Indefinite Integral

Chapter 21

SOME BASIC INTEGRALS

1. $\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + C \; for\; n \neq -1$. In particular, $\int x^n dx = \frac{x^{n+1}}{n+1} + C \; for\; n \neq -1$.

2. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$. In particular, $\int \frac{1}{x} dx = \log|x| + C$.

3. $\int \sin x dx = -\cos x + C$.

4. $\int \cos x dx = \sin x + C$.

5. $\int \tan x dx = -\log|\cos x| + C = \log|\sec x| + C$.

6. $\int \cot x dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$.

7. $\int \sec x dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

8. $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

9. $\int a^x dx = \frac{a^x}{\log_e a} + C$. In particular, $\int e^x dx = e^x + C$.

10. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$. In particular, $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$.

11. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$.

12. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$.

13. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$.

14. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}$

15. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$.

16. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$.

17. $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$.

18. $\int \sqrt{a^2 + x^2}\,dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$.

19. $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$.

20. $\int |x|^n dx = \frac{x|x|^n}{n+1} + C$, where $n \neq -1$

INTEGRALS OF THE FORM $\int \mathbf{e^x\{f(x) + f'(x)\}dx}$:

If the integral is of the form $\int e^x \{f(x) + f'(x)\} dx$, then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is, by doing so, we get

1. $\int e^x [f(x) + f'(x)] dx \;=\; e^x f(x) + c$

2. $\int e^{mx}\left[mf(x)+f'(x)\right]dx = e^{mx}f(x)+c$

3. $\int e^{mx}\left[f(x)+\frac{f'(x)}{m}\right]dx = \frac{e^{mx}f(x)}{m}+c$

INTEGRAL IS OF THE FORM $\int\left[\mathbf{xf'(x)+f(x)}\right]\mathbf{dx}$:

If the integral is of the form $\int\left[xf'(x)+f(x)\right]dx$ then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is , by doing so, we get,

$$\int\left[x\,f'(x)+f(x)\right]dx = x\,f(x)+c.$$

INTEGRALS OF THE FORM $\int \mathbf{e^{ax}sinbxdx},\ \int \mathbf{e^{ax}cosbx\,dx}$:

$$\int e^{ax}\sin bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

$$\int e^{ax}.\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

$$\int e^{ax}.\sin(bx+c)\ dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c)-b\cos(bx+c)\right]+k = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$$

$$\int e^{ax}.\cos(bx+c)\ dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c)+b\sin(bx+c)\right]+k = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left[(bx+c)-\tan^{-1}\left(\frac{b}{a}\right)\right]+k$$

INTEGRALS OF THE FORM $\int\frac{\mathbf{x^2+1}}{\mathbf{x^4+kx^2+1}}\mathbf{dx},\ \int\frac{\mathbf{x^2-1}}{\mathbf{x^4+kx^2+1}}\mathbf{dx},\ \int\frac{\mathbf{dx}}{\mathbf{x^4+kx^2+1}}$, where $\mathbf{k\in R}$

Working Method

(a) To evaluate these types of integrals divide the numerator and denominator by x^2

(b) Put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ as required.

$$\int\frac{x^2+a^2}{x^4+kx^2+a^4}dx,\quad \int\frac{x^2-a^2}{x^4+kx^2+a^4}dx,\quad \text{where } k \text{ is a constant}$$

These integrals can be obtained by dividing numerator and denominator by x^2, then putting

$x-\frac{a^2}{x}=t$ and $x+\frac{a^2}{x}=t$ respectively.

STANDARD SUBSTITUTIONS

(a) For terms of the form x^2+a^2 or $\sqrt{x^2+a^2}$, put $x=a\tan\theta$ or $a\cot\theta$ $\quad\theta\in(0,\pi/2)$

(b) For terms of the form x^2-a^2 or $\sqrt{x^2-a^2}$, put $x=a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$ $\quad\theta\in(0,\pi/2)$

(c) For terms of the form a^2-x^2 or $\sqrt{a^2-x^2}$, put $x=a\sin\theta$ or $a\cos\theta$ $\quad\theta\in[0,\pi/2]$

(d) If both $\sqrt{a+x}$, $\sqrt{a-x}$ are present, then put $x=a\cos\theta$.

(e) For the type $\sqrt{(x-a)(b-x)}$, $\sqrt{\frac{x-a}{b-x}}$, put $x=a\cos^2\theta+b\sin^2\theta$ $\quad\theta\in[0,\pi/2]$

(f) For the type $\sqrt{(x-a)(x-b)}$, $\sqrt{\frac{x-a}{x-b}}$, put $x=a\sec^2\theta-b\tan^2\theta$ $\quad\theta\in(0,\pi/2)$

(g) For the type $\left(\sqrt{x^2+a^2} \pm x\right)^n$ or $\left(x \pm \sqrt{x^2-a^2}\right)^n$, put the expression within the bracket $= t$.

(h) For the type $(x+a)^{-1-\frac{1}{n}}(x+b)^{-1+\frac{1}{n}}$ or $\left(\frac{x+b}{x+a}\right)^{\frac{1}{n}-1}\frac{1}{(x+a)^2}$ $(n \in N,\ n>1)$, put $\frac{x+b}{x+a}=t.$

(i) For $\frac{1}{(x+a)^{n_1}(x+b)^{n_2}}$, $n_1, n_2 \in N$ (and >1), again put $(x+a)=t(x+b)$

Euler's Substitution

The integral of the form $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$ are calculated with the aid of one of the three **Euler's substitution**.

(i) $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$, If $a>0$;

(ii) $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$, If $c>0$;

(iii) $\sqrt{ax^2+bx+c} = (x-\alpha)t$, If $ax^2+bx+c = a(x-\alpha)(x-\beta)$,

i.e., If α is a real root of the trinomial $ax^2+bx+c=0$.

INTEGRATION OF IRRATIONAL ALGEBRAIC FRACTIONS

1. Integrals of the form :

(i) $\int \frac{dx}{ax^2+bx+c}$ (ii) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ (iii) $\int \sqrt{ax^2+bx+c}\ dx$

Working Rule :

Write $ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[x^2+2x.\frac{b}{2a}+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right] = a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$

Thus, ax^2+bx+c will be reduced to the form A^2+X^2 or A^2-X^2 or X^2-A^2, where X is a linear expression in x and A is a constant.

2. Integral of the form :

(i) $\int \frac{px+q}{ax^2+bx+c}dx$ (ii) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

(iii) $\int (px+q)\sqrt{ax^2+bx+c}\ dx$ (iv) $\int \frac{px^2+qx+r}{ax^2+bx+c}dx$

Working Rule :

(a) In (i), (ii) and (iii), write $px+q = A$ [d.c. of $\left(ax^2+bx+c\right)$] $+B$ and find A and B equating the coefficient of similar powers of x and thus one part will be easily integrable and for other part proceed as in (1).

(b) In case (iv), write $px^2+qx+r = A\left(ax^2+bx+c\right)+B(2ax+b)+C$ and find A, B, C.

3. Integrals of the form $\int \frac{1}{X\sqrt{Y}}dx$ where X and Y are linear or quadratic expression in x.

Use the substitution as given in the following table and proceed.

X	Y	Substitution
Linear	Linear	$z^2 = Y$
Quadratic	Linear	$z^2 = Y$
Linear	Quadratic	$z = \frac{1}{X}$
Pure Quadratic	Pure Quadratic	$z^2 = \frac{Y}{X}$ or $z = \frac{1}{x}$

4. Integral of the form :

(i) $\int \left(x \pm \sqrt{a^2 + x^2}\right)^n dx$ (ii) $\int \frac{dx}{\left(x \pm \sqrt{a^2 + x^2}\right)^n}$ (iii) $\int \frac{\left(x \pm \sqrt{a^2 + x^2}\right)^n}{a^2 + x^2} dx$

where n is a positive rational number and $n \neq \pm 1$.

Working Rule : Put $z = x \pm \sqrt{a^2 + x^2}$.

5. Integrals of the form :

(i) $\int \frac{dx}{(x-a)^m (x-b)^n}$, where m and n are natural numbers and $a \neq b$.

Working Rule : Put $x - a = z(x - b)$

(ii) $\int R\left[x, (ax+b)^{\alpha/n}\right] dx$, where R means for a rational functional.

Working Rule : Put $z^n = ax + b$

(iii) $\int R\left[x, (ax+b)^{\alpha/n}, (ax+b)^{\beta/m}\right] dx$

Working Rule : Put $z^p = ax + b$, where $p =$ L.C.M. for m and n

(iv) $\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{\alpha}{n}}\right] dx$

Working Rule : Put $z^n = \frac{ax+b}{cx+d}$.

6. Integrals of the form $\int x^m \left(a + bx^n\right)^p dx$, Where m, n and p are rational numbers can be solved as follows :

(i) If $p \in N$, Expand the integral with the help of binomial theorem and then integrate.

(ii) If p is a negative integer, the integral reduces to the integral of a rational function by means of the substitution $x = t^s$, where s is the L.C.M. of denominators of the fractions m and n.

(iii) If $\frac{(m+1)}{n}$ is an integer, the integral can be rationalized by the substitution $a + bx^n = t^s$ where s is the denominator of the fraction P.

(iv) If $\frac{(m+1)}{n} + p$ is an integer, substitute $ax^{-n} + b = t^s$, where s is the denominator of the fraction P.

7. Integrals of the form $\int \frac{1}{(x-k)\sqrt{ax^2+bx+c}}\,dx$, the substitution $x-k=1/t$ reduces the integral $\int \frac{1}{(x-k)\sqrt{ax^2+bx+c}}\,dx$ to the problem of integrating an expression of the form $\frac{1}{\sqrt{At^2+Bt+C}}$.

8. $\int \frac{dx}{(x-k)^r\sqrt{ax^2+bx+x}}$. Here we substitute, $x-k=1/t$.

9. Integrals of the form $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$ are solved by isolating in the numerator, the derivative of the quadratic appearing under the root sign and expanding the integral into the sum of two integrals.

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx = \int \frac{(A/2a)(2ax+b)+B-(Ab/2a)}{\sqrt{ax^2+bx+c}}\,dx$$

$$= \frac{A}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \left(B-A\frac{b}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

10. Integrals of the form $\int \frac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+........+a_{n-1}x+a_n}{\sqrt{ax^2+bx+c}}\,dx$ are solved as follows

Here, we assume that

$$\int \frac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+......+a_{n-1}x+a_n}{\sqrt{ax^2+bx+c}}\,dx$$

$$=\left(C_0x^{n-1}+C_1x^{n-2}+.....+C_{n-1}\right)\sqrt{ax^2+bx+c}+C_n\int \frac{dx}{\sqrt{ax^2+bx+c}} \to \text{(I)}$$

Where $C_0, C_1, C_2, ..., C_n$ are arbitrary constants

Now differentiating both of (I) w.r.t x and multiplying by $\sqrt{ax^2+bx+c}$ we get.

$$a_0x^n+a_1x^{n-1}+.......a_{n-1}x+a_n$$

$$=\left[(n-1)C_0x^{n-2}+(n-2)C_1x^{n-3}+....+C_{n-2}\right]\left(ax^2+bx+c\right)+\frac{1}{2}\left(C_0x^{n-1}+C_1x^{n-2}+.......+C_{n-1}\right)(2ax+b)+C_n$$

Where constant $C_0, C_1, C_2, ..., C_n$ can be evaluate by comparity of like power of x four both sides.

Substituting the values of $C_0, C_1, C_2......C_n$ in (I) and evaluating $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ the given integral is determined completely.

11. $\int \frac{\left(ax^2+bx+c\right)dx}{(dx+e)\sqrt{fx^2+gx+h}}$.

Here, we write, $ax^2+bx+c=A_1(dx+e)(2fx+g)+B_1(dx+e)+C_1$ where A_1, B_1 and C_1 are constants which can be obtained by comparing the coefficient of like terms on both sides. And given integral will reduce to the form

$$A_1\int \frac{(2fx+g)}{\sqrt{fx^2+gx+h}}\,dx + B_1\int \frac{dx}{\sqrt{fx^2+gx+h}} + C_1\int \frac{dx}{(dx+e)\sqrt{fx^2+gx+h}}$$

INTEGRALS OF THE FORM $\int \frac{dx}{a + b\cos x}$, $\int \frac{dx}{a + b\sin x}$ **and** $\int \frac{dx}{a + b\sin x + c\cos x}$

To evaluate such form proceed as follows :

1. Put $\cos x = \dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\sin x = \dfrac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

2. Replace $1 + \tan^2 \frac{x}{2}$ in the numerator by $\sec^2 \frac{x}{2}$.

3. Put $\tan \frac{x}{2} = t$ so that $\frac{1}{2}\sec^2 \frac{x}{2} dx = dt$.

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

1. Integral of the form $\int \sin^m x \cos^n x\, dx$:

   (i) To evaluate the integrals of the form $I = \int \sin^m x \cos^n x\, dx$, where m and n are rational numbers.

   (a) Substitute $\sin x = t$, if n is odd ;

   (b) Substitute $\cos x = t$, if m is odd ;

   (c) Substitute $\tan x = t$, if m + n is a negative even integer; and

   The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$ However, in practice; it sometimes leads to very complex rational function. In some cases, the integral can be simplified by :

   (a) Substitute $\sin x = t$, if the integral is of the form $\int R(\sin x)\cos x\, dx$.

   (b) Substituting $\cos x = t$, if the integral is of the form $\int R(\cos x)\sin x\, dx$.

   (c) Substituting $\tan x = t, i.e. dx = \frac{dt}{1+t^2}$, if the integral is dependent only on $\tan x$.

   (d) If the given function is $R(\sin x + \cos x)\cos 2x$

   Put $\sin x + \cos x = t \quad \Rightarrow 1 + \sin 2x = t^2 \quad \Rightarrow \cancel{2}\cos 2x\, dx = \cancel{2}t\, dt$

   And if function is $R(\sin x - \cos x)\cos 2x$

   Put $\sin x - \cos x = t \quad \Rightarrow 1 - \sin 2x = t^2 \quad \Rightarrow -\cancel{2}\cos 2x\, dx = \cancel{2}t\, dt$ and proceed further.

INTEGRALS OF THE FORM :

(i) $\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c} dx$

In this integral express numerator as l (Denominator) + m (d.c. of denominator) + n.

Find l, m, n by comparing the coefficients of $\sin x$, $\cos x$ and constant term and split the integral into sum of three integrals.

$$l\int dx + m\int \frac{\text{d.c. of (Denominator)}}{\text{Denominator}} dx + n\int \frac{dx}{a\cos x + b\sin x}$$

(ii) $\int \frac{p\cos x + q\sin x}{a\cos x + b\sin x} dx$

Express numerator as l (denominator) + m (d.c. of denominator) and find l and m by comparing the coefficients of $\sin x$, $\cos x$.