SEQUENCE & SERIES

1. Let $7\overbrace{5...5}^{r}7$ denote the (r + 2) digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + ... + 7\overbrace{5...5}^{98}7$. If $S = \dfrac{7\overbrace{5...5}^{99}7 + m}{n}$, where m and n are natural numbers less than 3000, then the value of m + n is **[JEE(Advanced) 2023]**

2. Let $l_1, l_2, ..., l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1, and let $w_1, w_2, ..., w_{100}$ be consecutive terms of another arithmetic progression with common difference d_2, where $d_1 d_2 = 10$. For each $i = 1, 2, ..., 100$, let R_i be a rectangle with length l_i, width w_i and area A_i. If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is ____________. **[JEE(Advanced) 2022]**

3. Let $a_1, a_2, a_3, ...$ be an arithmetic progression with $a_1 = 7$ and common difference 8. Let $T_1, T_2, T_3, ...$ be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ for $n \geq 1$. Then, which of the following is/are TRUE ?

[JEE(Advanced) 2022]

(A) $T_{20} = 1604$

(B) $\sum_{k=1}^{20} T_k = 10510$

(C) $T_{30} = 3454$

(D) $\sum_{k=1}^{30} T_k = 35610$

4. Let m be the minimum possible value of $\log_3\left(3^{y_1} + 3^{y_2} + 3^{y_3}\right)$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is ______.

[JEE(Advanced) 2020]

5. Let $a_1, a_2, a_3, ...$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, ...$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$$2(a_1 + a_2 + ... + a_n) = b_1 + b_2 + ... + b_n$$

holds for some positive integer n, is ______ **[JEE(Advanced) 2020]**

6. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1$$

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$, $n \geq 2$.

Then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(B) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(C) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(D) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

7. Let AP (a; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If $AP(1; 3) \cap AP(2; 5) \cap AP(3; 7) = AP(a; d)$ then $a + d$ equals ______.

[JEE(Advanced) 2019]

8. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ... , and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is ______. **[JEE(Advanced) 2018]**

9. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ? **[JEE(Advanced) 2017]**

10. Let $b_i > 1$ for i = 1, 2, ..., 101. Suppose $\log_e b_1, \log_e b_2, ...,\log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$ then **[JEE(Advanced) 2016]**

(A) $s > t$ and $a_{101} > b_{101}$ (B) $s > t$ and $a_{101} < b_{101}$

(C) $s < t$ and $a_{101} > b_{101}$ (D) $s < t$ and $a_{101} < b_{101}$

11. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is **[JEE(Advanced) 2015]**

12. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is b + 2, then the value of $\frac{a^2 + a - 14}{a + 1}$ is **[JEE(Advanced) 2014]**

SOLUTIONS

1. Ans. (1219)

Sol. $S = 77 + 757 + 7557 + \ldots + 7\underbrace{5.....5}_{98}7$

$10S = 770 + 7570 + \ldots + 75 \ldots 570 + 755 \ldots.. 570$

$9S = -77 + \underbrace{13+13+.....+13}_{98\text{ times}} + 7\underbrace{5.....5}_{98}70$

$= -77 + 13 \times 98 + 7\underbrace{5.....5}_{99}7 + 13$

$$S = \frac{7\underbrace{5.....5}_{99}7 + 1210}{9}$$

$m = 1210$

$n = 9$

$m + n = 1219$

2. Ans. (18900.00)

Sol. Given

$A_{51} - A_{50} = 1000 \Rightarrow \ell_{51}w_{51} - \ell_{50}w_{50} = 1000$

$\Rightarrow (\ell_1 + 50d_1)(w_1 + 50d_2) - (\ell_1 + 49d_1)(w_1 + 49d_2) = 1000$

$\Rightarrow (\ell_1 d_2 + w_1 d_1) = 10 \quad \ldots.(1)$

(As $d_1 d_2 = 10$)

$\therefore\ A_{100} - A_{90} = \ell_{100}w_{100} - \ell_{90}w_{90}$

$= (\ell_1 + 99d_1)(w_1 + 99d_2) - (\ell_1 + 89d_1)(w_1 + 89d_2)$

$= 10(\ell_1 d_2 + w_1 d_1) + (99^2 - 89^2)d_1 d_2$

$= 10(10) + \underbrace{(99-89)}_{=10}(99+89)(10)$

(As, $d_1 d_2 = 10$)

$= 100\ (1 + 188) = 100\ (189) = 18900$

3. Ans. (B, C)

Sol. $a_1 = 7,\ d = 8$

$T_{n+1} - T_n = a_n \forall n \geq 1$

$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$

$S_n = T_1 + T_2 + T_3 + \ldots. + T_{n-1} + T_n$

on subtraction

$T_n = T_1 + a_1 + a_2 + \ldots. + a_{n-1}$

$T_n = 3 + (n - 1)(4n - 1)$

$T_n = 4n^2 - 5n + 4$

$$\sum_{k=1}^{n} T_k = 4\sum n^2 - 5\sum n + 4n$$

$T_{20} = 1504$

$T_{30} = 3454$

$$\sum_{k=1}^{30} T_k = 35615$$

$$\sum_{k=1}^{20} T_k = 10510$$

4. Ans. (8.00)

Sol. $\dfrac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq \left[3^{(y_1+y_2+y_3)}\right]^{\frac{1}{3}}$

$\Rightarrow \quad 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 3^4$

$\Rightarrow \quad \log_3(3^{y_1} + 3^{y_2} + 3^{y_3}) \geq 41$

$\Rightarrow \quad m = 4$

Also, $\dfrac{x_1 + x_2 + x_3}{3} \geq \sqrt[3]{x_1 x_2 x_3}$

$\Rightarrow \quad x_1 x_2 x_3 \leq 27$

$\Rightarrow \quad \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \leq 3$

$\Rightarrow \quad M = 3$

Thus, $\log_2(m^3) + \log_3(M^2) = 6 + 2 = 8$

5. Ans. (1.00)

Sol. Given $2(a_1 + a_2 + \ldots\ldots + a_n) = b_1 + b_2 + \ldots\ldots + b_n$

$\Rightarrow \quad 2 \times \dfrac{n}{2}(2c + (n-2)2) = c\left(\dfrac{2^n - 1}{2-1}\right)$

$\Rightarrow \quad 2n^2 - 2n = c(2^n - 1 - 2n)$

$\Rightarrow \quad c = \dfrac{2n^2 - 2n}{2^n - 1 - 2n} \in \mathbb{N}$

So, $\quad 2n^2 - 2n \geq 2^n - 1 - 2n$

$\Rightarrow \quad 2n^2 + 1 \geq 2^n \Rightarrow n < 7$

$\Rightarrow$ n can be 1,2,3,....,

Checking c against these values of n

we get c = 12 (when n = 3)

Hence number of such c = 1

6. Ans. (A, B, D)

Sol. α, β are roots of $x^2 - x - 1$

$a_{r+2} - a_r = \dfrac{(\alpha^{r+2} - \beta^{r+2}) - (\alpha^r - \beta^r)}{\alpha - \beta}$

$= \dfrac{(\alpha^{r+2} - \alpha^r) - (\beta^{r+2} - \beta^r)}{\alpha - \beta}$

$= \dfrac{\alpha^r(\alpha^2 - 1) - \beta^r(\beta^2 - 1)}{\alpha - \beta}$

$= \dfrac{\alpha^r\alpha - \beta^r\beta}{\alpha - \beta} = \dfrac{\alpha^{r+1} - \beta^{r+1}}{\alpha - \beta} = a_{r+1}$

$\Rightarrow a_{r+2} - a_{r+1} = a_r$

$$\Rightarrow \sum_{r=1}^{n} a_r = a_{n+2} - a_2 = a_{n+2} - \frac{\alpha^2-\beta^2}{\alpha-\beta}$$

$$= a_{n+2} - (\alpha+\beta) = a_{n+2} - 1$$

Now $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{\sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n - \sum_{n=1}^{\infty}\left(\frac{\beta}{10}\right)^n}{\alpha-\beta}$

$$= \frac{\frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}}}{\alpha-\beta} = \frac{\frac{\alpha}{10-\alpha} - \frac{\beta}{10-\beta}}{(\alpha-\beta)}$$

$$= \frac{10}{(10-\alpha)(10-\beta)} = \frac{10}{89}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum_{n=1}^{\infty} \frac{a_{n-1}+a_{n+1}}{10^n} = \frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}} = \frac{12}{89}$$

Further, $b_n = a_{n-1} + a_{n+1}$

$$= \frac{(\alpha^{n-1}-\beta^{n-1}) + (\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta}$$

(as $\alpha\beta = -1 \Rightarrow \alpha^{n-1} = -\alpha^n\beta$ & $\beta^{n-1} = -\alpha\beta^n$)

$$= \frac{\alpha^n(\alpha-\beta) + (\alpha-\beta)\beta^n}{\alpha-\beta} = \alpha^n + \beta^n$$

7. Ans. (157.00)

Sol. We equate the general terms of three respective A.P.'s as $1 + 3a = 2 + 5b = 3 + 7c$

$\Rightarrow$ 3 divides $1 + 2b$ and 5 divides $1 + 2c$

$\Rightarrow$ $1 + 2c = 5, 15, 25$ etc.

So, first such terms are possible when $1 + 2c = 15$ i.e. $c = 7$

Hence, first term = a = 52

$d = \text{lcm}(3, 5, 7) = 105 \quad \Rightarrow a + d = 157$

8. Ans. (3748)

Sol. X : 1, 6, 11, ..., 10086

Y : 9, 16, 23, ...,14128

$X \cap Y$: 16, 51, 86, ...

Let $m = n(X \cap Y)$

$\therefore \quad 16 + (m-1) \times 35 \le 10086$

$\Rightarrow \quad m \le 288.71$

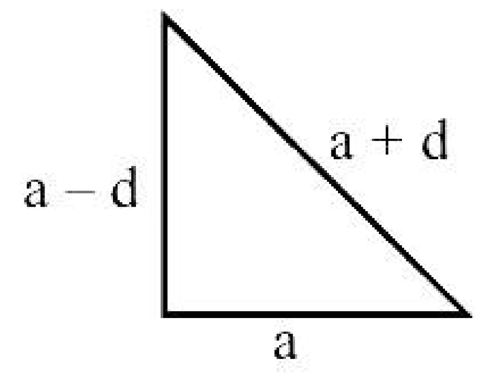
$\Rightarrow \quad m = 288$

$\therefore \quad n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$= 2018 + 2018 - 288 = 3748$

9. Ans. (6)

Sol.



where $d > 0, a > 0$

$\Rightarrow$ length of smallest side = $a - d$

Now $(a + d)^2 = a^2 + (a - d)^2$

$\Rightarrow a(a - 4d) = 0$

$\therefore a = 4d$ ...(1)

(As a = 0 is rejected)

Also, $\frac{1}{2}a.(a-d) = 24$

$\Rightarrow a(a - d) = 48$ ...(2)

$\therefore$ From (1) and (2), we get $a = 8, d = 2$

Hence, length of smallest side

$\Rightarrow (a - d) = (8 - 2) = 6$

10. Ans. (B)

Sol. If $\log_e b_1, \log_e b_2 ......\log_e b_{101} \rightarrow$ AP ; $D = \log_e 2$

$\Rightarrow b_1\, b_2\, b_3 ....b_{101} \rightarrow$ GP ; $\quad r = 2$

$\therefore \quad b_1, 2b_1, 2^2b_1, ..., 2^{100}b_1......$ GP

$a_1\, a_2\, a_3 ....a_{101}$ ..... AP

Given, $a_1 = b_1 \quad \& \quad a_{51} = b_{51}$

$\Rightarrow \quad a_1 + 50D = 2^{50}b_1$

$\therefore \quad a_1 + 50D = 2^{50}a_1 \quad$ (As $b_1 = a_1$)

Now, $t = b_1(2^{51} - 1)$; $s = \frac{51}{2}(2a_1 + 50D)$

$\Rightarrow t < a_1.2^{51}$ ....(i) ; $s = \frac{51}{2}(a_1 + a_1 + 50D)$

$$s = \frac{51}{2}(a_1 + 2^{50}a_1)$$

$$s = \frac{51a_1}{2} + \frac{51}{2}.2^{50}a_1$$

$\Rightarrow \quad s > a_1.2^{51}$ ....(ii)

clearly $s > t$ (from equation (i) and (ii))

Also $a_{101} = a_1 + 100D$; $b_{101} = b_1.2^{100}$

$\therefore\ a_{101} = a_1 + 100\left(\frac{2^{50}a_1 - a_1}{50}\right)$; b_{101}

$= 2^{100}a_1$ ....(iii)

$a_{101} = a_1 + 2^{51}a_1 - 2a_1 \Rightarrow a_{101} = 2^{51}a_1 - a_1$

$\Rightarrow a_{101} < 2^{51}a_1$ ....(iv)

clearly $b_{101} > a_{101}$ (from equation (iii) and (iv))

11. **Ans. (9)**

Sol. $\frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11} \Rightarrow \frac{7(2a+6d)}{(2a+10d)} = 6$

$\Rightarrow 2a = 18d$

$a = 9d$

also $130 < a + 6d < 140$

$\frac{26}{3} < d < \frac{28}{3} \Rightarrow d = 9$

12. **Ans. (4)**

Sol. Let a,b,c are a, ar, ar^2 where $r \in N$

also $\frac{a+b+c}{3} = b + 2$

$\Rightarrow a + ar + ar^2 = 3(ar) + 6$

$\Rightarrow ar^2 - 2ar + a = 6$

$\Rightarrow (r-1)^2 = \frac{6}{a}$

$\because \frac{6}{a}$ must be perfect square & $a \in N$

$\therefore$ a can be 6 only.

$\Rightarrow r - 1 = \pm 1 \Rightarrow r = 2$

& $\frac{a^2 + a - 14}{a+1} = \frac{36+6-14}{7} = 4$