M A T H E M A T I C S

FUNDAMENTALS OF MATHEMATICS

INTRODUCTION TO LOGARITHM

NOTES

What you already know

Exponents.

What you will learn

Introduction to logarithm.

Recap of Exponents

Solve for x: $2^x = 8$	**Solve for x:** $3^x = 81$
Step 1: In $2^x = 8$, 2 is the base, x is the exponent and 8 is the number.	**Step 1:** In $3^x = 81$, 3 is the base, x is the exponent and 81 is the number.
Step 2: $8 = 2 \times 2 \times 2$ implies $8 = 2^3$ Here 2 is the base, 3 is the exponent and 8 is the number.	**Step 2:** $81 = 3 \times 3 \times 3 \times 3 \Rightarrow 81 = 3^4$ Here 3 is the base, 4 is the exponent, and 81 is the number.
Step 3: $2^x = 8 \Rightarrow x = 3$ (as $2^3 = 8$)	**Step 3:** $3^x = 81 \Rightarrow x = 4$ (as $3^4 = 81$)
Step 4: $2^x = 2^3$ (as $8 = 2^3$) $\Rightarrow x = 3$	**Step 4:** $3^x = 3^4$ (as $81 = 3^4$) $\Rightarrow x = 4$

The logarithm of an INPUT ‘N’ to the BASE ‘a’

We calculate the logarithm of a certain quantity with respect to a base.
Logarithmic operator on an input ‘N’ with respect to the base ‘a’, is denoted by the notation ‘$log_a (N)$’ which is pronounced as the logarithm of the input ‘N’ to the base ‘a’ or log ‘N’ to the base ‘a’.

Relation between log and exponent -
$\log_a N = \text{Exponent} \Leftrightarrow a^{\text{Exponent}} = N$, i.e., when base ‘a’ is raised to the power “exponent”, it gives the number “N”

What is the logarithm of 8 to the base 2, i.e. $\log_2 8$?

Let $log_2 8 = x$
Since, $log_a N = Exponent \Leftrightarrow a^{Exponent} = N$
$log_2 8 = x \Rightarrow 2^x = 8$

$2^x = 2^3$ (as $8 = 2^3$)
As $2^3 = 8 \Rightarrow x = 3$

What is the logarithm of 81 to the base 3, i.e. $log_3 81$?

Let $log_3 81 = x$

Since, $log_a N = Exponent \Leftrightarrow a^{Exponent} = N$

$log_3 81 = x \Rightarrow 3^x = 81$

$3^x = 3^4$ (as $81 = 3^4$)

As $3^4 = 81 \Rightarrow x = 4$

If we need to find out '$log_a N$', consider, $\log_a N = k \Leftrightarrow a^k = N$

Concept Check

(1) $log_{1000} 100 = ?$

(a) -1 (b) $\frac{2}{3}$ (c) $\frac{-5}{4}$ (d) 2

(2) $log_{0.01} 10000 = ?$

(a) -2 (b) -1 (c) 2 (d) 1

(3) $log_{0.01} 0.0001 = ?$

(a) -1 (b) 1 (c) 2 (d) - 2

(4) $log_{2-\sqrt{3}} (2 + \sqrt{3}) = ?$

(a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 2

(5) $log_{\frac{1}{3}} (3\sqrt{3}) = ?$

(a) 1.5 (b) -1.5 (c) $\frac{3}{5}$ (d) 3

Restrictions on exponent, base, and input for the logarithm

$\text{Log}_a N = k \Longleftrightarrow a^k = N$

Logarithmic Equation ($\text{Log}_a N = k$) — Exponential Equation ($a^k = N$)

(i) $k \in R$ (ii) $a > 0$ and $a \neq 1$ (iii) $N > 0$

If a = 0,

$0^k = 0 = N \Rightarrow log_0 0 = k$, i.e, '$log_0 0$' has infinitely many solutions, which is not possible. Moreover, if $a = 0$, say $log_0 2 = m \Rightarrow 0^m = 2$, which is not possible.
Therefore, base a ≠ 0.

If $a = 1$,

$1^k = 1 = N \Rightarrow log_1 1 = k$, i.e., '$log_1 1$' has infinitely many solutions, which is not possible. Moreover, if a = 1, then $\log_2 1 = m \Rightarrow 1^m = 2$

If $a < 0$, for example, $a = -2$ (say), then $(-2)^{\frac{1}{2}} = N$ (say $k = \frac{1}{2}$) $\Rightarrow \sqrt{-2} = N \Rightarrow N \notin R$, which is not possible. Therefore, base $a \nless 0$.

For valid base ($a > 0$ and $a \neq 1$), a^k is always positive for $k < 0$, $k = 0$ and $k > 0$. So, $a^k = N > 0$.
Therefore, the logarithm of only strictly positive quantities exists. The logarithm of '0' and logarithm of negative quantity do not exist.

If $log_{(1-x)}(5 + x) = 2$, then find the value of x.

For $\log_n$ to be defined — $a > 0$ & $a \neq 1$

$\log_{(1-x)}(5 + x) = 2$

$\Rightarrow (5 + x) = (1 - x)^2$

$\Rightarrow 5 + x = 1 + x^2 - 2x$

$\Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0$

$\Rightarrow x = -1, 4$

When $x = 4$, $1 - x = -3 \Rightarrow$ Not possible.

When $x = -1$, $1 - x = 2$ and $5 + x = 4$

$\Rightarrow x = -1$ is the only solution.

Find the value of $log_{(tan\ 45°)}(sin^2\ 30° + cos^2\ 30°)$

For $\log_a n$ to be defined — $a > 0$ & $a \neq 1$

$log_{tan45°}(sin^2 30° + cos^2 30°)$

Base $a = tan45° = 1$

As $a \neq 1$, this logarithm does not exist.

Commonly used bases

$\log_a N = k \Leftrightarrow a^k = N$

Logarithm to the base 10 is called a decadic logarithm, common logarithm, or Briggs logarithm. log_{10} N is used for calculating values of huge multiplications or divisions. Mathematically, $y = log_{10}(x) = log(x)$.

The logarithm with base 'e = 2.718281828459...' (non-repeating, non-terminating, irrational number, strictly sandwiched between 2 and 3) is called Napier or natural logarithm. log_e N is used in theoretical applications. Mathematically, $y = log_e(x) = ln(x)$.

Note

log (*x*) and *ln* (*x*) are very different. $log(x) = log_{10}(x)$ and $ln(x) = log_e(x)$. If we change the base of the logarithm operator, then the value will drastically change.

As an example, $\log_4(4) = k \Rightarrow 4^k = 4 \Rightarrow k = 1 \Rightarrow \log_4(4) = 1$ and

$\log_2(4) = k \Rightarrow 2^k = 4 \Rightarrow k = 2 \Rightarrow \log_2(4) = 2$

Properties of Logarithm

Property i

$a > 0, a \neq 1$

$log_a 1 = 0$

$log_a N = k \Leftrightarrow a^k = N$

Proof: Let $log_a 1 = k$

$a^k = 1$

$k = 0$

$log_a 1 = 0$

Hence Proved

Property ii

a > 0 and a ≠ 1

$\log_a a = 1$

$\log_a N = k \Leftrightarrow a^k = N$

Proof:

Let $\log_a a = k$

$\Leftrightarrow a^k = a$

$\Rightarrow k = 1$

$\log_a a = 1$

Hence Proved

Property iii

a > 0 and a ≠ 1 and m, n > 0

$\log_a m + \log_a n = \log_a (mn)$

$\log_a N = k \Leftrightarrow a^k = N$

$a^k = N \Leftrightarrow \log_a N = k$

Proof:

Consider,

$\log_a m = k_1$ and $\log_a n = k_2$

LHS $\log_a m + \log_a n$

k_1 k_2

$\Rightarrow a^{k_1} = m$ $a^{k_2} = n$

$\Rightarrow mn = a^{k_1} \times a^{k_2}$

or $\Rightarrow mn = a^{k_1 + k_2}$

$\log_a mn = k_1 + k_2$

$\log_a mn = \log_a m + \log_a n$

Hence Proved

Which of the following is equal to $1 + \log_4 3$?

(a) $\log_{12} 3$

(b) $\log_3 12$

(c) $\log_4 12$

(d) $\log_5 15$

Solution: We have,

$1 + \log_4 3$

$= \log_4 4 + \log_4 3$

$= \log_4 (4 \times 3)$

$= \log_4 12$

$\log_a a = 1$

Corollary of Property iii

Corollary means derived from proof of a theorem or property.

Corollary 1
property - iii (valid for finitely many inputs)

$\log_a (m_1) + \log_a (m_2) + \ldots + \log_a (m_r) =$

$\log_a (m_1 \times m_2 \times \ldots . \times m_r)$,

where $m_1, m_2, \ldots, m_r > 0$ and $a > 0$ & $a \neq 1$

Corollary 2

$\log_a (m) + \log_a (n) = \log_a (mn)$,

where m, n > 0 and a > 0 & a ≠ 1

If m = n, then $\log_a (m) + \log_a (m) = \log_a (m^2)$

$2\log_a (m) = \log_a (m^2) \Rightarrow \log_a (m^2) = 2\log_a (m)$

Evaluate: $\log_{11}\left(1-\frac{1}{3}\right)+\log_{11}\left(1-\frac{1}{4}\right)+\log_{11}\left(1-\frac{1}{5}\right)+....+\log_{11}\left(1-\frac{1}{242}\right)$

(a) -2 (b) -1 (c) 1 (d) 2

Solution:

$\log_a m_1 + \log_a m_2 + ... + \log_a m_r = \log_a(m_1 \times m_2 \times ... \times m_r)$

$\text{LHS} = \log_{11}\left(\left(1-\frac{1}{3}\right)\times\left(1-\frac{1}{4}\right)\times\left(1-\frac{1}{5}\right)\times ....\times\left(1-\frac{1}{242}\right)\right)$

$= \log_{11}\left(\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times ....\times\frac{241}{242}\right)$

$= \log_{11}\frac{2}{242}$

$= \log_{11}\left(\frac{1}{121}\right)$

$= \log_{11}\left(\frac{1}{11^2}\right)$

$= \log_{11}(11^{-2})$

$= -2\,(\log_{11}11)$

$= -2 \times 1$

$= -2$

$\log_a N = k \Leftrightarrow N = a^k$

$\log_a a = 1$

Property iv

$\log_a m^\beta = \beta \log_a m;\ \beta \in \mathbb{R} - \{0\}$

Proof: Consider,

$\log_a m^\beta = k$

$\Rightarrow a^k = m^\beta$

$\Rightarrow (a^k)^{\frac{1}{\beta}} = (m^\beta)^{\frac{1}{\beta}}$

$\Rightarrow (a^k)^{\frac{1}{\beta}} = m$

$\Rightarrow a^{\frac{k}{\beta}} = m$

$\Rightarrow \log_a m = \frac{k}{\beta}$

$\Rightarrow \beta \log_a m = k = \log_a m^\beta$

$\therefore \log_a m^\beta = \beta \log_a m$

Hence proved

$\log_a N = k \Leftrightarrow a^k = N$

Property v

$\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$

Where m, n > 0 & a > 0, a ≠ 1

Proof: Consider,

$\log_a m - \log_a n$

$\downarrow \quad \downarrow$

$k_1 \quad k_2$

$\log_a N = k \Leftrightarrow a^k = N$

$\Rightarrow a^{k_1} = m \Rightarrow a^{k_2} = n$

$\Rightarrow \frac{a^{k_1}}{a^{k_2}} = \frac{m}{n}$

$\Rightarrow a^{k_1 - k_2} = \frac{m}{n}$

$\Rightarrow \log_a \frac{m}{n} = k_1 - k_2$

$\Rightarrow \log_a \frac{m}{n} = \log_a m - \log_a n$

Hence proved

Property vi

$\log_{a^\beta} m = \frac{1}{\beta}\log_a m;\ \beta \in \mathbb{R} - \{0\}$

Proof: Let, $\log_{a^\beta} m = k$

$\Rightarrow (a^\beta)^k = m$

$\Rightarrow a^{\beta k} = m$

$\Rightarrow \log_a m = \beta k$

$\Rightarrow \frac{1}{\beta}\log_a m = k = \log_{a^\beta} m$

$\therefore \log_{a^\beta} m = \frac{1}{\beta}\log_a m$

Hence proved

If $\log_7 2 = m$ then, $\log_{49} 28 = ?$

(a) $m(m+1)$ (b) $\frac{(m+1)(m-1)}{2}$ (c) $\frac{1+2m}{2}$ (d) $\frac{2m-1}{2}$

Solution: Consider,

$$\log_{49} 28 = \log_{7^2}(28)$$

$$= \frac{1}{2}\log_7(28)$$

$$= \frac{1}{2}\log_7(7 \times 4)$$

$$= \frac{1}{2}(\log_7 7 + \log_7 4)$$

$$= \frac{1}{2}(1 + \log_7 4)$$

$$= \frac{1}{2}(1 + \log_7 2^2)$$

$$= \frac{1}{2}(1 + 2m) = \frac{1+2m}{2}$$

$\log_{a^\beta} m = \frac{1}{\beta}\log_a m$

$\log_a mn = \log_a m + \log_a n$

$\log_a a = 1$

$\log_a m^\beta = \beta \log_a m$

Property vii

$$\log_a m = \frac{\log_b m}{\log_b a};\ a, b > 0\ \&\ m > 0,\ b \neq 1$$

Proof: Let, $\log_a m = k$

$\Rightarrow a^k = m$

$\Rightarrow \log_b a^k = \log_b m$

$\Rightarrow k \log_b a = \log_b m$

$\Rightarrow \frac{\log_b m}{\log_b a} = k$

$\Rightarrow \log_a m = \frac{\log_b m}{\log_b a}$

Hence proved

Note: $\log_a b\ \log_b c\ \log_c d = \log_a d$

Property viii

$$\log_a m = \frac{1}{\log_m a};\ m > 0\ \&\ a > 0\ a \neq 1$$

Proof: $\log_a m = \frac{\log_b m}{\log_b a}$ (Change of base formula)

Taking base b = m

$\Rightarrow \log_a m = \frac{\log_m m}{\log_m a} = \frac{1}{\log_m a}$

If $\log_5 a \times \log_a x = 2$ then, $x = ?$

(a) 15 (b) $\frac{3}{2}$ (c) 25 (d) 2

Solution: Consider,

$\Rightarrow \log_5 a \times \log_a x = 2$

$\Rightarrow \log_5 a \times \frac{\log_5 x}{\log_5 a} = 2$

$\Rightarrow \log_5 x = 2 \Rightarrow 5^2 = x$

$\Rightarrow x = 25$

$\log_a m = \frac{\log_b m}{\log_b a}$

Property ix

$a^{\log_a x} = x$

Proof: Let, $\log_a x = k$

$\Rightarrow a^k = x$

$\Rightarrow a^{\log_a x} = x$

Hence proved

Property x

$a^{\log_b c} = c^{\log_b a}$

$(a \longleftrightarrow c)$

Proof: Let, $a^{\log_b c} = k$

take $\log_b$ both sides:

$\log_a m^\beta = \beta \log_a m$

$\Rightarrow \log_b(a^{\log_b c}) = \log_b k$

$\Rightarrow (\log_b c)(\log_b a) = \log_b k$

$\Rightarrow (\log_b a)(\log_b c) = \log_b k$

$\Rightarrow \log_b(c^{\log_b a}) = \log_b k$

$\Rightarrow c^{\log_b a} = k$

$\Rightarrow c^{\log_b a} = a^{\log_b c}$

Hence proved

Summary

Key Takeaways

$\log_a N = k \Leftrightarrow a^k = N$

Logarithmic Equation — *Exponential Equation*

- (i) $k \in R$; (ii) $a > 0$ and $a \neq 1$; (iii) $N > 0$
- $y = \log_{10}(x) = \log(x)$
- $y = \log_e(x) = \ln(x)$

Key Results:

1. $\log_a 1 = 0$; $a > 0$ & $a \neq 1$

2. $\log_a a = 1$; $a > 0$ & $a \neq 1$

3. $\log_a m + \log_a n = \log_a(mn)$

   Where $m, n > 0$ & $a > 0, a \neq 1$

4. $\log_a m^\beta = \beta \log_a m$; $\beta \in \mathbb{R} - \{0\}$

5. $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$

   Where $m, n > 0$ & $a > 0, a \neq 1$

6. $\log_{a^\beta} m = \frac{1}{\beta} \log_a m$; $\beta \in \mathbb{R} - \{0\}$

7. Change of base formula

   $\log_a m = \frac{\log_b m}{\log_b a}$; $a, b > 0$ & $a, b \neq 1$

8. $\log_a m = \frac{1}{\log_m a}$; $m > 0$ & $m \neq 1$

9. $a^{\log_a x} = x$

10. $a^{\log_b c} = c^{\log_b a}$

    $(a \longleftrightarrow c)$

11. **Corollary 1 of property - 3 (valid for finitely many inputs)**

    $\log_a(m_1) + \log_a(m_2) + \ldots + \log_a(m_r)$

    $= \log_a(m_1 \times m_2 \times \ldots. \times m_r)$

Mind map

Logarithm

- $\log_a N = k \Leftrightarrow a^k = N$ where (i) $N > 0$; (ii) $a > 0$ and $a \neq 1$; (iii) $k \in R$
- Common log (base 10): log x
  Natural log (base e): ln x
- **Important Properties**
  - $\log_a m + \log_a n = \log_a(mn)$
    $m, n > 0,\ a > 0,\ a \neq 1$
  - $\log_a m^\beta = \beta \times \log_a m,\ \beta \in \{0\}$
  - $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$
    $m, n > 0,\ a > 0,\ a \neq 1$
  - $\log_{a^\beta}(m) = \frac{1}{\beta}\log_a m;\ \beta \in R - \{0\}$
  - Change of base:
    $\log_a m = \frac{\log_b m}{\log_b a}$; $a, b > 0$ & $a, b \neq 1$
  - Fundamental Property
    $a^{\log_a x} = x$
  - $a^{\log_b c} = c^{\log_b a}$

Self-Assessment

1. Express $6^3 = 216$ in logarithm form.

2. $\frac{\log\sqrt{8}}{\log 8}$ is equal to

   (a) $\frac{1}{\sqrt{8}}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

3. If log 9 = a, then $\log\left(\frac{1}{90}\right)$ is equal to

   (a) $-(1 + a)$ (b) $(1 + a)^{-1}$ (c) $\frac{a}{10}$ (d) $\frac{1}{10a}$

Answer

Concept Check:

1. Let $\log_{1000}100 = k$

$\Rightarrow 1000^k = 100$

$\Rightarrow (10^3)^k = 10^2$

$\Rightarrow 10^{3k} = 10^2$

$\Rightarrow 3k = 2$

$\Rightarrow k = \frac{2}{3}$

2. Let $\log_{0.01}10000 = k$

$\Rightarrow (0.01)^k = 10000$

$\Rightarrow \left(\frac{1}{100}\right)^k = 10000$

$\Rightarrow (10)^{-2k} = (10)^4$

$\Rightarrow -2k = 4 \Rightarrow k = -2$

3. Let $\log_{0.01}0.0001 = k$

$\Rightarrow (0.01)^k = 0.0001$

$\Rightarrow \left(\frac{1}{100}\right)^k = \frac{1}{10000}$

$\Rightarrow (10)^{-2k} = (10)^{-4}$

$\Rightarrow -2k = -4 \Rightarrow k = 2$

4. Let $\log_{2-\sqrt{3}}(2 + \sqrt{3}) = k$

$\Rightarrow (2 - \sqrt{3})^k = 2 + \sqrt{3}$

Rationalize

$\Rightarrow (2 - \sqrt{3})^k = \frac{(2 + \sqrt{3})(2 - \sqrt{3})}{(2 - \sqrt{3})}$

$\Rightarrow (2 - \sqrt{3})^k = \frac{1}{(2 - \sqrt{3})}$

$\Rightarrow (2 - \sqrt{3})^k = (2 - \sqrt{3})^{-1}$

$\Rightarrow k = -1$

5. Let, $\log_{\frac{1}{3}}3\sqrt{3} = k$

$\Rightarrow \left(\frac{1}{3}\right)^k = 3\sqrt{3}$

$\Rightarrow (3)^{-k} = (3)^{\frac{3}{2}}$

$\Rightarrow k = -\frac{3}{2}$

Self-Assessment

1. $6^3 = 216$

As we know $a^k = N \Leftrightarrow \log_a N = k$

$\log_6 216 = 3$

2. Option (C)

$\frac{\log\sqrt{8}}{\log 8} = \frac{\log\sqrt{8}}{\log 8} = \frac{1}{2}$

3. Option (a)

$\log\left(\frac{1}{90}\right) = \log 1 - \log 90$

$= 0 - \log(9 \times 10)$

$= -(\log 9 + \log 10)$

$= -(a + 1)$

FUNDAMENTALS OF MATHEMATICS

LOGARITHMIC FUNCTION

What you already know

What is logarithm?
Properties of logarithm

What you will learn

Logarithmic function,
Properties of logarithmic function

Logarithmic Function

The function **f(x) = $\log_a$x** where **x > 0** and '**a**' is a positive real number other than '**1**' is Known as a **logarithmic function**.

Base restrictions: $a > 0$ and $a \neq 1$. So, we get two logarithmic functions for a valid base.
(i) $\log_a(x)$ when $a > 1$ and (ii) $\log_a(x)$ when $0 < a < 1$

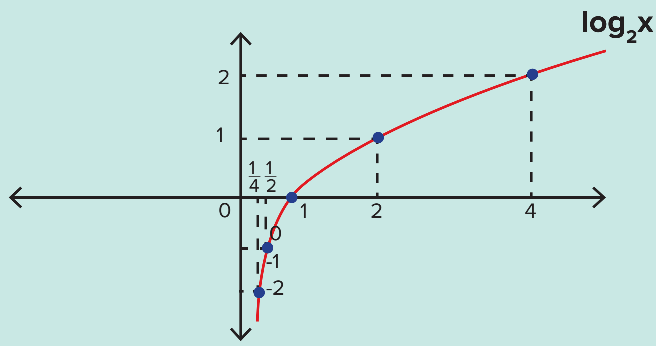
The behaviour of logarithmic functions is based on the base 'a'. The logarithmic functions $f(x)= \log_a(x)\ \forall\ x \in R^+$ when $0 < a < 1$ and $\log_a(x)\ \forall\ x \in R^+$ when $a > 1$ behave completely differently.

Characteristics of the graph of the function $f(x)= \log_a(x)\ \forall\ x \in \mathbb{R}^+$ and $a > 1$

Example: $f(x)= \log_2 x$

x	1	2	4	$\frac{1}{2}$	$\frac{1}{4}$
f(x)	0	1	2	-1	-2

Graph



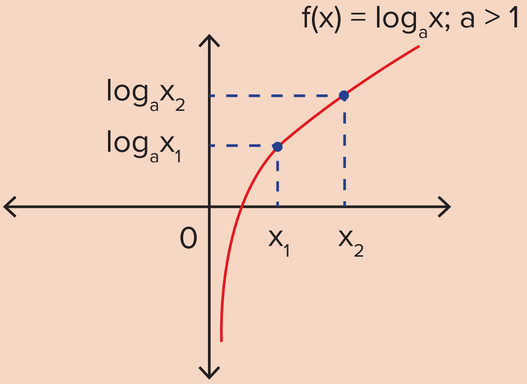
The following points can be inferred from the graph

1. $f(x) = \log_a(x) \in \mathbb{R}$ iff $x > 0$. $\log_a(0) \notin \mathbb{R}$ and $\log_a(-ve) \notin \mathbb{R}$

2. $f(x) = \log_a(x)$ with $a > 1$ is a continuous function.

3. Graph intersects the X-axis only at $x = 1$, i.e. $\log_a(x) = 0 \Leftrightarrow x = 1$

4.



If $x_2 > x_1$

$\Leftrightarrow \log_a x_2 > \log_a x_1$

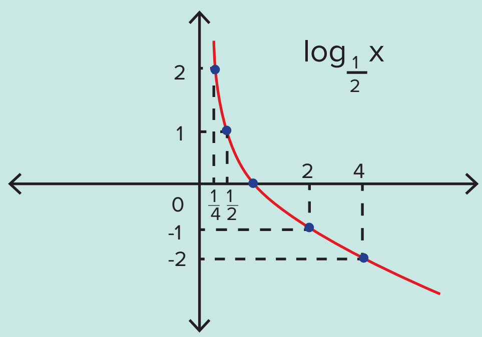
$\therefore \log_a x;\ a > 1$ is a strictly increasing function.

Characteristics of graph of the function $f(x) = \log_a(x)\ \forall\ x \in \mathbb{R}^+$ and $0 < a < 1$

Example: $f(x) = \log_{\frac{1}{2}} x$

x	1	2	4	$\frac{1}{2}$	$\frac{1}{4}$
f(x)	0	-1	-2	1	2

Graph



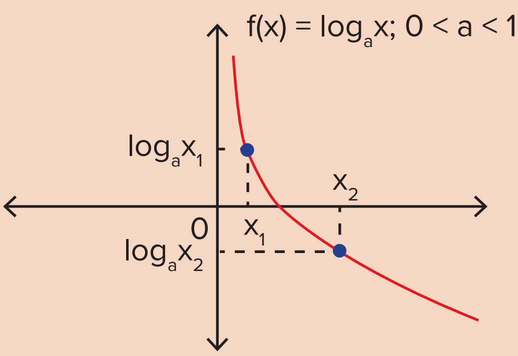
The following points can be inferred from the graph

1. $f(x) = \log_a(x) \in \mathbb{R}$ iff $x > 0$. $\log_a(0) \notin \mathbb{R}$ and $\log_a(\text{-ve}) \notin \mathbb{R}$

2. $f(x) = \log_a(x)$ with $0 < a < 1$ is a continuous function

3. Graph intersects the x-axis only at x = 1, i.e. $\log_a(x) = 0 \Leftrightarrow x = 1$

4.



If $x_2 > x_1$

$\Leftrightarrow \log_a x_2 < \log_a x_1$

$\therefore \log_a x;\ 0 < a < 1$ is a strictly decreasing function.

Properties for $f(x) = \log_a(x)$, (When base $a > 1$)

Property I

$\log_a(x) > 0$ iff $x > 1$

Property II

$\log_a(x) < 0$ iff $0 < x < 1$

Property III

$\log_a(x) = 0$ iff $x = 1$

Property IV

$\log_a(x) = 1$ iff $x = a$

Property V

$\log_a(x) > 1$ iff $x > a$

Property VI

$0 < \log_a(x) < 1$ iff $1 < x < a$

Property VII

$\log_a(x) < 0$ iff $0 < x < 1$

Property VIII

$\log_a(x) > k \Leftrightarrow x > a^k$

Properties for $f(x) = \log_a(x)$, (When base $0 < a < 1$)

Property I

$\log_a(x) > 0$ iff $0 < x < 1$

Property II

$\log_a(x) < 0$ iff $x > 1$

Property III

$\log_a(x) = 0$ iff $x = 1$

Property IV

$\log_a(x) = 1$ iff $x = a$

Property V

$\log_a(x) > 1$ iff $0 < x < a$

Property VI

$0 < \log_a(x) < 1$ iff $a < x < 1$

Property VII

$\log_a(x) < 0$ iff $x > 1$

Property VIII

$\log_a(x) > k \Leftrightarrow 0 < x < a^k$

For $f(x) = \log_a(x)$, $\forall\, x \in R^+$

Domain = $(0, \infty) = R^+$
Range = R
Continuous in its domain

Graph intersects x-axis at (1, 0)
$\log_a x = 0$ has only one solution ($x = 1$)

$\log_{\frac{1}{5}}(x - 3) > 2$, then solve for x.

Solution:

We have $\log_{\frac{1}{5}}(x - 3) > 2$

$0 < x - 3 < (\frac{1}{5})^2$ $[\log_a x > k \Leftrightarrow 0 < x < a^k]$

$3 < x < 3 + (\frac{1}{5})^2$

$3 < x < 3 + \frac{1}{25}$

$3 < x < \frac{76}{25}$

$\therefore$ Solution set: $x \in (3, \frac{76}{25})$

Solve for x, if $\log_{(x^2+3)} 2 > 1$

Solution:

Now $x^2 \geq 0 \ \forall \ x \in \mathbb{R}$

$\Rightarrow x^2 + 3 \geq 3 \ \forall \ x \in \mathbb{R}$

$\therefore$ Base $= x^2 + 3 > 1 \ \forall \ x \in \mathbb{R}$

$\log_a x > 1 \Leftrightarrow x > a; a > 1$

$\therefore \log_{(x^2+3)} 2 > 1$

$\Rightarrow 2 > x^2 + 3$

$\Rightarrow x^2 < -1$

(Not possible for any $x \in \mathbb{R}$)

$\therefore$ No solution

Concept Check

1. Find the value of x, if $\log_3 (2 - x) > 4$
2. Solve for x : $\log_x (6 - x) > 2$
3. If $x > 0$ and $\log_{(\frac{x}{x+1})} x^2 \geq \log_{(\frac{x}{x+1})} (2x + 3)$, then how many integral values of x exist?
   (a) 0 (b) 1 (c) 2 (d) 3

Summary Sheet

Key Takeaways

Logarithmic function: $f(x) = \log_a x : x \in R^+, a > 0 \ a \neq 1$

$f(x) = \log_a x \ \forall \ x \in R^+$ and $a > 1$

(i) Continuous function

(ii) Strictly increasing function

(iii) $\log_a x = 0 \Leftrightarrow x = 1$

(iv) If $x_2 > x_1$, $\Leftrightarrow \log_a (x_2) > \log_a (x_1)$

$f(x) = \log_a x \ \forall \ x \in R^+$ and $0 < a < 1$

(i) Continuous function

(ii) Strictly decreasing function

(iii) $\log_a x = 0 \Leftrightarrow x = 1$

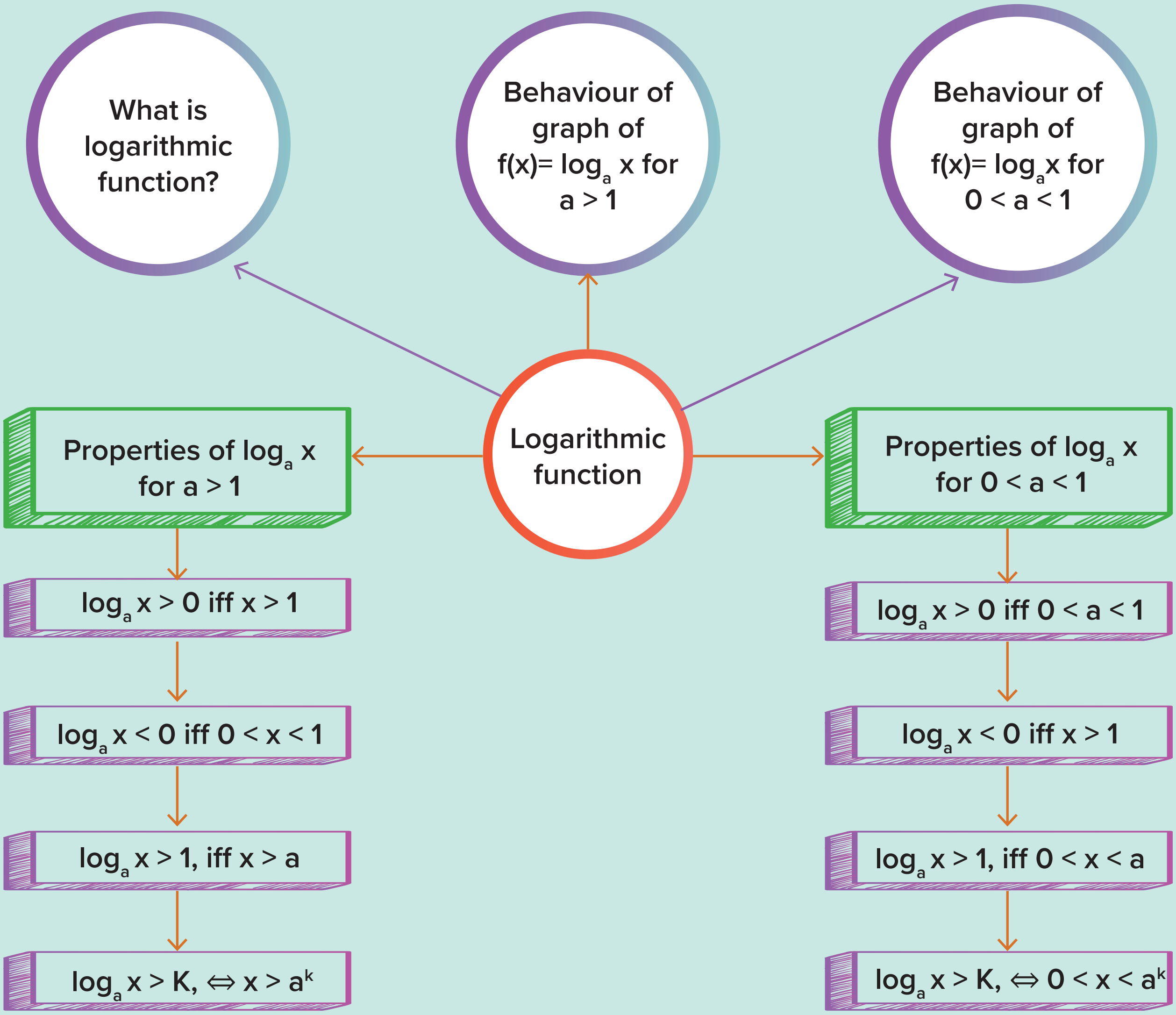
(IV) If $x_2 > x_1$, $\Leftrightarrow \log_a (x_2) < \log_a (x_1)$

Key Results

1. For $f(x) = \log_a(x)$, $a > 1$
   - $\log_a(x) > 0$ (strictly positive) if $x > 1$
   - $\log_a(x) < 0$ (strictly negative) if $0 < x < 1$
   - $\log_a(x) > 1$ iff $x > a$
   - $0 < \log_a(x) < 1$ iff $1 < x < a$
   - $\log_a(x) > k \Leftrightarrow x > a^k$

2. For $f(x) = \log_a(x)$, $0 < a < 1$
   - $\log_a(x) > 0$ (strictly positive) if $0 < x < 1$
   - $\log_a(x) < 0$ (strictly negative) if $x > 1$
   - $\log_a(x) > 1$ iff $0 < x < a$
   - $0 < \log_a(x) < 1$ iff $a < x < 1$
   - $\log_a(x) > k \Leftrightarrow 0 < x < a^k$

Mind Map



Self-Assessment

1. What values of x satisfy the following inequality?

$$\log_3(3x + 4) > \log_3(4x)$$

2. What are the values satisfying the given inequality?

$$\log_{\frac{1}{2}}(x + 2) > \log_{\frac{1}{4}}(x^2)$$

3. Find the values of x satisfying the following inequality.

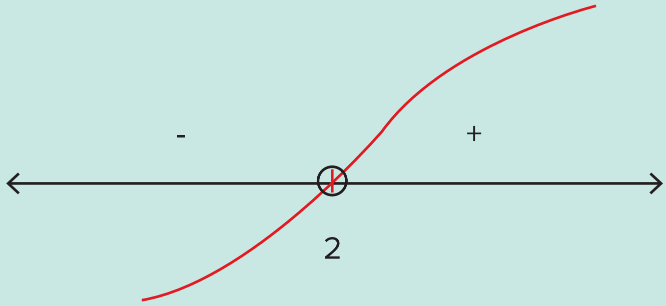
$$\log_{(x^2 + 2)}(12 - 3x) < 1$$

Answers

Concept Check:

1. We have, $\log_3 (2 - x) > 4$

(i) **$\log_a N$ exists if $N > 0$**

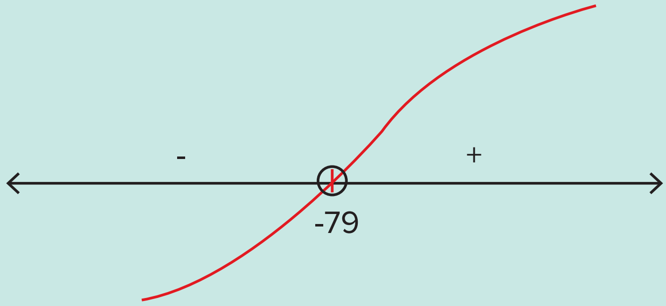


$\therefore 2 - x > 0$

or $x < 2$

i.e. $x \in (-\infty, 2) = A$

(ii) **$\log_a x > k \Leftrightarrow x > a^k;\ a > 1$**



$\therefore \log_3 (2 - x) > 4$

$\Rightarrow (2 - x) > 3^4$

$\Rightarrow 2 - x > 81$

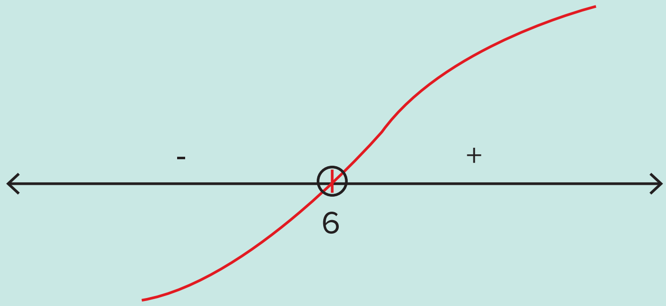
$\Rightarrow x < -79$

$\Rightarrow x \in (-\infty, -79) = B$

$\therefore A \cap B = (-\infty, -79) =$ **Solution set**

2. We have $\log_x (6 - x) > 2$.

(i) **$\log_a N$ exists iff $N > 0$**



$\therefore 6 - x > 0$

$\Rightarrow x < 6$

$\Rightarrow x \in (-\infty, 6)$

(ii) **$\log_a x$ exists iff $a > 0$ and $a \neq 1$**

$\therefore x > 1$ or $0 < x < 1$

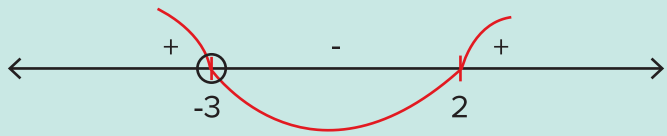
From (i) and (ii), $x \in (0, 1) \cup (1, 6)$

Case I: $x > 1$

$\log_a x > k$ iff $x > a^k;\ a > 1$

$\therefore \log_x (6 - x) > 2$

$\Rightarrow 6 - x > x^2$



$\Rightarrow x^2 + x - 6 < 0$

$\Rightarrow (x - 2)(x + 3) < 0$

$\Rightarrow x \in (-3, 2)$

$\Rightarrow x \in (1, 2)$

Case II: $0 < x < 1$

$\log_a x > k$ iff $x < a^k$; $a < 1$

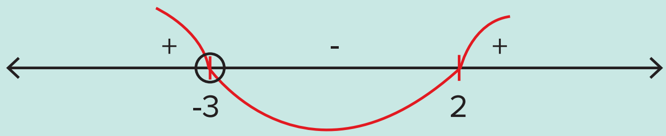
$\therefore \log_x (6 - x) > 2$

$\Rightarrow 6 - x < x^2$

$\Rightarrow x^2 + x - 6 > 0$

$\Rightarrow (x - 2)(x + 3) > 0$

$\Rightarrow x \in \Phi$ ($\because 0 < x < 1$)



$\therefore$ **Case I $\cup$ Case II:**

$(1, 2) \cup \Phi = (1, 2)$

$\therefore$ **Solution set:** $x \in (1, 2)$

3. Firstly, $x > 0 \Rightarrow x + 1 > 0$

$\therefore 0 < \dfrac{x}{x+1} < 1$

$\Rightarrow 0 <$ Base < 1

$f(x) = \log_a x$: $0 < a < 1$ is a strictly decreasing function
$\log_a x_1 \geq \log_a x_2$
$\Rightarrow x_1 \leq x_2$

$\therefore \log_{\frac{x}{x+1}} (x^2) \geq \log_{\frac{x}{x+1}} (2x + 3)$

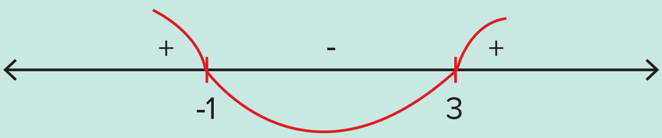
$\Rightarrow x^2 \leq 2x + 3$

$\Rightarrow x^2 - 2x - 3 \leq 0$

$\Rightarrow (x - 3)(x + 1) \leq 0$

$\Rightarrow x \in (0, 3]$ = Final solution set

$\Rightarrow x = 1, 2, 3$



Self-Assessment

1. For the inequation to be valid,

$3x + 4 > 0$ and $4x > 0$

$\Rightarrow x > \dfrac{-4}{3}$ and $x > 0$

$\therefore x > 0 \Rightarrow x \in (0, \infty)$

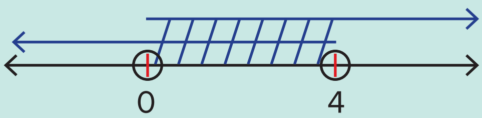
$\because$ Base is 3 which is > 1.

$\Rightarrow (3x + 4) > 4x$ (as $\log_a x_1 > \log_a x_2 \Leftrightarrow x_1 > x_2$ for $a > 1$)

$\Rightarrow 4x - 3x < 4 \Rightarrow x < 4$

$\therefore x \in (-\infty, 4)$

$\therefore$ Final solution is $(0, \infty) \cap (-\infty, 4)$



$x \in (0, 4)$

2. Given $\log_{\frac{1}{2}}(x+2) > \log_{\frac{1}{4}}(x^2)$

This inequation is valid only when, $x+2>0$ & $x^2>0$

$\Rightarrow x>-2$ and $x\neq 0 \Rightarrow x\in(-2,\infty)-\{0\}\rightarrow(1)$

Now $\log_{\frac{1}{2}}(x+2) > \log_{\left(\frac{1}{2}\right)^2}(x^2) \Rightarrow \log_{\frac{1}{2}}(x+2) > \frac{2}{2}\log_{\frac{1}{2}}|x| \quad \left(\text{As } |x|^2 = x^2\right)$

$\Rightarrow \log_{\frac{1}{2}}(x+2) > \log_{\frac{1}{2}}|x|$

$\Rightarrow x+2<|x| \quad \left(\text{As for } 0<a<1;\ \log_a x_1 > \log_a x_2 \Rightarrow x_1 < x_2\right)$

For $x>0; |x| = x$

$\Rightarrow x+2<x \Rightarrow x\in\varphi$

For $-2<x<0;\ |x| = -x$

$\Rightarrow x+2<-x \Rightarrow 2x<-2 \Rightarrow x<-1$

$\Rightarrow x\in(-2,-1)$

Solution Set : $x\in(-2,-1)$

3. Given $\log_{x^2+2}(12-3x)<1$

We know that $x^2\geq 0 \Rightarrow x^2+2\geq \forall\ x\in R$

Therefore, base$\geq 2\ \forall\ x\in R$

Also $12-3x>0$

$\Rightarrow 3x<12$

$\Rightarrow x<4\rightarrow(1)$

Now $\log_{x^2+2}(12-3x)<1$

$\Rightarrow 12-3x<x^2+2$

$\Rightarrow x^2+3x-10>0$

$\Rightarrow (x+5)(x-2)>0$

$\Rightarrow x\in(-\infty,-5)\cup(2,\infty)\rightarrow(2)$

From (1) and (2)

Solution Set : $(-\infty,-5)\cup(2,4)$

FUNDAMENTALS OF MATHEMATICS

CHARACTERISTIC AND MANTISSA

NOTES

What you will learn

- Characteristic and mantissa
- Applications of characteristic and mantissa
- Finding number of digits in a^n

What you already know

- Logarithm
- Properties of logarithm
- Logarithmic function
- Properties of logarithmic function

Relation Between Natural and Common Logarithm

We know that, the natural log is given by $\log_e x$ and the **common log** is given by $\log_{10} x$.

$\log_e 10 \cong 2.303 = 2.3$

And $\log_{10} e = \dfrac{1}{\log_e 10} = 0.43$ [Using the change of base rule]

$\log_{10} n = (\log_e n) \times (0.43)$

Characteristic and Mantissa

$\log_{10} N$, i.e. common logarithm of the valid input $N > 0$ always has an integral part (I) and a fractional part (F). Hence, we can write: $\log_{10} N = I + F$, where $I \in Z$ and $F \in (0, 1)$. I and F are known as the characteristic and mantissa of $\log_{10} N$, respectively.

$\log_{10} N$	Characteristic	Mantissa
5.799 (5 + 0.799)	**5**	**0.799**
0.4481 (0 + 0.4481)	**0**	**0.4481**
13 (13 + 0)	**13**	**0**
-0.5 (-1 + 0.5)	**-1**	**0.5**
-3.81 (-4 + 0.19)	**-4**	**0.19**

The integral part (I) can be 0 or any positive/negative integer, whereas the fractional part (F) is always a non-negative proper fraction, means F = 0 or 0 < F < 1.

Note: For $\log_{10} N$ with input n = 1, $\log_{10} 1 = 0 = 0 + 0$, where 0 is the characteristic I and 0 is also the mantissa F.

Characteristic of $\log_{10} N$ for N > 1

If N = 58.34, then what is the characteristic of $\log_{10} (58.34)$?

$\log_{10} 1 = 0$
$\log_{10} 10^1 = 1$
$\log_{10} 10^2 = 2$
$\log_{10} 10^3 = 3$

$10^1 \le 58.34 < 10^2$
$\log_{10} 10^1 \le \log_{10} 58.34 < \log_{10} 10^2$
$1 \le \log_{10} 58.34 < 2$

So, $\log_{10} (58.34)$ = 1.(something) = 1 + 0.(something), where 1 is the characteristic and 0.(something) is the mantissa. Therefore, the characteristic of $\log_{10} (58.34)$ is 1.

In general,
If the integral part of N contains k digits,
$10^{k-1} \le N < 10^k$
Taking log with base 10 on both sides,
$\Rightarrow k - 1 \le \log_{10} N < k$
$\therefore$ Characteristic of $\log_{10} N = k - 1$

Number of Digits in a^n

Logarithm can be extremely useful to calculate the number of digits in large numbers.
A number will have precisely k digits if and only if it is in the range $R = [10^{k-1}, 10^k - 1]$ For instance, the number 4,000,000, has 7 digits and is in the range $[10^6, 10^7 - 1] = [1{,}000{,}000,\ 9{,}999{,}999]$.

Given an integer a^n, one can determine k and the number of digits in a^n, by working with the inequality $a^n \in R = [10^{k-1}, 10^k - 1] \Rightarrow 10^{k-1} \le a^n \le 10^k - 1$
Taking log with base 10
$\Rightarrow (k - 1) \log 10 \le \log a^n \le \log (10^k - 1) \Rightarrow k - 1 \le \log a^n < k$ $(\because 10^k - 1 < 10^k \Rightarrow \log(10^k - 1) < \log 10^k)$

Here, k - 1 is nothing but the characteristic of $\log_{10} a^n$.

Find the number of digits in 2^{100}. ($\log_{10} 2 = 0.3010$).

Let $N = 2^{100}$
$\Rightarrow \log_{10} N = \log_{10} 2^{100}$
$\Rightarrow \log_{10} N = 100 \times \log_{10} 2$
$= 100 \times 0.3010$
$= 30.10$
$\Rightarrow \log_{10} N = 30 + 0.10$

Here, the characteristic is 30 and number of digits is (characteristic + 1).
$\therefore$ Number of digits in $2^{100} = 30 + 1 = 31$

Characteristic of $\log_{10} N$ for $0 < N < 1$

If N = 0.0093, then what is the characteristic of $\log_{10}(0.0093)$?

$\log_{10} 1 = 0.$
$\log_{10} 0.1 = -1$
$\log_{10} 0.01 = -2$
$\log_{10} 0.001 = -3$
$\log_{10} 0.0001 = -4$

$N = 0.0093$
$0.001 \leq 0.0093 < 0.01$
$10^{-3} \leq 0.0093 < 10^{-2}$
$\log_{10} 10^{-3} \leq \log_{10} 0.0093 < \log_{10} 10^{-2}$
$-3 \leq \log_{10} 0.0093 < -2$

So,
$\log_{10}(0.0093) = -2.$**(something)** $= -3 + 0.$**(something)**, where -3 is the characteristic and 0.(something) is the mantissa. Therefore, the characteristic of $\log_{10}(0.0093)$ is -3.

In general
If the number of zeroes between the decimal and the first non-zero (significant) digit in 'N' is 'k', then the characteristic of $\log_{10} N = -(k + 1)$.

Another way to solve is:
$$N = 0.0093 = \frac{93}{10000} = \frac{93}{10^4} = \frac{93}{10} \times 10^{-3} = 9.3 \times 10^{-3}$$
Taking log with base 10 on both sides,
$$\log_{10} N = \log_{10}(9.3 \times 10^{-3}) = \log_{10}(10^{-3}) + \log_{10}(9.3)$$
$$= -3 + \log_{10}(9.3) = -3 + 0.(\text{something})$$
-3 is the characteristic and 0.(something) $= \log_{10}(9.3)$ is the mantissa.

Applications: 1. If $\log_3\left(\frac{|x^2 - 4x| + 3}{x^2 + |x - 5|}\right) \geq 0$, then find the range of x.

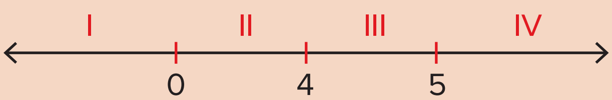
(a) $\left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, 2\right]$ (b) $(-\infty, 2]$ (c) $\left(-\infty, \frac{8}{5}\right]$ (d) $\mathbf{R} - \left\{-\frac{2}{3}, \frac{1}{2}, 2, \frac{8}{5}\right\}$

Solution:
(i) For inequality to be defined, the input of log $\frac{|x^2 - 4x| + 3}{x^2 + |x - 5|} > 0$

$|x^2 - 4x|$ is always $\geq 0 \Rightarrow |x^2 - 4x| + 3 > 0$
$x^2 \geq 0$ and $|x - 5| \geq 0 \Rightarrow x^2 + |x - 5| > 0$
$\therefore$ Input $\frac{|x^2 - 4x| + 3}{x^2 + |x - 5|}$ is always > 0
$\Rightarrow x \in R = A$

(ii) $\log_3(\text{input}) \geq 0 \Rightarrow \text{input} \geq 1$

$$\Rightarrow \frac{|x^2 - 4x| + 3}{x^2 + |x - 5|} \geq 1$$
$$\Rightarrow \frac{|x(x - 4)| + 3}{x^2 + |x - 5|} \geq 1$$
$$\Rightarrow \frac{|x|\,|(x - 4)| + 3}{x^2 + |x - 5|} \geq 1$$



$\Rightarrow |x|\,|(x - 4)| + 3 \geq x^2 + |x - 5|$ .... (1)

Case I: $x \leq 0$

$\Rightarrow |x| = -x$
$\Rightarrow |x - 4| = -(x - 4)$
$\Rightarrow |x - 5| = -(x - 5)$

$\therefore$ 1) becomes:

$(-x)\,(-(x - 4)) + 3 \geq x^2 - (x - 5)$
$\Rightarrow x\,(x - 4) + 3 \geq x^2 - (x + 5)$
$\Rightarrow x^2 - 4x + 3 - x^2 + x - 5 \geq 0$
$\Rightarrow -3x - 2 \geq 0$

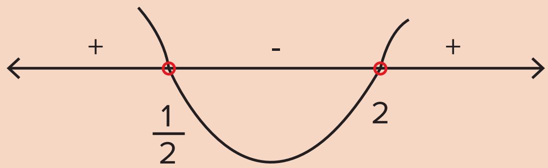
$\Rightarrow x \leq -\frac{2}{3} \Rightarrow x \in \left(-\infty, -\frac{2}{3}\right]$

Case II: $0 < x \leq 4$

$\Rightarrow |x| = x$
$\Rightarrow |x - 4| = -(x - 4)$
$\Rightarrow |x - 5| = -(x - 5)$

$\therefore$ 1) becomes:

$(x)\,(-(x - 4)) + 3 \geq x^2 - (x - 5)$
$\Rightarrow -x^2 + 4x + 3 - x^2 + x - 5 \geq 0$
$\Rightarrow -2x^2 + 5x - 2 \geq 0$
$\Rightarrow 2x^2 - 5x + 2 \leq 0$
$\Rightarrow (2x - 1)\,(x - 2) \leq 0$



$\Rightarrow x \in \left[\frac{1}{2}, 2\right]$

Case III: $4 < x \leq 5$

$\Rightarrow |x| = x$
$\Rightarrow |x - 4| = |x - 4| = x - 4$
$\Rightarrow |x - 5| = -(x - 5)$

$\therefore$ 1) becomes:

$x(x - 4) + 3 \geq x^2 - (x - 5)$
$\Rightarrow x^2 - 4x + 3 - x^2 + x - 5 \geq 0$
$\Rightarrow -3x - 2 \geq 0$
$\Rightarrow 3x + 2 \leq 0$

$\Rightarrow x \leq -\frac{2}{3} \rightarrow$ Not possible $\therefore x \in \Phi$

Case IV: $x > 5$

$\Rightarrow |x| = x$
$\Rightarrow |x - 4| = x - 4$
$\Rightarrow |x - 5| = x - 5$

$\therefore$ 1) becomes:

$x(x - 4) + 3 \geq x^2 + x - 5$
$\Rightarrow x^2 - 4x + 3 \geq x^2 + x - 5$
$\Rightarrow x^2 - 4x + 3 - x^2 - x + 5 \geq 0$
$\Rightarrow -5x + 8 \geq 0$

$\Rightarrow x \leq \frac{8}{5} \rightarrow$ Not possible $\therefore x \in \Phi$

Now,

B = (Case I) U (Case II) U (Case III) U (Case IV)

$= \left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, 2\right] \cup \Phi \cup \Phi$

$= \left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, 2\right]$

And, the final solution set = $A \cap B$

$= R \cap B$

$= B$

$= \left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, 2\right]$

Therefore, option (a) is the solution.

Find the solution of The solution of, $\frac{1}{\log_2 x} < \frac{1}{\log_2 \sqrt{x+2}}$.

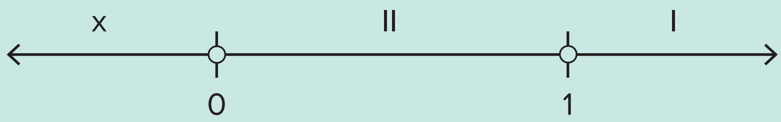
(a) (0, 1) (b) $(2, \infty)$ (c) $(0, 1) \cup (2, \infty)$ (d) $(0, \infty) - \{1, 2\}$

Solution:

For inequality to be defined, L.H.S. and R.H.S. should exist.

L.H.S. is defined when $\frac{1}{\log_2 x}$ is defined, which is defined for $x > 0$ and $x \neq 1$.

L.H.S. is defined when $\frac{1}{\log_2 \sqrt{x+2}}$ is defined, when is defined for $\sqrt{x+2} > 0$, $x + 2 > 0$ and $\sqrt{x+2} \neq 1$.



Case I: $x > 1$

$\Rightarrow \log_2 x > 0$

and $x > 1 \Rightarrow x + 2 > 3$

$\Rightarrow \sqrt{x+2} > 1$

$\therefore \log_2 \sqrt{x+2} > 0$

Then,

$\frac{1}{\log_2 x} < \frac{1}{\log_2 \sqrt{x+2}}$

$\Rightarrow \log_2 x > \log_2 \sqrt{x+2}$

$\Rightarrow x > \sqrt{x+2}$

$\Rightarrow x^2 > x + 2$

$\Rightarrow x^2 - x - 2 > 0$

$\Rightarrow (x - 2)(x + 1) > 0$

$\therefore x \in (2, \infty)$

Case II: $0 < x < 1$

$\Rightarrow \log_2 x < 0$

and $2 < x + 2 < 3$

$\Rightarrow \sqrt{x+2} > 1$

$\Rightarrow \log_2 \sqrt{x+2} > 0$

Then,

$\frac{1}{\log_2 x} < \frac{1}{\log_2 \sqrt{x+2}}$

$\downarrow$ **Negative** $\downarrow$ **Positive**

is true $\forall\, x \in (0, 1)$

Final solution set = (Case I) $\cup$ (Case II)

= $(0, 1) \cup (2, \infty)$

Hence, option (c) is the correct answer.

If $\log_3 x - \log_x 27 < 2$, then find the range of x.

(a) $\left(\frac{1}{3}, 27\right)$ (b) $\left(\frac{1}{27}, 3\right)$ (c) $\left(0, \frac{1}{3}\right) \cup (1, 27)$ (d) (0, 1)

Solution:

For inequality to be defined, L.H.S. and R.H.S. should exist.

R.H.S. is defined for $\mathbf{R}$ as it is a constant.

L.H.S. is defined when both $\log_3 x$ and $\log_x 27$ are defined.

$\log_3 x$ is defined for $x > 0$ and $\log_x 27$ is defined for $x > 0$ and $x \neq 1$.

The common region where both are defined is $x > 0$ and $x \neq 1$.

Consider,

$\log_3 x - \log_x 27 < 2$

$\Rightarrow \log_3 x - \log_x 3^3 < 2$

$\Rightarrow \log_3 x - 3\log_x 3 < 2$

$\Rightarrow \log_3 x - 3\left(\frac{1}{\log_2 x}\right) < 2$

$\Rightarrow \left(\frac{(\log_3 x)^2 - 3}{\log_3 x}\right) < 2$

Let $\log_3 x = t$

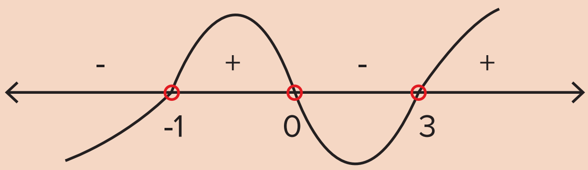
$\Rightarrow \frac{(t)^2 - 3}{t} < 2$

$\Rightarrow \frac{(t)^2 - 3}{t} - 2 < 0$

$\Rightarrow \frac{(t)^2 - 3 - 2t}{t} < 0$

$\Rightarrow \frac{t^2 - 2t - 3}{t} < 0$

$\Rightarrow \frac{(t - 3)(t + 1)}{t} < 0$



$\Rightarrow t \in (-\infty, -1) \cup (0, 3)$

$\Rightarrow \log_3 x < -1$ or $0 < \log_3 x < 3$

$\Rightarrow 0 < x < \frac{1}{3}$ or $1 < x < 3^3$

$\therefore x \in \left(0, \frac{1}{3}\right) \cup (1, 27)$

Therefore, option (c) is the correct answer.

If $(\log_{2x} 2)(\log_x 2)(\log_2 4x) > 1$. then find the range of x.

(a) $(1, 2^{\sqrt{2}})$ **(b)** $(1, \infty) - \{2^{\sqrt{2}}\}$ **(c)** $\left(2^{-\sqrt{2}}, \frac{1}{2}\right) \cup (1, 2^{\sqrt{2}})$ (d) $(0, 2^{\sqrt{2}})$

Solution:

We have, $(\log_{2x} 2)(\log_x 2)(\log_2 4x) > 1$

Clearly, $x > 0$ & $x \neq 1, \frac{1}{2}$

Then, $(\log_{2x} 2)(\log_x 2)(\log_2 4x) > 1$

$$\downarrow \qquad \downarrow$$

$$\left(\frac{1}{\log_2 2x}\right)\left(\frac{1}{\log_2 x}\right)$$

$$\left(\frac{1}{\log_2 2x}\right)\left(\frac{1}{\log_2 x}\right)(\log_2 4x) > 1$$

$$\Rightarrow \frac{\log_2(4 \cdot x)}{(\log_2 2x)(\log_2 x)} > 1$$

$$\Rightarrow \frac{(\log_2 4 + \log_2 x)}{(\log_2 2 + \log_2 x)(\log_2 x)} > 1$$

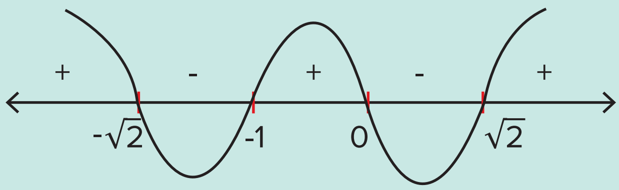
Let $\log_2 x = t$

$$\Rightarrow \frac{(2 + t)}{(1 + t)(t)} > 1$$

$$\Rightarrow \frac{2 + t}{t^2 + t} - 1 > 0$$

$$\Rightarrow \frac{t^2 - 2}{t(t + 1)} < 0$$

$$\Rightarrow \frac{(t + \sqrt{2})(t - \sqrt{2})}{t(t + 1)} < 0$$

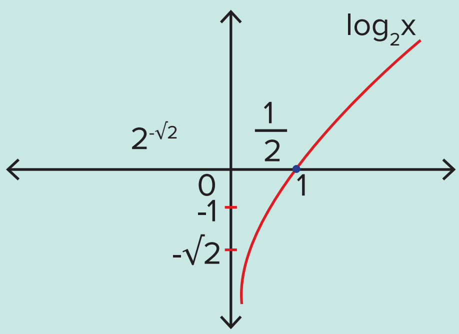


$\Rightarrow t \in (-\sqrt{2}, -1) \cup (0, \sqrt{2})$

$-\sqrt{2} < \log_2 x < -1$ or $0 < \log_2 x < \sqrt{2}$

$2^{-\sqrt{2}} < x < \frac{1}{2}$ or $1 < x < 2^{\sqrt{2}}$

$\therefore x \in \left(2^{-\sqrt{2}}, \frac{1}{2}\right) \cup (1, 2^{\sqrt{2}})$



Hence, option (c) is the correct answer.

Summary Sheet

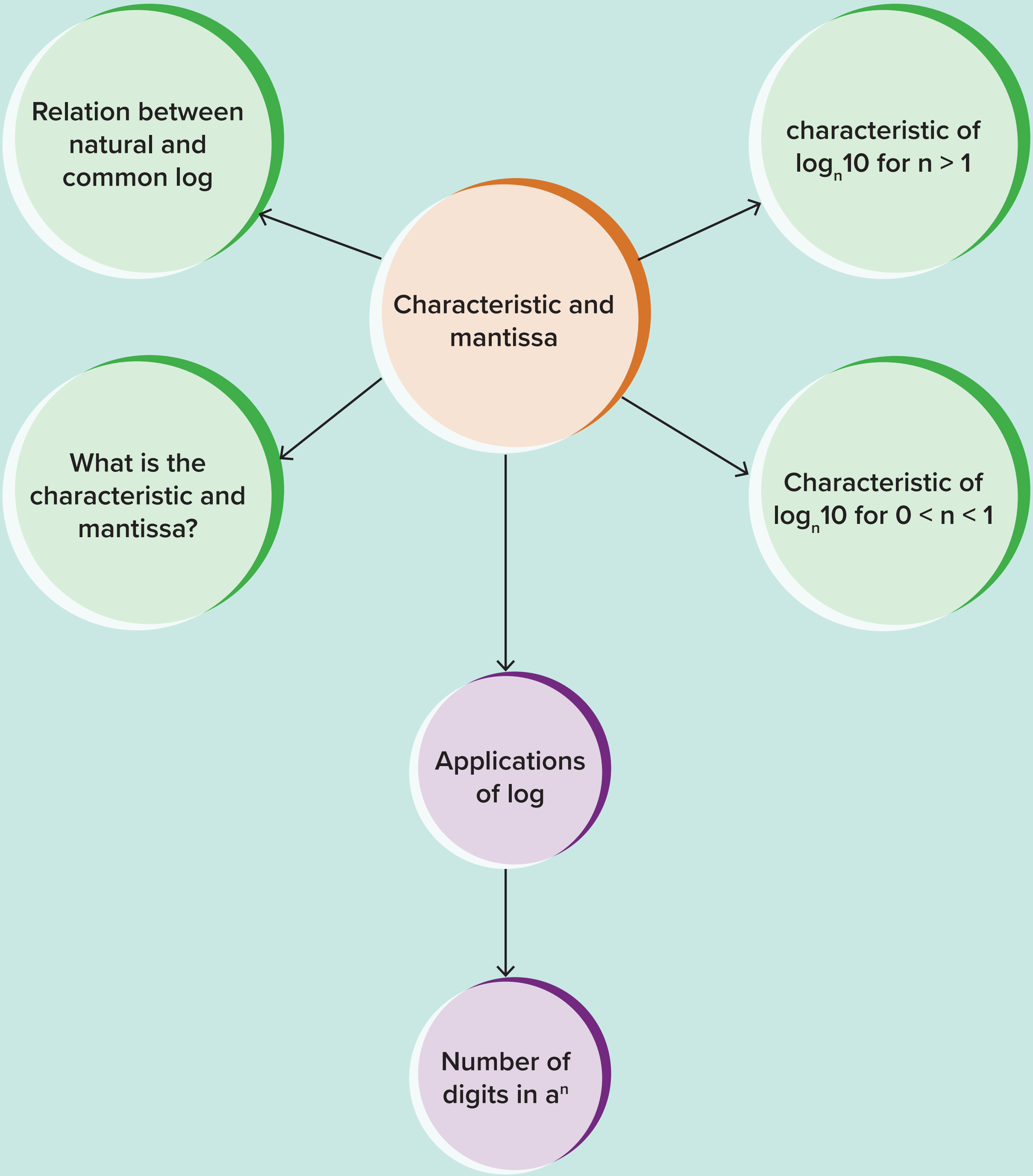
Key Takeaways

- $\log_{10} N = I + F$, where $I \in Z$ and $F \in [0, 1)$. I and F are known as **characteristic** and **mantissa**, respectively.
- If the integral part of N (for $n > 1$) contains k digits, characteristic of $\log_{10} N = k - 1$.
- If the number of zeroes between the decimal and the first non-zero (significant) digit in 'N' (for $0 < N < 1$) is 'k', then the characteristic of $\log_{10} N = -(k + 1)$.

Key Results

- $\log_{10} n = (\log_e n) \times (0.43)$
- The fractional part (F) of log of a number is always a non-negative proper fraction, means $F = 0$ or $0 < F < 1$
- If characteristic of $\log_{10} N = k$, then, the number of digits in $N = k + 1$.

Mind Map



Self-Assessment

1. If log 2 = 0.3010, then find the characteristic and mantissa of log400.
2. Find the characteristic of 0.000943.
3. Find the total number of digits in 18^{100}. Given, log 2 = 0.3010 and log 3 = 0.4771

Answers

Self-Assessment

1. $\log 400 = \log(4 \times 100) = \log 4 + \log 100$

$= \log 2^2 + \log 10^2 = 2\log 2 + 2\log 10$

$= 2 \times 0.3010 + 2 \times 1 = 0.6020 + 2$

$= 2.6020$

∴ Characteristic = 2 and mantissa = 0.6020

Characteristic = k - 1, where k is the number of digits in 'n'.

Here, n = 400 k = 3

∴ Characteristic = k - 1 = 3 - 1 = 2

2. n = 0.000943

$\Rightarrow 0 < n < 1$

∴ k = number of zeroes between decimal and first non-zero digit = 3

∴ Characteristic = -(k + 1)

= -(3 + 1) = -4

3. $n = 18^{100}$

Taking log on both sides,

$\log n = \log(18^{100})$

$= 100 \times \log 18$

$= 100 \times \log(2 \times 3^2)$

$= 100 \times [\log 2 + 2\log 3]$

$= 100 \times [0.3010 + 2 \times 0.4771]$

$= 100 \times [0.3010 + 0.9542]$

$= 100 \times 1.2552 = 125.52$

∴ Characteristic = 125

Number of digits in 18^{100} = characteristic + 1

= 125 + 1 = 126