CHAPTER 14

STRAIGHT LINE AND PAIR OF STRAIGHT LINE

14.1 DEFINITION

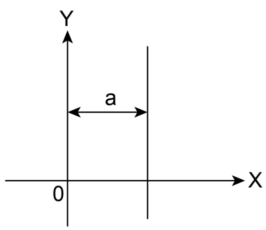
A straight line is a curve such that every point on the line segment joining any two points lie on it or in other words straight line is the locus of a point which moves such that the slope of line segment joining any two of its position remains constant.

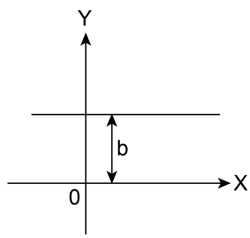
14.1.1 Equation of Straight Line

A relation between x and y which is satisfied by coordinates of every point lying on a line is called the equation of straight line. Every first degree equation in x, y, i.e., $ax + by + c = 0$ represents a line. Thus, a line which is also defined as the locus of a point satisfying the condition $ax + by + c = 0$, where a, b, c are constant.

- **Equation of straight line parallel to axes**

(i) Equation of a straight line which is parallel to x-axis and at a distance **b** units from it is given by $y = b$, $b >$ or < 0 according as it is in positive or negative side.





∴ equation of x-axis is $y = 0$.

(ii) Similarly, for any line parallel to the y axis and at a distance a unit from it is given by $x = a$, $a >$ or $a < 0$ according as the line lies on positive or negative sides of the x-axis.

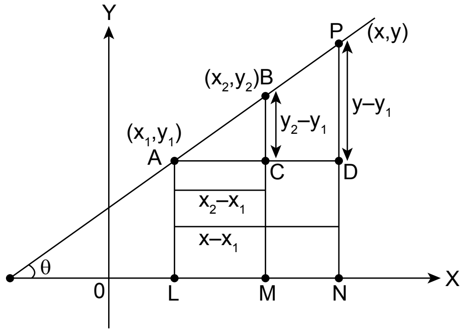
- The combined equation of the coordinate axis is $xy = 0$

14.1.2 Different Forms of the Equation of Straight Line

Two Point From: Straight line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$.

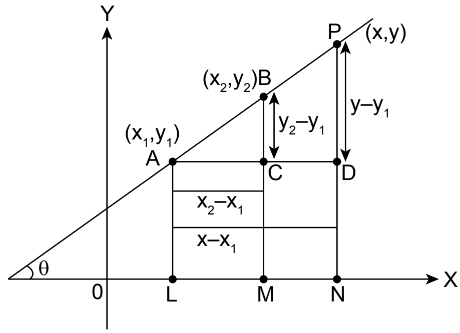
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ in determinant form



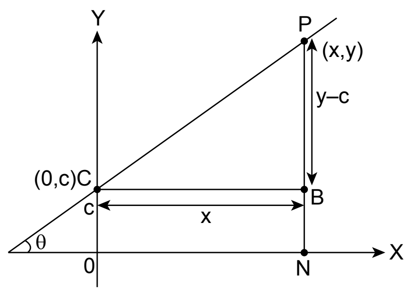
Slope Point From: Straight line passing through $A(x_1, y_1)$ and having slope m.

$$y - y_1 = m(x - x_1)$$



Slope Intercept Form: Equation of line having slope 'm' and making an intercept c on y-axis.

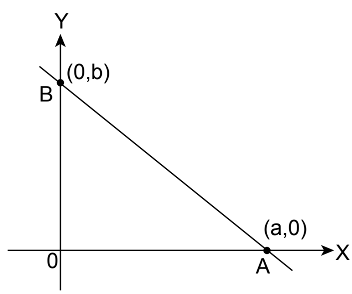
$y = mx + c$; where θ is the angle made by line with +ve direction of x-axis in counter-clockwise sense.

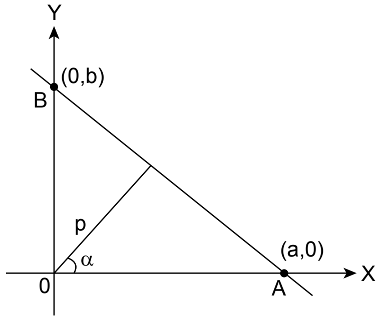
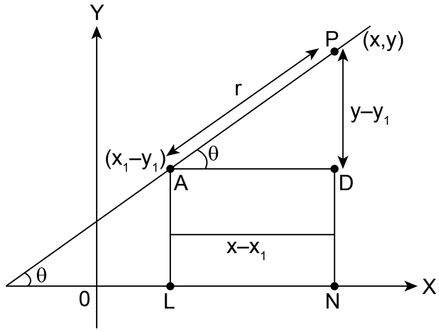


Two Intercept From: Equation of line making intercepts a and b respectively on x and y axis.

$\frac{x}{a} + \frac{y}{b} = 1$ **or** $\begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} = 0$

in determinant form.



Perpendicular/ Normal Form: Equation of line, upon which the length of perpendicular form origin is p and perpendicular makes α angle with +ve direction of x-axis.	$x \cos \alpha + y \sin \alpha = p$. $\Rightarrow$ If equation of line be $x \cos \alpha + y \sin \alpha = -p$, $(p > 0)$ the equation will not be in normal form, to convert it to normal form multiply both sides by -1 $\Rightarrow \quad x(-\cos\alpha) + y(-\sin\alpha) = p$ $\Rightarrow \quad x \cos(\pi + \alpha) + y\sin(\pi+\alpha) = p$	
Symmetric (Parametric) From: Straight line passing through $A(x_1, y_1)$ and making angle θ with positive x-axis.	$\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$; where r is distance of the point P(x, y) from the fixed point $A(x_1, y_1)$.	

$\Rightarrow$ Using symmetric form

- To find coordinate of any point P(x, y) from the fixed point $A(x_1 y_1)$ on the line if AP is given as r.

$\Rightarrow$ $x = x_1 + r \cos\theta$ and $y = y_1 + r \sin\theta$

- To find distance of a point from a fixed point on the line along the line.
- To find distance r if θ is known; θ if r is given.

14.1.3 Angle Between Two Lines

Given two lines:

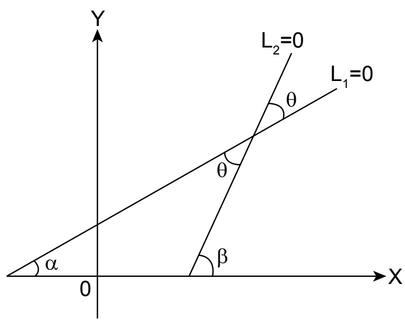
$$L_1 : a_1x + b_1y + c_1 = 0; \left(m_1 = -\frac{a_1}{b_1} = \tan\alpha \right);$$

$$L_2 : a_2x + b_2y + c_2 = 0; \left(m_2 = -\frac{a_2}{b_2} = \tan\beta \right)$$

The angle between $L_1 = 0$ and $L_2 = 0$; $\theta = \beta - \alpha$

$$\Rightarrow \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\Rightarrow \quad \tan\theta = \left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \right)$$



14.1.3.1 Conclusion

- There are two angles formed between any pair of line θ and π – θ (say), then tangent of acute angle θ.

  $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$ and $\tan(\pi - \theta) = -\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

- Lines are parallel if $\Rightarrow \tan\theta = 0 \Rightarrow m_1 = m_2$
- Lines are perpendicular $\Rightarrow \tan\theta \to \infty \Rightarrow m_1 . m_2 = -1$
- Lines are coincident if they have same slope and intercept

$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- Lines $L_1 = 0$ and $L_2 = 0$ are perpendicular when θ = 90.
- If $m_1.m_2 = 1$, then angle of L_1 with x-axis is same as angle of L_2 with y-axis.

  Hence, both lines make same angle with y = x + k and y = –x + k.
- If $m_1 + m_2 = 0$. Lines L_1 and L_2 make supplementary angles with x and y-axis, when extended to intersect they form an isosceles triangle with the coordinates axis (x or y).

14.1.4 Equation of a Line Perpendicular and Parallel to Given Line

⇒ Let m be the slope of the line ax + by + c = 0. Then m = –a/b.

∴ Since the required line is parallel to the given line. The slope of the required line is also m.

Let C_1 be the intercept by the line on y-axis. Then its equation is $y = mx + c_1$.

$\Rightarrow y = \frac{-a}{b}x + c_1$

$\Rightarrow ax + by - bc_1 = 0$

$\Rightarrow ax + by + \lambda = 0$; where $\lambda = -bc_1$ = constant

∴ The equation of line parallel to a given line is $ax + by + \lambda = 0$.

Note:

To find the equation of a line, parallel to a given line, keep the expression containing x and y same and simply replace the given constant by a new arbitrary constant λ. The value of a λ can be determined by same given condition.

⇒ *The equation of line perpendicular to given line ax+ by + cz = 0 is bx – ay + λ = 0.*

i.e., interchange the coefficient of x and y by reversing the sign of exactly of them one and replace the constant term by parameter λ.

14.1.5 Straight Line Through (x_1, y_1) Making an Angle α with y = mx + c

Equation of line passing through a point $A(x_1, y_1)$ and making a given angle θ with the line y = mx + c.

Let slope of the line be m'

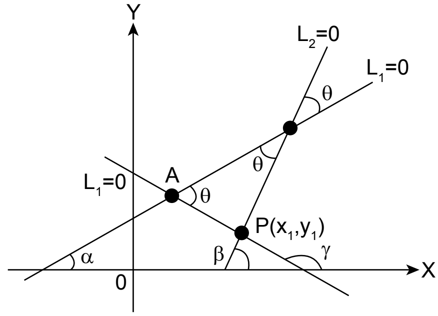
$\therefore \quad \tan\theta = \left|\dfrac{m-m'}{1+mm'}\right| \qquad \Rightarrow \dfrac{m-m'}{1+mm'} = \pm\tan\theta$

$\Rightarrow$ m – m' = ± tan θ ± m . m' tan θ

$\Rightarrow \; m \mp \tan\theta = m'(1 \pm m\tan\theta)$

$\Rightarrow \; m' = \dfrac{m \mp \tan\theta}{1 \pm m\tan\theta}$

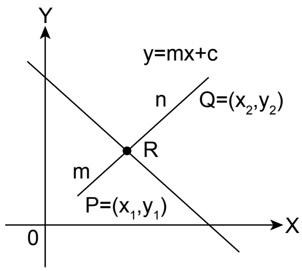
So equation of lines are $y - y_1 = \dfrac{m \mp \tan\theta}{1 \pm m\tan\theta}(x - x_1)$

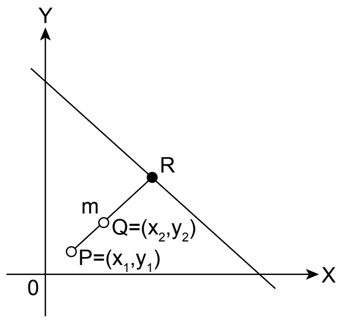


14.1.6 Position of Two Points w.r.t. a Straight Line

Two points P(x_1, y_1) and Q(x_2, y_2) lie on the same side or on the opposite side of the line ax + by + c = 0 according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign or opposite signs respectively. The coordinates of the point R which divides the line joining P and Q sides in the ratio m:n are $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$.

If this point lie on (i), then $a\left(\dfrac{mx_2 + nx_1}{m+n}\right) + b\left(\dfrac{my_2 + ny_1}{m+n}\right) + c = 0.$





$\Rightarrow$ $m(ax_2 + by_2 + c) + n(ax_1 + b_1 + c) = 0$

$\Rightarrow \; \dfrac{m}{n} = -\left(\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right)$

If the point R is between the points P and Q.

Then the ratio m : n is positive. So from the above equation, we get $\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite sign.

If point R is not between P and Q, then the ratio m : n is negative.

$\Rightarrow \; \dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$

$\Rightarrow$ $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same sign.

Note:

If the location of a single point is to be defined then the other point is taken as the origin and w.r.t. the origin. The location of the point w.r.t. the line is defined.

Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ will be located at the same side of the line. If they give the same sign of the expression when they are used in the line, otherwise they will lie on the opposite side of the line.

14.1.7 Distance of a Point From a Line

Let the given line be, ax + by + c = 0, then the distance of any point $P(x_1, y_1)$ from the given line be

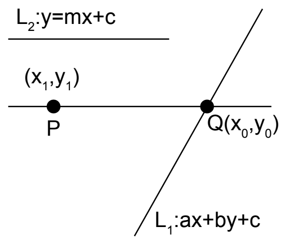
$$\Rightarrow \quad PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Note:

The length of the perpendicular from the origin to the line ax + by + c = 0 is $\frac{|c|}{\sqrt{a^2+b^2}}$.

14.1.8 Distance Between Two Parallel Straight Lines

Let ax + by + c = 0 and ax + by + c' = 0 be the parallel straight lines then the distance between them is given by $\left|\frac{c - c'}{\sqrt{a^2 + b^2}}\right|$



$\Rightarrow$ **Oblique distance of a point from a line:** Distance of a point $P(x_1, y_1)$ from a line L_1 = ax + by + c = 0 along L_2 = y = mx + c:

Method I: Let line parallel to y = mx + c through P cuts ax + by + c = 0 at $Q(x_0, y_0) \Rightarrow$ equation of PQ : $y - y_1 = m(x - x_1)$ ...(i)

Solving (i) and (ii) get coordinates of Q and applying distance formula: $d_{(p,q)} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Method II: Let m = tan θ; Equation of PQ is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$.

For Q : $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. Must satisfy L_1 : ax + by + c = 0

$\Rightarrow \quad a(x_1 + r\cos\theta) + b(y_1 + r\sin\theta) + c = 0$

$\Rightarrow \quad r = -\left(\frac{(ax_1 + by_1 + c)}{a\cos\theta + b\sin\theta}\right)$. The sign of r indicates the position of point w.r.t. Line and |r| is required distance.

14.1.9 Intersection of Two Lines

The point of intersection of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

- **Condition for concurrency of Lines:** Three lines are said to be concurrent if they pass through a common point. Thus, if three lines are concurrent, the point of intersection of two lines lies on the third line. Let $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$.

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$ which is the required condition for concurrency of lines.

Note:

Another condition of concurrency of three lines $L_1: a_1x + b_1y + c_1 = 0; L_2 : a_2x + b_2y + c_2 = 0$ and $L_3 : a_3x + b_3y + c_3 = 0$ are concurrent iff there exists constants $\lambda_1, \lambda_2, \lambda_3$ not all zero such that $\lambda_1L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0; \lambda_1(a_1x + b_1y + c_1) + \lambda_2 (a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$

14.1.10 Equation of the Bisectors of the Angles Between Lines

Method 1: Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ be two intersecting lines, then the equations of the lines bisecting the angles between L_1 and L_2 are given by $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$.

If $a_1a_2 + b_1b_2 = 0$, then the given lines are perpendicular to each other else they will contain acute and obtuse angle.

i.e., $a_1a_2 + b_1b_2 \neq 0$. Let θ be the angle between L_1 and L_2 which is bisected by one of the bisectors, say L_3. Then angle between L_1 and L_3 is θ/2. Now, find tan θ/2.

Two Cases Arise:

(i) If $\tan\frac{\theta}{2} < 1$, then $\theta < \frac{\pi}{2}$. Thus L_3 will be bisecting the acute angles between L_1 and L_2.

(ii) If $\tan\frac{\theta}{2} > 1$, then $\theta > \frac{\pi}{2}$. Thus L_3 will be bisecting the obtuse angle between L_1 and L_2.

Method 2: If $c_1 \neq 0$, $c_2 \neq 0$, then origin must lie in one of the angles between L_1 and L_2. Let us assume $c_1 c_2 > 0$. Then $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ is one of the bisectors of L_1 and L_2. If $a_1a_2 + b_1b_2 > 0$, the given equation represents obtuse angle bisector otherwise it represents acute angle bisector (if $a_1 a_2 + b_1b_2 < 0$).

14.1.10.1 Bisector of angle containing the origin

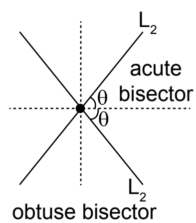
Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. To find the bisectors of the angle containing the origin the following steps are taken:

Step 1: See whether the constant terms c_1 and c_2 in the equations of two lines are positive or not. If not, then multiply both the sides of the equations by –1 to make the constant term positive.

Step 2: Now obtain the bisector corresponding to the positive sign. $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$.

This is the required bisector of the angle containing the origin and negative sign bisector of that angle which does not contain origin.

14.1.10.2 Bisector of acute and obtuse angle



Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_1x + b_2y + c_2 = 0$. To separate the bisectors of the obtuse and acute angles between the lines we proceed as follows:

Step 1: See whether the constant terms c_1 and c_2 in the equations of two lines are positive or not. If not, then multiply both the sides of the equations by –1 to make the constant term positive.

Step 2: Determine the sign of the expression $a_1a_2 + b_1b_2$.

Step 3: If $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to '+' sign gives the obtuse angle bisector and the bisector corresponding to '–' sign is the bisector of acute angle between the lines i.e., $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ and $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ are the bisectors of obtuse and acute angles respectively.

Step 4: If $a_1a_2 + b_1b_2 < 0$, then the bisector corresponding to '+' sign gives the acute and obtuse angle bisectors respectively $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = +\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ and $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ are the bisectors of acute and obtuse angles respectively.

14.1.10.3 Whether the origin lies in the obtuse angle or acute angle

Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. To determine whether the origin lies in the acute angle or obtuse angle between the lines we proceed as follows:

Step 1: See whether the constant terms c_1 and c_2 in the equations of two lines are positive, if not then multiply both the sides of the equations by –1 to make the constant term positive.

Step 2: Determine the sign of the expression $a_1a_2 + b_1b_2$.

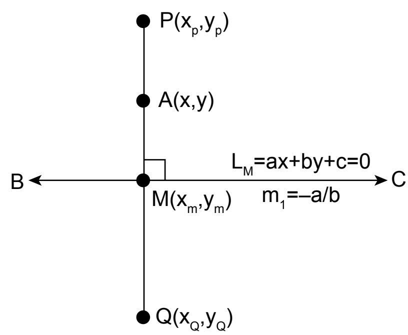
Step 3: If $a_1a_2 + b_1b_2 > 0$, then the origin lies in the obtuse angle and the '+' sign gives the bisector of obtuse angle. If $a_1a_2 + b_1b_2 < 0$, then the origin lies in the acute angle and the '+' sign gives the bisector of acute angle.

Tips and Tricks:

Equation of a Reflected Ray in a Mirror: Given a line mirror $L_M = ax + by + c = 0$ and a ray is incident along the line $L_1 = a_1x + b_1y + c_1 = 0$.

The equation of the reflected ray is $L_R = (y - b) - m_0 (x - \alpha) = 0$

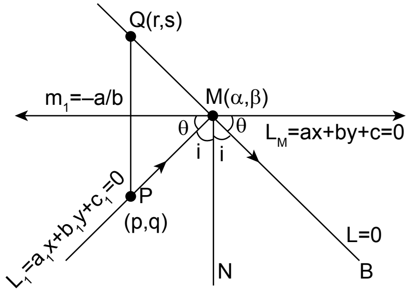
In general if a point (x_2, y_2) lies at a distance k times the distance of $P(x_1, y_1)$ from M (x_m, y_m) then $\frac{y_2-y_1}{b} = \frac{x_2-x_1}{a} = -(k+1)\frac{(ax_1+by_1+c)}{a^2+b^2}$.



Foot of perpendicular and image of a point in a line: If point P is reflected with respect to line L_m, then the coordinates of its reflection are given by Q $(x_Q = 2x_m - x_p;\ y_Q = 2y_m - y_p)$

- **Equation of a Reflected Ray in a Mirror:** Choose a point P(p, q) on the incident ray (preferably any one of p or q taken zero) and get the image in line mirror Q(r, s). In the line mirror.

$$\Rightarrow \frac{r-p}{a} = \frac{s-q}{b} = \frac{-2(ap+bq+c)}{a^2+b^2}$$



Equation of reflected ray is $y-\beta=\frac{s-\beta}{r-\alpha}(x-\alpha)$

⇒ **Yet another way** the equation of the reflected ray is given as LI + λLM = 0; i.e., $(a_1x + b_1y + c_1) + \lambda\,(ax + by + c) = 0$

⇒ $\left\{\lambda=0\ \text{(incident ray) or}\ \lambda=\frac{-2(aa_1+bb_1)}{a^2+b^2}\ \text{(reflected ray)}\right.$

Equation of reflected ray is $L_I+\frac{-2(aa_1+bb_1)}{a^2+b^2}L_M=0.$

14.1.11 Family of Straight Lines

The general equation of line has two effective parameters. Therefore, two conditions are needed to represent a line uniquely. But if only one condition is given, then the resulting equation consist of a parameter and termed as 'family of straight lines'.

- If $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ are two straight lines (not parallel), then $L_1 + \lambda L_2 \equiv a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$ represents family of lines passing through the point of intersection of $L_1 = 0$ and $L_2 = 0$. (Here, λ is a parameter).
- Family of straight lines parallel to the line ax + by + c = 0 is given by ax + by + k = 0; where k is a parameter.
- Family of straight lines perpendicular to the line ax + by + c = 0 is given by bx – y + k = 0; where k is a parameter.
- If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$, are concurrent, then $p\,(a_1x + b_1y + c_1) + q(a_2x + b_2y + c_2) + r(a_3x + b_3y + c_3) = 0 \Rightarrow p + q + r = 0$, i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & b_3 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$.

14.2 GENERAL EQUATION OF SECOND DEGREE AND PAIR OF STRAIGHT LINES

The general equation of pair of a straight lines is represented by the most general equation of second degree in x and y, but any equation in x and y in degree two does not always represent pair of straight lines.

Considering the following equation as a quadratic equation in y.

⇒ $by^2 + 2(hx + f)\,y + ax^2 + 2gx + c = 0$

⇒ $by=-(hx+f)\pm\sqrt{(hx+f)^2-b(ax^2+2gx+c)}$

$$hx+by+f=\pm\sqrt{(h^2-ab)}\sqrt{(x-\alpha)(x-\beta)} \quad \text{....(1)}$$

where α and β are roots of quadratic $(h^2 - ab)x^2 + 2(hf - bg)x + f^2 - bc$

This equation (1) represents pair of straight lines if α = β, i.e., D = 0.

⇒ $\Delta = 4\,(hf - bg)^2 - 4(h^2 - ab)\,(f^2 - bc) = 0$

$\Rightarrow$ $b^2g^2 - 2hfgb + h^2bc + abf^2 - ab^2c = 0$.

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \Rightarrow \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\Rightarrow$ The lines represented are given as: $hx + by + f = \pm\sqrt{h^2 - ab}(x - \alpha)$

Conclusions:

If $h^2 - ab > 0 \Rightarrow$ two real and distinct lines.
If $h^2 - ab < 0 \Rightarrow$ two imaginary lines.
If $h^2 - ab = 0 \Rightarrow$ two parallel lines if atleast one of $bg - hf \neq 0$, $af - gh \neq 0$
If $h^2 - ab = 0$ and $bg - hf = 0$, $af - gh = 0 \Rightarrow$ two coincident lines.
$a + b = 0 \Rightarrow$ both lines are perpendicular.

14.2.1 Pair of Straight Lines Through the Origin

The homogenous equation of second degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines through the origin $ax^2 + 2hxy + by^2 = 0$.

$\Rightarrow$ $b(y/x)^2 + 2h(y/x) + a = 0$ $\qquad \Rightarrow \dfrac{y}{x} = \dfrac{-2h \pm \sqrt{4h^2 - 4ab}}{2b}$

$\Rightarrow$ $y = m_1x$ or $y = m_2x$; where $m_1 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$ and $m_2 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$

Since $h^2 \leq ab$, therefore values of m_1 and m_2 are real. Clearly $y = m_1x$ and $y = m_2x$ are straight lines passing through the origin. Hence, $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.

$\Rightarrow$ According to the value of m_1 and m_2, then line are:

- Real and distinct, if $h^2 > ab = 0$ and $h^2 > ab$.
- If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, they can be found by considering the lines as $(lx + my + n)$; $(l'x + m'y + n') = 0$. After multiplying and comparing the coefficients of like power, we can find $\ell\ \ell'$, m, m′, n, n′ to find the required equations.

14.2.2 Angle Between the Pair of Straight Lines

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{...(i)}$$

$$\therefore \quad \theta = \tan^{-1}\left[\frac{2\sqrt{(h^2 - ab)}}{|a + b|}\right]$$

(i) Condition for the lines to be parallel : If the two lines are parallel, then $\theta = 0$, i.e., $\tan\theta = 0$. Hence, the two lines are parallel if $h^2 = ab$.

(ii) Condition for the lines to be perpendicular: If the two lines are perpendicular, then $\theta = 90°$, i.e., $\tan\theta = \infty$, $\therefore$ $a + b = 0$; i.e., coefficient of x^2 + coefficient of $y^2 = 0$.

Note:

The above conditions are also valid for general equation of second degree.

1. **Equation of angle bisector of the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$**

⇒ **Condition for coincidence of lines:** The lines will be coincident if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$. Taking the above ratios in pairs, the conditions are $h^2 - ab = 0$, $g^2 - ac = 0$ and $f^2 - bc = 0$.

⇒ **Point of intersection of the lines:** The point of intersection of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$ or $\sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\frac{g^2 - ca}{h^2 - ab}\right)}$.

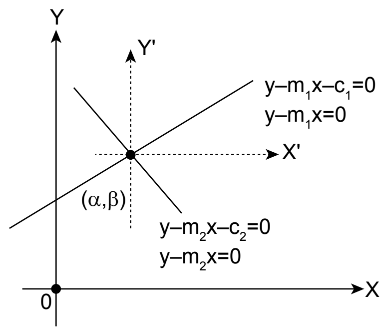
2. **Bisectors of the angles between the lines given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.**

If (x', y') be the point of intersection of the lines, then we shift the origin to the point (x', y'). The transformed equation will be $ax^2 + 2hxy + by^2 = 0$ of the bisectors which are given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

The above bisectors are referred to (x', y') as origin. Now, we have to write $x - x'$, from x and $y - y'$ for y. Hence, the equation of the bisectors of the angle between the lines is $\frac{(x - x')^2 - (y - y')^2}{a - b} = \frac{(x - x')(y - y')}{h}$

$$\begin{cases} ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \\ \frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h} \end{cases} \xrightarrow[(\alpha,\beta)]{\text{shifting origin to}} \begin{cases} ax^2 + 2hxy + by^2 = 0 \\ \frac{x^2 - y^2}{a - b} = \frac{xy}{h} \end{cases}.$$

Tips and Tricks:



⇒ **Point of Intersection:** Given a pair of straight lines $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Let (a,b) be the point of intersection of both lines represented by S = 0.

Shifting origin to (α, β), the equation S = 0 must transform to homogenous form.

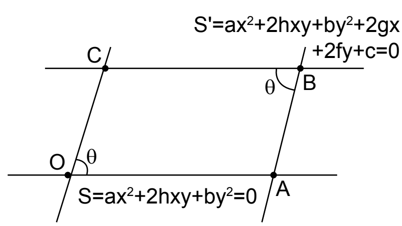
i.e., $a(x + \alpha)^2 + b(y + \beta)^2 + 2h(x + \alpha)(y + \beta) + 2g(x + \alpha) + 2f(y + \beta) + c = 0$

$$\begin{cases} \text{coefficient of } x = 0 \Rightarrow a\alpha + h\beta + g = 0 \\ \text{coefficient of } y = 0 \Rightarrow h\alpha + b\beta + f = 0 \end{cases}$$

⇒ $\left(\frac{\partial S}{\partial x}\right)_{(\alpha,\beta)} = 0$ and $\left(\frac{\partial S}{\partial y}\right)_{(\alpha,\beta)} = 0$

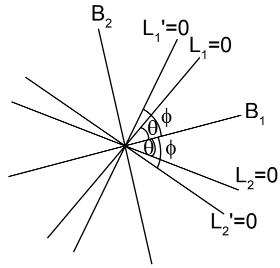
The point of intersection of P.O.S.L if D = 0

⇒ $\alpha = \frac{hf - bg}{ab - h^2}$ and $\beta = \frac{af - gh}{h^2 - ab}$.



❒ Homogeneous equation of degree 2 in x and $yax^2 + 2hxy + by^2 = 0$ always represents POSL (real or imaginary) passing through origin.

- A homogeneous equation of degree n represents n straight lines through origin.
- If two POSL have same homogeneous part of degree two in their equation, then they always construct a parallelogram.
- If two POSL, $S = 0$ ($L_1.L_2 = 0$) and $S' = 0$ ($L_1'.L_2' = 0$) have common angle bisectors ($B_1.B_2 = 0$) then their lines are iso-inclined to each other respectively, i.e., angle between L_1 and L_1' is equal to angle between L_2 and L_2' also angle; between L_1 and L_2' is equal to angle; between L_2 and L_1' ; angle between $L_1 = 0$ and $L_1' = 0$ = angle; between $L_2 = 0$ and $L_2' = 0 = \phi - \theta$ also angle; between $L_1 = 0$ and $L_2' = 0$ = angle between $L_2 = 0$ and $L_1' = 0 = \phi + \theta$.



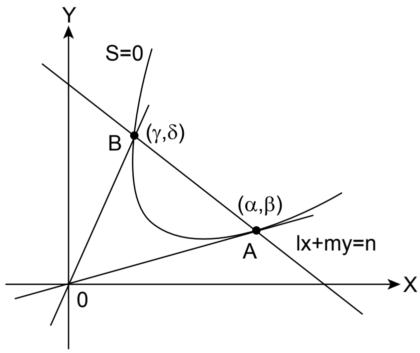
- Equation of POSL joining origin to the point of intersection of a curve and a straight line:

$$S = \underbrace{ax^2 + 2hxy + by^2}_{\text{Homogeneous}} + \underbrace{2\underbrace{(gx + fy)}_{\text{Linear}}\left(\frac{lx+my}{n}\right)}_{\text{Homogeneous}} + \underbrace{c\left(\frac{lx+my}{n}\right)^2}_{\text{Homogeneous}} = 0.$$

- Equation of POSL joining origin to the point of intersection of a curve and a straight line:

  Given a straight line $lx + my = n$ … (i)

  and a conic $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ … (ii)



Required a homogeneous equation of degree two that satisfies the coordinates of $A(\alpha, \beta)$ and $B(\gamma, \delta)$.

Since $l\alpha + m\beta = n$ and $S_{(\alpha, \beta)} = a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$.

- If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a ΔABC.

$\Rightarrow$ equation of median through A is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

$\Rightarrow$ equation of the internal bisector of angle A, is (where $b = AC$ and $c = AB$)

$$b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$