Chapter 13

Conic Sections-II

Solutions

SECTION - A

Objective Type Questions (One options is correct)

1. Three normals are drawn to the curve $y^2 = x$ from a point $(c, 0)$. Out of three, one is always the x-axis. If two other normals are perpendicular to each other, then 'c' is

(1) $\frac{3}{4}$ (2) $\frac{1}{2}$ (3) $\frac{3}{2}$ (4) 2

Sol. Answer (1)

Equation of normal to the curve $y^2 = x$

$y = mx - 2am - am^3$ Put $a = \frac{1}{4}$

$y = mx - \frac{1}{2}m - \frac{1}{4}m^3$ but these normals passes through $(c, 0)$

$\Rightarrow 0 = mc - \frac{1}{2}m - \frac{1}{4}m^3$

$m\left(\frac{m^2}{4} + \frac{1}{2} - c\right) = 0$

$m = 0$ and $\frac{m^2}{4} + \frac{1}{2} - c = 0$

$m = 0$ $m^2 + 2 - 4c = 0$

$\therefore$ Normal is x-axis

Remaining normals are perpendicular

$\therefore$ $m_1m_2 = -1$

$2 - 4c = 1$

$4c = 3$

$c = \frac{3}{4}$

2. If the line $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P, then eccentric angle of the point is equal to

(1) 0 (2) 90° (3) 45° (4) 60°

Sol. Answer (3)

Let point of contact be $P(a\cos\alpha, b\sin\alpha)$

$\therefore$ Equation of tangent

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}=1 \quad \ldots \text{(i)}$$

$$\frac{x}{a\sqrt{2}}+\frac{y}{b\sqrt{2}}=1 \quad \ldots \text{(ii)}$$

These lines are coincident lines

$$\therefore \quad \frac{\cos\alpha}{\frac{1}{\sqrt{2}}}=\frac{\sin\alpha}{\frac{1}{\sqrt{2}}}=1$$

$\cos\alpha = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \frac{1}{\sqrt{2}}$

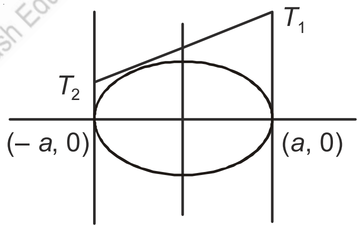
$\therefore \quad \alpha = 45°$

3. The tangent at any point on the ellipse $16x^2 + 25y^2 = 400$ meets the tangents at the ends of the major axis at T_1 and T_2. The circle on T_1T_2 as diameter passes through

(1) (3, 0) (2) (0, 0) (3) (0, 3) (4) (4, 0)

Sol. Answer (1)

Given $16x^2+25y^2=400 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1$



Any tangent to the ellipse is

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{4}=1$$

$$T_1 : \left(5, 4\tan\frac{\theta}{2}\right)$$

$$T_2 : \left(-5, 4\cot\frac{\theta}{2}\right)$$

Equation of a circle is

$$(x-5)(x+5)+\left(y-4\tan\frac{\theta}{2}\right)\left(y-4\cot\frac{\theta}{2}\right)=0$$

$$\Rightarrow \quad x^2-25+y^2+16-4\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)y=0$$

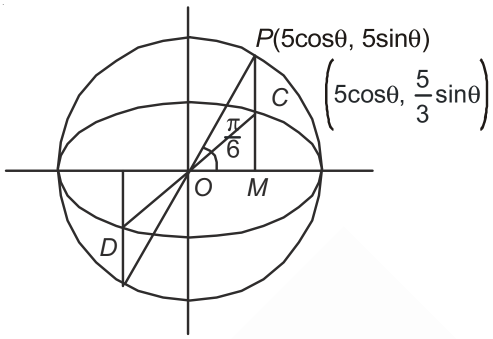
$$\Rightarrow \quad x^2+y^2-8y\operatorname{cosec}\theta-9=0$$

which passes through (3, 0)

4. If CD is a diameter of an ellipse $x^2 + 9y^2 = 25$ and the eccentric angle of C is $\frac{\pi}{6}$, then the eccentric angle of D is

(1) $\frac{5\pi}{6}$ (2) $-\frac{5\pi}{6}$ (3) $-\frac{2\pi}{3}$ (4) $\frac{2\pi}{3}$

Sol. Answer (2)



Clearly eccentric angle of $D = \frac{7\pi}{6} - 2\pi$

$= -\frac{5\pi}{6}$

5. If θ_1, θ_2, θ_3, θ_4 be eccentric angles of the four concyclic points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\theta_1 + \theta_2 + \theta_3 + \theta_4$ is (where $n \in I$)

(1) $(2n+1)\frac{\pi}{2}$ (2) $(2n + 1)\pi$ (3) $2n\pi$ (4) $n\pi$

Sol. Answer (3)

The equation of chords of the ellipse are

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=1$$

And $\frac{x}{a}\cos\left(\frac{\theta_2+\theta_3}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_3+\theta_4}{2}\right)=1$

If θ_1, θ_2, θ_3, θ_4 are concyclic points, then these lines will make equal angle with the axes

$\therefore \quad \tan\left(\frac{\theta_1+\theta_2}{2}\right)=-\tan\left(\frac{\theta_3+\theta_4}{2}\right)$

$\Rightarrow \quad \frac{\theta_1+\theta_2}{2}+\frac{\theta_3+\theta_4}{2}=n\pi$

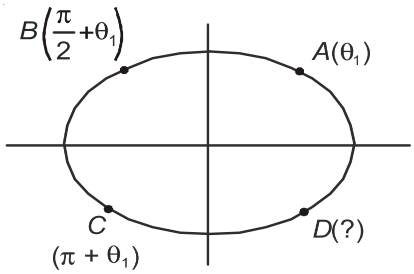
$\Rightarrow \quad \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2n\pi,\ n \in I$

6. Let the eccentric angles of three point A, B and C on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\theta_1, \frac{\pi}{2}+\theta_1, \pi+\theta_1$. A circle through A, B and C cuts the ellipse again at D. Then the eccentric angle of D is

(1) $\pi - 3\theta_1$ (2) $\frac{3\pi}{2}-3\theta_1$ (3) $\frac{\pi}{2}-3\theta_1$ (4) $3\theta_1-\frac{\pi}{2}$

Sol. Answer (3)

Let the eccentric angle of D be θ_2.



Then $\theta_1 + \frac{\pi}{2} + \theta_1 + \pi + \theta_1 + \theta_2 = 2\pi$

$\Rightarrow \frac{3\pi}{2} + 3\theta_1 + \theta_2 = 2\pi$

$\Rightarrow \theta_2 = 2\pi - \frac{3\pi}{2} - 3\theta_1$

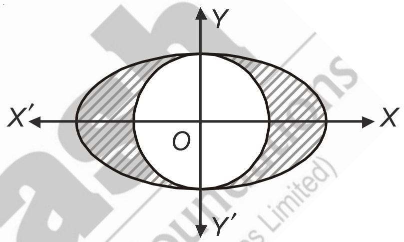
$\Rightarrow \theta_2 = \frac{\pi}{2} - 3\theta_1$

7. The area of the region bounded by the ellipse $\frac{x^2}{16}+\frac{y^2}{9} \le 1$ and the circle $x^2 + y^2 \ge 9$ is

(1) 12π (2) 3π (3) 9π (4) 6π

Sol. Answer (2)

Area of the shaded part



= Area of an ellipse – area of a circle

$= \pi \times 3 \times 4 - \pi \times 3^2$

$= 12\pi - 9\pi$

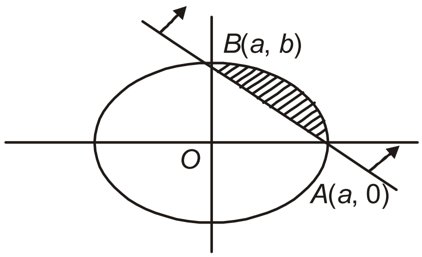
$= 3\pi$

8. Area of the region bounded by the curve $\left\{(x, y) : \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \le \frac{x}{a}+\frac{y}{b}\right\}$ is

(1) $\left(\frac{\pi}{4}-\frac{1}{2}\right)ab$ (2) $\left(\frac{\pi}{4}+\frac{1}{2}\right)ab$ (3) $\left(\frac{\pi}{4}-\frac{1}{3}\right)ab$ (4) $\frac{\pi}{4}ab$

Sol. Answer (1)

Area of the shaded part



$= \frac{1}{4}$th (area of the ellipse) – $ar(\Delta AOB)$

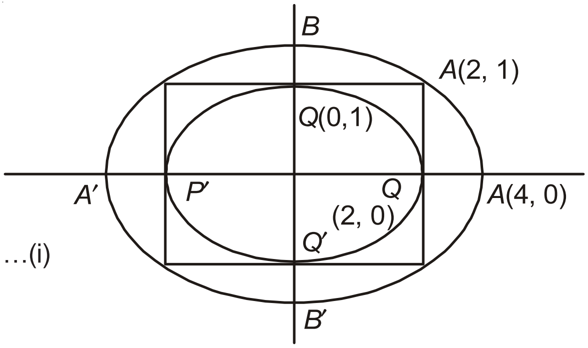
$= \frac{\pi}{4}ab - \frac{1}{2}ab$

$= \left(\frac{\pi}{4}-\frac{1}{2}\right)ab$ sq. units

9. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the co-ordinate axes which is turn in inscribed in another ellipse that passes through the (4, 0). Then the equation of the ellipse is

(1) $x^2 + 12y^2 = 16$ (2) $4x^2 + 48y^2 = 48$ (3) $4x^2 + 64y^2 = 48$ (4) $x^2 + 16y^2 = 16$

Sol. Answer (1)



$E : x^2 + 4y^2 = 4$

$\Rightarrow \dfrac{x^2}{4}+\dfrac{y^2}{1}=1$

Let the equation of the ellipse be

$$\frac{x^2}{16}+\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

Which is passing through $A(2, 1)$, so we get

$$\frac{4}{16}+\frac{1}{b^2}=1$$

$\Rightarrow \dfrac{1}{b^2}=1-\dfrac{4}{16}=\dfrac{12}{16}$

$\Rightarrow b^2=\dfrac{16}{12}$

Required equation is $\dfrac{x^2}{16}+\dfrac{y^2}{\frac{16}{12}}=1$

$\Rightarrow \dfrac{x^2}{16}+\dfrac{12y^2}{16}=1$

$\Rightarrow x^2 + 12y^2 = 16$

10. If LM is a double ordinate of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ such that OLM is an equilateral triangle, O being the centre of the hyperbola, then the eccentricity e of the hyperbola, satisfies

(1) $e>\dfrac{2}{\sqrt{3}}$ (2) $e=\dfrac{2}{\sqrt{3}}$ (3) $e<\dfrac{2}{\sqrt{3}}$ (4) $1<e<\dfrac{2}{\sqrt{3}}$

Sol. Answer (1)

$ON^2 = l^2-\dfrac{l^2}{4} = \dfrac{3}{4}l^2$

$ON = \dfrac{\sqrt{3}}{2}l$

$L\left(\dfrac{\sqrt{3}}{2}l, \dfrac{l}{2}\right)$ lies on the hyperbola

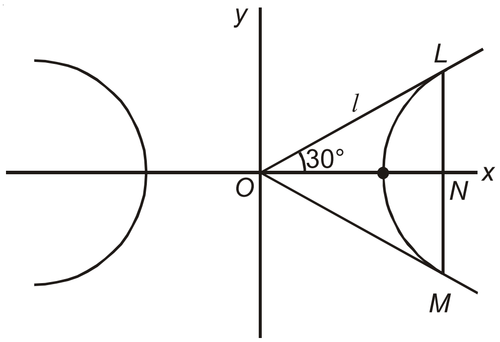
$\therefore \dfrac{3l^2}{4a^2}-\dfrac{l^2}{4b^2}=1$

But $b^2 = a^2(e^2 - 1)$

$\Rightarrow \dfrac{3l^2}{4a^2}-\dfrac{l^2}{4a^2(e^2-1)}=1$

$\Rightarrow \dfrac{l^2}{4a^2}\left[3-\dfrac{1}{e^2-1}\right]=1$

$\Rightarrow \dfrac{l^2}{a^2} = \dfrac{4(e^2-1)}{(3e^2-4)}$



ΔOLM is equilateral triangle.

$\therefore \quad l > a \qquad \dfrac{l^2}{a^2} > 1$

$\dfrac{4(e^2-1)}{(3e^2-4)} > 1 \qquad \Rightarrow \dfrac{4(e^2-1)}{3e^2-4} - 1 > 0$

$\Rightarrow \dfrac{e^2}{3e^2-4} > 0 \qquad \therefore \quad 3e^2 - 4 > 0$

$\Rightarrow e > \dfrac{2}{\sqrt{3}}$

11. A normal to the parabola $y^2 = 4px$ with slope m touches the rectangular hyperbola $x^2 - y^2 = p^2$, if

(1) $m^6 - 4m^4 - 3m^2 + 1 = 0$ (2) $m^6 + 4m^4 + 3m^2 + 1 = 0$

(3) $m^6 + 4m^4 - 3m^2 + 1 = 0$ (4) $m^6 - 4m^4 + 3m^2 - 1 = 0$

Sol. Answer (2)

$y^2 = 4px$

Equation of normal is

$y = mx - 2pm - pm^3$ … (i)

Equation of tangent of $x^2 - y^2 = p^2$ at (p secθ, p tan) is $x\sec\theta - y\tan\theta = p$ … (ii)

Comparing (i) and (ii),

$$\frac{-m}{\sec\theta} = \frac{1}{-\tan\theta} = \frac{-2pm - pm^3}{p} = -2m - m^3$$

$$\Rightarrow \frac{-m}{\sec\theta} = \frac{1}{-\tan\theta} = -2m - m^3$$

$$\Rightarrow \frac{m}{\sec\theta} = \frac{1}{\tan\theta} = 2m + m^3$$

$$\Rightarrow \sec\theta = \frac{m}{2m + m^3}$$

$$\Rightarrow \tan\theta = \frac{1}{2m + m^3}$$

$1 + \tan^2\theta = \sec^2\theta$

$$\Rightarrow 1 + \frac{1}{(2m + m^3)^2} = \frac{(m)^2}{(2m + m^3)^2}$$

$\Rightarrow (2m + m^3)^2 + 1 = m^2$

$\Rightarrow m^6 + 4m^2 + 4m^4 + 1 = m^2$

$\Rightarrow m^6 + 4m^4 + 4m^2 - m^2 + 1 = 0$

$\Rightarrow m^6 + 4m^4 + 3m^2 + 1 = 0$

12. The locus of a point $P(\alpha, \beta)$, such that the line $y = \alpha x + \beta$ is a tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is

(1) An ellipse (2) A circle (3) A parabola (4) A hyperbola

Sol. Answer (4)

Condition of tangent

$c^2 = a^2m^2 - b^2$

$b^2 = a^2\alpha^2 - b^2$

$\therefore$ Locus of $P(\alpha, \beta)$, $a^2x^2 - y^2 = b^2$ is a hyperbola

13. Let $P(a \sec\theta, b \tan\theta)$ and $Q(a \sec\phi, b \tan\phi)$, where $\theta+\phi=\frac{\pi}{2}$, be two points on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h, k) is the point of intersection of the normals at P and Q, then K is equal to

(1) $\frac{a^2+b^2}{a}$ (2) $-\left(\frac{a^2+b^2}{a}\right)$ (3) $\frac{a^2+b^2}{b}$ (4) $-\left(\frac{a^2+b^2}{b}\right)$

Sol. Answer (4)

Equation of normal at $(a \sec\theta, b \tan\theta)$

$ax \cos\theta + by \cot\theta = a^2 + b^2$ ... (i)

Equation of normal at ϕ,

$ax \cos\phi + by \cot\theta = a^2 + b^2$ $\phi = \frac{\pi}{2}-\theta$

$ax \sin\phi + by \tan\phi = a^2 + b^2$ ... (ii)

(i) $\times \sin\theta$ (ii) $\times \cos\theta$

$by[\cos\theta - \sin\theta] = (a^2 + b^2) [\sin\theta - \cos\theta]$

$\therefore$ $y=\frac{-(a^2+b^2)}{b}$

$\therefore$ $k=\frac{-(a^2+b^2)}{b}$

14. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2},1\right)$. Its one directrix is the common tangent, nearer to the point P, to $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse is

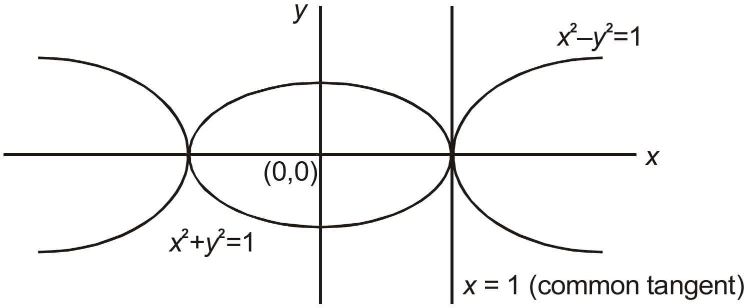
(1) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}}+\frac{(y-1)^2}{\frac{1}{12}}=1$ (2) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{8}}+\frac{(y-1)^2}{\frac{1}{12}}=1$

(3) $\frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}}+\frac{(y-1)^2}{\frac{1}{8}}=1$ (4) None of these

Sol. Answer (1)

The directrix of the ellipse is $x = 1$

Clearly the equation of common tangent is $x = 1$



Equation of ellipse

$Ps = ePM$

$\Rightarrow Ps^2 = e^2PM^2$

$$\Rightarrow \left(x-\frac{1}{2}\right)^2+(y-1)^2=\frac{1}{4}(x-1)^2$$

$$\Rightarrow \frac{\left(x-\frac{1}{3}\right)^2}{\frac{1}{9}}+\frac{(y-1)^2}{\frac{1}{12}}=1$$

15. If the polars of (x_1, y_1) and (x_2, y_2) w.r.t. the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are at right angles, then the value of $\frac{x_1x_2}{y_1y_2}$ is

(1) $\frac{a^4}{b^4}$ (2) $-\frac{a^4}{b^4}$ (3) $\frac{b^4}{a^4}$ (4) $-\frac{b^4}{a^4}$

Sol. Answer (2)

$H : \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Points : (x_1, y_1) and (x_2, y_2)

Polar at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

$$\Rightarrow \frac{yy_1}{b^2}=\frac{xx_1}{a^2}-1$$

$$\Rightarrow y=\frac{b^2xx_1}{a^2y_1}-\frac{b^2}{y_1}$$

$$\Rightarrow y=\left(\frac{b^2x_1}{a^2y_1}\right)x-\frac{b^2}{y_1}$$

$$\therefore m_1=\frac{b^2x_1}{a^2y_1}$$

Similarly $m_2 = \frac{b^2x_2}{b^2y_2}$

Given $m_1 \times m_2 = -1$

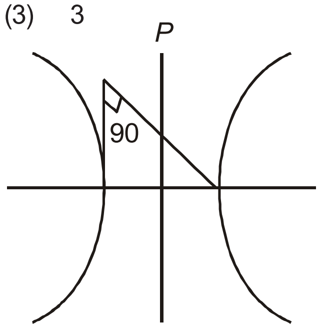
$$\Rightarrow \frac{b^2x_1}{a^2y_1}\times\frac{b^2x_2}{a^2y_2}=-1$$

$$\Rightarrow \frac{x_1x_2}{y_1y_2}=-\frac{a^4}{b^4}$$

16. If the locus of the point of intersection of two perpendicular tangents to a hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ is a circle with centre (0, 0), then the radius of a circle is

(1) 5 (2) 4 (3) 3 (4) 7

Sol. Answer (3)



Hyperbola : $\frac{x^2}{25}-\frac{y^2}{16}=1$

Clearly, locus of P is a director circle

$\Rightarrow$ $x^2 + y^2 = 25 - 16 = 9$

$\therefore$ $x^2 + y^2 = (3)^2$

Radius = 3

SECTION - B

Objective Type Questions (More than one options are correct)

1. Equation $y^2 - 2x - 2y + 5 = 0$ represents

(1) A pair of straight line (2) A circle with centre (1, 1)

(3) A parabola with vertex (2, 1) (4) A parabola with directrix $x=\frac{3}{2}$

Sol. Answer (3, 4)

$y^2 - 2y = 2x - 5$

$(y - 1)^2 = 2(x - 2)$ represents a parabola with vertex (2, 1) and directrix $x - 2 = -\frac{1}{2}$

$x=\frac{3}{2}$

2. The locus of the mid-point of the focal radii of a variable point moving on the parabola, $y^2 = 4ax$ is a parabola whose

(1) Focus has the coordinates $(a, 0)$

(2) Directrix is $x = 0$

(3) Vertex is $\left(\frac{a}{2},\ 0\right)$

(4) Latus rectum is half the latus rectum of the original parabola

Sol. Answer (1, 2, 3, 4)

$S(a, 0)$ is a focus

Moving point $Q(x, y)$

Let $P(h, k)$ be mid-point of SQ

$h=\frac{a+x}{2}, k=\frac{y}{2}$

$x = 2h - a, y = 2k$

$Q(x, y)$ will satisfy the $y^2 = 4ax$

$\therefore$ $4k^2 = 4a(2h - a)$

$\therefore$ Locus of $P(h, k)$

$$y^2 = 2a\left(x - \frac{a}{2}\right)$$

Shifting origin at $\left(\frac{a}{2}, 0\right)$

$\therefore$ $y^2 = 2ax$

Length of latus rectum = $2a$

Vertex $\left(\frac{a}{2}, 0\right)$

Equation of directrix $x = -\frac{a}{2}$

$\Rightarrow$ $x - \frac{a}{2} = -\frac{a}{2}$

$\Rightarrow$ $x = 0$

Focus of parabola

Focus with respect to original axis $(a, 0)$

3. If tangents PA and PB are drawn from $P(-1, 2)$ to $y^2 = 4x$, then

(1) Equation of AB is $y = x - 1$
(2) Length of AB is 8
(3) Length of AB is 4
(4) Equation of AB is $y = x + 1$

Sol. Answer (1, 2)

Equation of chord of contact $T = 0$

$\therefore$ $y.2 = 2(x - 1)$

$\Rightarrow$ $y = x - 1$ … (i)

$\Rightarrow$ $y^2 = 4x$ … (ii)

Solve equation (i) and (ii),

$$A(3 + 2\sqrt{2}, 2 + 2\sqrt{2})$$

$$B(3 - 2\sqrt{2}, 2 - 2\sqrt{2})$$

$$AB = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{64} = 8$$

4. Two parabolas have the same focus. If their directrices are the x-axis and the y-axis respectively, then the slope of their common chord is

(1) 1
(2) –1
(3) $\frac{3}{4}$
(4) $\frac{4}{3}$

Sol. Answer (1, 2)

Let focus be (a, b) and directrix $x = 0$

$\therefore$ Vertex of parabola $\left(\frac{a}{2}, b\right)$

∴ Equation of parabola

$(y-b)^2 = 2a\left(x-\frac{a}{2}\right)$ … (i)

Similarly equation of other parabola

$(x-a)^2 = 2b\left(y-\frac{b}{2}\right)$ … (ii)

Equation of common chord (i) – (ii),

$y^2 = x^2$

$y = \pm x$

∴ Slope of common chord are 1 and –1.

5. Equation of a common tangent to the circle $x^2 + y^2 = 18$ and $y^2 = 24x$ is

(1) $x + y + 6 = 0$ (2) $x - y + 6 = 0$ (3) $x + y - 6 = 0$ (4) $x - y - 6 = 0$

Sol. Answer (1, 2)

Equation of tangent to $y^2 = 24x$

be $y = mx + \frac{6}{m}$ … (i)

Line (i) is a tangent to the circle

∴ Distance from centre = radius

$\Rightarrow \left|\frac{\frac{6}{m}}{\sqrt{m^2+1}}\right| = \sqrt{18}$

$\Rightarrow \left(\frac{6}{m}\right)^2 = 18(m^2+1)$

$\Rightarrow 2 = m^2(m^2+1)$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow m = \pm 1$

∴ $x - y + 6 = 0$

$x + y + 6 = 0$

6. The normals to parabola $y^2 = 4ax$ from the point $(5a, -2a)$ are

(1) $y = x - 3a$ (2) $y + 2x = 12a$

(3) $y = -3x + 33a$ (4) $y = x + 3a$

Sol. Answer (1, 2)

Any normal to parabola $y^2 = 4ax$ may be

$y - 2at = -t\{x - at^2\}$

Or $y + tx - 2at - at^3 = 0$ ...(i)

$(5a, -2a)$ lies on (i), then

$t^3 - 3t + 2 = 0$

$\Rightarrow$ $t = -1, 2, -1$

$\therefore$ Equation of normals are

$y = x - 3a$ and $y + 2x = 12a$

7. Equation $x^2 - 2x - 2y + 5 = 0$ represents

(1) Parabola with vertex (1, 1)

(2) Parabola with vertex (1, 2)

(3) Parabola with directrix, $y = \frac{5}{2}$

(4) Parabola with directrix, $y = \frac{-1}{2}$

Sol. Answer (2, 3)

Given, $x^2 - 2x - 2y + 5 = 0$

$\Rightarrow$ $(x - 1)^2 = 2\{y - 2\}$ ...(i)

Equation (i) is a parabola whose vertex is (1, 2)

Its directrix is $y - 2 = a = \frac{1}{2}$ or $y = \frac{5}{2}$

8. The coordinates of a focus of the ellipse $4x^2 + 9y^2 = 1$ are

(1) $\left(\frac{\sqrt{5}}{6}, 0\right)$

(2) $\left(-\frac{\sqrt{5}}{6}, 0\right)$

(3) $\left(\frac{\sqrt{5}}{3}, 0\right)$

(4) $\left(-\frac{\sqrt{5}}{3}, 0\right)$

Sol. Answer (1, 2)

$$\frac{x^2}{\frac{1}{4}} = \frac{Y^2}{\frac{1}{9}} = 1 \qquad a^2 = \frac{1}{4},\ b^2 = \frac{1}{9}$$

$$\Rightarrow \quad e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow \quad e = \frac{\sqrt{5}}{3}$$

Foci $(\pm\ ae,\ 0) = \left(\pm\frac{\sqrt{5}}{6}, 0\right)$

9. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to $8x = 9y$ are

(1) $\left(\frac{2}{5}, \frac{1}{5}\right)$

(2) $\left(-\frac{2}{5}, \frac{1}{5}\right)$

(3) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$

(4) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

Sol. Answer (2, 4)

Equation of line parallel to $8x = 9y$ is $y = \frac{8}{9}x + c$

Condition of tangency,

$$c^2 = a^2m^2 + b^2$$

$$c^2 = \frac{1}{4}.\frac{64}{81} + \frac{1}{9} \quad \Rightarrow \quad c = \pm\frac{5}{9}$$

$\therefore \quad y = \frac{8}{9}x \pm \frac{5}{9}$

$8x - 9y \pm 5 = 0$ ... (i)

Let point of contact be $P(h, k)$

$\therefore$ Equation of tangent at $P(h, k)$

$4.x.h + 9.y.k - 1 = 0$ ... (ii)

Line (i) and (ii) are coincident line

$\therefore \quad \frac{4h}{8} = \frac{9k}{-9} = \frac{-1}{\pm 5}$

$$h = \mp\frac{2}{5}, k = \pm\frac{1}{5}$$

Point of contact $\left(\frac{-2}{5}, \frac{1}{5}\right)$ and $\left(\frac{2}{5}, -\frac{1}{5}\right)$.

10. The focal distances of the point $(4\sqrt{3}, 5)$ on the ellipse $25x^2 + 16y^2 = 1600$ may be

(1) 7 (2) 6 (3) 13 (4) 11

Sol. Answer (1, 3)

$$\frac{x^2}{64} + \frac{y^2}{100} = 1$$

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{64}{100} = \frac{36}{100}$$

$$e = \frac{3}{5}$$

Focal distance = $a \pm ey$ = $10 \pm \frac{3}{5}(5)$ = 13, 7

11. Let the ellipse be $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the circle $x^2 + y^2 = 16$. Then

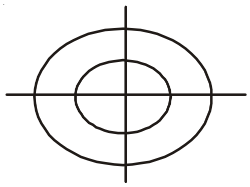
(1) Number of common tangents is zero

(2) Number of distinct common normals is two

(3) Area of the bounded region is 16π

(4) Angle between the two curves is 60°

Sol. Answer (1, 2)



$E : \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1;$

$C : x^2 + y^2 = 16$

From the figure, it is clear that, no common tangent can be drawn but 2 distinct normals X and Y axes can be drawn.

12. Let P be a variable point on the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ with foci at F_1 & F_2

(1) Area of ΔPF_1F_2 is $12\sin\theta$

(2) Area of ΔPF_1F_2 is maximum when $\theta = \dfrac{\pi}{2}$

(3) Co-ordinates of P are (0, 4)

(4) Centre of the ellipse is (1, 2)

Sol. Answer (1, 2, 3)

$E : \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

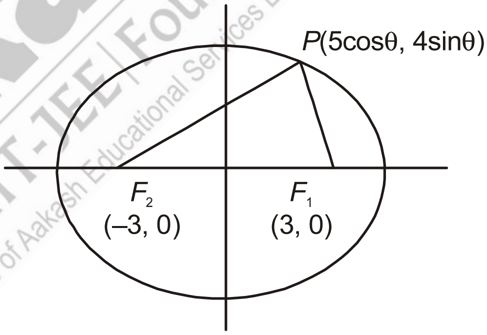
$e = \sqrt{1 - \dfrac{16}{25}} = \dfrac{3}{5}$

$ae = 5 \times \dfrac{3}{5} = 3$

$ar(\Delta PF_1F_2) = \left| \dfrac{1}{2} \begin{vmatrix} 5\cos\theta & 4\sin\theta & 1 \\ 3 & 0 & 1 \\ -3 & 0 & 1 \end{vmatrix} \right|$

$= \left| -\dfrac{1}{2} \times 6 \times 4\sin\theta \right|$

$= 12 \sin\theta$



Max. ar(ΔPF_1F_2) = 12

Co-ordinates of $P \equiv \left(5 \cdot \cos\dfrac{\pi}{2}, 4\sin\dfrac{\pi}{2}\right)$

$\equiv (0, 4)$

13. Tangents are drawn to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$ at the ends of a latus rectum then

(1) Co-ordinates of one end of a latus rectum are $\left(2, \dfrac{5}{3}\right)$

(2) Equation of tangent is $\dfrac{2}{9}x + \dfrac{y}{3} = 1$

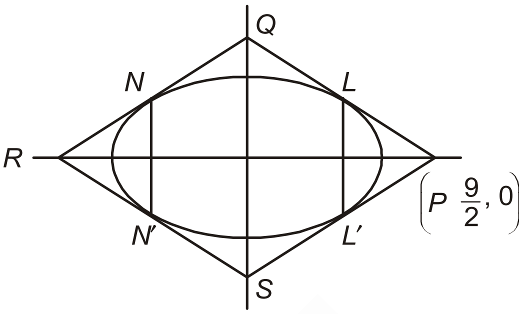
(3) Area of a quadrilateral so formed is 27 sq. units

(4) Centre of the ellipse is (2, 3)

Sol. Answer (1, 2, 3)

$$E : \frac{x^2}{9} + \frac{y^2}{5} = 1$$

$$\therefore \quad e - \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$



$$ae = 3 \times \frac{2}{3} = 2$$

$$\frac{b^2}{a} = \frac{5}{3}$$

$$L \equiv \left(ae, \frac{b^2}{a}\right) \equiv \left(2, \frac{5}{3}\right)$$

Tangent at L is $\frac{xx_1}{9} + \frac{yy_1}{5} = 1$

$$\Rightarrow \quad \frac{2x}{9} + \frac{y}{3} = 1$$

$$\therefore \quad P : \left(\frac{9}{2}, 0\right),\ Q : (0, 3)$$

Area ($\square PQRS$) = 4 × ar(ΔPOQ) = $4 \times \frac{1}{2} \times 3 \times \frac{9}{2} = 27$ sq. units.

14. The equation of common tangent of the curve $x^2 + 4y^2 = 8$ and $y^2 = 4x$ are

(1) $x - 2y + 4 = 0$ (2) $x + 2y + 4 = 0$ (3) $2x - y + 4 = 0$ (4) $2x + y + 4 = 0$

Sol. Answer (1, 2)

$$E : \frac{x^2}{8} + \frac{y^2}{2} = 1$$

$P : y^2 = 4x$

Equations of any tangent to the parabola is

$$y = mx + \frac{a}{m} = mx + \frac{1}{m} \qquad \ldots(i)$$

$$E : \frac{x^2}{8} + \frac{y^2}{2} = 1. \ ; \ T : y = mx + \frac{1}{m}$$

$c^2 = a^2m^2 + b^2$

$$\Rightarrow \quad \frac{1}{m^2} = 8m^2 + 2$$

$\Rightarrow \quad 8m^4 + 2m^2 - 1 = 0$

$\Rightarrow \quad 8m^4 + 4m^2 - 2m^2 - 1 = 0$

$\Rightarrow \quad 4m^2(2m^2 + 1) - 1\,(2m^2 + 1) = 0$

$\Rightarrow \quad 4m^2 = 1$

$\Rightarrow \quad m = \pm\frac{1}{2}$

$T : y = \pm\frac{1}{2}x \pm 2$

$\Rightarrow \quad 2y = \pm x \pm 4$

$\Rightarrow \quad x - 2y + 4 = 0,\ x + 2y + 4 = 0$

15. Let the equation of the ellipse be $2x^2 + y^2 = 4$ and the point is (2, 1)

(1) Equation of tangent is $4x + y = 4$

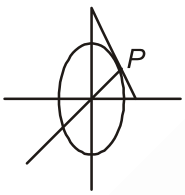
(2) Equation of normal is $x - 4y + 2 = 0$

(3) Equation of polar is $4x + y = 4$

(4) Equation of diameter is $x - 2y = 0$

Sol. Answer (1, 2, 3, 4)

$E : 2x^2 + y^2 = 4$



$\Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{4} = 1$

Tangent at $P : \frac{xx_1}{2} + \frac{yy_1}{4} = 1$

$\frac{2x}{2} + \frac{y}{4} = 1$

$\Rightarrow \quad 4x + y = 4$

Normal at $P : x - 4y = \lambda$

It passes through (2, 1), then $\lambda = 2 - 4 = -2$

$\therefore \quad x - 4y + 2 = 0$

Polar at $P : 4x + y = 4$

Diameter $y = \frac{1}{2}x$

$\Rightarrow \quad x - 2y = 0$

16. Chord of contact of tangents drawn from the point $M(h, k)$ to the ellipse $x^2 + 4y^2 = 4$ intersects at P & Q, subtends a right angle of the centre 'O'

(1) Locus of M is $x^2 + 16y^2 = 20$

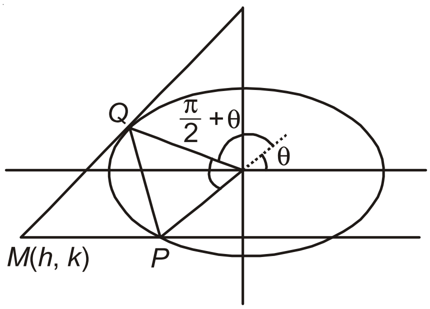
(2) $\frac{1}{OP^2} + \frac{1}{OQ^2} = \frac{5}{4}$

(3) Equation of chord of contact is $hx + 4ky = 4$

(4) Equation of normal is $3x + 2y + 5 = 0$

Sol. Answer (1, 2, 3)

$E : x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$



PQ : chord of contact

$hx + 4ky = 4$

$\Rightarrow \quad \frac{hx + 4ky}{4} = 1$

Now, $x^2 + 4y^2 = 4\left(\frac{hx+4ky}{4}\right)^2$

$\Rightarrow$ $4(x^2 + 4y^2) = (hx + 4ky)^2 = h^2x^2 + 16k^2y^2 + 8hkxy$

$\Rightarrow$ $(4 - h^2)x^2 + (16 - 16k^2)y^2 + (\ ) = 0$

Chord of contact subtends a right angle at the centre, so $4 - h^2 + 16 - 16k^2 = 0$

Locus of M is $x^2 + 16y^2 = 20$

$\Rightarrow$ $h^2 + 16k^2 = 20$

Let OP makes an angle θ with the x-axis then OQ makes an angle $\frac{\pi}{2}+\theta$

$\therefore$ $P : (OP\cos\theta,\ OP\sin\theta)$

$Q : (-OQ\sin\theta,\ OQ\cos\theta)$

Since P & Q lies on the ellipse, so we have

$OP^2\cos^2\theta + 4.OP^2\sin^2\theta = 4$

$\Rightarrow$ $\frac{4}{OP^2} = \cos^2\theta + 4\sin^2\theta$

Also, $OQ^2\sin^2\theta + 4.OQ^2\cos^2\theta = 4$

$\Rightarrow$ $\frac{4}{OQ^2} = \sin^2\theta + 4\cos^2\theta$

Now, $\frac{4}{OP^2} + \frac{4}{OQ^2} = (\cos^2\theta + 4\sin^2\theta) + (\sin^2\theta + 4\cos^2\theta) = 5(\cos^2\theta + \sin^2\theta) = 5$

$\Rightarrow$ $\frac{1}{OP^2} + \frac{1}{OQ^2} = \frac{5}{4}$

17. Let the ellipse be $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and P is one end of a minor axis of the ellipse. Then

(1) Equation of the incident ray is $4x - 3y - 12 = 0$

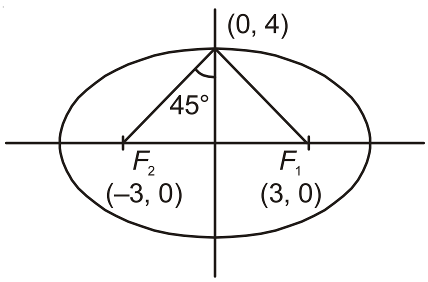
(2) Equation of the reflection ray is $4x + 3y + 12 = 0$

(3) Angle between the focal radii normal at P is 45°

(4) Distance between two foci is 6

Sol. Answer (1, 2, 3, 4)

$E : \frac{x^2}{25} + \frac{y^2}{16} = 1$



$\therefore$ $e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

$F_1 = (ae, 0)$

$= (3, 0)$

$F_2 = (-3, 0)$

$\therefore$ $F_1F_2 = 3 + 3 = 6$

Equation of the incident ray is

$$y = 0 = \frac{4}{3}(x - 3)$$

$\Rightarrow$ $3y = 4x - 12$

$\Rightarrow$ $4x - 3y = 12$

Equation of the reflection ray is

$$y - 0 = -\frac{4}{3}(x + 3)$$

$\Rightarrow$ $3y = -4x - 12$

$\Rightarrow$ $4x + 3y + 12 = 0$

Angle between the focal radii and normal is 45°.

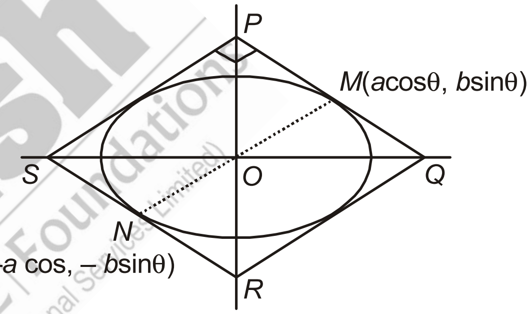
18. If a quadrilateral formed by four tangents to the ellipse $3x^2 + 4y^2 = 12$ is a square, then

(1) The vertices of the square lie on $y = \pm x$ (2) The vertices of the square lie on $x^2 + y^2 = 7^2$

(3) The area of all such squares is constant (4) Only two such squares are possible

Sol. Answer (2, 3)

$E : 3x^2 + 4y^2 = 12$



$\Rightarrow$ $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Clearly, the vertices of the squares will lie on the director circle of the ellipse

i.e., $x^2 + y^2 = 4 + 3 = 7$

Area of a square *PQRS*

$= 2(a^2 + b^2)$

$= 2(4 + 3)$

$= 14$ sq. units

Only one such square is possible.

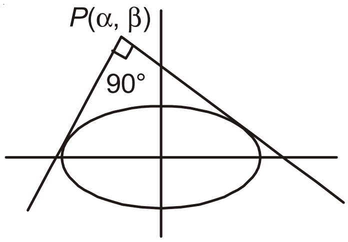
19. Let (α, β) be a point from which two perpendicular tangent can be drawn to the ellipse $4x^2 + 5y^2 = 20$. If $F = 4\alpha + 3\beta$, then

(1) Domain of F is [–3, 3] (2) Range of F is [–15, 15]

(3) Equation of the director circle is $x^2 + y^2 = 9$ (4) Maximum value of F is 15

Sol. Answer (1, 2, 3, 4)

$E : 4x^2 + 5y^2 = 20$



$\Rightarrow$ $\frac{x^2}{5} + \frac{y^2}{4} = 1$

Clearly locus of $P(\alpha, \beta)$ is a director circle

$\therefore$ $x^2 + y^2 = 5 + 4 = 9$

which is passes through $P(\alpha, \beta)$

$\therefore$ $\alpha^2 + \beta^2 = 9$

Now, $F = 4\alpha + 3\beta = 4\alpha + 3\sqrt{9-\alpha^2}$

$\Rightarrow D_F = [-3, 3]$

$R_F = [-15, 15]$

Maximum value of F is 15

20. Equation of a tangent passing through (2, 8) to the hyperbola $5x^2 - y^2 = 5$ is

(1) $3x - y + 2 = 0$ (2) $23x - 3y - 22 = 0$ (3) $3x - 23y + 178 = 0$ (4) $3x + y + 14 = 0$

Sol. Answer (1, 2)

Equation of a line through (2, 8)

$y - 8 = m(x - 2)$

$y = m.x + 8 - 2m$ ... (i)

Line (i) is a tangent to the hyperbola $\frac{x^2}{1} - \frac{y^2}{5} = 1$

$\therefore$ Condition of tangency $c^2 = a^2m^2 - b^2$

$(8 - 2m)^2 = m^2 - 5$

$3m^2 - 32m + 69 = 0$

Gives $m = 3$, $m = \frac{23}{3}$

$\therefore$ Tangents are $3x - y + 2 = 0$ and $23x - 3y - 22 = 0$

21. The circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points (x_1, y_1) (x_2, y_2) (x_3, y_3) and (x_4, y_4), then

(1) $\Sigma x_i = 0$ (2) $\Sigma y_i = 0$ (3) $x_1x_2x_3x_4 = c^4$ (4) $y_1y_2y_3y_4 = c^4$

Sol. Answer (1, 2, 3, 4)

$y = \frac{c^2}{x}$ put in $x^2 + y^2 = a^2$

$x^2 + \frac{c^4}{x^2} = a^2$

$x^4 - a^2x^2 + c^4 = 0$

Let roots are x_1, x_2, x_3, and x_4

$x_1 x_2 x_3 x_4 = c^4$

Similarly $y_1 y_2 y_3 y_4 = c^4$

$x_1 + x_2 + x_3 + x_4 = 0$

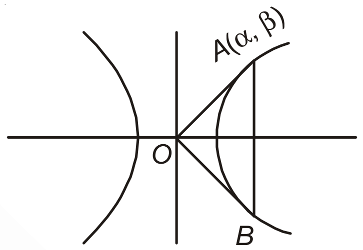
$\Sigma x_i = 0$

Similarly $\Sigma y_i = 0$

22. Let AB be a double ordinate of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If O be the centre of the hyperbola and OAB is an equilateral triangle, then which of the following is/are true, if $A(\alpha, \beta)$?

(1) $e>\frac{2}{\sqrt{3}}$ (2) $e>\frac{4}{\sqrt{3}}$ (3) $\alpha^2=3\beta^2$ (4) $\alpha^2>3\beta^2$

Sol. Answer (1, 3)



$$\tan 30^\circ=\frac{\beta}{\alpha}=\frac{1}{\sqrt{3}}$$

$$3\beta^2=\alpha^2$$

$$\Rightarrow \frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}=1$$

$$\Rightarrow \frac{3\beta^2}{a^2}-\frac{\beta^2}{b^2}=1$$

$$\Rightarrow \frac{3}{a^2}-\frac{1}{b^2}=\frac{1}{\beta^2}$$

$$\Rightarrow \frac{3}{a^2}-\frac{1}{b^2}>0$$

$$\Rightarrow \frac{3}{a^2}>\frac{1}{b^2}$$

$$\Rightarrow e>\frac{2}{\sqrt{3}}$$

23. The angle between a pair of tangent drawn from a point P to the parabola $y^2 = 4ax$ is 45°. If locus of point P is hyperbola, then its foci are

(1) $(a, 0)$ (2) $(-7a, 0)$ (3) $(4a, 0)$ (4) $(-4a, 0)$

Sol. Answer (1, 2)

$$y=mx+\frac{a}{m}$$

$$k = mh + \frac{a}{m}$$

$$m^2h - km + a = 0$$

$$\Rightarrow \tan 45=\frac{m_1-m_2}{1+m_1m_2}$$

$$\Rightarrow 1=\frac{\sqrt{\left(\frac{k}{h}\right)^2-4\frac{a}{h}}}{1+\frac{a}{h}}=\frac{\sqrt{k^2-4ah}}{h+a}$$

$\Rightarrow$ $k^2 - 4ah = h^2 + a^2 + 2ah$

$x^2 + 6ax - y^2 + a^2 = 0$

$(x + 3a)^2 - y^2 = 8a^2$

Its centre is $(-3a, 0)$

Distance of focus from centre is = $2\sqrt{2}a \cdot \sqrt{2}$ = 4 a

Hence focii are $(a, 0)$, $(-7a, 0)$

24. Which of the following is/are true for locus represented by $x = \frac{1}{2}a\left(t + \frac{1}{t}\right)$, $y = \frac{1}{2}a\left(t - \frac{1}{t}\right)$

(1) Locus is ellipse

(2) Locus is hyperbola

(3) Directrix are $x = \pm \frac{a}{\sqrt{2}}$

(4) Directrix are $x = \pm a\sqrt{2}$

Sol. Answer (2, 3)

$$x^2 - y^2 = \frac{1}{4}a^2\left(t^2 + \frac{1}{t^2} + 2 - t^2 - \frac{1}{t^2} + 2\right)$$

$x^2 - y^2 = a^2$

Its directrices are $x = \pm \frac{a}{\sqrt{2}}$

25. If line joining point (0, 3) and (5, –2) is tangent to curve $y = \frac{1}{x + c}$, then value of c is/are

(1) –5

(2) –1

(3) 5

(4) 1

Sol. Answer (1, 2)

Equation of line is

$$y - 3 = \frac{-2 - 3}{5 - 0}(x - 0)$$

$x + y = 3$

Since it is tangent

$(3 - x)(x + c) = 1$

$3x + 3c - x^2 - xc = 1$

$x^2 + (c - 3)x + 1 - 3c = 0$

$(c - 3)^2 - 4(1 - 3c) = 0$

$c^2 + 9 - 6c - 4 + 12c = 0$

$c^2 + 6c + 5 = 0$

$(c + 5)(c + 1) = 0$

$c = -5, -1$

26. Tangents at any point P is drawn to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intersects asymptotes at Q and R, if O is the centre of hyperbola then

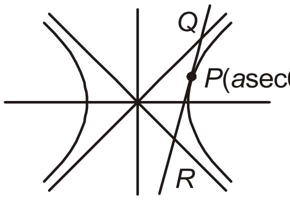
(1) Area of triangle OQR is ab

(2) Area of triangle OQR is $2ab$

(3) P is mid-point of QR

(4) P trisect QR

Sol. Answer (1, 3)



Tangent at P is $\frac{x-\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

Equation of asymptotes are $y=\pm\frac{b}{a}x$

$Q(a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta))$

$R(a(\sec\theta - \tan\theta), b(\sec\theta - \tan\theta))$

Area of OQR is = ab, P is mid-point of QR

27. The locus of a point whose chord of contact touches the circle described on the straight line joining the foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as diameter is/has

(1) Hyperbola

(2) Ellipse

(3) $e=\frac{\sqrt{a^4-b^4}}{a^2}$

(4) $e=\frac{\sqrt{a^4+b^4}}{a^2}$

Sol. Answer (2, 3)

Given circle is

$(x - ae)(x + ae) + y^2 = 0$

$x^2 + y^2 = a^2e^2 = a^2 + b^2$

Chord of contact is

$$\frac{xh}{a^2}-\frac{yk}{b^2}=1$$

$$\frac{-1}{\sqrt{\frac{h^2}{a^4}+\frac{k^2}{b^4}}}=\sqrt{a^2+b^2}$$

Locus is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$

Its eccentricity

$$\frac{b^4}{a^2+b^2}=\frac{a^4}{a^2+b^2}(1-e^2)$$

$$\Rightarrow \quad e^2=1-\frac{b^4}{a^4}=\frac{a^4-b^4}{a^4}$$

$$\Rightarrow \quad e=\frac{\sqrt{a^4-b^4}}{a^2}$$

28. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N, and lines MP and NP are drawn at right angle to the axes then locus of P is a hyperbola with eccentricity e', if eccentricity of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is e, then

(1) e' is eccentricity of conjugate of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(2) $\frac{1}{e^2}+\frac{1}{e'^2}=1$

(3) $e^2 + e'^2 = 3$

(4) $e^2 + e'^2 = 4$

Sol. Answer (1, 2)

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

$$M\left(\frac{(a^2+b^2)}{a}\sec\theta, 0\right)$$

$$N\left(0, \frac{(a^2+b^2)}{b}\tan\theta\right)$$

Locus is $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$

$\Rightarrow$ $a^2 = b^2(e'^2 - 1)$

which is same as conjugate hyperbola.

29. If the axis of a varying central hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be fixed in magnitude and position, then locus of point of contact of a tangent drawn to it from a fixed point on x-axis is

(1) Parabola

(2) Ellipse

(3) Hyperbola

(4) Straight line

Sol. Answer (3, 4)

Points of contact is $\left(\frac{-a^2m}{\sqrt{a^2m^2-b^2}}, \frac{-b^2}{\sqrt{a^2m^2-b^2}}\right)$

If tangents are drawn from $(\alpha, 0)$, then

$$0 = m\alpha + \sqrt{a^2m^2-b^2}$$

$$\Rightarrow \quad m=\frac{b^2}{a^2-\alpha^2}$$

$$h=-\frac{a}{\alpha}$$

$$k=\frac{-b(a^2-\alpha^2)}{\alpha^2}$$

Hence locus may be straight line or hyperbola.

30. If e_1 and e_2 are the eccentricity of hyperbola $xy = c^2$ and $x^2 - y^2 = c^2$, then

(1) $\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$ (2) $e_1^2 + e_2^2 = 1$ (3) $e_1^2 - e_2^2 = 0$ (4) $e_1 \cdot e_2 = 2$

Sol. Answer (1, 3, 4)

Eccentricity of rectangular hyperbola is constant and it is equal to $\sqrt{2}$

31. If equation of hyperbola is $xy + 3x - 2y - 10$, then

(1) Eccentricity = $\sqrt{2}$ (2) Centre = (2, –3)

(3) Lengths of latus rectum = $4\sqrt{2}$ (4) Asymptotes are $x = 2$ and $y = -3$

Sol. Answer (1, 2, 3, 4)

Given equation $xy + 3x - 2y - 10 = 0$

$x(y + 3) - 2y - 6 + 6 - 10 = 0$

$x(y + 3) - 2(y + 3) - 4 = 0$

$(x - 2)(y + 3) = 4$

Length of latus rectum = $\frac{2b^2}{a} = 2a$

And $c^2 = \frac{a^2}{2} = 4 \Rightarrow a = 2\sqrt{2}$

$\therefore$ Length of latus rectum = $4\sqrt{2}$

32. If (5, 12) and (24, 7) are the focii of a conic passing through the origin, then eccentricity of conic is

(1) $\frac{\sqrt{386}}{12}$ (2) $\frac{\sqrt{386}}{13}$ (3) $\frac{\sqrt{386}}{38}$ (4) $\frac{\sqrt{386}}{25}$

Sol. Answer (1, 3)

Conic may be ellipse or hyperbola

Case –I: If conic is ellipse, then

$SP + S'P = 2a$ and $2ae = SS'$

Where S and S' are the focii and e is the eccentricity

Let P is origin

Now, $SP + S'P = \sqrt{5^2+12^2} + \sqrt{24^2+7^2} = 13 + 25 = 38$

$\therefore \quad 2ae = SS' = \sqrt{386}$

$\Rightarrow \quad e = \frac{\sqrt{386}}{38}$

Case-II: If conic is hyperbola, then

$|SP - S'P| = 2a$ and $2ae = SS'$

$|SP - S'P| = |13 - 25| = 12$

$\therefore \quad |SP - S'P| = |13 - 25| = 12$

Now, $e = \frac{SS'}{2a} = \frac{\sqrt{386}}{12}$

33. If equation of hyperbola is $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$ then

(1) Conjugate hyperbola is $2x^2 + 5xy + 2y^2 - 11x - 7y + 4 = 0$

(2) Conjugate hyperbola is $2x^2 + 5xy + 2y^2 - 11x - 7y + 14 = 0$

(3) Asymptotes is $2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$

(4) Asymptotes is $2x^2 + 5xy + 2y^2 - 11x - 7y - 5 = 0$

Sol. Answer (2, 3)

Given equation of hyperbola is $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$

Then equation of asymptotes $2x^2 + 5xy + 2y^2 - 11x - 7y + \lambda = 0$

It will represent two straight line, then $\Delta = 0$

$\therefore \quad a = 2, b = 2, c = \lambda, f = \dfrac{-7}{2}, g = \dfrac{-11}{2}$ and $h = \dfrac{5}{2}$

$\Rightarrow \quad 2\times 2\times \lambda + 2\left(\dfrac{-7}{2}\right)\left(\dfrac{-11}{2}\right)\dfrac{5}{2} - 2\left(\dfrac{-7}{2}\right)^2 - 2\left(\dfrac{-11}{2}\right)^2 - \lambda\left(\dfrac{5}{2}\right)^2 = 0$

$\Rightarrow \lambda = 5$

Then equation of asymptotes is $2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$

For conjugate hyperbola

$\boxed{H + C.H = 2ASM}$

$\therefore$ Conjugate hyperbola $2(2x^2 + 5xy + 2y^2 - 11x - 7y + 5) - (2x^2 + 5xy + 2y^2 - 11x - 7y - 4) = 0$

$\Rightarrow 2x^2 + 5xy + 2y^2 - 11x - 7y + 14 = 0$

34. For the equation of rectangular hyperbola $xy = 18$

(1) Length of transverse axis = length of conjugate axis = 12

(2) Vertices are $(3\sqrt{2}, 3\sqrt{2})$ or $(-3\sqrt{2}, -3\sqrt{2})$

(3) Focii are (6, + 6), (–6, –6)

(4) Equation of tangent with slope 1 cannot be possible

Sol. Answer (1, 2, 3, 4)

(1) Equation of rectangular hyperbola $xy = \dfrac{a^2}{2}$

By comparing with given hyperbola $\dfrac{a^2}{2} = 18 \Rightarrow a = 6$

$\therefore$ Length of transverse axis = $2a$ = 12

And length of transverse axis = length of conjugate axis

(2) Vertices are $\left(\dfrac{+a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$ or $\left(\dfrac{-a}{\sqrt{2}}, \dfrac{-a}{\sqrt{2}}\right)$

$\Rightarrow \quad (+3\sqrt{2}, 3\sqrt{2})$ or $(-3\sqrt{2}, -3\sqrt{2})$

(3) Focii are (a, a) or $(-a, -a)$

$\Rightarrow$ (6, 6) or (–6, –6)

(4) If $m < 0$, then tangent can possible to the hyperbola.

35. If the normals at (x_i, y_i), i = 1, 2, 3, 4 on the rectangular hyperbola $xy = c^2$ meet at the point (α, β), then

(1) $\Sigma x_i = \alpha$ (2) $\Sigma y_i = \beta$ (3) $\Sigma x_i^2 = \alpha^2$ (4) $\Sigma y_i^2 = \beta^2$

Sol. Answer (1, 2, 3, 4)

Let $(x_i, y_i) = \left(ct_i, \dfrac{c}{t_i}\right)$, i = 1, 2, 3, 4 are the points on the rectangular hyperbola $xy = c^2$,

Equation of normal to the hyperbola $xy = c^2$ at $\left(ct, \dfrac{c}{t}\right)$ is

$ct^4 - t^3x + ty - c = 0$

It passes through (α, β), then

$ct^4 - t^3\alpha + t\beta - c = 0$

Its biquadratic equation in$t$. Let the roots of equation are t_1, t_2, t_3, t_4, then

$$\Sigma t_1 = \frac{\alpha}{c}, \Sigma t_1t_2 = 0, \Sigma t_1t_2t_3 = \frac{-\beta}{c}, t_1t_2t_3t_4 = -1$$

(1) $\Sigma x_i = \Sigma ct_i = c\Sigma t_i = c\left(\dfrac{\alpha}{c}\right) = \alpha$

(2) $\Sigma y_i = \Sigma \dfrac{c}{t_i} = c\Sigma \dfrac{1}{t_i} = \dfrac{c\Sigma t_1t_2t_3}{t_1t_2t_3t_4} = \dfrac{c\left(\dfrac{-B}{C}\right)}{-1} = B$

(3) $\Sigma x_i^2 = (\Sigma x_i)^2 - 2\Sigma x_i x_2 = \alpha^2 - 0 = \alpha^2$

(4) $\Sigma y_i^2 = (\Sigma y_i)^2 - 2\Sigma y_1 y_2 = \beta^2 - 0 = \beta^2$

36. The feet of the normals to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ from (h, k) lie on

(1) $a^2y(x - h) + b^2x(y - k) = 0$ (2) $b^2x(x - h) + a^2y(y - k) = 0$

(3) $(a^2 + b^2)xy - a^2hy - b^2xk = 0$ (4) None of these

Sol. Answer (1, 3)

Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ be co-normal points so that normals drawn from them meet in $T(h, k)$. The equation of normal at $p(x_1, y_1)$

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

Or $a^2xy_1 + b^2yx_1 = (a^2 + b^2)x_1y_1$

Or $(a^2 + b^2)x_1y_1 - a^2xy_1 - b^2yx_1 = 0$

The point $T(h, k)$ lies on it

$(a^2 + b^2)x_1y_1 - a^2hy_1 - b^2x_1k = 0$

Similarly, for points Q, R and S are

$(a^2 + b^2)x_2y_2 - a^2hy_2 - b^2x_2k = 0$

$(a^2 + b^2)x_3y_3 - a^2hy_3 - b^2x_3k = 0$

$(a^2 + b^2)x_4y_4 - a^2hy_4 - b^2x_4k = 0$

Hence P, Q, R, S lie on the curve

$(a^2 + b^2)xy - a^2hy - b^2xk = 0$

Or $a^2y(x - h) + b^2x(y - k) = 0$

37. The equation of the asymptotes of a hyperbola are $4x - 3y + 8 = 0$ and $3x + 4y - 7 = 0$, then

(1) Eccentricity is $\sqrt{2}$

(2) Centre is $\left(\frac{-11}{25}, \frac{52}{25}\right)$

(3) Centre is $\left(\frac{11}{25}, \frac{-52}{25}\right)$

(4) Equation of axes are $x - 7y + 15 = 0$ and $7x + y + 1 = 0$

Sol. Answer (1, 2, 4)

(1) Asymptotes are perpendicular, then hyperbola is rectangular hyperbola

$\therefore \quad e = \sqrt{2}$

(2) Centre of rectangular hyperbola is the point of intersection of asymptotes

$\therefore \quad 4x - 3y + 8 = 0$ and $3x + 4y - 7 = 0$

(4) Axes are along the bisector of angles between the asymptotes

Then equation of axes are

$$\frac{4x - 3y + 8}{\sqrt{4^2 + (-3)^2}} = \pm \frac{3x + 4y - 7}{\sqrt{3^2 + 4^2}}$$

$\therefore \quad x - 7y + 15 = 0$ and $7x + y + 1 = 0$

38. If the tangent at the point $(a \sec\alpha, b \tan\alpha)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at T. Then the distance of T form a focus of the hyperbola is

(1) $a(e - \cos\alpha)$

(2) $b(e + \cos\alpha)$

(3) $a(e + \cos\alpha)$

(4) $\sqrt{a^2e^2 + b^2\cot^2\alpha}$

Sol. Answer (1, 3)

Equation of tangent at $(a\sec\alpha, b\tan\alpha)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\sec\alpha - \frac{y}{b}\tan\alpha = 1$ which meets the transverse axis $y = 0$ at the point $T\ (a\cos\alpha, 0)$ whose distance from the focus $ae = 0$ is $ae - a\cos\alpha$ and from the focus $(-ae, 0)$ is $(ae + a\cos\alpha)$

Note that $ae > a\cos\alpha$

Since $e > 1$

39. If $ax + by + c = 0$ is the normal to the hyperbola $xy + 1 = 0$, then

(1) $a > 0, b > 0$

(2) $a < 0, b > 0$

(3) $a > 0, b < 0$

(4) $a < 0, b < 0$

Sol. Answer (1, 4)

Slope of normal = $-\dfrac{a}{b}$

And $xy = -1 \Rightarrow y = -\dfrac{1}{x}$

$\Rightarrow \dfrac{dy}{dx} = +\dfrac{1}{x^2}$

Slope of normal = $-x^2 = -\dfrac{a}{b}$ (given)

$\dfrac{a}{b} > 0$

It can be possible if a and b are same sign.

40. If equation of hyperbola is $4(2y - x - 3)^2 - 9(2x + y - 1)^2 = 80$, then

(1) Eccentricity is $\dfrac{\sqrt{13}}{3}$

(2) Centre of hyperbola is $\left(-\dfrac{1}{5}, -\dfrac{7}{5}\right)$

(3) Transverse axis is $2x + y - 1 = 0$

(4) Conjugate axis is $x - 2y + 3 = 0$

Sol. Answer (1, 2, 3, 4)

$$4.5\left(\frac{2y - x - 3}{\sqrt{5}}\right)^2 - 9.5\left(\frac{2x + y - 1}{\sqrt{5}}\right)^2 = 80$$

$\Rightarrow 20x^2 - 9.5y^2 = 80$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{\left(\frac{16}{9}\right)} = 1$$

(1) Eccentricity = $\sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1 + \dfrac{16}{9.(4)}} = \dfrac{\sqrt{13}}{3}$

(2) Centre: $2y - x - 3 = 0$

And $2x + y - 1 = 0$

(3) Transverse axis : $Y = 0$

(4) Conjugate axis : $X = 0$

SECTION - C

Linked Comprehension Type Questions

Comprehension-I

Let $C : y = x^2 - 3$, $D : y = kx^2$ be two parabolas and $L_1 : x = a$, $L_2 : x = 1$ ($a \neq 0$) be two straight lines.

1. If C and D intersect at a point A on the line L_1, then the equation of the tangent line L at A to the parabola D is

(1) $2(a^3 - 3)x - ay + (a^3 - 3a) = 0$

(2) $2(a^2 - 3)x - ay - a^3 + 3a = 0$

(3) $(a^3 - 3)x - 2ay - 2a^3 + 6a = 0$

(4) None of these

Sol. Answer (2)

Put $x = a$ on both parabolas $y = a^2 - 3$ and $y = ka^2$

$a^2 - 3 = ka^2$

$k = \dfrac{a^2 - 3}{a^2}$

Point of intersection $A(a, a^2 - 3)$

$\therefore$ Equation of tangent L to the curve

$y = k \cdot x^2$ at A

$\frac{1}{2}(y + a^2 - 3) = k.x.a$

$\frac{1}{2}(y + a^2 - 3) = \left(\dfrac{a^2 - 3}{a^2}\right).xa$

$a(y + a^2 - 3) = 2(a^2 - 3)x$

$2(a^2 - 3)x - ay - a^3 + 3a = 0$

2. If the line L meets the parabola C at a point B on the line L_2, other than A, then a may be equal to

(1) –3 (2) –2 (3) 2 (4) None of these

Sol. Answer (2)

On L_2 , $x = 1$

C : $y_2 = x^2 - 3 = 1 - 3 = -2$

$B(1, -2)$

$B(1, -2)$ lie on L

$2(a^2 - 3) + 2a - a^3 + 3a = 0$

$a^3 - 2a^2 - 5a + 6 = 0$

$(a - 1)(a + 2)(a - 3) = 0$

$a = 1, -2, 3$

3. If $a > 0$, the angle subtended by the chord AB at the vertex of the parabola C is

(1) $\tan^{-1}\left(\frac{5}{7}\right)$ (2) $\tan^{-1}\left(\frac{1}{2}\right)$ (3) $\tan^{-1}(2)$ (4) $\tan^{-1}\left(\frac{1}{8}\right)$

Sol. Answer (2)

$a > 0$, a may be 1 or 3

$a = 3$, [$a \neq 1$, if $a = 1$, A and B coincidence]

$A(3, 6)$, $B(1, -2)$

Vertex of $e(0, -3)$

Slope of AC = $\dfrac{6+3}{3} = 3 = m_1$ say

Slope of $BC = \frac{1}{1} = 1 = m_2$ say

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3-1}{1+3}\right| = \frac{2}{4}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$

Comprehension-II

Let $P_1 : y^2 = 4ax$ and $P_2 : y^2 = -4ax$ be two parabolas and $L : y = x$ be a straight line.

1. If $a = 4$, then the equation of the ellipse having the line segment joining the foci of the parabolas P_1 and P_2 as the major axis and eccentricity equal to $\frac{1}{2}$ is

(1) $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (2) $\frac{x^2}{3} + \frac{y^2}{4} = 1$ (3) $\frac{x^2}{16} + \frac{y^2}{12} = 1$ (4) $\frac{x^2}{12} + \frac{y^2}{16} = 1$

Sol. Answer (3)

Foci $P_1(4, 0)$ and $P_2(4, 0)$

$2a = 8 \Rightarrow a = 4$

$b^2 = a^2(1 - e^2) = 16\left(1 - \frac{1}{4}\right) = 12$

Equation of ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$

2. Equation of the tangent at the point on the parabola P_1 where the line L meets the parabola is

(1) $x - 2y + 4a = 0$ (2) $x + 2y - 4a = 0$ (3) $x + 2y - 8a = 0$ (4) $x - 2y + 8a = 0$

Sol. Answer (1)

Solve equations $y = x$ and $y^2 = 4ax$

Points of intersections are $O(0, 0)$ and $A(4a, 4a)$

Equation of tangent at $O(0, 0)$ is $x = 0$

and equation of tangent at $A(4a, 4a)$

$y.4a = 2a(x + 4a)$

$x - 2y + 4a = 0$

3. The co-ordinates of the other extremity of a focal chord of the parabola P_2, one of whose extremity is the point of intersection of L and P_2 is

(1) $(-a, 2a)$ (2) $\left(-\frac{a}{4}, a\right)$ (3) $\left(-\frac{a}{4}, -a\right)$ (4) $(-a, -2a)$

Sol. Answer (2)

Solve equation $y = x$ and $y^2 = -4ax$

$O(0, 0)$, $B(-4a, -4a)$

Let B, $B(-at_1^2, -2at_1)$

Focal chord again intersect parabola at t_2 where $t_2 = -\frac{1}{t_1}$

$\therefore$ $t_1 = 2$ and $t_2 = -\dfrac{1}{2}$

$B'(-at_2^2, -2at_2)$

$\Rightarrow$ $B'\left(-\dfrac{a}{4}, a\right)$

Comprehension-III

The equation of the curve represented by

$C \equiv 9x^2 - 24xy + 16y^2 - 20x - 15y - 60 = 0$, then

1. The locus of curve C given in the above statement is

(1) Circle (2) Pair of straight line

(3) Parabola (4) Ellipse

2. The equation of axis of curve C is

(1) $x = 4y$ (2) $3x = y$

(3) $3x + 4y = 0$ (4) $3x = 4y$

3. The equation of directrix of curve C is

(1) $16x + 9y = 53$ (2) $16x + 12y + 53 = 0$

(3) $16x + 12y = 53$ (4) $16x + y = 53$

4. The length of latus rectum of curve C is

(1) 1 (2) 2

(3) 4 (4) 8

1. Answer (3)

2. Answer (4)

3. Answer (2)

4. Answer (1)

Solution of Comprehension-III

Comparing given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 9,\ b = 16,\ c = -60,\ h = -12,\ g = -10,\ f = -\dfrac{15}{2}$

$\Rightarrow$ $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= -8640 - 1800 - 0.9\left\{\dfrac{225}{4}\right\} - 16\{1000\} + 8640$

$\neq 0$

And $h^2 - ab = 144 - 144 = 0$

So, c represents a parabola

Now, $9x^2 - 24xy + 16y^2 - 20x - 15y - 60 = 0$

$\Rightarrow \{3x - 4y\}^2 = 5\{4x + 3y + 12\}$

$$\Rightarrow \left\{\frac{3x-4y}{\sqrt{3^2+4^2}}\right\}^2 = \frac{4x+3y+12}{\sqrt{4^2+3^2}}$$

Let $\frac{3x-4y}{\sqrt{3^2+4^2}} = Y$ and $\frac{4x+3y+12}{\sqrt{4^2+3^2}} = X$

$\therefore$ Equation reduces to, $Y^2 = X$

Comparing $Y^2 = X$ with $y^2 = 4ax$, we get

$$4a = 1 \Rightarrow a = \frac{1}{4}$$

So, axis of parabola is $y = 0$

$\Rightarrow 3x - 4y = 0$

Directrix of parabola is $X = -\frac{1}{4}$

$$\Rightarrow \frac{4x+3y+12}{\sqrt{3^2+4^2}} = -\frac{1}{4}$$

$$\Rightarrow 4x + 3y + \frac{53}{4} = 0 \Rightarrow 16x + 12y + 53 = 0$$

Hence, length of latus rectum = $4a = 1$

Comprehension-IV

Let $C : x^2 + y^2 = 9$, $E : \frac{x^2}{9} + \frac{y^2}{4} = 1$ and $L : y = 2x$ be three curves. P be a point on C and PL be the perpendicular to the major axis of ellipse E. PL cuts the ellipse at point M.

1. $\frac{ML}{PL}$ is equal to

(1) $\frac{1}{3}$ (2) $\frac{2}{3}$ (3) $\frac{1}{2}$ (4) 1

Sol. Answer (2)

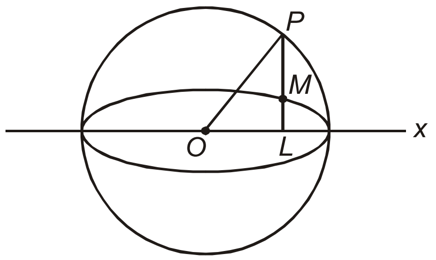
$M(3\cos\theta, 2\sin\theta)$

$P(3\cos\theta, 3\sin\theta)$

$L(3\cos\theta, 0)$

$ML = 2\sin\theta$

$PL = 3\sin\theta$



$$\frac{ML}{PL} = \frac{2}{3}$$

2. If equation of normal to C at point P be $L : y = 2x$ then the equation of the tangent at M to the ellipse E is

(1) $x+3y\pm 3\sqrt{5}=0$ (2) $4x+3y\pm\sqrt{5}=0$ (3) $x + y \pm 3 = 0$ (4) None of these

Sol. Answer (1)

OP is a normal

$\therefore$ $P(3\cos\theta, 3\sin\theta)$ lie on $y = 2x$

$3\sin\theta = 2 \,.\, 3\cos\theta$

$$\frac{\sin\theta}{2}=\frac{\cos\theta}{1}=\frac{\sqrt{\sin^2\theta+\cos^2\theta}}{\pm\sqrt{4+1}}=\frac{1}{\pm\sqrt{5}}$$

$\therefore$ $\sin\theta = \pm\dfrac{2}{\sqrt{5}}$ and $\cos\theta = \pm\dfrac{1}{\sqrt{5}}$

Equation of tangent at M

$$\frac{x}{3}.\cos\theta+\frac{y}{2}.\sin\theta=1$$

$$\frac{x}{3}.\frac{1}{\pm\sqrt{5}}+\frac{y}{2}.\frac{2}{\pm\sqrt{5}}=1$$

$x + 3y \pm 3\sqrt{5} = 0$

3. If R is the point of intersection of the line L with the line $x = 1$, then

(1) R lies inside both C and E

(2) R lies outside both C and E

(3) R lies on both C and E

(4) R lies inside C but outside E

Sol. Answer (4)

Put $x = 1$ in $y = 2x$

$\therefore$ $R(1, 2)$

$C \equiv 1 + 4 - 9 < 0$

$E \equiv \dfrac{1}{9}+\dfrac{4}{4}-1>0$

$\therefore$ R lies inside C and outside E.

Comprehension-V

An ellipse has its centre $C(1, 3)$ focus at $S(6, 3)$ and passing through the point $P(4, 7)$ then

1. The product of the lengths of perpendicular segments from the focii on tangent at point P is

(1) 20

(2) 45

(3) 40

(4) Cannot be determined

Sol. Answer (1)

Equation of ellipse will be assumed as

$$\frac{(x-1)^2}{a^2}+\frac{(y-3)^2}{b^2}=1$$

$$ae=\sqrt{(6-1)^2+0}=5$$

$P(4,7)$ lies on ellipse, hence $\frac{9}{a^2}+\frac{16}{b^2}=1$

Hence, $e=\frac{\sqrt{5}}{3}$

$a^2=45$

$b^2=20$

Since product of length of perpendicular segments from the foci on tangents at P is b^2

2. The point of the lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at point P is

(1) $\left(\frac{5}{3},5\right)$ (2) $\left(\frac{4}{3},3\right)$ (3) $\left(\frac{8}{3},3\right)$ (4) $\left(\frac{10}{3},5\right)$

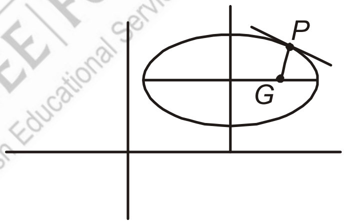
Sol. Answer (4)

Lines joining each focus to the foot of perpendicular to other focus bisects the normal at P, hence required point is mid-point of normal PG. Equation of normals at $P(4, 7)$ is

$3x-y-5=0$

Point $G\left(\frac{8}{3},3\right)$

Required point is $\left(\frac{10}{3},5\right)$



3. If the normal at a variable point on the ellipse meets its axes in Q and R then the locus of the mid-point of QR is a conic with an eccentricity(e') then

(1) $e'=\frac{3}{\sqrt{10}}$ (2) $e'=\frac{\sqrt{5}}{3}$ (3) $e'=\frac{3}{\sqrt{5}}$ (4) $e'=\frac{\sqrt{10}}{3}$

Sol. Answer (2)

Locus of mid-point of QR will be ellipse of same eccentricity $e'=\frac{\sqrt{5}}{3}$

Comprehension-VI

Equation of chord of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can be written as $T=S_1$ if its mid point is (x_1, y_1) which is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

1. Tangents drawn from any point on circle $x^2 + y^2 = c^2$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then locus of mid point of chord of contact is

(1) $x^2 - y^2 = c^2\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$

(2) $x^2 + y^2 = c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$

(3) $c^2(x^2 - y^2) = \frac{x^2}{a^2}-\frac{y^2}{b^2}$

(4) $c^2(x^2 + y^2) = \frac{x^2}{a^2}+\frac{y^2}{b^2}$

Sol. Answer (2)

Equation of chord of contact is

$$\frac{xc\cos\theta}{a^2}+\frac{yc\sin\theta}{b^2}=1$$

which is same as

$$\frac{xh}{a^2}+\frac{yk}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}\quad \text{mid point is } (h,\ k)$$

$$\frac{\frac{c\cos\theta}{a^2}}{\frac{h}{a^2}}=\frac{\frac{c\sin\theta}{b^2}}{\frac{k}{b^2}}=\frac{1}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}$$

$$c\cos\theta=\frac{h}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}\quad c\sin\theta=\frac{k}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}$$

$$c^2=\frac{h^2+k^2}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}$$

$$x^2 + y^2 = c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

2. Locus of mid-point of chord if it subtend right angle at origin

(1) $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^4}\right)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

(2) $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

(3) $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

(4) $\frac{x^2}{a^4}-\frac{y^2}{b^4}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

Sol. Answer (1)

$$\frac{xh}{a^2}+\frac{yk}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{\frac{xh}{a^2}+\frac{yk}{b^2}}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2}$$

$$\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\left(\frac{xh}{a^2}+\frac{yk}{b^2}\right)^2$$

For 90°,

$$\frac{1}{a^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2-\frac{h^2}{a^4}+\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2\frac{1}{b^2}-\frac{k^2}{b^4}=0$$

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

3. If $\left(\frac{1}{2},\frac{2}{5}\right)$ be the mid point of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, then its length is

(1) $7\sqrt{41}$ (2) $\frac{7}{5}\sqrt{41}$ (3) $\frac{7}{10}\sqrt{41}$ (4) $\frac{7}{15}\sqrt{41}$

Sol. Answer (2)

The chord whose mid-point is $\left(\frac{1}{2},\frac{2}{5}\right)$

Chord is $4x + 5y = 4$

Solving with ellipse we get points

$\left(4,\frac{-12}{5}\right)$ and $\left(-3,\frac{16}{5}\right)$

Length = $\frac{7}{5}\sqrt{41}$

Comprehension-VII

Two tangents can be drawn from any external point on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which can be observed by equation $y=mx+\sqrt{a^2m^2+b^2}$, since this is quadratic.

1. The equation of tangent from the point (2, 2) to the ellipse $4x^2 + 9y^2 = 36$

(1) $y = 0$ (2) $x = 2$ (3) $8x + 5y = 26$ (4) $8x + 5y = 13$

Sol. Answer (3)

$\frac{x^2}{9}+\frac{y^2}{4}=1$

$y=mx+\sqrt{9m^2+4}$

$2=2m+\sqrt{9m^2+4}$

$4 + 4m^2 - 8m = 9m^2 + 4$

$5m^2 + 8m = 0,\ m=0,-\frac{8}{5}$

Tangents are $y = 2$

$8x + 5y = 26$

2. The equation of the locus of a point from which two tangents can be drawn to the ellipse making angles θ_1, θ_2 with the major axis such that $\tan^2\theta_1 + \tan^2\theta_2 = 0$

(1) $2x^2y^2 = (x^2 - a^2)(y^2 - b^2)$
(2) $x^2y^2 = (x^2 - a^2)(y^2 - b^2)$
(3) $2x^2y^2 = (x^2 + a^2)(y^2 + b^2)$
(4) $x^2y^2 = (x^2 + a^2)(y^2 + b^2)$

Sol. Answer (1)

$$y = mx + \sqrt{a^2 + m^2 + b^2}$$

$$(y - mx)^2 = a^2m^2 + b^2$$

$$k^2 + m^2h^2 - 2hkm = a^2m^2 + b^2$$

$$m^2(h^2 - a^2) - 2khm + k^2 - b^2 = 0$$

$$m_1^2 + m_2^2 = c$$

$$(m_1 + m_2)^2 - 2m_1m_2 = c$$

$$\left(\frac{2kh}{h^2 - a^2}\right)^2 - 2\frac{k^2 - b^2}{h^2 - a^2} = 0$$

$$2x^2y^2 = (x^2 - a^2)(y^2 - b^2)$$

3. The equation of the locus of a points from which two tangents can be drawn to the ellipse making angles θ_1, θ_2 wtih the major axis such that $\theta_1 + \theta_2 = 2\alpha$(constant)

(1) $x^2 + y^2 - 2xy\cot2\alpha = a^2 - b^2$
(2) $x^2 - y^2 - 2xy\cot2\alpha = a^2 - b^2$
(3) $x^2 - y^2 + 2xy\cot2\alpha = a^2 - b^2$
(4) $x^2 - y^2 + 2xy\cot2\alpha = a^2 + b^2$

Sol. Answer (2)

$$\theta_1 + \theta_2 = 2\alpha$$

$$\tan(\theta_1 + \theta_2) = \tan2\alpha$$

$$\frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = \tan 2\alpha$$

$$\frac{m_1 + m_2}{1 - m_1m_2} = \tan 2\alpha$$

$$\frac{\frac{2kh}{h^2 - a^2}}{1 - \frac{k^2 - b^2}{h^2 - a^2}} = \tan 2\alpha$$

$$\frac{2kh}{h^2 - k^2 + b^2 - a^2} = \tan 2\alpha$$

$$\cot2\alpha.\ 2xy = x^2 - y^2 + b^2 - a^2$$

$$x^2 - y^2 - 2xy\cot2\alpha = a^2 - b^2$$

Comprehension-VIII

Rectangular hyperbola is the hyperbola whose asymptotes are perpendicular hence its equation is $x^2 - y^2 = a^2$, if axes are rotated by 45° in clockwise direction then its equation becomes $xy = c^2$.

1. Focus of hyperbola $xy = 16$, is

(1) $(4\sqrt{2}, 4\sqrt{2})$ (2) $(4\sqrt{2}, 0)$ (3) $(0, 4\sqrt{2})$ (4) $(4, 0)$

Sol. Answer (1)

a = distance of vertex from centre

$= 4\sqrt{2}$

$e = \sqrt{2}$

$ae = 4\sqrt{2}\cdot\sqrt{2} = 8$

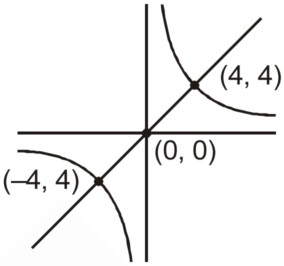
Distance of focii from centre = 8

Hence its coordinates are

$x^2 + x^2 = 8^2$

$2x^2 = 64$

$x = 4\sqrt{2}$



2. Directrix of hyperbola $xy = 16$ are

(1) $x + y = 4\sqrt{2}$ (2) $x - y = 4\sqrt{2}$ (3) $x + y = 4$ (4) $x + y = -4$

Sol. Answer (1)

Distance of direction from centre = $\dfrac{a}{e} = \dfrac{4\sqrt{2}}{\sqrt{2}} = 4$

Equation of directrix is $x + y = \pm\lambda$

$\dfrac{\pm\lambda}{\sqrt{2}} = 4$

$\lambda = \pm 4\sqrt{2}$

$x + y = \pm 4\sqrt{2}$

3. Length of minor axis of hyperbola $xy = 16$ is

(1) $4\sqrt{2}$ (2) 4 (3) $8\sqrt{2}$ (4) 8

Sol. Answer (3)

Length of minor axis is same as major axis in rectangular hyperbola

$L = 8\sqrt{2}$

Comprehension-IX

If P is a variable point and F_1 and F_2 are two fixed points such that $(PF_1 - PF_2) = 2a$. Then the locus of the point P is a hyperbola with points F_1 and F_2 as the two focii ($F_1F_2 > 2a$). If $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is a hyperbola, then its conjugate hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Let $P(x, y)$ is a variable point such that

$|\sqrt{(x-1)^2 + (y-2)^2} - \sqrt{(x-5)^2 + (y-5)^2}| = 3$

1. If the locus of the point P represents a hyperbola of eccentricity e, then the eccentricity e' of the corresponding conjugate hyperbola is

(1) $\frac{5}{3}$ (2) $\frac{4}{3}$ (3) $\frac{5}{4}$ (4) $\frac{3}{\sqrt{7}}$

Sol. Answer (3)

$2ae = 5$

$2a = 3$

$e = \frac{5}{3}$, $b^2 = \frac{9}{4}\left(\frac{25}{9} - 1\right) = 4$

For eccentricity of conjugate hyperbola

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\frac{1}{\frac{25}{9}} + \frac{1}{e'^2} = 1$$

$$e' = \frac{5}{4}$$

2. Locus of intersection of two perpendicular tangent to the given hyperbola is

(1) $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{55}{4}$ (2) $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{25}{4}$

(3) $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{7}{4}$ (4) None of these

Sol. Answer (4)

Equation of directrix circle will be

$$(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = a^2 - b^2$$

$$= \frac{9}{4} - 4 = \text{negative}$$

Hence, no points.

3. If origin is shifted to point $\left(3, \frac{7}{2}\right)$ and the axes are rotated through an angle θ in anticlockwise sense so that equation of given hyperbola changes to the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then θ is

(1) $\tan^{-1}\left(\frac{4}{3}\right)$ (2) $\tan^{-1}\left(\frac{3}{4}\right)$ (3) $\tan^{-1}\left(\frac{5}{3}\right)$ (4) $\tan^{-1}\left(\frac{3}{5}\right)$

Sol. Answer (2)

Rotation must be equal to slope of major axis.

$$\text{Slope} = \frac{5-2}{5-1} = \frac{3}{4}$$

$$\tan\theta = \frac{3}{4}$$

SECTION - D

Matrix-Match Type Questions

1. Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$, which intersect at (3, 0). Then

Column-I	Column-II
(A) Area of ΔPQR	(p) 2
(B) Radius of circumcircle of ΔPQR	(q) $\frac{5}{2}$
(C) Centroid of ΔPQR	(r) $\left(\frac{5}{2}, 0\right)$
(D) Circumcentre of ΔPQR	(s) $\left(\frac{2}{3}, 0\right)$

Sol. Answer : A(p), B(q), C(s), D(r)

Equation of normal at $(at^2, 2at)$

$y + tx = 2at + at^3$ Put $a = 1$

$\therefore$ Normal at $(t^2, 2t)$

$y + tx = 2t + t^3$ passes through (3, 0)

$3t = 2t + t^3$

$t^3 = t$ gives $t = 0, 1, -1$

$P(0, 0)$ $Q(1, 2)$ $R(1, -2)$

Area of $\Delta PQR = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \left|\frac{1}{2}(-4)\right| = 2$ sq. unit

Let $c(a, b)$ be the circumcentre of ΔPQR

$\therefore$ $CP^2 = CQ^2 = CR^2$

$a^2 + b^2 = (a - 1)^2 + (b - 2)^2 = (a - 1)^2 + (b + 2)^2$

$\therefore$ $b = 0,\ a = \frac{5}{2}$

Circumcentre $c\left(\frac{5}{2}, 0\right)$

Circum radius = $CP = \sqrt{\frac{25}{4}} = \frac{5}{2}$

Centroid of $\Delta PQR = \left(\frac{2}{3}, 0\right)$

2. Match column I to column II according to the given condition.

Column I	Column II
(A) Tangents at the ends of focal chord for a parabola	(p) Meet at directrix
(B) Let P is a point on the parabola and tangent at P meets the directrix at Q. If S is the focus of parabola then SP and SQ	(q) Meet at an angle of 90°
(C) If $A\left(at_1^2, 2at_1\right)$ and $B\left(at_2^2, 2at_2\right)$ are the ends of a focal chord of $y^2 = 4ax$ then	(r) $t_1\, t_2 = -1$
(D) If the normal at $\left(at_1^2, 2at_1\right)$ meets the parabola $y^2 = 4ax$ again at $\left(at_2^2, 2at_2\right)$ then	(s) $t_2 = -t_1 - \dfrac{2}{t_1}$
	(t) $t_1t_2 + t_1^2 + 2 = 0$

Sol. Answer : A(p, q), B(q), C(r), D(s, t)

(A) Tangent at the each of focal chord meet at 90° and meet at directrix

(B) The angle between SP and SQ is 90°

(C) Condition for focal chord $t_1t_2 = -1$

(D) Condition for normal chord

$$t_2 = -t_1 - \frac{2}{t_1} \qquad \Rightarrow t_1t_2 = -t_1^2 - 2, \qquad t_1\,t_2 + t_1^2 + 2 = 0$$

3. Match the following

Column I	Column II
(A) Product of the perpendicular from foci of any tangent to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is	(p) $x = \pm\dfrac{16}{\sqrt{7}}$
(B) The equation $\dfrac{x^2}{3-a} + \dfrac{y^2}{2-a} = 1$ represents an ellipse, then the value of a is	(q) $(-\infty, 2)$
(C) The common tangent of $x^2 + y^2 = 4$ and $x^2 + 2y^2 = 2$ is	(r) 4
(D) Polar of the focus to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ is	(s) Not defined
	(t) $2x - 3y + 4 = 0$

Sol. Answer A(r), B(q), C(s), D(p)

(A) $E:\frac{x^2}{9}+\frac{y^2}{4}=1$

Product of the perpendicular from the foci of any tangent to an ellipse

= square of the semi-minor axis

= 4

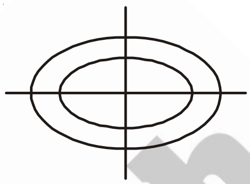
(B) $3-a>0,\ 2-a>0$

$\Rightarrow\ a<3,\ a<2 \Rightarrow a\in(-\infty,2)$

(C) $C: x^2+y^2=4,\ E:\ ax^2+2y^2=2$

$\Rightarrow\ \frac{x^2}{2}=\frac{y^2}{1}=1$

(D) $E:\ \frac{x^2}{16}+\frac{y^2}{9}=1$



Not possible

$\therefore\ \ e=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$

$F:\ (ae,0)=\left(4\cdot\frac{\sqrt{7}}{4},0\right)=(\sqrt{7},0)$

Polar of the focus $(\sqrt{7},0)$ is

$\frac{xx_1}{16}+\frac{yy_1}{9}=1$

$\Rightarrow\ \frac{x}{16}\times\sqrt{7}+0=1$

$\Rightarrow\ x=\frac{16}{\sqrt{7}}$

4. Consider the ellipse $x^2+2y^2-4x-4y+4=0$ and match column I to column II according to the given condition.

Column I		Column II
(A) The eccentricity of the ellipse is	(p)	$\frac{1}{\sqrt{2}}$
(B) The length of latus rectum	(q)	$\sqrt{2}$
(C) The area of parallelogram formed by the tangents at the end of conjugate diameters is always greater than	(r)	4
(D) The sum of squares of two semi-conjugate diameters is greater than	(s)	$2\sqrt{2}$
	(t)	2

Sol. Answer : A(p), B(q), C(p, q, r, s, t), D(p, q, s, t)

The given ellipse can be written as

$$\frac{(x-2)^2}{2}+\frac{(y-1)^2}{1}=1$$

$\Rightarrow$ $a^2 = 2$, $b^2 = 1$

(A) Eccentricity = $e=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}$

(B) The length of latus rectum = $\frac{2b^2}{a}=\frac{2\times 1}{\sqrt{2}}=\sqrt{2}$

(C) Area = $4ab=4\sqrt{2}$

(D) Sum = $a^2 + b^2 = 2 + 1 = 3$

5. Match the following

	Column I		**Column II**
(A)	Angle between the pair of tangents drawn from the point (1, 2) to the ellipse $2x^2 + 3y^2 = 6$ is	(p)	30°
(B)	If the locus of a point p to the ellipse $3x^2 + 4y^2 = 12$ is $x^2 + y^2 = 7$, then the angle between the tangents drawn at p is	(q)	60°
(C)	If θ and ϕ are the eccentric angles of the extremities of two conjugate diameters then $\theta - \phi$ is	(r)	45°
(D)	If the angle between the focal radii of an ellipse at a point is 90°, then the angle between a normal and focal radii at that point is	(s)	90°
		(t)	120°

Sol. Answer A(s), B(s), C(s), D(r)

(A) E : $2x^2 + 3y^2 = 6$

$$\Rightarrow \quad \frac{x^2}{3}=\frac{y^2}{2}=1$$

Any tangent to the ellipse be

$$y=mx+\sqrt{a^2m^2+b^2}$$

$$\Rightarrow \quad y=mx+\sqrt{3m^2+2}$$

Which is passing through (1, 2)

$$\therefore \quad 2=m+\sqrt{3m^2+2}$$

$\Rightarrow$ $(2 - m)^2 = 3m^2 + 2$

$\Rightarrow$ $4 - 4m + m^2 - 3m^2 - 2 = 0$

$\Rightarrow$ $-2m^2 - 4m + 2 = 0$

$\Rightarrow$ $m^2 + 2m - 1 = 0$

m_1, m_2

$\therefore$ $m_1 m_2 = -1$

(B) $E : 3x^2 + 4y^2 = 12$

$\Rightarrow$ $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

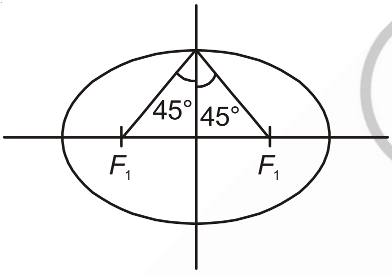
Equation of D.C. is $x^2 + y^2 = 7$

$\Rightarrow$ Angle between them is 90°

(C) $\theta - \phi = 90°$

(D) The tangent and normal at any point of an ellipse bisects the angles between the focal radii of that point

$\therefore$ Angle is 45°



6. Match the following

Column I		Column II
(A) Equation of the diameter passes through (0, 0) w.r.t. an ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ is	(p)	$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
(B) For all real value of m, the straight line $y = mx + \sqrt{9m^2 + 4}$ is a tangent to the curve is	(q)	$y = 3$
(C) Equation of common tangents to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ and the circle $x^2 + y^2 = 9$ are	(r)	$2x + 3y = 6$
(D) From a point $P(3, 2)$ tangents are drawn to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and intersects the ellipse at Q & R. Then the locus of P is	(s)	$y = -\dfrac{16x}{25m}$
	(t)	$y = -3$

Sol. Answer A(s), B(p), C(q, t), D(r)

(A) Equation of diameter w.r.t.

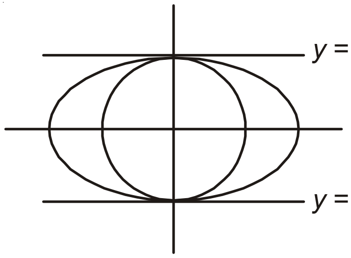
An ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is

$$y=-\frac{b^2x}{a^2m}=-\frac{16}{25m}x$$

(B) T : $y=mx+\sqrt{9m^2+4}$

E : $\frac{x^2}{9}+\frac{y^2}{4}=1$

(C)



(D) Locus of P is a chord of contact

$\therefore \quad \frac{x}{3}+\frac{y}{2}=1$

$\Rightarrow \quad 2x + 3y = 6$

7. Match the following

Column-I	Column-II
(A) Locus of point of intersection of $x = at^2$, $y = 2at$	(p) $x^2 + y^2 = 2a^2$
(B) Director circle of $x^2 + y^2 = a^2$	(q) $y^2 = 4ax$
(C) Locus of point of intersection of the lines $x \cos\theta = y \cot\theta = a$	(r) $x^2 + y^2 = ax$
(D) The locus of mid point of the chords of the circle $x^2 + y^2 - 2ax = 0$ passing through the origin	(s) $x^2 - y^2 = a^2$

Sol. Answer : A(q), B(p), C(s), D(r)

(A) $x = at^2$ and $y = 2at$ eliminate t

$\therefore \quad x=a\cdot\left(\frac{y}{2a}\right)^2 \Rightarrow 4ax=y^2$

(B) Director circle of $x^2+y^2=r^2$ is $x^2+y^2=2r^2$.

(C) $\cos\theta=\frac{a}{x},\ \cot\theta=\frac{a}{y}$

$\sec\theta=\frac{x}{a},\ \tan\theta=\frac{y}{a}$

w.r.t. $\sec^2\theta-\tan^2\theta=1$

$\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$

$x^2-y^2=a^2$ locus

(D) Let $P(h, k)$ be mid-point of chord AB.

Equal of chord $T = S_1$

$$x \cdot h + y \cdot k - a(x+h) = h^2 + k^2 - 2ah$$

Chord passes through (0, 0).

$\therefore \quad -a(h) = h^2 + k^2 - 2ab$

$\therefore$ Locus of $P(h, k)$

$$x^2 + y^2 = ax$$

$$S = x^2 + y^2 + x - y - z = 0$$

SECTION - E

Assertion-Reason Type Questions

1. STATEMENT-1 : If the parabola $y = (a - b)x^2 + (b - c)x + (c - a)$ touches the x-axes in the interval (0, 1) then the line $ax + by + c = 0$ always passes through a fixed point.

and

STATEMENT-2 : The equation $L_1 + \lambda L_2 = 0$ or $\mu L_1 + vL_2 = 0$ represents a line passing through intersection of lines $L_1 = 0$ and $L_2 = 0$ which is a fixed point, when λ, μ, v are constants.

Sol. Answer (1)

If $x \in (0, 1)$, a linear equation in a, b and c will be obtained

$\therefore \quad ax + by + c = 0$ will pass through a fixed point.

2. STATEMENT-1 : Two tangents are drawn from the point $(\sqrt{24}, 1)$ to $\frac{x^2}{16} + \frac{y^2}{9} = 1$, then they must be perpendicular.

and

STATEMENT-2 : The equation of the director circle to $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is given by $x^2 + y^2 = 25$.

Sol. Answer (1)

Pair of tangents from any point on director circle are perpendicular to each other.

3. STATEMENT-1 : A hyperbola and its conjugate hyperbola have the same asymptotes.

and

STATEMENT-2 : In a second degree curve, equation of asymptotes, if exists differ by constant only.

Sol. Answer (1)

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ on fixing a, b, h, g and f can be give unique pair of lines for single value of c.

4. STATEMENT-1 : The line $3x + 4y = 5$ intersects the hyperbola $9x^2 - 16y^2 = 144$ only at one point.

and

STATEMENT-2 : Given line is parallel to an asymptotes of the hyperbola.

Sol. Answer (1)

Line parallel to asymptote intersect hyperbola at unique point.

5. STATEMENT-1 : If lines $y = m_1x$ and $y = m_2x$ are the conjugate diameter of the hyperbola $xy = c^2$, then $m_1 + m_2 = 0$.

and

STATEMENT-2 : Two lines are called conjugate diameter of hyperbola if they bisect the chords parallel to each other.

Sol. Answer (1)

Let a diameter of $xy = c^2$ has slope m.

$\therefore \quad T = S_1 \quad \Rightarrow \quad \frac{kx + hy}{2} = \frac{hk}{2}$

$\Rightarrow \quad m = \frac{-k}{h}$

Diameter $\Rightarrow \quad y = -mx$

$\therefore$ Slope of conjugate diameter $\equiv -m$

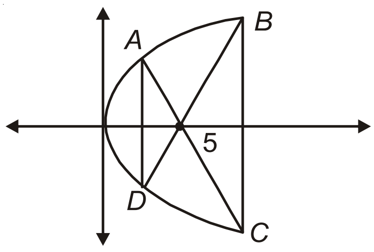
$\Rightarrow \quad m - m = 0 \quad \Rightarrow$ statement-1 is true and statement-2 is correct explanation.

SECTION - F

Integer Answer Type Questions

1. A trapezium is inscribed in parabola $y^2 = 4x$ such that its diagonal pass through the point (1, 0) and each has length $\frac{25}{4}$. If area of trapezium be P then $\frac{4P}{25}$ equals ____.

Sol. Answer (3)



Focus of parabola, $y^2 = 4x$ is (1, 0)

So, diagonals are focal chord

$AS = 1 + t^2 = c$ (say)

$\because \quad \frac{1}{c} + \frac{1}{\frac{25}{4} - c} = 1 \left\{ \because \frac{1}{AS} + \frac{1}{CS} = \frac{1}{a} \right\}$

$\Rightarrow \quad 4c^2 - 25c + 25 = 0 \Rightarrow c = \frac{5}{4}, 5$

For $c = \frac{5}{4}, 1 + t^2 = \frac{5}{4} \Rightarrow t^2 = \frac{1}{4} \Rightarrow t = \pm\frac{1}{2}$

For $c = 5$, $1 + t^2 = 5 \Rightarrow t = \pm 2$

$\therefore \quad A \equiv \left(\frac{1}{4}, 1\right), B \equiv (4, 4), C \equiv (4, -4) \text{ and } D \equiv \left(\frac{1}{4}, -1\right)$

$AD = 2$ and $BC = 8$, distance between AD and $BC = \dfrac{15}{4}$

$\therefore$ Area of trapezium, $P = \dfrac{1}{2}(2+8)\cdot\dfrac{15}{4} = \dfrac{75}{4} \Rightarrow \dfrac{4P}{25} = 3$

2. Three normals drawn from any point to parabola $y^2 = 4ax$ cut the line $x = 2a$ in points whose ordinates are in AP. If slopes of normals be m_1, m_2 and m_3 then $\left(\dfrac{m_1}{m_2}\right)\left(\dfrac{m_3}{m_2}\right)$ equals ____.

Sol. Answer (1)

Equation of parabola $\rightarrow y^2 = 4\,ax.$

If feet of normals passing through a point be $(am_1^2, -2am_1), (am_2^2, -2am_2)$ and $(am_3^l - 2am_3)$

$\therefore$ Equation of normal at $(am_1^2, -2am_1)$ is $y = m_1x - 2am_1 - am_1^3$

Solving with line $x = 2a$, we get point of intersection as $(2a, m_1^3)$ similarly solving with other normals the ordinates of points of intersection with $x = 2a$ are $-am_2^2$ and $-am_3^2$.

$\therefore$ According to the question,

$-\{am_1^3 + am_3^3\} = -2am_2^2$ ...(i)

But $m_1 + m_3 = -m_2$ ...(ii)

$\therefore$ $2m_2^3 = (m_1 + m_3)\{m_1^2 + m_3^2 - m_1m_3\} = 2m_2^2$

$\therefore$ $2m_2^3 = (m_1^2 - m_3^2 + m_1m_3\}$ ...(iii) {using (ii)}

Also $m_2^2 = m_1^2 + m_3^2 + 2m_1m_3$ ...(iv)

(iv) – (iii) $\Rightarrow 3m_2^2 = 3m_1, m_3 \Rightarrow \left\{\dfrac{m_1}{m_2}\right\}\left\{\dfrac{m_3}{m_2}\right\} = 1$

3. A chord is drawn from a point $P(1, t)$ to parabola $y^2 = 4x$ which cuts the parabola at A and B. If $PA.PB = 3|t|$, then maximum value of 't' equals ____

Sol. Answer (4)

Let equation of line passing through $P(1, t)$ be

$$\frac{x-1}{\cos\theta} = \frac{y-t}{\sin\theta} = r$$

$x = r\cos\theta = 1$

$y = r\sin\theta + t$

Line meets parabola at A and B

$\Rightarrow \{r\sin\theta + t\}^2 = 4\{r\cos\theta + 1\}$

$r^2\sin^2\theta + 2r\{t\sin\theta - 2\cos\theta\} + t^2 - 4 = 0$

$\therefore$ $PA \cdot PB = \dfrac{t^2 - 4}{\sin^2\theta} = 3\,|\,t\,|$

$\Rightarrow \left|\frac{t^2-4}{3t}\right| = \sin^2\theta \le 1 \Rightarrow \frac{|t^2-4|}{3|t|} \le 1$

$\Rightarrow t^2 - 3|t| - 4 \le 0$

$\Rightarrow (|t| + 1)(|t| - 4) \le 0$

$\Rightarrow |t| \le 4 \Rightarrow$ maximum value of it is 4

4. If the locus of the feet of perpendicular from the foci on any tangent to an ellipse $\frac{x^2}{4}+\frac{y^2}{2}=1$ is $x^2 + y^2 = k$, then the value of k is ______.

Sol. Answer (4)

The general equation of tangent of slop m is

$y = mx \pm \sqrt{a^2m^2 + b^2}$ ...(i)

Equation of perpendicular is

$y - 0 = -\frac{1}{m}(x - ae)$ (let focus is $(ae, 0)$)

$ym + x - ae = 0$

$my + x = ae$ ...(ii)

Locus of point of intersection of (i) and (ii) is obtained by eliminating m between (i) and (ii)

That is given by

$(y - xm)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2e^2$

$\Rightarrow (1 + m^2)(x^2 + y^2) = a^2m^2 + (a^2 - a^2e^2) + a^2e^2$

$\Rightarrow x^2 + y^2 = a^2$

Here $a^2 = 4 \Rightarrow k = 4$

5. The number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, is ______ .

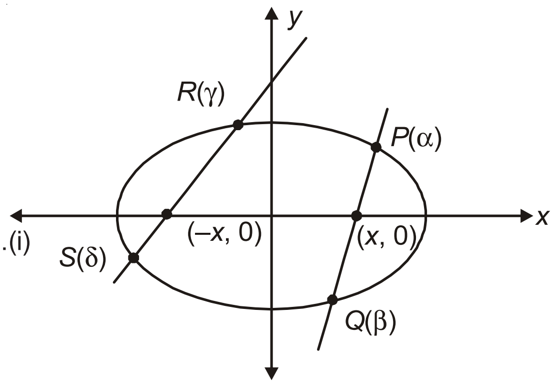
Sol. Answer (4)

For ellipse, $\frac{x^2}{16}+\frac{y^2}{9}=1$, equation of director circle is $x^2 + y^2 = 25$

This director circle will cut the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ at 4 points.

6. If any two chords be drawn through two points on major axis of an ellipse equidistant from centre, then $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right)\tan\left(\frac{\delta}{2}\right)$ = _______ , (where α, β, γ, δ are ecentric angles of extremities of chords)

Sol. Answer (1)



From figure,

$$PQ = \frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

If passes through $(x_1, 0)$ so $\dfrac{x_1}{a} = \dfrac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$ …(i)

Similarly for RS, which passes through $(–x, 0)$

$$\frac{-x_1}{a} = \frac{\cos\left(\frac{\gamma-\delta}{2}\right)}{\cos\left(\frac{\gamma+\delta}{2}\right)} \quad \text{…(ii)}$$

From (i) and (ii),

$$\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} = -\frac{\cos\left(\frac{\gamma-\delta}{2}\right)}{\cos\left(\frac{\gamma+\delta}{2}\right)}$$

Using componendo and dividendo and solving, we get

$$\frac{2\cos\frac{\alpha}{\beta}\cos\frac{\beta}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} = \frac{2\sin\frac{\gamma}{2}\sin\frac{\delta}{2}}{2\cos\frac{\gamma}{2}\cos\frac{\delta}{2}} \Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} = 1$$

7. If M_1 and M_2 are feet of perpendiculars from foci S_1 and S_2 of ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ on the tangent at any point P an ellipse, then $\sqrt{(S_1M_1)(S_2M_2)}$ equals _______

Sol. Answer (4)

We know that product of perpendicular drawn from two foci S_1 and S_2 of an ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ on the tangent at any point P on ellipse is equal to square of semi-minor axis

i.e., $\sqrt{(S_1M_1)(S_2M_2)} = \sqrt{16} = 4$

8. An ellipse intersects the hyperbola $2x^2 – 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes and the equation of the ellipse is $x^2 + ky^2 = k$ then the value of k is ______.

Sol. Answer (2)

Let the equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b \quad \text{…(i)}$$

$$e_1 = \sqrt{1-\frac{b^2}{a^2}}$$

$$x^2 - y^2 = \frac{1}{2} \quad \text{…(ii)}$$

$$e_2 = \sqrt{1+\frac{1}{1}} = \sqrt{2}$$

But $e_2 = \frac{1}{e_1}$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \sqrt{1-\frac{b^2}{a^2}} \;\Rightarrow\; \frac{1}{2} = 1-\frac{b^2}{a^2} \;\Rightarrow\; \frac{b^2}{a^2} = \frac{1}{2}$$

$$a = b\sqrt{2} \qquad \text{...(iii)}$$

by (i) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \left(\frac{-\frac{x}{a^2}}{\frac{y}{b^2}}\right) = -\frac{b^2}{a^2}.\frac{x}{y} = -\frac{x}{2y} \qquad \text{...(iv)}$$

(ii) $2x - 2y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{x}{y} \qquad \text{...(v)}$$

by but $\left(\frac{x}{y}\right)\left(-\frac{x}{2y}\right) = -1$

$\Rightarrow \quad x^2 = 2y^2$ ...(vi)

by (ii) and (vi),

$$y^2 = \frac{1}{2}$$

$$x^2 = 1$$

Putting $x^2 = 1$, $y^2 = \frac{1}{2}$ in (i)

$$\frac{1}{a^2} + \frac{1}{2b^2} = 1 \qquad \text{...(vii)}$$

by (iii) and (vii),

$a^2 = 2$, $b^2 = 1$

Hence the equation of ellipse

$$\frac{x^2}{2} + \frac{y^2}{1} = 1,\; x^2 + 2y^2 = 2 \quad \Rightarrow \quad k = 2$$

9. The equation of directrix of a hyperbola is $3x + 4y + 8 = 0$. The focus of the hyperbola is (1, 1). If eccentricity of the hyperbola is 2 and the length of conjugate axis is k then $[k]$, where [ ] represents the greatest integer function, is equal to _____.

Sol. Answer (6)

The distance between focus and directrix is $= \left|\frac{3\times1+4\times1+8}{\sqrt{9+16}}\right| = 3$

e = eccentricity, $2a$ = transverse axis,

$2b$ = conjugate axis

Then $\left|\frac{a}{e} - ae\right| = 3$

$\Rightarrow \quad ae - \frac{a}{e} = 3 \qquad \Rightarrow a\left(2 - \frac{1}{2}\right) = 3 \qquad \Rightarrow a\left(\frac{3}{2}\right) = 3, \qquad a = 2$

Between know that $e^2 = 1 + \frac{b^2}{a^2}$

$\Rightarrow 4 = 1 + \frac{b^2}{4} \qquad \Rightarrow b^2 = 12, \qquad b = 2\sqrt{3}$

$k = 2b = 4\sqrt{3}$

$[k] = 6$

10. Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. If the locus of middle point of segment of any normal intercepted between the coordinate axis is $36x^2 - 16y^2 = k$, then the value of $\frac{k}{169}$, is _____.

Sol. Answer (1)

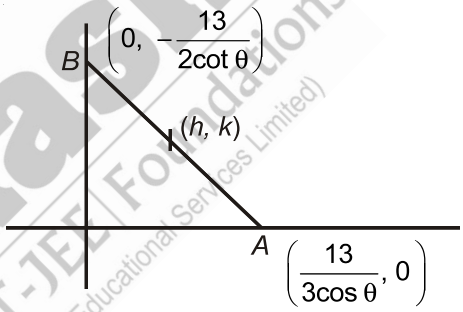
Equation of normal

$ax \cos\theta - by \cot\theta = a^2 + b^2$

$\Rightarrow \quad 3x \cos\theta - 2y \cot\theta = 13$

at $y = 0$, $x = \frac{13}{3\cos\theta}$

at $x = 0$, $y = -\frac{13}{2\cot\theta}$



Let middle point of (h, k) is

$h = \frac{13}{6\cos\theta}$

$k = -\frac{13}{4\cot\theta}$

$\Rightarrow \quad \cos\theta = \frac{13}{6h} \quad \Rightarrow \quad \sec\theta = \frac{6h}{13}$

$\Rightarrow \quad \cot\theta = -\frac{13}{4k} \quad \Rightarrow \quad \tan\theta = -\frac{4k}{13}$

For locus of (h, k), we use

$1 + \tan^2\theta = \sec^2\theta$

$\Rightarrow \quad 1 + \frac{16k^2}{169} = \frac{36h^2}{169}$

$\Rightarrow \quad 36h^2 - 16y^2 = 169$

$\Rightarrow \quad k = 169 \Rightarrow \frac{k}{169} = 1$

11. The area of the triangle formed by any arbitrary tangents of the hyperbola $xy = 4$, with the coordinate axis is _____ units.

Sol. Answer (8)

Let any tangent $\frac{x}{t} + yt = 4$

Area of $\Delta = \frac{1}{2} \times \frac{4}{t} \times 4t = 8$ units

12. If the circle $x^2 + y^2 = 4$ intersects the hyperbola $xy = 4$ in four points (x_i, y_i), $i = 1, 2, 3, 4$, then value of $\frac{x_1x_2x_3x_4}{4}$ is ____.

Sol. Answer (4)

Solving circle and hyperbola, we get

$$x^2 + \frac{16}{x^2} = 4 \Rightarrow x^4 - 4x^2 + 16 = 0$$

Let its root be x_1, x_2, x_3 and x_4

$\therefore \quad x_1x_2x_3x_4 = 16$

$\Rightarrow \quad \frac{x_1x_2x_3x_4}{4} = 4$

13. The locus of the point of intersection of two perpendicular tangents of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is a circle of radius R, then $[R]$ equals ____. (where [·] represents greatest integer function)

Sol. Answer (6)

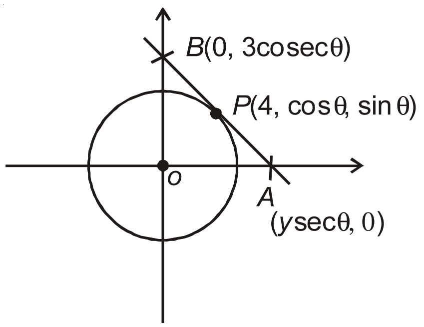
Required circle is directs circle, hence it's equation is $x^2 + y^2 = 25 + 16$

$\Rightarrow \quad R = \sqrt{41}$

$\Rightarrow \quad [R] = 6$

14. The minimum area of the triangle formed by the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the coordinate axes is k units, then $\frac{k}{2}$ equals _____.

Sol. Answer (6)



Tangent at point P is $\frac{x}{4}\cos\theta + \frac{x}{4}\sin\theta = 1$

Area of triangle $OAB = \frac{1}{2} \times 4\sec\theta \times 3\cos\theta$

$$= \frac{12}{\sin 2\theta}$$

for minimum $\sin 2\theta = 1$

$\Rightarrow \quad (\text{Area})_{min} = 12 = k$

$\Rightarrow \quad \frac{k}{2} = 6$

15. The number of integral values of k for which $\frac{x^2}{12-k}+\frac{y^2}{8-k}$ = 1 represents hyperbola is _____.

Sol. Answer (3)

For hyperbola,

$12 - k > 0$ and $8 - k < 0$

$\Rightarrow$ $k < 12$ and $k > 8$

$\Rightarrow$ $k \in (8, 12)$

Number of integral values 9, 10, 11 = 3 values

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