Session 2

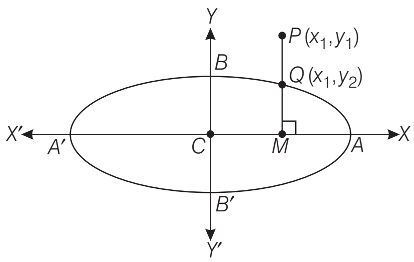
Position of a Point with Respect to an Ellipse, Intersection of a Line and an Ellipse, Equation of Tangent in Different Forms, Equations of Normals in Different Forms, Properties of Eccentric Angles of the Co-normal Points, Co-normal Points Lie on a Fixed Curve

Position of a Point with Respect to an Ellipse

Theorem : Prove that the point $P(x_1, y_1)$ lies outside, on, or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >, = \text{or}, < 0$$

Proof : From point $P(x_1, y_1)$ draw perpendicular PM on AA' to meet the ellipse at $Q(x_1, y_2)$.



Since, $Q(x_1, y_2)$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

then, $$\frac{x_1^2}{a^2}+\frac{y_2^2}{b^2}=1$$

$$\Rightarrow \quad \frac{y_2^2}{b^2}=1-\frac{x_1^2}{a^2} \quad \text{...(i)}$$

Now, point P lies outside, on or inside the ellipse according as

$$PM >, = \text{or}, < QM$$

$$\Rightarrow \quad y_1 >, = \text{or}, < y_2$$

$$\Rightarrow \quad \frac{y_1^2}{b^2} >, = \text{or}, < \frac{y_2^2}{b^2}$$

$$\Rightarrow \quad \frac{y_1^2}{b^2} >, = \text{or}, < 1-\frac{x_1^2}{a^2} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} >, = \text{or}, < 1,$$

or $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >, = \text{or}, < 0$$

Hence, the point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 > 0, = \text{or}, < 0$$

Remark

Let $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$, then $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

The point (x_1, y_1) lies outside, on or inside the ellipse $S=0$ according as $S_1 >, =$ or <0.

Example 16 Find the position of the point $(4, -3)$ relative to the ellipse $5x^2+7y^2=140$.

Sol. The given ellipse can be written as $\frac{x^2}{28}+\frac{y^2}{20}-1=0$

Let $$S=\frac{x^2}{28}+\frac{y^2}{20}-1$$

$$\therefore \quad S_1=\frac{(4)^2}{28}+\frac{(-3)^2}{20}-1=\frac{3}{140}>0$$

So, the point $(4, -3)$ lies outside the ellipse $5x^2 + 7y^2 = 140$.

Example 17 Find the integral value of α for which the point $\left(7 - \frac{5\alpha}{4}, -\alpha\right)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Sol. Since, the point $\left(7 - \frac{5\alpha}{4}, -\alpha\right)$ lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, then $\frac{1}{25}\left(7 - \frac{5\alpha}{4}\right)^2 + \frac{1}{16}(-\alpha)^2 - 1 < 0$

$$\Rightarrow \quad (28 - 5\alpha)^2 + 25\alpha^2 - 400 < 0$$

$$\Rightarrow \quad 50\alpha^2 - 280\alpha + 384 < 0$$

$$\Rightarrow \quad 25\alpha^2 - 140\alpha + 192 < 0$$

$$\Rightarrow \quad (5\alpha - 12)(5\alpha - 16) < 0$$

$$\therefore \quad \frac{12}{5} < \alpha < \frac{16}{5}$$

Hence, integral value of α is 3

Intersection of a Line and an Ellipse

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

and the given line be $y = mx + c$ ...(ii)

Eliminating y from Eqs. (i) and (ii), then

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad (a^2m^2 + b^2)x^2 + 2mca^2x + c^2a^2 - a^2b^2 = 0 \quad \text{...(iii)}$$

Above equation being a quadratic in x gives two values of x. Shows that every straight line will cut the ellipse in two points may be real, coincident or imaginary according as Discriminant of Eq. (iii) $>, =, < 0$

i.e. $\quad 4m^2c^2a^4 - 4(a^2m^2 + b^2)(c^2a^2 - a^2b^2) >, =, < 0$

or $\quad -a^2b^2c^2 + a^4b^2m^2 + a^2b^4 >, =, < 0$

or $\quad a^2m^2 + b^2 >, =, < c^2$ ...(iv)

Condition of Tangency : If the line Eq. (ii) touches the ellipse Eq. (i), then Eq. (iii) has equal roots.

$\therefore$ Discriminant of Eq. (iii) $= 0$

$$\Rightarrow \quad c^2 = a^2m^2 + b^2 \quad \text{or} \quad c = \pm\sqrt{(a^2m^2 + b^2)} \quad \text{...(v)}$$

so, the line $y = mx + c$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{if} \quad c^2 = a^2m^2 + b^2$$

(which is condition of tangency)

Substituting the value of c from Eq. (v) in Eq. (ii), then

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}$$

Hence, the lines $y = mx \pm \sqrt{(a^2m^2 + b^2)}$ will always tangents to the ellipse.

Point of contact : Substituting $c = \pm\sqrt{(a^2m^2 + b^2)}$ in Eq. (iii), then

$$(a^2m^2 + b^2)x^2 \pm 2ma^2x\sqrt{(a^2m^2 + b^2)} + (a^2m^2 + b^2)a^2 - a^2b^2 = 0$$

or $\quad (a^2m^2 + b^2)x^2 \pm 2ma^2x\sqrt{(a^2m^2 + b^2)} + a^4m^2 = 0$

or $\quad (x\sqrt{(a^2m^2 + b^2)} \pm a^2m)^2 = 0$

$$\therefore \quad x = \pm\frac{a^2m}{\sqrt{(a^2m^2 + b^2)}} = \pm\frac{a^2m}{c}$$

From Eq. (i), $\quad \frac{a^4m^2}{c^2} \cdot \frac{1}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \frac{a^2m^2}{c^2} = \frac{c^2 - a^2m^2}{c^2} = \frac{b^2}{c^2}$$

$$y = \pm\frac{b^2}{c}$$

Hence, the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$ this known as m-point on the ellipse.

Remark

If $m = 0$, then Eq. (iii) gives $b^2x^2 + c^2a^2 - a^2b^2 = 0$

or $\quad b^2x^2 + (a^2m^2 + b^2)a^2 - a^2b^2 = 0$

$$\therefore \quad x = \pm\frac{a^2m}{b}$$

which gives two values of x.

Example 18 Prove that the straight line $lx + my + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $a^2l^2 + b^2m^2 = n^2$.

Sol. The given line is

$$lx + my + n = 0$$

or $\quad y = -\frac{l}{m}x - \frac{n}{m}$ ...(i)

Comparing this line with $y = Mx + c$

$$\therefore \quad M = -\frac{l}{m} \quad \text{and} \quad c = -\frac{n}{m}$$

The line Eq. (i) will touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$c^2 = a^2M^2 + b^2$$

$$\frac{n^2}{m^2} = \frac{a^2l^2}{m^2} + b^2$$

$$a^2l^2 + b^2m^2 = n^2$$

Example 19 ***Show that the line*** $x\cos\alpha + y\sin\alpha = p$ ***touches the ellipse*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ***if*** $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ ***and that point of contact is*** $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$.

Sol. The given line is $x\cos\alpha + y\sin\alpha = p$

$$y = -x\cot\alpha + p\operatorname{cosec}\alpha$$

Comparing this line with $y = mx + c$

$\therefore \quad m = -\cot\alpha \quad$ and $\quad c = p\operatorname{cosec}\alpha$

Hence, the given line touches the ellipse, then

$$c^2 = a^2m^2 + b^2$$

$\Rightarrow \quad p^2\operatorname{cosec}^2\alpha = a^2\cot^2\alpha + b^2$

$\Rightarrow \quad p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$

and point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

i.e. $\left(-\frac{a^2(-\cot\alpha)}{p\operatorname{cosec}\alpha}, \frac{b^2}{p\operatorname{cosec}\alpha}\right)$

i.e. $\left(\frac{a^2\cos\alpha}{p}, \frac{b^2\sin\alpha}{p}\right)$

Example 20 For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. $\because$ Equation of ellipse is

$$9x^2 + 16y^2 = 144 \quad \text{or} \quad \frac{x^2}{16}+\frac{y^2}{9}=1$$

Comparing this with $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

then, we get $\quad a^2 = 16 \quad$ and $\quad b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore \quad m = 1$

and $\quad c = \lambda$

If the line $y = x + \lambda$ touches the ellipse

$$9x^2 + 16y^2 = 144$$

then $\quad c^2 = a^2m^2 + b^2$

$\Rightarrow \quad \lambda^2 = 16 \times 1^2 + 9$

$\Rightarrow \quad \lambda^2 = 25$

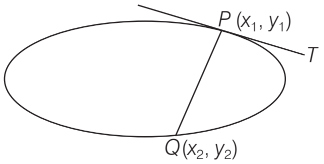
$\therefore \quad \lambda = \pm 5$

Equation of Tangent in Different Forms

1. Point Form

Theorem : The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Proof : (By first Principal Method)



$\because$ Equation of ellipse is $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad$ ...(i)

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two point on Eq. (i), then

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad \text{...(ii)}$$

and

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \quad \text{...(iii)}$$

Subtracting Eqs. (ii) from (iii), then

$$\frac{1}{a^2}(x_2^2 - x_1^2) + \frac{1}{b^2}(y_2^2 - y_1^2) = 0$$

$$\Rightarrow \quad \frac{(x_2+x_1)(x_2-x_1)}{a^2} + \frac{(y_2+y_1)(y_2-y_1)}{b^2} = 0$$

$$\Rightarrow \quad \frac{y_2-y_1}{x_2-x_1} = -\frac{b^2(x_1+x_2)}{a^2(y_1+y_2)} \quad \text{...(iv)}$$

Equation of PQ is

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1) \quad \text{...(v)}$$

From Eqs. (iv) and (v), then

$$y - y_1 = -\frac{b^2(x_1+x_2)}{a^2(y_1+y_2)}(x - x_1) \quad \text{...(vi)}$$

Now, for tangent at $P, Q \to P$ *i.e.*, $x_2 \to x_1$ and $y_2 \to y_1$, then Eq. (vi) becomes

$$y - y_1 = -\frac{b^2(2x_1)}{a^2(2y_1)}(x - x_1)$$

or

$$\frac{yy_1 - y_1^2}{b^2} = -\left(\frac{xx_1 - x_1^2}{a^2}\right)$$

or

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} \text{ or } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{[from (ii)]}$$

which is required equation of tangent at (x_1, y_1).

Remark

The equation of tangent at (x_1, y_1) can be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$.

This method is applicable only when the equation of ellipse is a polynomial of second degree in x and y.

2. Parametric form

Theorem : The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

Proof : The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ (by point form)

Replacing x_1 by $a\cos\phi$ and y_1 by $b\sin\phi$, then we get

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$$

Remark

Point of intersection of tangent at 'θ' and 'φ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right)$

Remembering method : ∵ Equation of chord joining $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \quad \frac{x}{a}\left\{\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}+\frac{y}{b}\left\{\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}=1$$

$$\text{or} \quad \frac{x}{a^2}\left\{\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}+\frac{y}{b^2}\left\{\frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}\right\}=1$$

3. Slope form

Theorem : The equations of tangents of slope m to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $y=mx\pm\sqrt{(a^2m^2+b^2)}$ and the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(a^2m^2+b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2+b^2)}}\right)$$

Proof : Let $y=mx+c$ be a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then the equation $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$$\Rightarrow \quad x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0 \quad \text{...(i)}$$

must have equal roots

$$4a^4m^2c^2-4(a^2m^2+b^2)a^2(c^2-b^2)=0$$

$$\{\because B^2-4AC=0\}$$

$$\Rightarrow \quad a^2m^2c^2-(a^2m^2+b^2)(c^2-b^2)=0$$

$$\Rightarrow \quad a^2m^2c^2-a^2m^2c^2+a^2b^2m^2-b^2c^2+b^4=0$$

$$\Rightarrow \quad a^2b^2m^2-b^2c^2+b^4=0$$

$$\Rightarrow \quad c^2=a^2m^2+b^2$$

$$\therefore \quad c=\pm\sqrt{(a^2m^2+b^2)}$$

Substituting this value of c in $y=mx+c$, we get

$$y=mx\pm\sqrt{(a^2m^2+b^2)}$$

as the required equations of tangent of ellipse in terms of slope, putting $c=\pm\sqrt{(a^2m^2+b^2)}$ in (i), we get

$$x^2(a^2m^2+b^2)\pm 2a^2m\sqrt{(a^2m^2+b^2)}\,x+a^4m^2=0$$

$$\Rightarrow \quad (\sqrt{(a^2m^2+b^2)}\,x\pm a^2m)^2=0$$

$$\Rightarrow \quad x=\mp\frac{a^2m}{\sqrt{(a^2m^2+b^2)}}$$

Substituting this value of x in

$$y=mx\pm\sqrt{a^2m^2+b^2}$$

we obtained $\quad y=\pm\frac{b^2}{\sqrt{(a^2m^2+b^2)}}$

Thus, the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(a^2m^2+b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2+b^2)}}\right)$$

Example 21 If the line $3x+4y=\sqrt{7}$ touches the ellipse $3x^2+4y^2=1$, then find the point of contact.

Sol. Let the given line touches the ellipse at point $P(x_1, y_1)$.

The equation of tangent at P is

$$3xx_1+4yy_1=1 \quad \text{...(i)}$$

Comparing Eq. (i) with the given equation of line $3x + 4y = \sqrt{7}$, we get

$$\frac{3x_1}{3} = \frac{4y_1}{4} = \frac{1}{\sqrt{7}}$$

$$\therefore \quad x_1 = y_1 = \frac{1}{\sqrt{7}}$$

Hence, point of contact (x_1, y_1) is $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$.

Example 22 Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$$\therefore \quad m \times -2 = -1$$

$$\Rightarrow \quad m = \frac{1}{2}$$

Since $\quad 3x^2 + 4y^2 = 12$

or $\quad \frac{x^2}{4} + \frac{y^2}{3} = 1$

Comparing this with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 4$

and $\quad b^2 = 3$

So the equations of the tangents are

$$y = \frac{1}{2}x \pm \sqrt{4 \times \frac{1}{4} + 3}$$

$$\Rightarrow \quad y = \frac{1}{2}x \pm 2$$

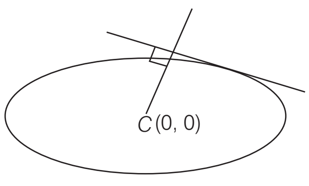
or $\quad x - 2y \pm 4 = 0$

Example 23 Find the locus of the foot of the perpendicular drawn from centre upon any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Any tangent of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = mx + \sqrt{(a^2m^2 + b^2)} \quad \text{...(i)}$$

Equation of the line perpendicular to Eq. (i) and passing through (0, 0) is



$$y = -\frac{1}{m}x \text{ or } m = -\frac{x}{y} \quad \text{...(ii)}$$

Substituting the value of m from Eq. (ii) in Eq. (i), then

$$y = -\frac{x^2}{y} + \sqrt{\left(a^2\frac{x^2}{y^2} + b^2\right)}$$

$$\Rightarrow \quad (x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

or changing to polars by putting $x = r\cos\theta$, $y = r\sin\theta$ it becomes

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

Example 24 ***Find the point on the ellipse $16x^2 + 11y^2 = 256$, where the common tangent to it and the circle $x^2 + y^2 - 2x = 15$ touch.***

Sol. The given ellipse is $\frac{x^2}{16} + \frac{y^2}{(256/11)} = 1$

Equation of tangent to it at point $\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ is

$$\frac{x}{4}\cos\theta + y\frac{\sqrt{11}}{16}\sin\theta = 1$$

It also touch the circle $(x-1)^2 + (y-0)^2 = 4^2$

Therefore,

$$\frac{\left|\frac{1}{4}\cos\theta - 1\right|}{\sqrt{\left(\frac{\cos^2\theta}{16} + \frac{11}{256}\sin^2\theta\right)}} = 4$$

$$\Rightarrow \quad |\cos\theta - 4| = \sqrt{(16\cos^2\theta + 11\sin^2\theta)}$$

or $\quad 4\cos^2\theta + 8\cos\theta - 5 = 0$

or $\quad (2\cos\theta - 1)(2\cos\theta + 5) = 0$

or $\quad \cos\theta = \frac{1}{2} \qquad \left(\because \cos\theta \neq -\frac{5}{2}\right)$

$$\therefore \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Therefore, points are $\left(2, \pm\frac{8\sqrt{3}}{11}\right)$.

Example 25 Find the maximum area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which touches the line $y = 3x + 2$.

Sol. $\because$ Line $y = 3x + 2$ touches ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Here, $\quad m = 3$ and $c = 2$

Substituting in $c^2 = a^2m^2 + b^2$

or $\quad 4 = 9a^2 + b^2 \quad \text{...(i)}$

Now, $AM \geq GM$

$$\Rightarrow \quad \frac{9a^2 + b^2}{2} \geq \sqrt{(9a^2)\, b^2} \quad \Rightarrow \quad \frac{9a^2 + b^2}{2} \geq 3ab$$

$$\Rightarrow \quad 2 \geq 3ab \qquad \text{[from Eq. (i)]}$$

or $$\frac{2\pi}{3} \geq \pi\, ab$$

or $$\frac{2\pi}{3} \geq \text{Area of ellipse}$$

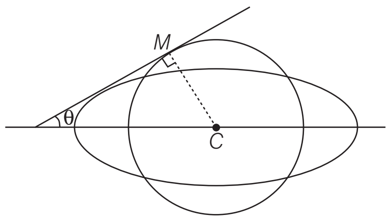
Therefore, the maximum area of the ellipse is $\frac{2\pi}{3}$.

Example 26 A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$.

Sol. Equation of the circle of radius r and concentric with ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = r^2 \qquad \text{...(i)}$$

any tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is



$$y = mx + \sqrt{(a^2m^2 + b^2)} \qquad (\text{where } m = \tan\theta)$$

If it is a tangent to circle, then perpendicular from (0, 0) is equal to radius r,

$$\therefore \quad \frac{\sqrt{(a^2m^2 + b^2)}}{\sqrt{(m^2 + 1)}} = |\, r \,| \text{ or } a^2m^2 + b^2 = m^2r^2 + r^2$$

$$(a^2 - r^2)\, m^2 = r^2 - b^2$$

$$\therefore \quad m = \sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$$

$$\tan\theta = \sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$$

$$\theta = \tan^{-1}\sqrt{\left(\frac{r^2 - b^2}{a^2 - r^2}\right)}$$

Example 27 Show that the product of the perpendiculars from the foci of any tangent to an ellipse is equal to the square of the semi minor axis, and the feet of a these perpendiculars lie on the auxiliary circle.

Sol. Let equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{... (i)}$$

Equation of any tangent in term of slope (m) of (i) is

$$y = mx + \sqrt{(a^2m^2 + b^2)}$$

or $$y - mx = \sqrt{a^2m^2 + b^2} \qquad \text{...(ii)}$$

Equation of a line perpendicular to Eq. (ii) and passing through $S\,(ae, 0)$ is

$$y - 0 = -\frac{1}{m}(x - ae)$$

or $$x + my = ae \qquad \text{...(iii)}$$

The lines Eq. (ii) and Eq. (iii) will meet at the foot of perpendicular whose locus is obtained by eliminating the variable m between Eq. (ii) and Eq. (iii), then squaring and adding Eq. (ii) and Eq. (iii), we get

$$(y - mx)^2 + (x + my)^2 = a^2m^2 + b^2 + a^2e^2$$

$$\Rightarrow \quad (1 + m^2)(x^2 + y^2) = a^2m^2 + b^2 + a^2 - b^2$$

$$\Rightarrow \quad (1 + m^2)(x^2 + y^2) = a^2\,(1 + m^2)$$

or $$x^2 + y^2 = a^2$$

which is auxiliary circle of ellipse, similarly we can show that the other foot drawn from second focus also lies on $x^2 + y^2 = a^2$.

Again if p_1 and p_2 be perpendiculars from foci $S(ae, 0)$ and $S'\,(-ae, 0)$ on (ii), then

$$p_1 = \frac{|\sqrt{(a^2m^2 + b^2)} + mae|}{\sqrt{(1 + m^2)}}$$

and $$p_2 = \frac{|\sqrt{(a^2m^2 + b^2)} - mae|}{\sqrt{(1 + m^2)}}$$

$$\therefore \quad p_1p_2 = \frac{|\, a^2m^2 + b^2 - a^2e^2m^2|}{(1 + m^2)}$$

$$= \frac{|\, a^2m^2 + b^2 - (a^2 - b^2)\, m^2|}{(1 + m^2)}$$

$$= \frac{b^2\,(1 + m^2)}{(1 + m^2)}$$

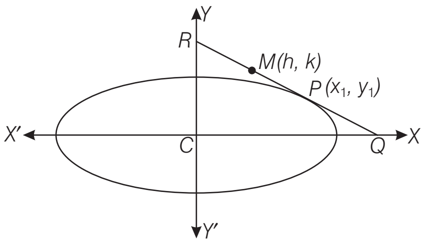
$$= b^2 = (\text{semi minor axis })^2$$

Example 28 ***Prove that the locus of mid-points of the portion of the tangents to the ellipse*** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ***intercepted between the axes is*** $a^2y^2+b^2x^2=4x^2y^2$.

Sol. Let $P(x_1, y_1)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

$\therefore$ Equation of tangent at (x_1, y_1) to (i) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$



This meet the coordinate axes at

$$Q\left(\frac{a^2}{x_1}, 0\right) \text{ and } R\left(0, \frac{b^2}{y_1}\right)$$

Let $M(h, k)$ be the mid-point of QR then,

$$h=\frac{\frac{a^2}{x_1}+0}{2}, k=\frac{0+\frac{b^2}{y_1}}{2}$$

$$\Rightarrow \quad x_1=\frac{a^2}{2h}, y_1=\frac{b^2}{2k}$$

Since, (x_1, y_1) lies on Eq. (i)

$$\therefore \quad \frac{\left(\frac{a^2}{2h}\right)^2}{a^2}+\frac{\left(\frac{b^2}{2k}\right)^2}{b^2}=1$$

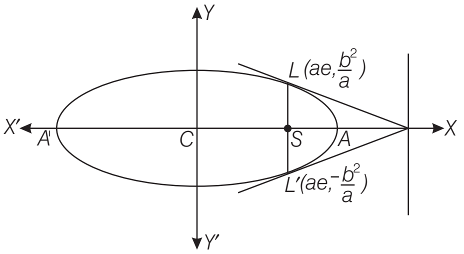
$$\Rightarrow \quad \frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$$

$$\Rightarrow \quad a^2k^2+b^2h^2=4h^2k^2$$

Hence, the locus of $M(h, k)$ is $a^2y^2+b^2x^2=4x^2y^2$

Example 29 Prove that the tangents at the extremities of latusrectum of an ellipse intersect on the corresponding directrix.

Sol. Let LSL' be a latusrectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



$\therefore$ The coordinates of L and L' are

$$\left(ae, \frac{b^2}{a}\right) \text{ and } \left(ae, -\frac{b^2}{a}\right) \text{ respectively}$$

$\therefore$ Equation of tangent at $L\left(ae, \frac{b^2}{a}\right)$ is

$$\Rightarrow \quad \frac{x(ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1$$

$$\Rightarrow \quad xe+y=a \quad \text{...(i)}$$

The equation of the tangent at $L'\left(ae, -\frac{b^2}{a}\right)$ is

$$\frac{x(ae)}{a^2}+\frac{y\left(-\frac{b^2}{a}\right)}{b^2}=1$$

$$\Rightarrow \quad ex-y=a \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$x=\frac{a}{e} \text{ and } y=0$$

Thus, the tangents at L and L' intersect at $(a/e, 0)$ which is a point lying on the corresponding directrix i.e. $x=\frac{a}{e}$.

Equations of Normals in Different Forms

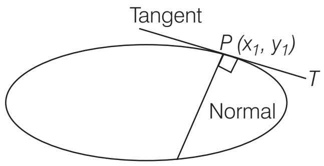
1. Point form

Theorem : The equation of normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

Proof : Since the equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$



The slope of the tangent at $(x_1, y_1)=-\frac{b^2x_1}{a^2y_1}$

$\therefore$ Slope of Normals at $(x_1, y_1) = \frac{a^2 y_1}{b^2 x_1}$

Hence, the equation of normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

Remark

The equation of normal at (x_1, y_1) can also be obtained by this method

$$\frac{x - x_1}{a' x_1 + h y_1 + g} = \frac{y - y_1}{h x_1 + b' y_1 + f} \quad \text{...(i)}$$

a', b', g, f, h are obtained by comparing the given ellipse with

$$a' x^2 + 2hxy + b' y^2 + 2gx + 2fy + c = 0 \quad \text{...(ii)}$$

The denominors of (i) can easily remembered by the first two rows of this determinant

i.e. $$\begin{vmatrix} a' & h & g \\ h & b' & f \\ g & f & c \end{vmatrix}$$

Since, first row, $a'(x_1) + h(y_1) + g(1)$

and second row, $h(x_1) + b'(y_1) + f(1)$

Here ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad \text{... (iii)}$$

Comparing Eqs. (ii) and (iii), then we get

$$a' = \frac{1}{a^2}, b' = \frac{1}{b^2}, g = 0, f = 0, h = 0$$

From, Eq. (i), equation of normal of Eq. (iii) at (x_1, y_1) is

$$\frac{x - x_1}{\frac{1}{a^2} x_1 + 0 + 0} = \frac{y - y_1}{0 + \frac{1}{b^2} y_1 + 0}$$

or $$\frac{a^2 (x - x_1)}{x_1} = \frac{b^2 (y - y_1)}{y_1}$$

or $$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

2. Parametric form

Theorem : The equation of normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (a\cos\phi, b\sin\phi) \text{ is}$$

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$$

Proof : Since, the equation of normal of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad \text{...(i)}$$

Replacing x_1 by $a\cos\phi$ and y_1 by $b\sin\phi$, then Eq.(i) becomes

$$\frac{a^2 x}{a\cos\phi} - \frac{b^2 y}{b\sin\phi} = a^2 - b^2$$

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$$

is the equation of normal at $(a\cos\phi, b\sin\phi)$

3. Slope form

Theorem : The equations of the normals of slope m to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by

$$y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2 m^2)}}$$

at the points $\left(\pm \frac{a^2}{\sqrt{(a^2 + b^2 m^2)}}, \pm \frac{mb^2}{\sqrt{(a^2 + b^2 m^2)}} \right)$.

Proof : The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad \text{...(i)}$$

Since, 'm' is the slope of the normal, then

$$m = \frac{a^2 y_1}{b^2 x_1}$$

$$y_1 = \frac{b^2 x_1 m}{a^2} \quad \text{...(ii)}$$

Since, (x_1, y_1) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

or $$\frac{x_1^2}{a^2} + \frac{b^4 x_1^2 m^2}{a^4 b^2} = 1$$

or $$\frac{x_1^2}{a^2} + \frac{b^2 x_1^2 m^2}{a^4} = 1 \text{ or } x_1^2 = \frac{a^4}{(a^2 + b^2 m^2)}$$

$$\therefore \quad x_1 = \pm \frac{a^2}{\sqrt{a^2 + b^2 m^2}}$$

From Eq. (ii),

$$y_1 = \pm \frac{mb^2}{\sqrt{(a^2 + b^2m^2)}}$$

$\therefore$ Equation of normal in terms of slope is

$$y - \left(\pm \frac{mb^2}{\sqrt{a^2 + b^2m^2}} \right) = m\left(x - \left(\pm \frac{a^2}{\sqrt{a^2 + b^2m^2}} \right) \right)$$

$$\Rightarrow \qquad y = mx \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}} \qquad \text{...(iii)}$$

Thus $y = mx \pm \dfrac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}}$ is a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where m is the slope of the normal.

The coordinates of the point of contact are

$$\left(\pm \frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm \frac{mb^2}{\sqrt{a^2 + b^2m^2}} \right)$$

Comparing Eq. (iii) with,

$$y = mx + c$$

$$\therefore \qquad c = \mp \frac{m(a^2 - b^2)}{\sqrt{(a^2 + b^2m^2)}}$$

or

$$c^2 = \frac{m^2(a^2 - b^2)^2}{(a^2 + b^2m^2)}$$

which is condition of normality, when $y = mx + c$ is the normal of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 30 ***If the normal at an end of a latusrectum of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by***

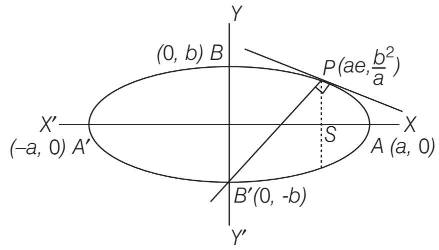
$$e^4 + e^2 - 1 = 0 \quad \textbf{or} \quad e^2 = \frac{\sqrt{5} - 1}{2}$$

Sol. The coordinates of an end of the latusrectum are $(ae, b^2/a)$. The equation of normal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae} - \frac{b^2(y)}{b^2/a} = a^2 - b^2$$

or

$$\frac{ax}{e} - ay = a^2 - b^2$$



It passes through one extremity of the minor axis whose coordinates are $(0, -b)$

$$\therefore \qquad 0 + ab = a^2 - b^2$$

or $\qquad (a^2b^2) = (a^2 - b^2)^2$

or $\qquad a^2a^2(1 - e^2) = (a^2e^2)^2$

or $\qquad 1 - e^2 = e^4$

or $\qquad e^4 + e^2 - 1 = 0$

or $\qquad (e^2)^2 + e^2 - 1 = 0$

$$\therefore \qquad e^2 = \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$\Rightarrow \qquad e^2 = \frac{\sqrt{5} - 1}{2} \qquad \text{(taking +ve sign)}$$

Example 31 Prove that the straight line $lx + my + n = 0$ is a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ if $\dfrac{a^2}{l^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2 - b^2)^2}{n^2}$.

Sol. The equation of any normal to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2 \qquad \text{... (i)}$$

The straight line $lx + my + n = 0$ will be a normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Therefore, Eq. (i) and $lx + my + n = 0$ represent the same line

$$\frac{a \sec \phi}{l} = \frac{-b \operatorname{cosec} \phi}{m} = \frac{a^2 - b^2}{-n}$$

$$\cos \phi = \frac{-na}{l(a^2 - b^2)}$$

and $\qquad \sin \phi = \dfrac{nb}{m(a^2 - b^2)}$

$$\because \qquad \sin^2 \phi + \cos^2 \phi = 1$$

$$\therefore \qquad \frac{n^2b^2}{m^2(a^2 - b^2)^2} + \frac{n^2a^2}{l^2(a^2 - b^2)^2} = 1$$

$$\Rightarrow \qquad \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$$

Example 32 A normal inclined at an angle of 45° to x-axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn. It meets the major and minor axes in P and Q respectively. If C is the centre of the ellipse, prove that area of ΔCPQ is $\frac{(a^2-b^2)^2}{2(a^2+b^2)}$ sq units.

Sol. Let $R(a\cos\phi, b\sin\phi)$ be any point on the ellipse, then equation of normal at R is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2$$

or
$$\frac{x}{\frac{\cos\phi\,(a^2-b^2)}{a}}+\frac{y}{\frac{-\sin\phi\,(a^2-b^2)}{b}}=1$$

If meets the major and minor axes at $P\left(\frac{(a^2-b^2)}{a}\cos\phi, 0\right)$ and $Q\left(0, -\frac{(a^2-b^2)}{b}\sin\phi\right)$ are respectively

$$\therefore \quad CP=\left(\frac{a^2-b^2}{a}\right)|\cos\phi|$$

and
$$CQ=\left(\frac{a^2-b^2}{b}\right)|\sin\phi|$$

$$\therefore \quad \text{Area of } \Delta CPQ=\frac{1}{2}\times CP\times CQ$$

$$=\frac{(a^2-b^2)^2\,|\sin\phi\cos\phi|}{2ab} \quad \text{... (i)}$$

But slope of normal $=\frac{a}{b}\tan\phi=\tan 45°$ (given)

$$\frac{a}{b}\tan\phi=1$$

$$\tan\phi=\frac{b}{a}$$

$$\therefore \quad \sin 2\phi=\frac{2\tan\phi}{1+\tan^2\phi}=\frac{2ab}{a^2+b^2}$$

$$\therefore \text{ From Eq. (i), Area of } \Delta CPQ=\frac{(a^2-b^2)^2\left|\frac{\sin 2\phi}{2}\right|}{2ab}$$

$$=\frac{(a^2-b^2)^2\,\frac{ab}{(a^2+b^2)}}{2ab}$$

$$=\frac{(a^2-b^2)^2}{2(a^2+b^2)} \text{ sq units.}$$

Example 33 Any ordinate MP of an ellipse meets the auxiliary circle in Q. Prove that the locus of the point of intersection of the normals at P and Q is the circle $x^2+y^2=(a^2+b^2)^2$.

Sol. Let equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ its auxiliary circle is

$$x^2+y^2=a^2$$

Coordinates of P and Q are $(a\cos\phi, b\sin\phi)$ and $(a\cos\phi, a\sin\phi)$ respectively. Equation of normal at P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2 \quad \text{...(i)}$$

and equation of normal at Q to the circle $x^2+y^2=a^2$ is

$$y = x\tan\phi \quad \text{...(ii)}$$

From Eq. (ii), $\tan\phi=\frac{y}{x}$

$$\therefore \quad \sin\phi=\frac{y}{\sqrt{(x^2+y^2)}} \quad \text{and} \quad \cos\phi=\frac{x}{\sqrt{(x^2+y^2)}}$$

or
$$\text{cosec}\,\phi=\frac{\sqrt{(x^2+y^2)}}{y}$$

and
$$\sec\phi=\frac{\sqrt{(x^2+y^2)}}{x} \quad \text{...(iii)}$$

Substituting the values of $\sec\phi$ and $\text{cosec}\,\phi$ from Eq. (iii) in Eq. (i)

$$\therefore \quad ax\times\frac{\sqrt{(x^2+y^2)}}{x}-by\times\frac{\sqrt{(x^2+y^2)}}{y}=a^2-b^2$$

or
$$(a-b)\sqrt{(x^2+y^2)}=(a+b)(a-b)$$

or
$$\sqrt{x^2+y^2}=a+b$$

$$\therefore \quad x^2+y^2=(a+b)^2$$

which is required locus.

Properties of Eccentric Angles of the Co-normal Points

1. **In general, four normals can be drawn to an ellipse from any point and if $\alpha, \beta, \gamma, \delta$ the eccentric angles of these four co-normal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.**

   Let $Q(h, k)$ be any given point and let $P(a\cos\phi, b\sin\phi)$ be any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

   Equation of normal at $P(a\cos\phi, b\sin\phi)$ is

$$ax\sec\phi - by\,\text{cosec}\,\phi = a^2-b^2$$

it passes through $Q(h, k)$

$\therefore \quad ah \sec\phi - bk \operatorname{cosec}\phi = a^2 - b^2$

or $$\frac{ah}{\cos\phi} - \frac{bk}{\sin\phi} = a^2 - b^2 \quad \text{...(i)}$$

or $$\frac{ah}{\left(\frac{1-\tan^2(\phi/2)}{1+\tan^2(\phi/2)}\right)} - \frac{bk}{\left(\frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)}\right)} = a^2 - b^2 \quad \text{...(ii)}$$

Let $\tan\phi/2 = t$

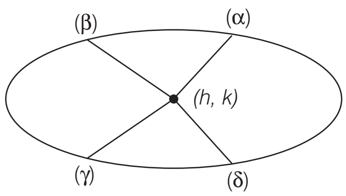
then, Eq. (ii) reduced to

$$bkt^4 + 2\{ah + (a^2 - b^2)\}t^3 + 2\{ah - (a^2 - b^2)\}t - bk = 0 \quad \text{... (iii)}$$

Which is a fourth degree equation in t, hence four normals can be drawn to an ellipse from any point.

Consequently, it has four values of ϕ say $\alpha, \beta, \gamma, \delta$ ($\because t = \tan\phi/2$).

Now, $$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4}$$



$$= \frac{\frac{-2\{ah + (a^2 - b^2)\}}{bk} + \frac{2\{(ah - (a^2 - b^2)\}}{bk}}{1 - 0 - 1}$$

$= \infty$ (From trigonometry) ($\because a \neq b$)

or $$\cot\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = 0$$

or $$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} = \text{an odd multiple of } \pi/2$$

$\Rightarrow \quad \alpha + \beta + \gamma + \delta = \text{an odd multiple of } \pi$

Aliter :

Let $z = e^{i\phi} = \cos\phi + i\sin\phi$

$\therefore \quad \frac{1}{z} = e^{-i\phi} = \cos\phi - i\sin\phi$

$\therefore \quad \cos\phi = \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 1}{2z}$

and $\quad \sin\phi = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$

Now, Eq. (i), reduces to

$$\frac{ah}{\left(\frac{z^2+1}{2z}\right)} - \frac{bk}{\left(\frac{z^2-1}{2iz}\right)} = a^2 - b^2$$

$$\Rightarrow \quad (a^2 - b^2)z^4 - 2(ah - ibk)z^3 + 2(ah + ibk)z - (a^2 - b^2) = 0 \quad \text{...(iv)}$$

Consequently $z = e^{i\phi}$

gives four values of ϕ, say $\alpha, \beta, \gamma, \delta$ (Here, sum of four angles)

$\therefore \quad z_1 . z_2 . z_3 . z_4 = -1$

$\Rightarrow \quad e^{i\alpha} \cdot e^{i\beta} \cdot e^{i\gamma} \cdot e^{i\delta} = -1$

$\Rightarrow \quad e^{i(\alpha + \beta + \gamma + \delta)} = -1$

$\cos(\alpha + \beta + \gamma + \delta) + i\sin(\alpha + \beta + \gamma + \delta) = -1$

or $\quad \cos(\alpha + \beta + \gamma + \delta) = -1$

and $\quad \sin(\alpha + \beta + \gamma + \delta) = 0$

$\alpha + \beta + \gamma + \delta = (2n + 1)\pi$

and $\quad \alpha + \beta + \gamma + \delta = r\pi$

where, $n\, r, \in I$

Hence, $\alpha + \beta + \gamma + \delta$ = odd multiple of π

2. If α, β, γ are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the normals at which are concurrent, then

$\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0$

Here, in each term sum of two eccentric angles

$\therefore$ From Eq. (iv),

$$\Sigma z_1 z_2 = 0$$

or $\quad z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 = 0$

$$\Rightarrow \quad e^{i(\alpha+\beta)} + e^{i(\alpha+\gamma)} + e^{i(\alpha+\delta)} + e^{i(\beta+\gamma)} + e^{i(\beta+\delta)} + e^{i(\gamma+\delta)} = 0$$

$$\Rightarrow \quad [\cos(\alpha + \beta) + \cos(\alpha + \gamma) + \cos(\alpha + \delta) + \cos(\beta + \gamma) + \cos(\beta + \delta) + \cos(\gamma + \delta)] + i[(\sin(\alpha + \beta) + \sin(\alpha + \gamma) + \sin(\alpha + \delta) + \sin(\beta + \gamma) + \sin(\beta + \delta) + \sin(\gamma + \delta)] = 0$$

Comparing the imaginary part, then

$$\sin(\alpha + \beta) + \sin(\alpha + \gamma) + \sin(\alpha + \delta) + \sin(\beta + \gamma) + \sin(\beta + \delta) + \sin(\gamma + \delta) = 0 \quad \text{...(v)}$$

Since, from property Eq. (i)

$\alpha + \beta + \gamma + \delta$ = odd multiple of π

$(\alpha + \delta)$ = odd multiple of $\pi - (\beta + \gamma)$

$(\beta + \delta)$ = odd multiple of $\pi - (\alpha + \gamma)$

$$(\gamma+\delta) = \text{odd multiple of } \pi - (\alpha+\beta)$$

$$\left.\begin{array}{l}\sin(\alpha+\delta) = \sin(\beta+\gamma)\\ \sin(\beta+\delta) = \sin(\alpha+\gamma)\\ \sin(\gamma+\delta) = \sin(\alpha+\beta)\end{array}\right\}$$

$$\{\because \sin(n\pi-\alpha) = \sin\alpha, \text{ if } n \text{ is integer}\} \quad \text{...(vi)}$$

From Eqs. (v) and (vi), we get

$$2\sin(\alpha+\beta) + 2\sin(\beta+\gamma) + 2\sin(\gamma+\alpha) = 0$$

Hence, $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$

Aliter :

From Eq. (iii), $\Sigma t_1 t_2 = 0$ ...(vii)

and $t_1 t_2 t_3 t_4 = -1$ ...(viii)

Now, $\Sigma t_1 t_2 = 0$

$$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = -t_4(t_1+t_2+t_3)$$

$$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = \frac{t_1+t_2+t_3}{t_1t_2t_3} \quad \{\text{from (viii)}\}$$

$$\Rightarrow \quad t_1t_2 + t_2t_3 + t_3t_1 = \frac{1}{t_2t_3} + \frac{1}{t_3t_1} + \frac{1}{t_1t_2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2}$$

$$= \cot\frac{\beta}{2}\cot\frac{\gamma}{2} + \cot\frac{\gamma}{2}\cot\frac{\alpha}{2} + \cot\frac{\alpha}{2}\cot\frac{\beta}{2}$$

$$\Rightarrow \quad \sum\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2} - \cot\frac{\alpha}{2}\cot\frac{\beta}{2}\right) = 0$$

$$\Rightarrow \quad \sum\left(\frac{\sin^2(\alpha/2)\sin^2(\beta/2) - \cos^2(\alpha/2)\cos^2(\beta/2)}{\sin(\alpha/2)\sin(\beta/2)\cos(\alpha/2)\cos(\beta/2)}\right) = 0$$

$$\Rightarrow \quad \sum -4\left(\frac{\{\cos(\alpha/2)\cos(\beta/2) + \sin(\alpha/2)\sin(\beta/2)\}\{\cos(\alpha/2)\cos(\beta/2) - \sin(\alpha/2)\sin(\beta/2)\}}{\sin\alpha\sin\beta}\right) = 0$$

$$\Rightarrow \quad \sum -4\left(\frac{\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\alpha\sin\beta}\right) = 0$$

$$\Rightarrow \quad \sum -2\frac{[\cos\alpha + \cos\beta]}{\sin\alpha\sin\beta} = 0$$

$$\Rightarrow \quad \sum \frac{\sin\gamma(\cos\alpha+\cos\beta)}{\sin\alpha\sin\beta\sin\gamma} = 0$$

$$\Rightarrow \quad \sum \sin\gamma(\cos\alpha+\cos\beta) = 0$$

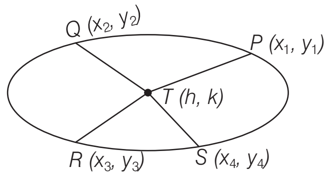
$$\Rightarrow \quad \sin\gamma(\cos\alpha+\cos\beta) + \sin\alpha(\cos\beta+\cos\gamma) + \sin\beta(\cos\gamma+\cos\alpha) = 0$$

$$\Rightarrow \quad \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$$

Co-normal Points Lie on a Fixed Curve

Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $T(x_4, y_4)$ be conormal points so that normal drawn from them meet in $T(h, k)$.

Then, equation of normal at $P(x_1, y_1)$ is



$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

or $\quad (a^2-b^2)\,x_1y_1 + b^2yx_1 - a^2xy_1 = 0$

but the point $T(h, k)$ lies on it

$$\therefore \quad (a^2-b^2)\,x_1y_1 + b^2kx_1 - a^2hy_1 = 0$$

Similarly, for points Q, R and S are

$$(a^2-b^2)x_2y_2 + b^2kx_2 - a^2hy_2 = 0$$

$$(a^2-b^2)\,x_3y_3 + b^2kx_3 - a^2hy_3 = 0$$

and $\quad (a^2-b^2)\,x_4y_4 + b^2kx_4 - a^2hy_4 = 0$

Hence, P, Q, R, S all lie on the curve

$$(a^2-b^2)\,xy + b^2kx - a^2hy = 0$$

This curve is called **Apollonian rectangular hyperbola**.

Remark

The feet of the normals from any fixed point to the ellipse lie at the intersections of the **Apollonian rectangular hyperbola** with the ellipse.

Exercise for Session 2

1. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$, is

(a) 0 (b) 1 (c) 2 (d) infinite

2. If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts off intercepts of length h and k on the axes, then $\frac{a^2}{h^2} + \frac{b^2}{k^2}$ is equal to

(a) – 1 (b) 0 (c) 1 (d) None of these

3. The equations of the tangents to the ellipse $3x^2 + y^2 = 3$ making equal intercepts on the axes are

(a) $y = \pm x \pm 2$ (b) $y = \pm x \pm 4$ (c) $y = \pm x \pm \sqrt{30}$ (d) $y = \pm x \pm \sqrt{35}$

4. If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then its, eccentric angle θ is equal to

(a) 0 (b) 45° (c) 60° (d) 90°

5. The number of values of $\theta \in [0, 2\pi]$ for which the line $2x \cos\theta + 3y \sin\theta = 6$ touches the ellipse $4x^2 + 9y^2 = 36$ is

(a) 1 (b) 2 (c) 4 (d) infinite

6. The common tangent of $x^2 + y^2 = 4$ and $2x^2 + y^2 = 2$ is

(a) $x + y + 4 = 0$ (b) $x - y + 7 = 0$ (c) $2x + 3y + 8 = 0$ (d) None of these

7. If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes in G and g respectively, then $PG \cdot Pg =$

(a) $a : b$ (b) $a^2 : b^2$ (c) $b : a$ (d) $b^2 : a^2$

8. Number of distinct normal lines that can be drawn to the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$, from the point $(0, 6)$ is

(a) one (b) two (c) three (d) four

9. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

(a) 4 (b) 2 (c) 1 (d) None of these

10. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$, intersect it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

11. The line $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If '$\theta$' be eccentric angle of the foot of this normal, then 'θ' is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

12. If the tangent drawn at point $(\lambda^2, 2\lambda)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of λ and θ.

13. If the normal at any point P of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, meets the major and minor axes in G and H respectively and C in the centre of the ellipse, then prove that

$$a^2 (CG)^2 + b^2 (CH)^2 = (a^2 - b^2)^2$$

14. If the normal at the point $P(\theta)$ to the ellipse $5x^2 + 14y^2 = 70$ intersects it again at the point $Q(2\theta)$, show that $\cos\theta = -\frac{2}{3}$.

15. The tangent and normal at any point P of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut its major axis in point Q and R respectively. If $QR = a$ prove that the eccentric angle of the point P is given by

$$e^2 \cos^2\phi + \cos\phi - 1 = 0$$

Answers

Exercise for Session 2

1. (c) **2.** (c) **3.** (a) **4.** (b) **5.** (d)
6. (d) **7.** (d) **8.** (c) **9.** (a) **10.** (b)
11. (b) **12.** $\lambda = -\frac{1}{\sqrt{5}}$; $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$