\pi}{n}} = \dfrac{1}{\sin\dfrac{2\pi}{n}}$

$\Rightarrow \dfrac{\sin\dfrac{3\pi}{n} - \sin\dfrac{\pi}{n}}{\sin\dfrac{\pi}{n}.\sin\dfrac{3\pi}{n}} = \dfrac{1}{\sin\dfrac{2\pi}{n}}$

$\Rightarrow 2\cos\left(\dfrac{2\pi}{n}\right).\sin\dfrac{\pi}{n} = \dfrac{\sin\dfrac{\pi}{n}.\sin\dfrac{3\pi}{n}}{\sin\dfrac{2\pi}{n}}$

$\Rightarrow \dfrac{4\pi}{n} = \pi - \dfrac{3\pi}{n}$ $\qquad \Rightarrow \dfrac{7\pi}{n} = \pi ; \Rightarrow n = 7$

Sol 14: $(y+z)\cos 3\theta = xyz \sin 3\theta$ .....(i)

$xyz \sin 3\theta = 2z\cos 3\theta + 2y\sin 3\theta$ .....(ii)

$xyz \sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$ ....(iii)

From (i), (ii) and (iii), we get

$(y+z)\cos 3\theta = 2z\cos 3\theta + 2y\sin 3\theta$

$= (y+2z)\cos 3\theta + y\sin 3\theta$

$\Rightarrow y(\cos 3\theta - 2\sin 3\theta) = z\cos 3\theta$ and

$y(\cos 3\theta - \sin 3\theta) = 0$

$\Rightarrow \cos 3\theta = \sin 3\theta$

$\Rightarrow \tan 3\theta = 1 = \tan\dfrac{\pi}{4}$

$\Rightarrow 3\theta = n\pi + \dfrac{\pi}{4}$

$\Rightarrow \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{4}$

$\Rightarrow \dfrac{\pi}{4}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12}$

Total three solution in $(0, \pi)$

Sol 15: $\tan\theta = \cot 5\theta$

$\tan\theta = \tan\left(\dfrac{\pi}{2} - 5\theta\right)$

$\Rightarrow \theta = n\pi + \frac{\pi}{2} - 5\theta$

$\Rightarrow 6\theta = n\pi + \frac{\pi}{2}$

$\Rightarrow \theta = \frac{n\pi}{6} + \frac{\pi}{12} \Rightarrow \theta = \pm\frac{\pi}{12}, \frac{\pm\pi}{4}, \frac{\pm 5\pi}{12}.....$ .... (i)

$\sin 2\theta = \cos 4\theta$

$\Rightarrow \cos 4\theta = \cos\left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow 4\theta = 2m\pi \pm \left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \theta = \frac{m\pi}{3} + \frac{\pi}{12}, m\pi - \frac{\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{12}, -\frac{\pi}{4}, \frac{5\pi}{12}$ ....(ii)

In (i) and (ii) only (iii) solutions are common

$-\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$

Sol 16: Given

$$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$$

Let

$A = \sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta$

$= 1 + 4\cos^2\theta + 3\sin\theta\cos\theta$

$= 1 + 2(1 + \cos 2\theta) + \frac{3}{2}\sin 2\theta$

$= 3 + 2\cos 2\theta + \frac{3}{2}\sin 2\theta$

$\left[a\cos\theta + b\sin\theta \ge -\sqrt{a^2 + b^2}\right]$

$A_{min} = 3 - \sqrt{2^2 + \left(\frac{3}{2}\right)^2}$

$= 3 - \sqrt{4 + \frac{9}{4}}$

$= 3 - \sqrt{\frac{25}{4}} = 3 - \frac{5}{2} = \frac{1}{2}$

The maximum value of given expression

$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is $\frac{1}{1/2} = 2$

Sol 17: (D) $P = \left\{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\right\}$

$\Rightarrow \sin\theta = \left(\sqrt{2} + 1\right)\cos\theta$

$\Rightarrow \tan\theta = \sqrt{2} + 1$

$Q = \left\{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\right\}$

$\Rightarrow \cos\theta = \left(\sqrt{2} - 1\right)\sin\theta$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$

$\therefore P = Q$

Sol 18: $\frac{1}{\sin\frac{\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}} + \frac{1}{\sin\frac{3\pi}{n}}$

$\Rightarrow \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin 2\frac{\pi}{n}}$

$\Rightarrow \frac{2\cos\frac{\left(\frac{3\pi}{n} + \frac{\pi}{n}\right)}{2}\sin\frac{\left(\frac{3\pi}{n} - \frac{\pi}{n}\right)}{2}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$

$\Rightarrow \frac{2\cos\frac{2\pi}{n}\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\sin\frac{3\pi}{n}} = \frac{1}{\sin 2\frac{\pi}{n}}$

$= 2\sin\frac{2\pi}{n}\cos\frac{2\pi}{n} = \sin\frac{3\pi}{n}$

$\Rightarrow \sin\frac{4\pi}{n} = \sin\frac{3\pi}{n}$

$\Rightarrow \sin\frac{4\pi}{n} - \sin\frac{3\pi}{n} = 0$

$\Rightarrow 2\cos\frac{7\pi}{2n}\sin\frac{\pi}{2n} = 0$

$\Rightarrow \cos\frac{7\pi}{2n} = 0$ or $\sin\frac{\pi}{2n} = 0$

$\Rightarrow n = \dfrac{7}{2m+1}$

For integral values of n > 3

Sol 19: (A, C, D) $\theta \in [0, 2\pi]$

$2\cos\theta(1-\sin\phi) = \sin^2\theta\left(\tan\dfrac{\theta}{2} + \cot\dfrac{\theta}{2}\right)\cos\phi - 1$

$\Rightarrow 2\cos\theta(1-\sin\phi) = \dfrac{\sin^2\theta\left(\tan^2\dfrac{\theta}{2}+1\right)}{\tan\dfrac{\theta}{2}}\cos\phi - 1$

$\Rightarrow 2\cos\theta(1-\sin\phi) = \dfrac{\sin^2\theta.\sec^2\dfrac{\theta}{2}}{\tan\dfrac{\theta}{2}}\cos\phi - 1$

$\Rightarrow 2\cos\theta(1-\sin\phi) = 2\sin\theta(\cos\phi) - 1$

$\Rightarrow 1 + 2\cos\theta = 2\sin(\theta+\phi)$ ....(i)

Now, given $\tan(2\pi-\theta) > 0$, $-1 < \sin\theta < -\dfrac{\sqrt{3}}{2}$

$\Rightarrow \tan\theta < 0$ and $\dfrac{3\pi}{2} < \theta < \dfrac{5\pi}{3}$

$\Rightarrow \dfrac{3\pi}{2} < \theta < \dfrac{5\pi}{3}$

From (i), as $\theta \in \left(\dfrac{3\pi}{2}, \dfrac{5\pi}{3}\right)$

$\Rightarrow 1 < 2\cos\theta + 1 < 2$

$\Rightarrow 1 < 2\sin(\theta+\phi) < 2$

$\Rightarrow \dfrac{1}{2} < \sin(\theta+\phi) < 1 \Rightarrow \theta+\phi \in \left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right), \theta+\phi \neq \dfrac{\pi}{2}$

Or

$\theta+\phi \in \left(\dfrac{13\pi}{6}, \dfrac{17\pi}{6}\right), \theta+\phi \neq \dfrac{5\pi}{2} \Rightarrow \dfrac{\pi}{6} - \theta < \phi < \dfrac{5\pi}{6} - \theta$

or $\dfrac{13\pi}{6} - \theta < \phi < \dfrac{17\pi}{6} - \theta$

As, $\theta \in \left(-\dfrac{3\pi}{2}, -\dfrac{2\pi}{3}\right)$ or $\left(\dfrac{2\pi}{3}, \dfrac{7\pi}{6}\right)$

Sol 20: (D) sin x + 2 sin 2x – sin 3x = 3

$\Rightarrow \sin x - \sin 3x + 2\sin 2x = 3$

$\Rightarrow 2\cos 2x \sin(-x) + 2\sin 2x = 3$

$\Rightarrow -2\sin x[2\cos^2 x - 1] + 2\times 2\sin x\cos x = 3$

$\Rightarrow -4\sin x\left[\cos^2 x - \dfrac{1}{2} - \cos x\right] = 3$

$\Rightarrow -4\sin x\left[\left(\cos x - \dfrac{1}{2}\right)^2 - \dfrac{3}{4}\right] = 3$

$\Rightarrow \left(\cos x - \dfrac{1}{2}\right)^2 - \dfrac{3}{4} = -\dfrac{3}{4\sin x}$

$\Rightarrow \left(\cos x - \dfrac{1}{2}\right)^2 = \dfrac{3}{4} - \dfrac{3}{4\sin x}$

$\Rightarrow \left(\cos x - \dfrac{1}{2}\right)^2 = \dfrac{3}{4}\left(1 - \dfrac{3}{\sin x}\right)$

L.H.S. > 0 and R.H.S. < 0

∴ No solution

Sol 21: $\dfrac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$

$\Rightarrow \dfrac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^4 x$

$[1-\sin^2 x] + \sin^4 x[1-\cos^2 x] = 2$

$\Rightarrow \dfrac{5}{4}\cos^2 2x + 2(\cos^4 x + \sin^4 x) - \sin^2 x\cos^2 x$

$(\sin^2 x + \cos^2 x) = 2$

$\Rightarrow \dfrac{5}{4}\cos^2 2x + 2\left[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x\right]$

$-\sin^2 x\cos^2 x = 2$

$\Rightarrow \dfrac{5}{4}\cos^2 2x - 5\sin^2 x\cos^2 x = 0$

$\Rightarrow \dfrac{5}{4}\cos^2 2x = \dfrac{5}{4}\sin^2 2x$

$\Rightarrow \tan^2 2x = 1$

$\Rightarrow \tan 2x = \pm = \tan\left(\pm\dfrac{\pi}{4}\right)$

$\Rightarrow 2x = n\pi \pm \dfrac{\pi}{4}$

$\Rightarrow x = \dfrac{n\pi}{2} \pm \dfrac{\pi}{8}$

$\Rightarrow x = \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{13\pi}{8}, \dfrac{15\pi}{8} \rightarrow$

Total 8 solution.

Sol 22: (C) $\sqrt{3}\sec x+\operatorname{cosec} x+2(\tan x-\cot x)=0$

$\Rightarrow \frac{\sqrt{3}}{\cos x}+\frac{1}{\sin x}+2\left[\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}\right]=0$

$\Rightarrow \sqrt{3}\sin x+\cos x+2\left[\sin^2-\cos^2 x\right]=0$

$\Rightarrow \sqrt{3}\sin x+\cos x=2\cos 2x$

$\Rightarrow \frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x=\cos 2x$

$\Rightarrow \sin\frac{\pi}{3}\sin x+\cos\frac{\pi}{3}\cos x=\cos 2x$

$\Rightarrow \cos\left(x-\frac{\pi}{3}\right)=\cos 2x$

$\Rightarrow x-\frac{\pi}{3}=2n\pi\pm 2x$

$\Rightarrow x=-\left(2n\pi+\frac{\pi}{3}\right)$ or $x=\frac{2n\pi}{3}+\frac{\pi}{9}$

$\Rightarrow x=-\frac{\pi}{3},\frac{\pi}{9},-\frac{5\pi}{9},\frac{7\pi}{9}$

Sum = 0