Binomial Theorem

INTRODUCTION TO BINOMIAL THEOREM — Section - 1

1.1 Properties of ${}^nC_r$

(i)	${}^nC_0 = {}^nC_n = 1$	**(ii)**	${}^nC_1 = {}^nC_{n-1} = n$
(iii)	${}^nC_r = {}^nC_{n-r}$	**(iv)**	${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
(v)	$r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$	**(vi)**	$r(r-1)\,{}^nC_r = n(n-1)\,{}^{n-2}C_{r-2}$
(vii)	$\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$		

1.2 Binomial Theorem : (Positive Integral Exponent)

The following formula which expands $(x+y)^n$ in powers of x and y is known as the Binomial Theorem.

(A) $(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 + \ldots\ldots + {}^nC_n\, x^0 y^n$

or $$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

In this formula, n is a positive integer, x and y are real or complex numbers and

$${}^nC_r = \frac{n!}{r!(n-r)!},\quad r = 0, 1, 2, 3, \ldots\ldots\ldots\ldots\ldots, n \text{ are binomial coefficients}$$

(B) By replacing y by $-y$, we can also find expansion of $(x-y)^n$

i.e. $(x-y)^n = {}^nC_0\, x^n y^0 - {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 - \ldots\ldots + (-1)^n\, {}^nC_n\, x^0 y^n$

or $$(x-y)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r} y^r$$

1.3 Some important observations in Binomial Theorem

(i) The expansion of $(x+y)^n$ can also be taken as identity in 'x' and 'y'.

(ii) The number of terms in the expansion are $n+1$.

(iii) The expansion contains decreasing powers of x and increasing powers of y. The sum of the powers of x and y in each term is equal to n.

(iv) The binomial coefficients : ${}^nC_0, {}^nC_1, {}^nC_2,$ .... equidistant from beginning and end are equal i.e. ${}^nC_r = {}^nC_{n-r}$.

(v) The corresponding terms in the expansion of $(x + y)^n$ and $(x - y)^n$ are numerically equal.

(vi) The terms in the expansion of $(x - y)^n$ are alternately positive and negative. The last term in the expansion is positive or negative accordingly as n is even or odd integer.

(vi) Put $x = 1$ and $y = x$ in 1.2 (A) we get,

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2\,x^2 + {}^nC_3\,x^3 + \ldots\ldots + {}^nC_r\,x^r + \ldots\ldots + {}^nC_n\,x^n$$

i.e. $$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\,x^r$$

(vii) Put $y = 1$ in 1.2 (A) we get,

$$(x + 1)^n = {}^nC_0\,x^n + {}^nC_1x^{n-1} + {}^nC_2\,x^{n-2} + \ldots\ldots + {}^nC_r\,x^{n-r} + \ldots\ldots + {}^nC_n\,x^0$$

i.e. $$(x + 1)^n = \sum_{r=0}^{n} {}^nC_r\,x^{n-r}$$

(viii) Put $x = 1$ and $y = -x$ in 1.2 (A) we get,

$$(1 - x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2\,x^2 - {}^nC_3\,x^3 + \ldots\ldots + (-1)^r\,{}^nC_r\,x^r + \ldots\ldots + (-1)^n\,{}^nC_n\,x^n$$

i.e. $$(1 - x)^n = \sum_{r=0}^{n} (-1)^r\,{}^nC_r\,x^r$$

(ix) The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$

(x) If we have,

$$(x + y)^n + (x - y)^n = 2\,[{}^nC_0\,x^n\,y^0 + {}^nC_2\,x^{n-2}\,y^2 + \ldots\ldots]$$

Now, the number of terms in $(x + y)^n + (x - y)^n$ is

(A) If 'n' is odd then number of terms is $\dfrac{n+1}{2}$

(B) If 'n' is even then number of terms is $\dfrac{n}{2} + 1$.

(xi) If we have

$$(x + y)^n - (x - y)^n = 2\,[{}^nC_1\,x^{n-1}\,y^1 + {}^nC_3\,x^{n-3}\,y^3 + \ldots\ldots]$$

Now, the number of terms in $(x + y)^n - (x - y)^n$ is

(A) If 'n' is odd, then the number of terms is $\frac{n+1}{2}$.

(B) If 'n' is even, then the number of terms is $\frac{n}{2}$.

(xii) Sometimes ^nC_r is written as C_r.

1.4 General Term

The general term in the expansion is $(r + 1)^{st}$ term. It is represented as T_{r+1}.

In the expansion $(x + y)^n$, $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

In the expansion $(x - y)^n$, $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}\, y^r$

The binomial expansions of $(x + y)^n$ and $(x - y)^n$ can also be represented as

$$\sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, y^r \quad \text{and} \quad \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r}\, y^r \text{ respectively.}$$

(i) In binomial expansion is $(1 + x)^n$ we have,

$$T_{r+1} = {}^nC_r\, x^r$$

(ii) In binomial expansion of $(1 - x)^n$, we have

$$T_{r+1} = (-1)^r\, {}^nC_r\, x^r$$

1.5 Middle Term

The middle term in the expansion depends upon the value of n.

If n is even, then total number of terms in the expansion is odd. So there is only one middle term

i.e. $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

If n is odd, then total number of terms in the expansion is even. So there are two middle terms

i.e. $\left(\frac{n+1}{2}\right)^{th}$ term and the next are two middle terms.

Illustration - 1 *If ${}^{15}C_r : {}^{15}C_{r-1} = 11 : 5$, find r.*

SOLUTION :

$${}^{15}C_r : {}^{15}C_{r-1} = 11 : 5$$

$$\Rightarrow \frac{15!}{r!(15-r)!} \times \frac{(r-1)!(15-(r-1))!}{15!} = \frac{11}{5}$$

$$\Rightarrow \frac{(r-1)!(16-r)!}{r!(15-r)!} = \frac{11}{5}$$

$$\Rightarrow \frac{(r-1)!(16-r)(15-r)!}{r(r-1)!(15-r)!} = \frac{11}{5}$$

$$\Rightarrow (16-r) \times 5 = 11(r)$$

$$\Rightarrow r = 5$$

Illustration - 2 *Show that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.*

SOLUTION :

Consier L.H.S. : $\quad {}^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!(n-r+1) + r(n!)}{r!(n-r+1)!} = \frac{n!(n-r+1+r)}{r!(n-r+1)!}$$

$$= \frac{(n+1)(n!)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!}$$

$$= {}^{n+1}C_r = \text{R.H.S.}$$

Illustration - 3 (i) *Find the 7th term in the expansion of*

$$\left(\frac{4x}{5} - \frac{5}{2x}\right)^9.$$

(ii) *Find the coefficient of x^7 in* $\left(ax^2 + \frac{1}{bx}\right)^{11}$.

SOLUTION :

(i) In the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

The general terms is T_{r+1}

$$= {}^9C_r \left(\frac{4x}{5}\right)^{9-r} \left(-\frac{5}{2x}\right)^r.$$

For 7th term (T_7), Put $r = 6$.

$$\Rightarrow T_7 = T_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6} \left(-\frac{5}{2x}\right)^6$$

$$\Rightarrow T_7 = \frac{9 \times 8 \times 7}{3!} \left(\frac{4}{5}\right)^3 x^3 \left(-\frac{5}{2}\right)^6 \frac{1}{x^6}$$

$\Rightarrow \quad T_7 = \frac{9 \times 8 \times 7}{3!} 5^3 \frac{1}{x^3}$

$\Rightarrow \quad T_7 = \frac{10500}{x^3}$

(ii) In $\left(ax^2 + \frac{1}{bx}\right)^{11}$ general term is

$T_{r+1} = {}^{11}C_r (ax^2)^{11-r} (1/bx)^r$

$T_{r+1} = {}^{11}C_r\, a^{11-r}\, b^{-r}\, x^{22-3r}$

For term involving x^7, $22 - 3r = 7$

$\Rightarrow \quad r = 5$

Hence T_{5+1} or the 6th term will contain x^7.

$T_6 = {}^{11}C_5 (ax^2)^{11-5} \left(\frac{1}{bx}\right)^5$

$= \frac{11 \times 10 \times 9 \times 8 \times 7}{5!} \frac{a^6}{b^5} x^7 = \frac{462\, a^6}{b^5} x^7$

Hence the coefficient of x^7 is $\frac{462\, a^6}{b^5}$.

Illustration - 4

Find the term independent of x in $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$.

SOLUTION :

$T_{r+1} = {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(-\frac{1}{3x}\right)^x$

$= {}^9C_r (-1)^r \frac{3^{9-2r}}{2^{9-r}} x^{18-3r}$

For term independent of x, $18 - 3r = 0$

$\Rightarrow \quad r = 6$

Hence T_{6+1} or 7th term is independent of x.

$T_7 = {}^9C_6 (-1)^6 \frac{3^{-3}}{2^3} x^0$

$= \frac{9 \times 8 \times 7}{3!} \frac{1}{27 \times 8} = \frac{7}{18}$.

Illustration - 5

Find the coefficient of x^{11} *in the expansion of* $(2x^2 + x - 3)^6$.

SOLUTION :

$(2x^2 + x - 3)^6 = (x - 1)^6 (2x + 3)^6$

Term containing x^{11} in $(2x^2 + x - 3)^6$

$(x - 1)^6 = {}^6C_0 x^6 - {}^6C_1 x^5 + {}^6C_2 x^4 - {}^6C_3 x^3 + \ldots$

$(2x + 3)^6 = {}^6C_0 (2x)^6 + {}^6C_1 (2x)^5 3 + {}^6C_2 (2x)^4 3^2 + \ldots\ldots\ldots$

Term containing x^{11} in the product

$(x - 1)^6 (2x + 3)^6$

$= \left[{}^6C_0 x^6\right]\left[{}^6C_1 (2x)^5 3\right] - \left[{}^6C_1 x^5\right]\left[{}^6C_0 (2x)^6\right]$

$= 32 (18 x^{11}) - 6 (64) x^{11} = 192 x^{11}$

The coefficient of x^{11} is 192.

Illustration - 6 *Find the coefficient of x^3 in the expansion $(1 + x + x^2)^n$.*

SOLUTION :

$(1 + x + x^2)^n = [1 + x(1 + x)]^n$

$= {}^nC_0 + {}^nC_1\, x(1 + x)$

$+ {}^nC_2\, x^2 (1 + x)^2 + \ldots$

Coefficient of $x^3 = {}^nC_2$ [coeff of x in $(1 + x)^2$] $+ {}^nC_3$ [coeff of x^0 in $(1 + x)^3$]

$= {}^nC_2 (2) + {}^nC_3 (1)$

$$= \frac{2n(n-1)}{2} + \frac{n(n-1)(n-2)}{3!}$$

$$= \frac{n(n-1)}{6}[6 + n - 2]$$

$$= \frac{n(n-1)(n+4)}{6}$$

Illustration - 7 *The coefficient of x^{20} in the expansion of* $(1 + x^2)^{40} \cdot \left(x^2 + 2 + \frac{1}{x^2}\right)^{-5}$ *is :*

(A) ${}^{30}C_{10}$ **(B)** ${}^{30}C_{25}$ **(C)** 1 **(D)** *None of these*

SOLUTION : (B)

Expression $= (1 + x^2)^{40} \cdot \left(x + \frac{1}{x}\right)^{-10} = (1 + x^2)^{30} \cdot x^{10}$

The coefficient of x^{20} in $x^{10}(1 + x^2)^{30}$

= the coefficient of x^{10} in $(1 + x^2)^{30}$ [By using coefficient of x^r in $(1 + x^2)^n$ is ${}^nC_{r/2}$]

$= {}^{30}C_5 = {}^{30}C_{25}$.

Illustration - 8 *The coefficient of x^6 in $\{(1 + x)^6 + (1 + x)^7 + \ldots + (1 + x)^{15}\}$ is :*

(A) ${}^{16}C_9$ **(B)** ${}^{16}C_5 - {}^6C_5$ **(C)** ${}^{16}C_6 - 1$ **(D)** *None of these*

SOUTION :

Given expression is G.P. with first term $(1 + x)^6$ and common ratio $(1 + x)$.

$$\text{Expression} = \frac{(1+x)^6\{1 - (1+x)^{10}\}}{1 - (1 - x)}$$

i.e. sum of 10 term of G.P. $= \dfrac{(1+x)^{16} - (1+x)^6}{x}$

$\therefore$ the required coefficient

= the coefficient of x^7 in $\{(1 + x)^{16} - (1 + x)^6\}$

$= {}^{16}C_7 = {}^{16}C_9$.

Illustration - 9 *The coefficient of x^{13} in the expansion of $(1-x)^5(1+x+x^2+x^3)^4$ is :*

(A) 4 **(B)** – 4 **(C)** 0 **(D)** *None of these*

SOLUTION : (A)

The given expression is $(1-x)^5(1+x)^4(1+x^2)^4$

$= (1-x^2)^4(1+x^2)^4(1-x) = (1-x^4)^4(1-x)$

$\therefore$ Coeff. of x^{13} in $\{(1-x)(1-4x^4+6x^8-4x^{12}+x^{16})\} = 4.$

Illustration - 10 *The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^n$ is :*

(A) ${}^nC_n$ **(B)** ${}^nC_4 + {}^nC_2$ **(C)** ${}^nC_4 + {}^nC_1 + {}^nC_4 - {}^nC_2$ **(C)** ${}^nC_4 + {}^nC_2 + {}^nC_1 \cdot {}^nC_2$

SOLUTION : (D)

The given expression is G.P. with first term 1 and the common ratio x

$(1+x+x^2+x^3)^n = ((1+x)(1+x^2))^n$

$= (1+x)^n(1+x^2)^n$

$= ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + \ldots) \times ({}^nC_0 + {}^nC_1x^2 + {}^nC_2x^4 + \ldots)$

$\therefore$ Coeff. of $x^4 = {}^nC_0 \cdot {}^nC_2 + {}^nC_2 \cdot {}^nC_1 + {}^nC_4 \cdot {}^nC_0$

$= {}^nC_2 + {}^nC_1 \cdot {}^nC_2 + {}^nC_4.$

Illustration - 11 *If the 6th term in the expansion of* $\left(\frac{1}{x^{8/3}} + x^2 \log_{10} x\right)^8$ *is 5600, find the value of x.*

SOLUTION :

We have $T_6 = 5600$

$\Rightarrow T_{5+1} = 5600$

$\Rightarrow {}^8C_5\left(\frac{1}{x^{8/3}}\right)^{8-5}(x^2\log_{10} x)^5 = 5600$

$\Rightarrow 56\, x^2(\log_{10} x)^5 = 5600$

$\Rightarrow x^2(\log_{10} x)^5 = 100$

$\Rightarrow x^2(\log_{10} x)^5 = 10^2$

$\Rightarrow x^2(\log_{10} x)^5 = 10^2(\log_{10} 10)^5$

$\Rightarrow x = 10.$

Illustration - 12 *Find the number of terms in the expansions of the following :*

(i) $(2x - 3y)^9$ **(ii)** $(\sqrt{x} + \sqrt{y})^{10} + (\sqrt{x} - \sqrt{y})^{10}$ **(iii)** $(2x + 3y - 4z)^n$

(iv) $[(3x + y)^8 - (3x - y)^8]$ **(v)** $(1 + 2x + x^2)^{20}$

SOLUTION :

(i) The expansion of $(x + a)^n$ has $(n + 1)$ terms. So, the expansion of $(2x - 3y)^9$ has 10 terms.

(ii) If n is even, then the expansion of $\{(x + a)^n + (x - a)^n\}$ has $\left(\frac{n}{2} + 1\right)$ terms. So,

$$(\sqrt{x} + \sqrt{y})^{10} + (\sqrt{x} - \sqrt{y})^{10} \text{ has 6 terms.}$$

(iii) We have, $(2x + 3y - 4z)^n = \{2x + (3y - 4z)\}^n$

$$= {}^nC_0\,(2x)^n\,(3y - 4z)^0 + {}^nC_1\,(2x)^{n-1}\,(3y - 4z)^1 + {}^nC_2\,(2x)^{n-2}\,(3y - 4z)^2 + \ldots + {}^nC_{n-1}\,(2x)^1\,(3y - 4z)^{n-1} + {}^nC_n\,(3y - 4z)^n$$

Clearly, the first term in the above expansion gives one term, second term gives two terms, third term gives three terms and so on.

So, Total number of terms $= 1 + 2 + 3 + \ldots + n + (n + 1) = \dfrac{(n+1)(n+2)}{2}$.

(iv) If n is even, then $\{(x + a)^n - (x - a)^n\}$ has $\frac{n}{2}$ terms. So, $(3x + y)^8 - (3x - y)^8$ has 4 terms.

(v) We have $(1 + 2x + x^2)^{20} = [(1 + x)^2]^{20} = (1 + x)^{40}$.

So, there are 41 terms in the expansion of $(1 + 2x + x^2)^{20}$.

Illustration - 13 *The number of non-zero terms in the expansion of* $(1 + 3\sqrt{2}x)^9 + (1 - 3\sqrt{2}x)^9$ *is :*

(A) 9 **(B)** 0 **(C)** 5 **(D)** 10

SOLUTION : (C)

The given expression

$$= 2\,[1 + {}^9C_2\,(3\sqrt{2}x)^2 + {}^9C_4\,(3\sqrt{2}x)^4 + {}^9C_6\,(3\sqrt{2}x)^6 + {}^9C_8\,(3\sqrt{2}\,x)^8]$$

[i.e. all the terms which contains odd power of x will be cancel out]

$\therefore$ The number of non-zero terms is 5.

Illustration - 14 *In the expansion of* $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$, *the number of terms is :*

(A) 7 **(B)** 14 **(C)** 6 **(D)** 4

SOLUTION : (D)

On expansion and simplification,

expression = $2\ \{{}^6C_0\ x^6 + {}^6C_2\ x^4\ (x^2-1) + {}^6C_4\ x^2\ (x^2-1)^2 + {}^6C_6\ (x^2-1)^3\}$

$= 2\ \{({}^6C_0 + {}^6C_2 + {}^6C_4 + {}^6C_6)\ x^6 + (-\ {}^6C_2 - {}^6C_2 - {}^6C_4 \times 2 - {}^6C_6 \times 3)\ x^4 + ({}^6C_4 + {}^6C_6 \times 3)\ x^2 - {}^6C_6\}.$

Aliter :

As number of terms in $(x+y)^n+(x-y)^n$ is $\left(\frac{n}{2}+1\right)$, if n is even.

$\therefore$ number of term in,

$\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$ is $\left(\frac{6}{2}+1\right)=4$.

Thus number of terms = 4.

Illustration - 15 *The middle term in the expansion of* $\left(x^2+\frac{1}{x^2}+2\right)^n$ is :

(A) $\frac{n!}{[(n/2)!]^2}$ **(B)** $\frac{2n!}{[(n/2)!]^2}$ **(C)** $\frac{1.3.5....(2n+1)}{n!}2^n$ **(D)** $\frac{(2n)!}{(n!)^2}$

SOLUTION : (D)

The given expression $=\left\{\left(x+\frac{1}{x}\right)^2\right\}^n=\left(x+\frac{1}{x}\right)^{2n}$.

The number of terms = $2n+1$, which is odd. The middle term $= t_{\frac{(2n+1)+1}{2}} = t_{n+1}$.

$= {}^{2n}C_n\ x^{2n-n}\cdot\left(\frac{1}{x}\right)^n=\frac{(2n)!}{(n)!\,(n)!}$

$=\frac{1.2.3.4.5.6....(2n-1).2n}{(n)!\,(n)!}$

$=\frac{[1.3.5...(2n-1)]\,2^n\,[1.2.3...n]}{(n)![1.2.3...n]}$ $=\frac{1.3.5...(2n-1)}{(n)!}\cdot 2^n$.

Illustration - 16

Find the middle terms in the expansion of $\left(3x - \dfrac{x^3}{6}\right)^7$.

SOLUTION :

The given expression = $\left(3x - \dfrac{x^3}{6}\right)^7$. Here $n = 7$, which is an odd number.

So, $\left(\dfrac{7+1}{2}\right)^{th}$ and $\left(\dfrac{7+1}{2}+1\right)^{th}$ i.e. 4th and 5th terms are two middle terms.

$$\text{Now, } T_4 = T_{3+1} = {}^7C_3\,(3x)^{7-3}\left(-\frac{x^3}{6}\right)^3 = (-1)^3\,{}^7C_3\,(3x)^4\left(\frac{x^3}{6}\right)^3$$

$$= -35 \times 81\,x^4 \times \frac{x^9}{216} = -\frac{105x^{13}}{8}$$

$$\text{and } T_5 = T_{4+1} = {}^7C_4\,(3x)^{7-4}\left(-\frac{x^3}{6}\right)^4 = {}^7C_4\,(3x)^3\left(-\frac{x^3}{6}\right)^4$$

$$= 35 \times 27\,x^3 \times \frac{x^{12}}{1296} = \frac{35\,x^{15}}{48}.$$

Hence, the middle terms are $-\dfrac{105\,x^{13}}{8}$ and $\dfrac{35\,x^{15}}{48}$.

Illustration - 17

Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\dfrac{1\cdot3\cdot5\ldots(2n-1)}{n!}\,2^n\cdot x^n$.

SOLUTION :

The given expression = $(1 + x)^{2n}$. Here, the index $2n$ is even.

So, $\left(\dfrac{2n}{2}+1\right)^{th}$ i.e. $(n+1)^{th}$ term is the middle term.

Hence, the middle term = $T_{n+1} = {}^{2n}C_n\,(1)^{2n-n}\,x^n$

$$= {}^{2n}C_n\,x^n = \frac{(2n)!}{(2n-n)!\,n!}x^n = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\ldots(2n-3)(2n-2)(2n-1)(2n)}{n!\,n!}x^n$$

$$= \frac{\{1\cdot3\cdot5\ldots(2n-3)\,(2n-1)\}\,\{2\cdot4\cdot6\ldots(2n-2)\,(2n)\}}{n!\,n!}x^n$$

$$= \frac{\{1\cdot3\cdot5\ldots(2n-3)\,(2n-1)\}\,\{1\cdot2\cdot3\ldots(n-1)\,(n)\}\,2^n}{n!\,n!}x^n$$

$$= \frac{\{1\cdot3\cdot5\ldots(2n-3)(2n-1)\}\, n!\cdot 2^n \cdot x^n}{n!\,n!} \qquad = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!} 2^n\, x^n.$$

Illustration - 18 *Find the relation between r and n so that coefficient of $3r^{th}$ and $(r + 2)^{th}$ terms of $(1 + x)^{2n}$ are equal.*

SOLUTION :

In $(1 + x)^n$, $\quad T_{r+1} = {}^{2n}C_r\, x^r$

$T_{3r} = {}^{2n}C_{3r-1}\, x^{3r-1}$

$T_{r+2} = {}^{2n}C_{r+1}\, x^{r+1}$

If the coefficients are equal then ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

There are two possibilities

Case 1 :

$3r - 1 = r + 1$

$\Rightarrow \quad r = 1$

$\Rightarrow \quad T_{3r} = T_3$ and $T_{r+2} = T_3$

$\Rightarrow \quad T_{3r}$ and T_{r+2} are same terms.

Case 2 :

${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow \quad {}^{2n}C_{3r-1} = {}^{2n}C_{2n-(r+1)}$

$\Rightarrow \quad 3r - 1 = 2n - (r + 1)$

$\Rightarrow \quad r = n/2$

Illustration - 19 *If a_1, a_2, a_3, and a_4 are the coefficients of any four consecutive terms in the expansion of $(1 + x)^n$, prove that* $\dfrac{a_1}{a_1 + a_2} + \dfrac{a_3}{a_3 + a_4} = \dfrac{2a_2}{a_2 + a_3}$.

SOLUTION :

Let a_1 = coefficient of $T_{r+1} = {}^nCr$

$\Rightarrow \quad a_2 = {}^nC_{r+1}, \quad a_3 = {}^nC_{r+2}, \quad a_4 = {}^nC_{r+3}$

$$\Rightarrow \quad \frac{a_1}{a_1 + a_2} + \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \frac{{}^nC_r}{{}^{n+1}C_{r+1}} = \frac{r+1}{n+1}$$

$$\text{and} \quad \frac{a_3}{a_3 + a_4} + \frac{{}^nC_{r+2}}{{}^nC_{r+2} + {}^nC_{r+3}} = \frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} = \frac{r+3}{n+1}$$

$$\text{L.H.S.} = \frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2(r+2)}{n+1}$$

$$\text{R.H.S.} = \frac{2a_2}{a_2 + a_3} = \frac{2\,{}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}} = \frac{2^n\, C_{r+1}}{{}^{n+1}C_{r+2}} = \frac{2(r+2)}{n+1},$$

Hence R.H.S. = L.H.S.

1.6 Greatest Term

To find the numerically greatest term in the expansion of $(1+x)^n$:

(i) Calcualate $m = \left|\dfrac{x(n+1)}{x+1}\right|$

(ii) If m is an integer, then T_m and T_{m+1} are equal and both are greatest terms.

(iii) If m is not an integer, then $T_{[m]+1}$ is the greatest term, where $[m]$ is the integral part of m.

Some observations :

(a) Numerically the greatest term in the expansion of $(1-x)^n$, $x > 0$, $n \in N$ is the same as the greatest term in $(1+x)^n$.

(b) To find greatest term in the expansion of $(x+y)^n$, write it as $x^n(1+y/x)^n$ and then find greatest term in $(1+y/x)^n$.

Illustration - 20 *The greatest term* (nemerically) *in the expansion of* $(2+3x)^9$, *when* $x = 3/2$, *is :*

(A) $\dfrac{5\times 3^{11}}{2}$ (B) $\dfrac{5\times 3^{13}}{2}$ (C) $\dfrac{7\times 3^{13}}{2}$ (D) *None of these*

SOLUTION : (C)

We have,

$$(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9 = 2^9\left(1+\frac{9}{4}\right)^9$$

$$\left(\text{As } x = \frac{3}{2}\right)$$

$$\therefore \quad r = \left|\frac{x(n+1)}{(x+1)}\right| = \left|\frac{\left(\frac{9}{4}\right)(9+1)}{\left(\frac{9}{4}\right)+1}\right|$$

$$= \frac{90}{13} = 6\frac{12}{13} \neq \text{integer}$$

The greatest term in the expansion is

$$T_{[r]+1} = T_{6+1} = T_7.$$

Hence the greatest term $= 2^9 . T_7$

$$= 2^9 \cdot T_{6+1} = 2^9 \cdot {}^9C_6\left(\frac{9}{4}\right)^6$$

$$= 2^9 \cdot {}^9C_3\left(\frac{9}{4}\right)^6 = 2^9 \cdot \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{3^{12}}{2^{12}} = \frac{7\times 3^{13}}{2}.$$

Illustration - 21 *The greatest term* (numerically) *in the expansion of* $(3 - 5x)^{11}$, *when* $x = 1/5$ *is :*

(A) 55×3^9 **(B)** 46×3^9 **(C)** 55×3^6 **(D)** *None of these*

SOLUTION : (A)

We have,

$$(3-5x)^{11} = 3^{11}\left(1-\frac{5x}{2}\right)^{11} = 3^{11}\left(1-\frac{1}{3}\right)^{11}$$

$$\left(\text{As } x = \frac{1}{5}\right)$$

$$\therefore \quad r = \frac{|x|(n+1)}{(|x|+1)} \qquad \left(-\frac{1}{3} < 0\right)$$

$$= \frac{\left(\left|-\frac{1}{3}\right|\right)(11+1)}{\left(\left|-\frac{1}{3}\right|+1\right)} = 3$$

The greatest terms in the expansion are T_3 and T_4

$\therefore$ Greatest term (when $r = 2$) = $3^{11}\,|T_{2+1}|$

$$= 3^{11}\left|{}^{11}C_2\left(-\frac{1}{3}\right)^2\right| = 3^{11}\left|\frac{11.10}{1.2}\times\frac{1}{9}\right| = 55\times 3^9$$

and greatest term (when $r = 3$) = $3^{11}\,|T_{3+1}|$

$$= 3^{11}\left|{}^{11}C_3\left(-\frac{1}{3}\right)^3\right| = 3^{11}\left|\frac{11.10.9}{1.2.3}\times\left(-\frac{1}{27}\right)\right|$$

$= 55 \times 3^9$

From above we see that the values of both greatest terms are equal.

1.7 Greatest Coefficient

(i) When n is even, greatest coefficient = ${}^nC_{\frac{n}{2}}$.

(ii) When n is odd, greatest coefficient = ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$ (Note : both of them are equal)

SERIES INVOLVING BINOMIAL COEFFICIENTS & THEIR PROPERTIES Section - 2

2.1 (A) $C_0 + C_1 + C_2 + C_3 + \ldots\ldots + C_n = 2^n$ or $\sum_{r=0}^{n} {}^nC_r = 2^n$ $[C_r = {}^nC_r]$

Proof :

$$\text{LHS} = \sum_{r=0}^{n} {}^nC_r = C_0 + C_1 + C_2 + C_3 + \ldots + C_n$$

Now, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$

Putting $x = 1$, we get : $(1+1)^n = C_0 + C_1(1) + C_2(1)^2 + \ldots + C_n(1)^n$

$\Rightarrow$ $C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n = \text{RHS}$. Hence proved.

$\Rightarrow$ Sum of binomial coefficients in expansion $(1+x)^n$ is 2^n.

(B) $1\,C_1 + 2\,C_2 + 3\,C_3 + \ldots\ldots + nC_n = n \cdot 2^{n-1}$ or $\sum_{r=1}^{n} r \cdot {}^nC_r = n \cdot 2^{n-1}$

Proof :

$$\text{LHS} = \sum_{r=1}^{n} r\,C_r = (1)\,C_1 + (2)\,C_2 + (3)\,C_3 + \ldots + (n)\,C_n$$

$\Rightarrow$ $\sum_{r=1}^{n} r\,C_r = \sum_{r=1}^{n} n\,({}^{n-1}C_{r-1})$ [using $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$]

$= n\left[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots + {}^{n-1}C_{n-1}\right]$

$= n\,(1+1)^{n-1}$ [In 2.1(A), replace n by $n-1$]

$= n\,2^{n-1} = \text{RHS}$. Hence proved.

★ **Calculus Method :** *Note that it involves knowledge of Calculus, you can leave this now and do it later after finish Calculus.*

Given series is $C_1 + 2\,C_2 + 3\,C_3 + \ldots + n\,C_n$ [Here last term of the series is $n \cdot {}^nC_n$]

Consider Binomial identity,

$$(1+x)^n = C_0 + C_1 \cdot x + C_2 \cdot x^2 + \ldots + C_n \cdot x^n$$

Differentiating both sides w.r.t. x, we get :

$$n\,(1+x)^{n-1} = 0 + C_1 \cdot 1 + C_2 \cdot 2x + \ldots + C_n \cdot nx^{n-1}$$

The above expression is an identity so, putting $x = 1$, we get :

$$n \cdot 2^{n-1} = C_1 + 2 \cdot C_2 + 3 \cdot C_3 + \ldots + n \cdot C_n$$

$$\sum_{r=1}^{n} r \cdot {}^{n}C_r = n \cdot 2^{n-1}$$ Hence proved.

(C) $1^2\,C_1 + 2^2\,C_2 + 3^2\,C_3 + \ldots\ldots + n^2\,C_n = n\,(n+1)\,2^{n-2}$

or $$\sum_{r=1}^{n} r^2 \cdot C_r = n\,(n+1)\,2^{n-2}$$

Proof :

$$\text{LHS} = \sum r^2\,C_r = \sum_{r=1}^{n} (r^2 - r + r)\,C_r$$

$$= \sum_{r=1}^{n} r\,(r-1)\,C_r + \sum_{r=1}^{n} r\,C_r$$

$$= \sum_{r=1}^{n} n\,(n-1)\,{}^{n-2}C_{r-2} + \sum_{r=1}^{n} n\,{}^{n-1}C_{n-1}$$ [Using $r\,{}^{n}C_r = n\,{}^{n-1}C_{r-1}$]

$$= n\,(n-1)\sum_{r=2}^{n} {}^{n-2}C_{r-2} + n\sum_{r=1}^{n} {}^{n-1}C_{n-1}$$

[In first $\sum$, $r = 1$ is rejected as it makes expression meaningless]

$= n\,(n-1)\,2^{n-2} + n\,2^{n-1}$ [Replace n by $n-2$ and $n-1$ in 2.1(A)]

$= n\,(n+1)\,2^{n-2}$.

★ **Calculus Method :** *Note that it involves knowledge of Calculus, you can leave this now and do it later after finish Calculus.*

Given series is $1^2 \cdot C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots\ldots + n^2 \cdot C_n$

Given, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots\ldots + C_n x^n$

Differentiating both sides w.r.t. x, we get :

$$n\,(1+x)^{n-1} = C_1 + C_2 \cdot 2x + C_3 \cdot 3x^2 + \ldots\ldots + C_n \cdot nx^{n-1}$$

Multiplying both sides by x, we get :

$$n\,x\,(1+x)^{n-1} \cdot x = C_1 x + C_2 \cdot 2x^2 + C_3 \cdot 3x^3 + \ldots + C_n \cdot nx^n$$

Differentiating both sides w.r.t. x, we get :

$$n\,([1 \cdot (1+x)^{n-1} + x\,(n-1)\,(1+x)^{n-2} \cdot 1]) = C_1 + C_2 \cdot 2^2 x + C_3 \cdot 3^2\,x^2 + \ldots\ldots + C_n \cdot n^2\,x^{n-1}$$

The above expression is an identity so putting $x = 1$, we get :

$$n\,[2^{n-1} + (n-1)2^{n-2}] = C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots\ldots + n^2 \cdot C_n$$

$$n\,2^{n-2}\,(2 + n - 1) = n\,(n+1)\,2^{n-2} = C_1 + 2^2 \cdot C_2 + 3^2 \cdot C_3 + \ldots\ldots + n^2 \cdot C_n$$

2.2 **(A)** $C_0 - C_1 + C_2 - C_3 + \ldots\ldots\ldots + (-1)^n C_n = 0$ or $\sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$ $[C_r = {}^nC_r]$

Proof :

We have, $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$

The above expression is an identity so, putting $x = -1$, we get :

$$0 = C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n \quad \text{or} \quad \sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$$

(B) $1\,C_1 - 2\,C_2 + 3\,C_3 - \ldots\ldots + (-1)^n\, nC_n = 0$ or $\sum_{r=1}^{n} (-1)^r\, r \cdot {}^nC_r = 0$

The proof of this result is similar to the proof of the result 2.1(B). You should replace $x = -1$ instead of $x = 1$ just before the last step.

(C) $1^2\,C_1 - 2^2\,C_2 + 3^2\,C_3 - \ldots\ldots + (-1)^n n^2\,C_n = 0$ or $\sum_{r=1}^{n} (-1)^r\, r^2 \cdot C_r = 0$

The proof of this result is similar to the proof of the result 2.1(C). You should replace $x = -1$ instead of $x = 1$ just before the last step.

Illustration - 22 *If $C_0, C_1, C_2, \ldots . C_n$ denote the coefficients in the binomial expansion of $(1 + x)^n$, prove that:*

(i) $C_0 + 2\,C_1 + 3\,C_2 + \ldots + (n+1)\,C_n = (n+2) \cdot 2^{n-1}$.

(ii) $C_0 + 3\,C_1 + 5\,C_2 + \ldots + (2n+1)\,C_n = (n+1) \cdot 2^n$

SOLUTION :

(i) First Method : We will apply formulae given in section 2.1 to find this summation of series.

We have :

$C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \ldots + (n+1)\,C_n$

$$= \sum_{r=0}^{n} (r+1)\,{}^nC_r = \sum_{r=0}^{n} (r \cdot {}^nC_r + {}^nC_r)$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r \quad = n \cdot 2^{n-1} + 2^n = (n+2) \cdot 2^{n-1}$$

[Using 2.1 (A) and 2.1 (B)]

★ **Calculus Method :** *Note that it involves knowledge of Calculus, you can leave this now and do it later after finish Calculus.*

Given series is $C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \ldots + (n+1)\, C_n$

[Here last term of the series is $(n+1) \cdot {}^nC_n$]

According to question,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_n \cdot x^n$$

Multiplying both sides by x, we get :

$$x\,(1+x)^n = C_0\, x + C_1\, x^2 + C_2\, x^3 + \ldots + C_n\, x^{n+1}$$

Diff. both sides w.r. to x, we get :

$$1 \cdot (1+x)^n + x \cdot n\,(1+x)^{n-1} = C_0 + C_1 \cdot 2x + C_2 \cdot 3x^2 + \ldots + C_n \cdot (n+1)\, x^n$$

The above expression is an identity so, putting $x = 1$, we get :

$$C_0 + 2 \cdot C_1 + 3 \cdot C_2 + \ldots + (n+1)\, C_n = 2^n + 1 \cdot n \cdot 2^{n-1} = 2^{n-1}\,(n+2)$$

(ii) First Method : We will apply formulae given in section 2.1 to find this summation of series.

We have :

$C_0 + 3\, C_1 + 5\, C_2 + \ldots + (2n+1)\, C_n$

$$= \sum_{r=0}^{n} (2r+1)\, {}^nC_r = \sum_{r=0}^{n} (2r \cdot {}^nC_r + {}^nC_r)$$

$$= 2 \sum_{r=0}^{n} r \cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r$$

$$= 2n \cdot 2^{n-1} + 2^n = (n+1)2^n$$

★ **Calculus Method :** *Note that it involves knowledge of Calculus, you can leave this now and do it later after finish Calculus.*

Given series is $C_0 + 3 \cdot C_1 + 5 \cdot C_2 + \ldots + (2n+1)\, C_n$

[Here last term of the series is $(2n+1) \cdot {}^nC_n$]

According to question, $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n \cdot x^n$

Putting x^2 in place of x, we get : $(1 + x^2)^n = C_0 + C_1 x^2 + C_2 x^4 + ... + C_n x^{2n}$

Multiplying both sides by x, we get : $x (1 + x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + ... + C_n x^{2n+1}$

Diff. both sides w.r. to x, we get : $1 \cdot (1 + x^2)^n + x \cdot n (1 + x^2)^{n-1} 2x$

$$= C_0 + C_1 \cdot 3x^2 + C_2 \cdot 5x^4 + ... + C_n \cdot (2n + 1) x^{2n}$$

The above expression is an identity so, putting $x = 1$, we get :

$$C_0 + 3 \cdot C_1 + 5 \cdot C_2 + ... + (2n + 1) C_n = 2^n + n2^{n-1} \cdot 2 = 2^n(1 + n)$$

Note : It is advisable to use formula method instead of calculus method as formula method takes less time to apply.

Illustration - 23 *If $C_0, C_1, C_2, ...\ C_n$ denote the coefficients in the binomial expansion of $(1 + x)^n$, prove that: $1^3 \cdot C_1 + 2^3 \cdot C_2 + 3^3 \cdot C_3 + ... + n^3 \cdot C_n = n^2 (n + 3) 2^{n-3}$.*

SOLUTION :

We have :

$$1^3 \cdot C_1 + 2^3 \cdot C_2 + 3^3 \cdot C_3 + ... + n^3 \cdot C_n = n^2 (n + 3) 2^{n-3}.$$

$$= \sum_{r=1}^{n} r^3 \cdot {}^nC_r$$

$$= \sum_{r=1}^{n} (r^3 - 3r^2 + 2r + 3r^2 - 2r)\ {}^nC_r$$

$$= \sum_{r=1}^{n} [r (r-1)(r-2) + 3r (r-1) + r]\ {}^nC_r$$

$$= \sum_{r=1}^{n} r (r-1)(r-2)\ {}^nC_r + \sum_{r=1}^{n} 3r (r-1)\ {}^nC_r + \sum_{r=1}^{n} r \cdot {}^nC_r$$

$$= \sum_{r=3}^{n} r (r-1)(r-2)\ {}^nC_r + 3\sum_{r=2}^{n} r (r-1)\ {}^nC_r + \sum_{r=1}^{n} r \cdot {}^nC_r$$

$$= n (n-1)(n-2) \sum_{r=3}^{n} {}^{n-3}C_{r-3} + 3n (n-1) \sum_{r=2}^{n} {}^{n-2}C_{r-2} + \sum_{r=1}^{n} r \cdot {}^nC_r$$

[By using : $r\ {}^nC_r = n\ {}^{n-1}C_{r-1}$]

$= n\,(n-1)\,(n-2)\,(1+1)^{n-3} + 3 \cdot n\,(n-1)\,(1+1)^{n-2} + n - 2^{n-1}$

$= [(n-1)\,(n-2) + 6\,(n-1) + 4] \cdot n \cdot 2^{n-3} = n\,(n^2 + 3n)\,2^{n-3}$

$= n^2\,(n+3)\,2^{n-3}$

Note : We have applied formula given in Section 2.1 to solve this problem. Had we used calculus method to solved this problem, it would have taken much more time to solve.

Illustration - 24 *If $C_0, C_1, C_2, \ldots C_n$ denote the coefficients in the binomial expansion of $(1 + x)^n$, prove that:*

(i) $a - (a-1)\,C_1 + (a-2)\,C_2 - (a-3)\,C_3 + \ldots + (-1)^n\,(a-n)\,C_n = 0$

(ii) $a\,C_0 - (a+d)\,C_1 + (a+2d)\,C_2 - (a+3d)\,C_3 + \ldots + (-1)^n\,(a+nd)\,C_n = 0$

SOLUTION :

(i) We have :

$a - (a-1)\,C_1 + (a-2)\,C_2 - (a-3)\,C_3 + \ldots + (-1)^n\,(a-n)\,C_n$

$$= \sum_{r=0}^{n} (-1)^r\,(a-r)\,{}^nC_r$$

$$= a \sum_{r=0}^{n} (-1)^r\,{}^nC_r - \sum_{r=0}^{n} (-1)^r \cdot r \cdot {}^nC_r$$

i.e. $= 0 + 0$ [By using 2.2 (A) and 2.2 (B)]

(ii) We have :

$a\,C_0 - (a+d)\,C_1 + (a+2d)\,C_2 - (a+3d)\,C_3 + \ldots + (-1)^n\,(a+nd)\,C_n$

$$= \sum_{r=0}^{n} (-1)^r\,(a+rd)\,C_r$$

$$= \sum_{r=0}^{n} (-1)^r\,(a+rd)\,{}^nC_r$$

$$= a \sum_{r=0}^{n} (-1)^r\,{}^nC_r + d \sum_{r=0}^{n} (-1)^r\,{}^nC_r\,(r)$$

i.e. $= 0 + 0$ [By using 2.2 (A) and 2.2 (B)]

SERIES INVOLVING PRODUCT OF TWO BINOMIAL COEFFICIENTS — Section - 3

3.1 **(A)** $C_0^2 + C_1^2 + C_2^2 + \dots C_n^2 = {}^{2n}C_n$ or $\sum_{r=0}^{n} C_r^2 = {}^{2n}C_n$

Working Rule :

1. Identify the two series which we want to multiply. Express one of them as increasing power of x and other in terms of decreasing power of x.
2. Multiply them and decide the appropriate power of 'x' which has to be compared on both sides.
3. Find the coefficient of the corresponding term in LHS.

Proof :

$$\sum_{r=0}^{n} C_r{}^2 = C_0^2 + C_1^2 + C_2^2 + C_3^2 + \dots C_n^2$$

$$C_0 \cdot C_0 + C_1 \cdot C_1 + C_2 \cdot C_2 + C_3 \cdot C_3 + \dots + C_n \cdot C_n$$

Consider the identities

$$(1 + x)^n = C_0 + C_1 x + C_2 x_2 + C_3 x_3 + \dots + C_n x_n \qquad \dots (i)$$

$$(x + 1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + C_3 x^{n-3} \dots + C_n \qquad \dots (ii)$$

Multiply these identities we get another identities

$$(1 + x)^n (x + 1)^n = (C_0 + C_1 x + C_2 x^2 + \dots\dots + C_n x^n) \times (C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots\dots + C_n)$$

$$(1 + x)^{2n} = (C_0 + C_1 x + C_2 x^2 + \dots\dots + C_n x^n) \times (C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots\dots + C_n)$$

Compare coefficients of x^n on both sides.

In LHS, coeff. of x^n = coeff. of x^n in $(1 + x)^{2n} = {}^{2n}C_n$

In RHS., terms containing x^n are

$$C_0{}^2 x^n + C_1{}^2 x^n + C_2{}^2 x^n + \dots\dots\dots + C_n{}^2 x^n$$

$\Rightarrow$ Coeff. of x^n on RHS $= C_0{}^2 + C_1{}^2 + C_2{}^2 + \dots\dots\dots + C_n{}^2$

Equating the coefficients,

$$C_0{}^2 + C_1{}^2 + C_2{}^2 + \dots\dots\dots C_n{}^2 = {}^{2n}C_n$$ Hence proved.

(B) $1 \cdot C_1^2 + 2 \cdot C_2^2 + 3 \cdot C_3^2 + \dots + n\, C_n^2 = n\ {}^{2n-1}C_{n-1}$ or $\sum_{r=1}^{n} r \cdot C_r^2 = n \cdot {}^{2n-1}C_{n-1}$

$$= \sum_{r=1}^{n} r\, C_r^2 = 1 \cdot C_1^2 + 2 \cdot C_2^2 + 3 \cdot C_3^2 + \dots + n\, C_n^2$$

$$= 1 \cdot C_1 \cdot C_1 + 2 \cdot C_2 \cdot C_2 + 3 \cdot C_3 \cdot C_3 + \dots + nC_n \cdot C_n$$

Now consider those identities which contain all these coefficient,

i.e. $n(1+x)^{n-1} = 1 \cdot C_1 + 2\,C_2\,x + 3 \cdot C_3\,x^2 + \ldots + nC_n\,x^{n-1}$

$(x+1)^n = C_0\,x^n + C_1\,x^{n-1} + C_2\,x^{n-2} + \ldots + C_n$

Multiplying these identities we get another identities :

i.e. $n(1+x)^{n-1}(x+1)^n$

$= (1 \cdot C_1 + 2\,C_2\,x + 3 \cdot C_3\,x^2 + \ldots + {}^nC_n\,x^{n-1}) \times (C_0\,x^n + C_1\,x^{n-1} + \ldots + C_n)$

$= n(1+x)^{2n-1}$

$= (1 \cdot C_1 + 2\,C_2\,x + 3 \cdot C_3\,x^2 + \ldots + {}^nC_n\,x^{n-1}) \times (C_0\,x^n + C_1\,x^{n-1} + \ldots + C_n)$

Compare coefficient of x^{n-1} on both sides.

In LHS, coefficient of x^{n-1} = coefficient of x^{n-1} in $n \cdot (1+x)^{2n-1} = n \cdot {}^{2n-1}C_{n-1}$.

In RHS term containing x^{n-1} are $1 \cdot C_1^2\,x^{n-1} + 2 \cdot C_2^2\,x^{n-1} + 3 \cdot C_3^2\,x^{n-1} + \ldots + n \cdot C_n^2\,x^{n-1}$

= Coefficient of x^{n-1} on RHS = $1 \cdot C_1^2 + 2 \cdot C_2^2 + 3 \cdot C_3^2 + \ldots + n\,C_n^2$

Equating the coefficients

$= 1 \cdot C_1^2 + 2 \cdot C_2^2 + 3 \cdot C_3^2 + \ldots + n\,C_n^2 = n\,{}^{2n-1}C_{n-1}$ Hence proved.

(C) $1^2 \cdot C_1^2 + 2^2 \cdot C_2^2 + 3^2 \cdot C_3^2 + \ldots + n^2 \cdot C_n^2 = n^2\,{}^{2n-2}C_{n-1}$

or $\sum_{r=1}^{n} r^2\,C_r^2 = n^2\,{}^{2n-2}C_{n-1}$

$= \sum_{r=1}^{n} r^2\,C_r^2 = 1^2 \cdot C_1^2 + 2^2 \cdot C_2^2 + 3^3 \cdot C_3^2 + \ldots + n^2\,C_n^2$

$= 1 \cdot C_1 \cdot 1 \cdot C_1 + 2 \cdot C_2 \cdot 2 \cdot C_2 + 3 \cdot C_3 \cdot 3 \cdot C_3 + \ldots + nC_n \cdot n \cdot C_n$

Now consider those identities which contain all these coefficient,

i.e. $n(1+x)^{n-1} = 1 \cdot C_1 + 2\,C_2\,x + 3 \cdot C_3\,x^2 + \ldots + {}^nC_n\,x^{n-1}$

$n(x+1)^{n-1} = C_1\,x^{n-1} + 2 \cdot C_2\,x^{n-2} + 3 \cdot C_3\,x^{n-3} + \ldots + n \cdot C_n$

Multiplying these identities we get another identities :

i.e. $n^2(1+x)^{n-1}(x+1)^{n-1} = (1 \cdot C_1 + 2\,C_2\,x + 3 \cdot C_3\,x^2 + \ldots + {}^nC_n\,x^{n-1})$

$\times (C_1\,x^{n-1} + 2 \cdot C_2\,x^{n-2} + 3 \cdot C_3\,x^{n-3} + \ldots + n \cdot C_n)$

$n^2(1+x)^{2n-2} = (1 \cdot C_1 + 2\,C_2\,x + 3 \cdot C_3\,x^2 + \ldots + {}^nC_n\,x^{n-1})$

$\times (C_1\,x^{n-1} + 2 \cdot C_2\,x^{n-2} + 3 \cdot C_3\,x^{n-3} + \ldots + n \cdot C_n)$

Compare coefficient of x^{n-1} on both sides.

In LHS, coefficient of x^{n-1} = coefficient of x^{n-1} in $n^2 \cdot (1+x)^{2n-2} = n^2 \cdot {}^{2n-2}C_{n-1}$.

In RHS term containing x^{n-1} are $1^2 \cdot C_1^2\ x^{n-1} + 2^2 \cdot C_2^2\ x^{n-1} + 3^2 \cdot C_3^2\ x^{n-1} + \ldots + n^2 \cdot C_n^2\ x^{n-1}$

= Coefficient of x^{n-1} on RHS = $1^2 \cdot C_1^2 + 2^2 \cdot C_2^2 + 3^2 \cdot C_3^2 + \ldots + n^2\ C_n^2$

Equating the coefficients

$$=1^2 \cdot C_1^2 + 2^2 \cdot C_2^2 + 3^2 \cdot C_3^2 + \ldots + n^2\ C_n^2 = n^2\ {}^{2n-2}C_{n-1}$$ Hence proved.

Illustration - 25 *If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots C_nx^n$. Prove that :*

(i) $C_0 C_1 + C_1C_2 + C_2C_3 + \ldots\ldots\ldots\ldots + C_{n-1} C_n = \dfrac{(2n)!}{(n-1)!(n+1)!}$

(ii) $C_0 C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots\ldots\ldots\ldots + C_{n-r} C_n = \dfrac{(2n)!}{(n-r)!(n+r)!}$

(iii) $1C_1 C_r + 2C_2 C_{r+1} + 3C_3 C_{r+2} + \ldots\ldots\ldots\ldots + (n-r+1)\ C_{n-r+1} C_n = n\ {}^{2n-1}C_{n-r}$.

SOLUTION :

(i) Consider the identities

$$(1+x)^n = C_0 + C_1\ x + C_2\ x^2 + \ldots\ldots\ldots\ldots\ldots + C_n\ x^n$$
$$(1+x)^n = C_0\ x^n + C_1\ x^{n-1} + C_2\ x^{n-2} + \ldots\ldots\ldots\ldots + C_n$$

Multiplying these we get another identity.

$$(1+x)^n (x+1)^n = (C_0 + C_1\ x + C_2\ x^2 + \ldots\ldots\ldots + C_nx^n) \times (C_0\ x^n + C_1\ x^{n-1} + C_2\ x^{n-2} + \ldots\ldots\ldots + C_n)$$

Compare the coefficients of x^{n-1} on both sides.

In L.H.S., coefficient of $x^{n-1} = {}^{2n}C_{n-1}$

In R.H.S., term containing x^{n-1} is $C_0\ C_1\ x^{n-1} + C_1\ C_2\ x^{n-1} + \ldots\ldots$

Hence coefficient of x^{n-1} in R.H.S. = $C_0\ C_1 + C_1\ C_2 + C_2\ C_3 + \ldots\ldots.$

Equation the coefficients,

$$C_0\ C_1 + C_1\ C_2 + \ldots\ldots + C_{n-1}\ C_n = {}^{2n}C_{n-1} = \frac{(2n)!}{(n-1)!(n+1)!}$$

(ii) Consider the identities

$$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots\ldots\ldots + C_n x^n$$

$$(1 + x)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots. + C_r x^{n-r} + C_{r+1} x^{n-r-1} + C_{r+2} x^{n-r-2} + \ldots\ldots. + C_n$$

Multiplying these we get another identity.

$$(1 + x)^n (x + 1)^n = (C_0 + C_1 x + C_2 x^2 + \ldots\ldots.. + C_n x^n) \times (C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots. + C_r x^{n-r} + C_{r+1} x^{n-r-1} + C_{r+2} x^{n-r-2} + \ldots\ldots. + C_n)$$

Compare the coefficients of x^{n-r} on both sides.

In L.H.S., coefficient of $x^{n-r} = {}^{2n}C_{n-r}$

In R.H.S., term containing x^{n-r} is $C_0 C_r x^{n-r} + C_1 C_{r+1} x^{n-r} + \ldots\ldots$

Hence coefficient of x^{n-r} in R.H.S. $= C_0 C_r x^{n-r} + C_1 C_{r+1} x^{n-r} + \ldots\ldots$

Equation the coefficients,

$$C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \ldots\ldots\ldots. C_{n-r} C_n = \frac{(2n)!}{(n-r)!(n+r)!}$$

(iii) Consider the identities

$$n(1 + x)^{n-1} = 1 \cdot C_1 + 2 C_2 x + 3 \cdot C_3 x^2 + \ldots. + {}^nC_n x^{n-1} \quad (1 + x)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots. + C_r x^{n-r} + C_{r+1} x^{n-r-1} + C_{r+2} x^{n-r-2} + \ldots\ldots. + C_n$$

Multiplying these we get another identity.

$$n(1 + x)^{n-1} (x + 1)^n = (1 \cdot C_1 + 2 C_2 x + 3 \cdot C_3 x^2 + \ldots. + {}^nC_n x^{n-1}) \times (C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \ldots. + C_r x^{n-r} + C_{r+1} x^{n-r-1} + C_{r+2} x^{n-r-2} + \ldots\ldots. + C_n)$$

Compare the coefficients of x^{n-r} on both sides.

In L.H.S., coefficient of $x^{n-r} = n\,{}^{2n-1}C_{n-r}$

In R.H.S., term containing x^{n-r} is $1C_1 C_r + 2C_2 C_{r+1} + 3C_3 C_{r+2} + \ldots.. + (n - r + 1) C_{n-r+1} C_n$

Hence coefficient of of x^{n-r} in R.H.S. $= 1C_1 C_r + 2C_2 C_{r+1} + 3C_3 C_{r+2} + \ldots.. + (n - r + 1) C_{n-r+1} C_n$

Equation the coefficients,

$$1C_1 C_r + 2C_2 C_{r+1} + 3C_3 C_{r+2} + \ldots.. + (n - r + 1) C_{n-r+1} C_n = n\,{}^{2n-1}C_{n-r}$$

Illustration - 26 *If ^nC_r is denoted as C_r, show that*

(A) $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \ldots\ldots\ldots (C_{n-1} + C_n) = \dfrac{C_0 C_1 \ldots. C_{n-1}}{n!}$.

(B) $\dfrac{C_1}{C_0} + 2\dfrac{C_2}{C_1} + 3\dfrac{C_3}{C_2} + \ldots\ldots + n\dfrac{C_n}{C_{n-1}} = \dfrac{n(n+1)}{2}$.

SOLUTION :

(A) L.H.S. $= (C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \ldots\ldots\ldots (C_{n-1} + C_n)$

Multiply and Divide by $C_0\, C_1\, C_2 \ldots\ldots C_{n-1}$

$$= C_0\, C_1\, C_2 \ldots\ldots C_{n-1}\left(1+\frac{C_1}{C_0}\right)\left(1+\frac{C_2}{C_1}\right)\ldots\ldots\left(1+\frac{C_n}{C_{n-1}}\right) \qquad \left[\text{using } \frac{C_r}{C_{r-1}} = \frac{n-r+1}{r}\right]$$

$$= C_0\, C_1\, C_2\, C_3 \ldots\ldots\ldots C_{n-1}\left(1+\frac{n-1+1}{1}\right)\times\left(1+\frac{n-2+1}{2}\right)+\ldots\ldots+\left(1+\frac{n-n+1}{n}\right)$$

$$= C_0\, C_1\, C_2 \ldots\ldots\ldots C_{n-1}\left(\frac{n+1}{1}\right)\left(\frac{n+1}{2}\right)\times\ldots\ldots\times\left(\frac{n+1}{n}\right)$$

$$= C_0\, C_1\, C_2\, C_3 \ldots\ldots\ldots C_{n-1}\,\frac{(n+1)^n}{n!} = \text{R.H.S.}$$

(B) L.H.S. $= \dfrac{C_1}{C_0} + 2\dfrac{C_2}{C_1} + 3\dfrac{C_3}{C_2} + \ldots\ldots + n\dfrac{C_n}{C_{n-1}}$ $\qquad \left[\text{using } \dfrac{C_r}{C_{r-1}} = \dfrac{n-r+1}{r}\right]$

$$= \left(\frac{n-1+1}{1}\right) + 2\left(\frac{n-2+1}{2}\right) + \ldots\ldots + n\frac{(n-n+1)}{n}$$

$$= n + (n-1) + (n-2) + \ldots\ldots\ldots\ldots\ldots\ldots + 1$$

$=$ Sum of first n natural numbers

$$= \frac{n(n+1)}{2} = \text{R.H.S}$$

SERIES INVOLVING THE FORM ^nC_r / integer — Section - 4

Working Rule :

1. Write down the rth term of the series and then use the formula $\frac{^nC_r}{r+1} = \frac{^{n+1}C_{r+1}}{n+1}$.
2. Or Calculus Method.

Illustration - 27 *If* $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, *show that*

(i) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(ii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - ... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

(iii) $2 \cdot C_0 + 2^2 \cdot \frac{C_1}{2} + 2^3 \cdot \frac{C_2}{3} + ... + 2^{n+1} \cdot \frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(iv) $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \frac{C_7}{8} + ... = \frac{2^n - 1}{n+1}$

SOLUTION :

(i) First Method :

Given series is $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1}$

$$T_r = \sum_{r=0}^{n} \frac{^nC_r}{r+1}$$

$$T_r = \sum_{r=1}^{n} \frac{^{n+1}C_{r+1}}{n+1} \qquad \left[\text{As } \frac{^nC_r}{r+1} = \frac{^{n+1}C_{r+1}}{n+1}\right]$$

$$= \frac{1}{n+1}(^{n+1}C_1 + {^{n+1}C_2} + ... + {^{n+1}C_{n+1}})$$

$$= \frac{1}{n+1}(^{n+1}C_0 + {^{n+1}C_1} + {^{n+1}C_2} + ... + {^{n+1}C_{n+1}} - {^{n+1}C_0})$$

[Adding and subtracting $^{n+1}C_0$]

$$= \frac{1}{n+1}(2^{n+1} - 1) = \frac{2^{n+1}-1}{n+1}$$

[Using $^{n+1}C_0 = 1$ and formula given in section 2.1(A)]

(ii) First Method :

Given series is $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \frac{C_n}{n+1}$

$$T_r = \sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{r+1}$$

$$T_r = \sum_{r=1}^{n} (-1)^r \frac{{}^{n+1}C_{r+1}}{n+1} \qquad \left[\text{As } \frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}\right]$$

$$= \frac{1}{n+1}[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \ldots + (-1)^n \cdot {}^{n+1}C_{n+1}]$$

$$= \frac{1}{n+1}[- {}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 \cdot + {}^{n+1}C_3 - \ldots + (-1)^n \cdot {}^{n+1}C_{n+1} + {}^{n+1}C_0]$$

[Adding and subtracting ${}^{n+1}C_0$]

$$= \frac{1}{n+1}[-\{{}^{n+1}C_0 - {}^{n+1}C_1 + \ldots + (-1)^{n+1} \cdot {}^{n+1}C_{n+1}\} + {}^{n+1}C_0]$$

$$= \frac{1}{n+1}[-(1-1)^{n+1} + {}^{n+1}C_0] = \frac{1}{n+1}$$

[Using ${}^{n+1}C_0 = 1$ and formula given in section 2.1(A)]

(iii) First Method :

Given series is $2 \cdot C_0 + 2^2 \cdot \frac{C_1}{2} + 2^3 \cdot \frac{C_2}{3} + \ldots\ldots + 2^{n+1} \cdot \frac{C_n}{n+1}$

$$T_r = \sum_{r=0}^{n} \frac{{}^nC_r}{r+1} 2^{r+1}$$

$$T_r = \sum_{r=1}^{n} \frac{{}^{n+1}C_{r+1}}{n+1} 2^{r+1} \qquad \left[\text{As } \frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}\right]$$

$$= \frac{1}{n+1}[{}^{n+1}C_1 \cdot 2^1 + {}^{n+1}C_2 \cdot 2^2 + {}^{n+1}C_3 \cdot 2^3 + \ldots + {}^{n+1}C_{n+1} \cdot 2^{n+1}]$$

$$= \frac{1}{n+1}[\{{}^{n+1}C_0 + {}^{n+1}C_1 \cdot 2 + {}^{n+1}C_2 \cdot 2^2 + \ldots + {}^{n+1}C_{n+1} \cdot 2^{n+1} - {}^{n+1}C_0\}]$$

[Adding and subtracting ${}^{n+1}C_0$]

$$= \frac{1}{n+1}[\{1+2)^{n+1} - 1] = \frac{3^{n+1} - 1}{n+1}$$

(iv) First Method :

Given series is $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+\ldots..$

$$=\frac{1}{2}\left[\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+\ldots\right]-\frac{1}{2}\left[\frac{C_0}{1}-\frac{C_1}{2}+\frac{C_2}{3}-\ldots\right]$$

$$\left[\text{Adding and substracting } \frac{1}{2}\left[\frac{C_0}{1}-\frac{C_2}{3}+\ldots\right]\right]$$

$$=\frac{1}{2}\sum_{r=0}^{n}\frac{{}^nC_r}{r+1}-\frac{1}{2}\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{r+1}$$

$$=\frac{1}{2}\sum_{r=0}^{n}\frac{{}^{n+1}C_{r+1}}{n+1}-\frac{1}{2}\sum_{r=0}^{n}(-1)^r\frac{{}^{n+1}C_{r+1}}{n+1}$$

$$=\frac{1}{2(n+1)}\left[\left(2^{n+1}-1\right)-1\right]=\frac{1}{2(n+1)}\left[2^{n+1}-2\right]=\frac{2^n-1}{n+1}$$

Calculus Method : *Note that is involves knowledge of Calculus, you can leave this now and do it later after you finish Calculus.*

(i) $(1+x)^n=C_0+C_1x+C_2x^2+\ldots\ldots+C_nx^n$ **. . . (i)**

Integrating between limits 0 and 1, we get :

$$\int_0^1(1+x)^n\,dx=\int_0^1 C_0\,dx+\int_0^1 C_1\,x\,dx+\int_0^1 C_2\,x^2\,dx+\ldots+\int_0^1 C_n\,x^n\,dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1=\frac{C_0x}{1}\Big|_0^1+C_1\frac{x^2}{2}\Big|_0^1+C_2\cdot\frac{x^3}{3}\Big|_0^1+\ldots+C_n\cdot\frac{x^{n+1}}{n+1}\Big|_0^1$$

$$\Rightarrow\quad \frac{2^{n+1}}{n+1}-\frac{1}{n+1}=C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}$$

$$\Rightarrow\quad C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$$ **. . . (A)**

(ii) $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_n x^n$, . . . (i)

Integrating between limits 0 and – 1, we get :

$$\int_0^{-1} (1+x)^n \, dx = \int_0^{-1} \left[C_0 + C_1 x + \ldots + C_n \, x^n \right] dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^{-1} = \left[C_0 x + C_1 \cdot \frac{x^2}{2} + \frac{C_2 x^3}{3} + \ldots + \frac{C_n x^{n+1}}{n+1} \right]_0^{-1}$$

$$\Rightarrow \quad \frac{(1-1)^{n+1}}{n+1} - \frac{1}{n+1} = -C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \ldots + (-1)^{n+1}, \frac{C_n}{n+1}$$

$$\Rightarrow \quad -\frac{1}{n+1} = -\left[C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \cdot \frac{C_n}{n+1} \right]$$

$$\Rightarrow \quad C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \ldots + (-1)^n \cdot \frac{C_n}{n+1} = \frac{1}{n+1} \qquad \ldots \text{(B)}$$

(iii) $(1+x)^n = C_0 + C_1x + C_2 x^2 + \ldots + C_n x^n$,

Integrating between limits 0 and 2, we get :

$$\int_0^{2} (1+x)^n \, dx = \int_0^{2} \left[C_0 + C_1 x + \ldots + C_n \, x^n \right] dx$$

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^{2} = \left[C_0 x + C_1 \cdot \frac{x^2}{2} + C_2 \cdot \frac{x^3}{3} + \ldots C_n \cdot \frac{x^{n+1}}{n+1} \right]_0^{2}$$

$$\Rightarrow \quad \frac{3^{n+1} - 1}{n+1} = C_0 \cdot 2 + 2^2 \cdot \frac{C_1}{2} + 2^3 \cdot \frac{C_2}{3} + \ldots + 2^{n+1} \cdot \frac{C_n}{n+1}$$

(iv) Subtracting (B) from (A), we get :

$$2\left[\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots \right] = \frac{2^{n+1} - 1 - 1}{n+1} = \frac{2^{n+1} - 2}{n+1} = \frac{2\,(2^n - 1)}{n+1}$$

$$\Rightarrow \quad \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots = \frac{2^n - 1}{n+1}$$

★★

Illustration - 28 *If* $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_n x^n$ *then evaluate :*

(i) $\frac{C_0}{1.2} + \frac{C_1}{2.3} + \frac{C_2}{3.4} + \ldots \frac{C_n}{n(n+1)}$ (ii) $\frac{C_0}{1.2.3} + \frac{C_1}{2.3.4} + \frac{C_2}{3.4.5} + \ldots \frac{C_n}{n(n+1)(n+2)}$

SOLUTION :

(i) First Method :

We have $\frac{C_0}{1.2} + \frac{C_1}{2.3} + \frac{C_2}{3.4} + \ldots \frac{C_n}{n(n+1)}$

Therefore $T_r = \frac{{}^nC_r}{(r+1)(r+2)}$

The sum of $(n+1)$ terms is :

$$\sum_{r=0}^{n} T_r = \sum_{r=0}^{n} \frac{C_r}{(r+1)(r+2)}$$

$$= \sum_{r=0}^{n} \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)} \quad \text{[By using : } (r+1)(r+2)\,{}^{n+2}C_{r+2} = (n+1)(n+2)\,{}^nC_r]$$

$$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} {}^{n+2}C_{r+2}$$

$$= \frac{1}{(n+1)(n+2)} [{}^{n+2}C_2 + {}^{n+2}C_3 + {}^{n+2}C_4 + \ldots + {}^{n+2}C_{n+2}]$$

$$\Rightarrow \quad = \frac{1}{(n+1)(n+2)} [(1+1)^{n+2} - {}^{n+2}C_0 - {}^{n+2}C_1]$$

$$= \frac{2^{n+2} - (n+3)}{(n+1)(n+2)}$$

(ii) First Method :

We have $\frac{C_0}{1.2.3} + \frac{C_1}{2.3.4} + \frac{C_2}{3.4.5} + \ldots \frac{C_n}{n(n+1)(n+2)}$

Therefore $T_r = \frac{C_r}{(r+1)(r+2)(r+3)}$

The sum of $(n+0)$ terms is :

$$\sum_{r=0}^{n} T_r = \sum_{r=0}^{n} \frac{C_r}{(r+1)(r+2)(r+3)}$$

$$= \sum_{r=0}^{n} \frac{{}^{n+3}C_{r+3}}{(n+1)(n+2)(n+3)}$$

[By using : $(r + 1)(r + 2)(r + 3)\ {}^{n+3}C_{r+3} = (n + 1)(n + 2)(n + 3)\ {}^{n}C_{r}$]

$$= \frac{1}{(n+1)(n+2)(n+3)} \sum_{r=0}^{n} {}^{n+3}C_{r+3}$$

$$= \frac{1}{(n+1)(n+2)(n+3)} [{}^{n+3}C_3 + {}^{n+3}C_4 + {}^{n+3}C_5 + \ldots + {}^{n+3}C_{n+3}]$$

$$\Rightarrow \quad \frac{1}{(n+1)(n+2)(n+3)} [(1+1)^{n+3} - {}^{n+3}C_0 - {}^{n+3}C_1 - {}^{n+3}C_2]$$

$$= \frac{2^{n+1} - (n+4)(n+3) - 2}{2(n+1)(n+2)(n+3)} = \frac{2^{n+4} - n^2 - 7n - 14}{2(n+1)(n+2)(n+3)}$$

Calculus Method : *Note that it involves knowledge of Calculus, you can leave this now and do it later after you finish Calculus.*

(i) Given series is $\frac{C_0}{1.2} + \frac{C_1}{2.3} + \frac{C_2}{3.4} + \ldots. \frac{C_n}{n(n+1)}$

Given, $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_n x^n$

Integrating both sides w.r.t x between 0 to x, we get :

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_0^x = \left.\frac{C_0 x}{1}\right|_0^x + \left.\frac{C_1 x^2}{2}\right|_0^x + \left.\frac{C_2 x^3}{3}\right|_0^x + \ldots. + \left.\frac{C_n x^{n+1}}{n+1}\right|_0^x$$

$$\frac{(1+x)^{n+1} - 1}{n+1} = \frac{C_0 x}{1} + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots. \frac{C_n x^{n+1}}{n+1}$$

Again integrating both side w.r.t x between 0 to 1, we get :

$$\left.\frac{(1+x)^{n+2}}{n+1} - \frac{x}{n+1}\right|_0^1 = \left.\frac{C_0 x^2}{1.2}\right|_0^1 + \left.\frac{C_1 x^3}{2.3}\right|_0^1 + \left.\frac{C_2 x^4}{3.4}\right|_0^1 + \ldots. + \left.\frac{C_n x^{n+2}}{(n+1)(n+2)}\right|_0^1$$

$$\frac{2^{n+1} - (n+3)}{(n+1)(n+2)} = \frac{C_0}{1.2} + \frac{C_1}{2.3} + \frac{C_2}{3.4} + \ldots. \frac{C_n}{n(n+1)}$$

(ii) Given series is $\dfrac{C_0}{1.2.3} + \dfrac{C_1}{2.3.4} + \dfrac{C_2}{3.4.5} + \ldots \dfrac{C_n}{n(n+1)(n+2)}$

Given, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots\ldots + C_n x^n$

Integrating both sides w.r.t x between 0 to x, we get :

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_0^x = \left.\frac{C_0 x}{1}\right|_0^x + \left.\frac{C_1 x^2}{2}\right|_0^x + \left.\frac{C_2 x^3}{3}\right|_0^x + \ldots + \left.\frac{C_n x^{n+1}}{n+1}\right|_0^x$$

$$\frac{(1+x)^{n+1} - 1}{n+1} = \frac{C_0 x}{1} + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + \ldots \frac{C_n x^{n+1}}{n+1}$$

Again integrating both side w.r.t x between 0 to x, we get :

$$\left.\frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{(n+1)}\right|_0^x = \left.\frac{C_0 x^2}{1.2}\right|_0^x + \left.\frac{C_1 x^3}{2.3}\right|_0^x + \left.\frac{C_2 x^4}{3.4}\right|_0^x + \ldots + \left.\frac{C_n x^{n+2}}{(n+1)(n+2)}\right|_0^x$$

$$\left[\frac{(1+x)^{n+2}}{(n+1)(n+2)} - \frac{x}{(n+1)}\right] - \left[\frac{1}{(n+1)(n+2)}\right] = \frac{C_0 x^2}{1.2} + \frac{C_1 x^3}{2.3} + \frac{C_2 x^4}{3.4} + \ldots + \frac{C_n x^{n+2}}{(n+1)(n+2)}$$

Again integrating both side w.r.t x between 0 to 1,
we get

$$\left.\left[\frac{(1+x)^{n+3}}{(n+1)(n+2)(n+3)} - \frac{x^2}{2(n+1)}\right] - \frac{x}{(n+1)(n+2)}\right|_0^1$$

$$= \left.\frac{C_0 x^3}{1.2.3}\right|_0^1 + \left.\frac{C_1 x^4}{2.3.4}\right|_0^1 + \ldots + \left.\frac{C_n x^{n+3}}{(n+1)(n+2)(n+3)}\right|_0^1$$

$$\Rightarrow \frac{2^{n+4} - (n+3)(n+4) - 2}{2(n+1)(n+2)(n+3)} = \frac{C_0}{1.2.3} + \frac{C_1}{2.3.4} + \frac{C_2}{3.4.5} + \ldots \frac{C_n}{n(n+1)(n+2)}$$

Note : Above expression can be solved using partial fraction method.

★★

Illustration - 29 *If* $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_n x^n$ *then evaluate :*

(i) $\dfrac{C_0}{2} + \dfrac{C_1}{3} + \dfrac{C_2}{4} + \ldots \dfrac{C_n}{n+2}$ **(ii)** $\dfrac{C_0}{3} + \dfrac{C_1}{4} + \dfrac{C_2}{5} + \ldots \dfrac{C_n}{n+3}$

Note that is involves knowledge of Calulus, you can leave this now and do it later after you finish Calculus.

SOLUTION :

(i) Consider the given series : $(1 + x)^n = C_0 + C_1x + C_2x^2 + . . . + C_nx^n$

Multiply both side by 'x' and integrate between limit 0 to 1.

$$\Rightarrow \int_0^1 x.(1+x)^n .dx = \int_0^1 C_0 x + \int_0^1 C_1 x^2 + \int_0^1 C_2 x^3 + \ldots + \int_0^1 C_n x^{n+1}$$

$$\Rightarrow \int_0^1 x.(1+x)^n .dx = \left.\frac{C_0 x^2}{2}\right|_0^1 + \left.\frac{C_1 x^3}{3}\right|_0^1 + \ldots + \left.\frac{C_n}{n+2} x^{n+1}\right|_0^1$$

$$\Rightarrow \int_0^1 x.(1+x)^n .dx = \frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \ldots + \frac{C_n}{n+2}$$

Now consider $\int_0^1 x.(1+x)^n .dx$

Put $1 + x = t$ $\Rightarrow$ $dx = dt$ and $x = 0, t = 1$ and $x = 1, t = 2$

$$\Rightarrow \int_0^2 (t-1).t^n\, dt \Rightarrow \int_0^2 (t^{n+1} - t^n).dt$$

$$\Rightarrow \left.\frac{t^{n+2}}{n+2}\right|_1^2 - \left.\frac{t^{n+1}}{n+1}\right|_1^2 = \left[\frac{2^{n+2}}{n+2} - \frac{1}{n+2}\right] - \left[\frac{2^{n+1}}{n+1} - \frac{1}{n+1}\right]$$

$$= \frac{2^{n+2}}{n+2} - \frac{2^{n+1}}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} = \frac{n.2^{n+1}+1}{(n+1)(n+2)}$$

(ii) Consider the given series : $(1 + x)^n = C_0 + C_1x + C_2x^2 + . . . + C_nx^n$

Multiply both side by 'x^2' and integrate between limit 0 to 1.

$$\Rightarrow \int_0^1 x^2 .(1+x)^n .dx = \int_0^1 C_0 x^2 + \int_0^1 C_1 x^3 + \int_0^1 C_2 x^4 + \ldots + \int_0^1 C_n x^{n+2}$$

$$\Rightarrow \int_0^1 x^2 .(1+x)^n .dx = \left.\frac{C_0 x^3}{3}\right|_0^1 + \left.\frac{C_1 x^4}{4}\right|_0^1 + \ldots + \left.\frac{C_n x^4}{n+3} x^{n+3}\right|_0^1$$

$$\Rightarrow \int_0^1 x^2 .(1+x)^n .dx = \frac{C_0}{3} + \frac{C_1}{4} + \frac{C_2}{5} + \ldots + \frac{C_n}{n+3}$$

Now consider $\int_0^1 x^2 .(1+x)^n .dx$

Put $\quad 1+x=t \quad \Rightarrow \quad dx=dt$ and $x=0,\ t=1$ and $x=1,\ t=2$

$$\Rightarrow \int_0^1 (t-1)^2 .t^n\ dt \Rightarrow \left.\frac{t^{n+3}}{n+3}+\frac{t^{n+1}}{n+1}-\frac{2t^{n+2}}{n+2}\right|_1^2 = \frac{2^{n+3}-1}{n+3}-2\left(\frac{2^{n+2}-1}{n+2}\right)+\left(\frac{2^{n+1}-1}{n+1}\right)$$

Illustration - 30

Prove that $\sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1}=0,$ *where* $k=3n/2$ *and* n *is an even positive integer.*

SOLUTION :

Let $n=2n \Rightarrow k=3m$

$$\text{LHS}=\sum_{r=1}^{3m}(-3)^{r-1}\ {}^{6m}C_{2r-1}={}^{6m}C_1-3\,{}^{6m}C_3+9\,{}^{6m}C_5-\ldots\ldots\ldots\ldots+(-3)^{3m-1}\ {}^{6m}C_{6m-1} \qquad \text{...(i)}$$

Consider $\quad (1+x)^{6m}={}^{6m}C_0+{}^{6m}C_1x+{}^{6m}C_2x^2+\ldots\ldots\ldots+{}^{6m}C_{6m}x^{6m}$ and

$$(1-x)^{6m}={}^{6m}C_0-{}^{6m}C_1x+{}^{6m}C_2x^2+\ldots\ldots\ldots\ldots+{}^{6m}C_{6m}x^{6m}$$

On subtracting the above two relationships, we get :

$$(1+x)^{6m}-(1+x)^{6m}=2({}^{6m}C_1x+{}^{6m}C_3x^3+{}^{6m}C_5x^5+\ldots\ldots\ldots+{}^{6m}C_{6m-1}x^{6m-1})$$

Divide both side by $2x$ to get :

$$\frac{(1+x)^{6m}-(1-x)^{6m}}{2x}={}^{6m}C_1+{}^{6m}C_3x^2+\ldots\ldots+{}^{6m}C_{6m-1}x^{6m-2}$$

Put $x=\sqrt{3}\,i$ in the above identity to get :

$$\frac{(1+i\sqrt{3})^{6m}-(1-i\sqrt{3})^{6m}}{2\sqrt{3}\,i}={}^{6m}C_1-3\,{}^{6m}C_3+\ldots\ldots+(-3)^{3m-1}\ {}^{6m}C_{6m-1} \qquad \text{...(ii)}$$

Comparing (i) and (ii), we get :

$$\text{LHS}=\frac{2^{6m}\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{6m}-\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^{6m}\right]}{2\sqrt{3}\,i}$$

$$\Rightarrow \quad \text{LHS} = \frac{2^{6m}[(\cos 2\pi m + i\sin 2\pi m) - (\cos 2\pi n - i\sin 2\pi n)]}{2\sqrt{3}\,i}$$ [using De morvie's Law]

$$\Rightarrow \quad \text{LHS} = \frac{2^{6m} 2i\sin 2\pi m}{2\sqrt{3}\,i} = \frac{2^{6m}\sin 2\pi m}{\sqrt{3}} = 0$$ [because $\sin 2\pi m = 0$]

Illustration - 31 *If* $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, *Show that*

(i) $a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$

(ii) $a_0 - a_1 + a_2 - a_3 + ... + a_{2n} = 1$

(iii) $a_0 + a_3 + a_6 + ... = 3^{n-1}$

(iv) *The value of* a_r *when* $0 \le r \le 2n$ *is* a_{2n-r}

(v) $a_0 + a_1 + a_2 + ... + a_{n-1} = \dfrac{3^n - a_n}{2}$

(vi) $a_0^2 - a_1^2 + a_2^2 - + ... + a_{2n}^2 = a_n$.

SOLUTION :

Given, $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$ *or* $\sum_{r=0}^{2n} a_r x^r$ ...(i)

(i) Putting $x = 1$, we get

$$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + ... + a_{2n} \quad ...(ii)$$

(ii) Putting $x = -1$, we get

$$1^n = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - ... + (-1)^{2n} \cdot a_{2n}$$

$$\Rightarrow \quad 1 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - ... + a_{2n} \quad ...(iii)$$

(iii) Putting $x = 1, \omega$ and ω^2, in (i), we get :

$$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + ... + a_{2n} \quad ...(A)$$

$$0 = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + a_5\omega^2 + a_6 + ... + a_{2n}\omega^{2n} \quad ...(B)$$

$$0 = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + a_5\omega + a_6 + ... + a_{2n}\omega^{4n} \quad ...(C)$$

Adding (A), (B) and (C), we get :

$$3^n = 3(a_0 + a_3 + a_6 + ...)$$

$$\therefore \qquad a_0 + a_3 + a_6 + \ldots = \frac{3^n}{3} = 3^{n-1}$$

(iv) In (i) replace $x \to \dfrac{1}{x}$ to get,

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_{2n}}{x^{2n}}$$

$$\Rightarrow \qquad \frac{(1+x+x^2)^n}{x^{2n}} = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_{2n}}{x^{2n}} = \sum_{r=0}^{2n} a_r\, x^{-r}$$

$$\Rightarrow \qquad (1+x+x^2) = a_0 x^{2n} + a_1 x^{2n-1} + a_2 x^{2n-2} + \ldots + a_{2n} = \sum_{r=0}^{2n} a_r\, x^{2n-r} \qquad \ldots(D)$$

$$\Rightarrow \qquad (1+x+x^2)^n = a_{2n} + a_{2n-1}x + a_{2n-2}x^2 + \ldots + a_0 x^{2n} = \sum_{r=0}^{2n} a_{2n-r} x^r$$

[Reversing the above series]

From (i) and (D)

$$\sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_{2n-r} x^r \qquad \Rightarrow \qquad a_{2n-r} = a_r$$

(v) As $a_r = a_{2n-r}$ *for* $0 \le r \le 2n$

$$\Rightarrow \qquad \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_{2n-r}$$

$$\Rightarrow \qquad a_0 + a_1 + a_2 + a_3 + \ldots + a_{n-1} = a_{2n} + a_{2n-1} + \ldots + a_{n+1}$$

$$\Rightarrow \qquad 2(a_0 + a_1 + a_2 + \ldots + a_{n-1}) + a_n = a_0 + a_1 + a_2 + \ldots + a_{2n}$$

[by adding $a_0 + a_1 + a_2 + a_3 + \ldots + a_n$ on both side]

(vi) Consider the given identity : $(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_{2n}x^{2n}$ ...(i)

Replace x by $-1/x$ in this identity to get :

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - + ... + \frac{a_{2n}}{x^{2n}}$$

$\Rightarrow \quad (1-x+x^2)^n = a_0 x^{2n} - a_1 x^{2n-1} + a_2 x^{2n-2} - + ... + a_{2n}$ ...(ii)

$\Rightarrow \quad 2(a_0 + a_1 + a_2 + ... + a_{n-1}) + a_n = 3^n$ [using (ii)]

$\Rightarrow \quad = a_0 + a_1 + a_2 + ... + a_{n-1} = \dfrac{3^n - a_n}{2}$

Multiply (i) and (ii) and also compare coefficient of x^{2n} on both sides to get :

$$a_0^2 - a_1^2 + a_2^2 - + ... + a_{2n}{}^2 = \text{coefficient of } x^{2n} \text{ in } (1+x+x^2)^n (1-x+x^2)^n$$

$\Rightarrow \quad \text{LHS} = \text{coefficient of } x^{2n} \text{ in } (1+x^2+x^4)^n$

$\Rightarrow \quad \text{LHS} = \text{coefficient of } x^{2n} \text{ in } a_0 + a_1 x^2 + a_2 x^4 + ... + a_n x^{2n} + ... + a_{2n} x^{4n}$

[Replace x by x^2 in (i)]

$\Rightarrow \quad \text{LHS} = a_n$

Hence $a_0^2 - a_1^2 + a_2^2 - + ... + a_{2n}^2 = a_n$

PROBLEM BASED ON DIRECTION EXPANSION — Section - 5

Illustration - 32 *If $(2+\sqrt{3})^n = I + f$ where I and n are positive integers and 0 < f < 1, show that I is an odd integer and (1 – f) (I + f) = 1.*

Working Rule :

Step I : Write the given expression equal to $I + F$, where I is its integral part and F is the fractional part.

Step II: Define G by replacing ‘+’ sign in the given expression by ‘–’. Note that G always lies between 0 and 1.

Step III : Either add G to the expression in Step-I or substrac G from the expression in Step-I so that RHS is an integer.

Step IV : If G is added to the expression in Step-I, then $G + F$ will always come out to the equal to 1 i.e. $G = 1 - F$. If G is subtracted from the expression in Step-I, then G will always come out to be equal to F.

Step V : Obtain the value of the desired expression after getting F interms of G.

SOLUTION :

$(2+\sqrt{3})^n = I + f$

Let $(2-\sqrt{3})^n = f'$

[where $0 < f' < 1$ because $2-\sqrt{3}$ is between 0 and 1.]

Adding the expansions of $(2+\sqrt{3})^n$ and $(2-\sqrt{3})^n$, we get ;

$I + f + f' = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

$= 2\,[C_0\, 2^n + C_2\, 2^{n-2}(\sqrt{3})^2 + \ldots\ldots]$

= even integer ... **(i)**

$\Rightarrow$ $f + f'$ is also an integer

Now $0 < f < 1$ and $0 < f' < 1$

$\Rightarrow$ $0 < f + f' < 2$

The only integer between 0 and 2 is 1.

Hence $f + f' = 1$ ... **(ii)**

Consider (i) :

$I + f + f'$ = even integer

$\Rightarrow$ $I + 1$ = even integer [using (ii)]

$\Rightarrow$ I = odd integer

Also $(I + f)(1 - f) = (I + f)(f')$

$= (2+\sqrt{3})^n (2-\sqrt{3})^n = 1.$

Illustration - 33 *If $(6\sqrt{6}+14)^{2n+1} = P$, prove that the integral part of P is an even integer and $P(f) = 20^{2n+1}$ where f is the fractional part of P.*

SOLUTION :

Let I be the integral part of P.

$\Rightarrow$ $P = I + f = (6\sqrt{6}+14)^{2n+1}$

Let $f' = (6\sqrt{6}-14)^{2n+1}$ ... **(i)**

as $(6\sqrt{6}-14)$ lies between 0

and 1, $0 < f' < 1$

Subtracting f' from $I + f$ to eliminate the irrational terms in R.H.S. of (i)

$I + f - f'$

$= (6\sqrt{6}+14)^{2n+1} - (6\sqrt{6}-14)^{2n+1}$

$= 2[{}^{2n+1}C_1\,(6\sqrt{6})^{2n}(14)$

$+\ {}^{2n+1}C_3\,(6\sqrt{6})^{2n-2}(14)^3 + \ldots\ldots]$

= even integer ... **(ii)**

$\Rightarrow$ $f - f'$ is an integer.

Now $0 < f < 1$ and $0 < f' < 1$

$\Rightarrow$ $0 < f < 1$ and $-1 < -f' < 0$

Adding these two, we get ;

$-1 < f - f' < 1$

$\Rightarrow$ $f - f' = 0$ ... **(iii)**

Consider (ii) :

$I + f - f'$ = even integer

$\Rightarrow$ $I + 0$ = even integer [using (3)]

$\Rightarrow$ integral part of P is even.

Also $Pf = (I + f)\,f = (I + f)\,f'$

$= (6\sqrt{6}+14)^{2n+1}(6\sqrt{6}-14)^{2n+1}$

$= (216-196)^{2n+1} = 20^{2n+1}$

Illustration - 34 *The greatest integer less than or equal to* $(\sqrt{2}+1)^6$ *is :*

(A) 196 **(B)** 197 **(C)** 198 **(D)** 199

SOLUTION :

Let $(\sqrt{2}+1)^6 = I + F$, where I is an integer and $0 < F < 1$. Let $G = (\sqrt{2}-1)^6$. Then,

$I + F + G = (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2[{}^6C_0(\sqrt{2})^6 + \ldots]$ = an integer ... **(i)**

$\therefore$ $F + G = 1$

$\therefore$ Substituting $F + G = 1$ in (i), we get :

$$I + 1 = 2[{}^6C_0(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4 + {}^6C_4(\sqrt{2})^2 + {}^6C_6(\sqrt{2})^0]$$

$\Rightarrow$ I + 1 = 2 [8 + 60 + 30 + 1] $\Rightarrow$ I = 197

PROBLEMS FOR ON SHOWING THAT GIVEN EXPRESSION IS DIVISIBLE BY AN INTEGER — Section - 6

Working Rule :

1. First of all write down the given expression in such a way that there is a term containing n^{th} power of an integer a.
2. If an occurs, then go on subtracting 1, 2, 3, . . . from a and find integer r such that some power of $(a - r)$ is divisible by the number k from which the given expression is to be shown to be divisible.
3. Now write $a_n = [r + (a - r)]^n$ and expand and then collect the terms containing $(a - r)^m$ and higher powers of $(a - r)$ in one bracket if $(a - r)^m$

Illustration - 35 *Show that* $3^{2n+2} - 8n - 9$ *is divisible by* 64 *if* $n \in N$.

SOLUTION :

$3^{2n+2} - 8n - 9 = (1 + 8)^{n+1} - 8n - 9$

$= [1 + (n + 1)\,8 + {}^{n+1}C_2\,8^2 + \ldots] - 8n - 9$

$= {}^{n+1}C_2\,8^2 + {}^{n+1}C_3\,8^3 + {}^{n+1}C_4\,8^4 + \ldots$

$= 64\,[{}^{n+1}C_2 + {}^{n+1}C_3\,8 + {}^{n+1}C_4\,8^2 + \ldots\ldots]$

which is clearly divisible by 64.

Illustration - 36 *Show that $2^{4n} - 2^n(7n + 1)$ is some multiple of the square of 14, where n is a positive intetger.*

SOLUTION :

$2^{4n} - 2^n(7n + 1) = (16)^n - 2^n(7n + 1)$

$= (2 + 14)^n - 2^n \cdot 7n - 2^n$

$= (2^n + {}^nC_1\, 2^{n-1} \cdot 14 + {}^nC_2\, 2^{n-2} . 14^2 + \ldots + 14^n) - 2^n . 7n - 2^n$

$= 14^2 ({}^nC_2\, 2^{n-2} + {}^nC_3\, 2^{n-3}\, 14 + \ldots + 3\, 14^{n-2}) + (2^n + {}^nC_1 . 2^{n-1} . 14 - 2^n . 7n - 2^n)$

$= 14^2 ({}^nC_2\, 2^{n-2} + {}^nC_3\, 2^{n-3} . 14 + \ldots + 14^{n-2}) + (2^n + n2^{n-1} . 2^1 . 7 - 2^n . 7n - 2^n)$

$= 14^2 ({}^nC_2 . 2^{n-2} + {}^nC_3 . 2^{n-3} . 14 + \ldots + 14^{n-2})$ . . . (i)

This is divisible by 14^2 i.e. by 196 for all positive integral values of n.

Note : If $n = 1$, ${}^nC_2 = 0$, ${}^nC_3 = 0$ etc.

$\therefore$ given expression $= 14^2 \times 0 = 0$, which is divisible by 196.

THINGS TO REMEMBER

1. Properties of ${}^nC_r$

(i) ${}^nC_0 = {}^nC_n = 1$ (ii) ${}^nC_1 = {}^nC_{n-1} = n$ (iii) ${}^nC_r = {}^nC_{n-r}$

(iv) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (v) $r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$

(vi) $r(r-1)\,{}^nC_r = n(n-1)\,{}^{n-2}C_{r-2}$

(vii) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

2. $(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 + \ldots + {}^nC_n\, x^0 y^n$ or $(x + y)^n$

$= \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, y^r$

In this formula, n is a positive integer, x and y are real or complex numbers and

${}^nC_r = \dfrac{n!}{r!(n-r)!}$ where ${}^nC_r$ are binomial coefficients $r = 0, 1, 2, 3, \ldots\ldots\ldots, n$

3. By replacing y by $-y$, we can also find expansion of $(x - y)^n$

i.e. $(x - y)^n = {}^nC_0\, x^n y^0 - {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 - \ldots + (-1)^n\, {}^nC_n\, x^0 y^n$

or $(x - y)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r}\, y^r$

4. Some important observations in Binomial Theorem

(i) The expansion of $(x + y)^n$ can also be taken as identity in 'x' and 'y'.

(ii) The number of terms in the expansion are $n + 1$.

(iii) The expansion contains decreasing powers of x and increasing powers of y. The sum of the powers of x and y in each term is equal to n.

(iv) The binomial coefficients : ${}^nC_0, {}^nC_1, {}^nC_2,$ .... equidistant from beginning and end are equal i.e. ${}^nC_r = {}^nC_{n-r}$.

(v) The corresponding terms in the expansion of $(x + y)^n$ and $(x - y)^n$ are numerically equal.

(vi) The terms in the expansion of $(x - y)^n$ are alternately positive and negative. The last term in the expansion is positive or negative accordingly as n is even or odd integer.

(vi) Put $x = 1$ and $y = x$ in 1.2 (A) we get,

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots + {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n$$

i.e. $$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

(vii) Put $y = 1$ in 1.2 (A) we get,

$$(x + 1)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1} + {}^nC_2\, x^{n-2} + \ldots\ldots + {}^nC_r\, x^{n-r} + \ldots\ldots + {}^nC_n\, x^0$$

i.e. $$(x + 1)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}$$

(viii) Put $x = 1$ and $y = -x$ in 1.2 (A) we get,

$$(1 - x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2\, x^2 - {}^nC_3\, x^3 + \ldots\ldots + (-1)^r\, {}^nC_r\, x^r + \ldots\ldots + (-1)^n\, {}^nC_n\, x^n$$

i.e. $$(1 - x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$

(ix) The coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$

(x) If we have, $(x + y)^n + (x - y)^n = 2\,[{}^nC_0\, x^n y^0 + {}^nC_2\, x^{n-2} y^2 + \ldots\ldots]$

Now, the number of terms in $(x + y)^n + (x - y)^n$ is

(A) If 'n' is odd then number of terms is $\dfrac{n+1}{2}$

(B) If 'n' is even then number of terms is $\dfrac{n}{2} + 1$.

(xi) If we have $(x + y)^n - (x - y)^n = 2\,[{}^nC_1\, x^{n-1} y^1 + {}^nC_3\, x^{n-3} y^3 + \ldots\ldots]$

Now, the number of terms in $(x + y)^n - (x - y)^n$ is

(A) If 'n' is odd, then the number of terms is $\frac{n+1}{2}$.

(B) If 'n' is even, then the number of terms is $\frac{n}{2}$.

(xii) Sometimes ${}^nC_r$ is written as C_r.

5. General Term

The general term in the expansion is $(r + 1)^{st}$ term. It is represented as T_{r+1}.

In the expansion $(x + y)^n$, $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

In the expansion $(x - y)^n$, $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}\, y^r$

The binomial expansions of $(x + y)^n$ and $(x - y)^n$ can also be represented as

$$\sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, y^r \quad \text{and} \quad \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r}\, y^r \text{ respectively.}$$

(i) In binomial expansion is $(1 + x)^n$ we have, $T_{r+1} = {}^nC_r\, x^r$

(ii) In binomial expansion of $(1 - x)^n$, we have $T_{r+1} = (-1)^r\, {}^nC_r\, x^r$

6. Middle Term

The middle term in the expansion depends upon the value of n.

If n is even, then total number of terms in the expansion is odd. So there is only one middle term

i.e. $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

If n is odd, then total number of terms in the expansion is even. So there are two middle terms

i.e. $\left(\frac{n+1}{2}\right)^{th}$ term and the next are two middle terms.

7. Greatest Term

To find the numerically greatest term in the expansion of $(1 + x)^n$:

(i) Calculate $m = \left|\frac{x\,(n+1)}{x+1}\right|$

(ii) If m is an integer, then T_m and T_{m+1} are equal and both are greatest terms.

(iii) If m is not an integer, then $T_{[m]+1}$ is the greatest term, where $[m]$ is the integral part of m.

8. Greatest Coefficient

(i) When n is even, greatest coefficient = ${}^{n}C_{\frac{n}{2}}$.

(ii) When n is odd, greatest coefficient = ${}^{n}C_{\frac{n-1}{2}}$ or ${}^{n}C_{\frac{n+1}{2}}$ (Note : both of them are equal)

9. $C_0 + C_1 + C_2 + C_3 + \ldots\ldots.. + C_n = 2^n$ or $\sum_{r=0}^{n} {}^{n}C_r = 2^n$ $[C_r = {}^{n}C_r]$

10. $1\,C_1 + 2\,C_2 + 3\,C_3 + \ldots.. + nC_n = n \cdot 2^{n-1}$ or $\sum_{r=1}^{n} r \cdot {}^{n}C_r = n \cdot 2^{n-1}$

11. $1^2\,C_1 + 2^2\,C_2 + 3^2\,C_3 + \ldots.. + n^2\,C_n = n\,(n+1)\,2^{n-2}$ or $\sum_{r=1}^{n} r^2 \cdot C_r = n\,(n+1)\,2^{n-2}$

12. $C_0 - C_1 + C_2 - C_3 + \ldots\ldots.. + (-1)^n\,C_n = 0$ or $\sum_{r=0}^{n} (-1)^r\,{}^{n}C_r = 0$ $[C_r = {}^{n}C_r]$

13. $1\,C_1 - 2\,C_2 + 3\,C_3 - \ldots.. + (-1)^n\;nC_n = 0$ or $\sum_{r=1}^{n} (-1)^r\;r \cdot {}^{n}C_r = 0$

14. $1^2\,C_1 - 2^2\,C_2 + 3^2\,C_3 - \ldots.. + (-1)^n\,n^2\,C_n = 0$ or $\sum_{r=1}^{n} (-1)^r\;r^2 \cdot C_r = 0$

15. $C_0^2 + C_1^2 + C_2^2 + \ldots.\,C_n^2 = {}^{2n}C_n$ or $\sum_{r=0}^{n} C_r^2 = {}^{2n}C_n$

16. $1 \cdot C_1^2 + 2 \cdot C_2^2 + 3 \cdot C_3^2 + \ldots. + n\,C_n^2 = n\;{}^{2n-1}C_{n-1}$ or $\sum_{r=1}^{n} r \cdot C_r^2 = n \cdot {}^{2n-1}C_{n-1}$

17. $1^2 \cdot C_1^2 + 2^2 \cdot C_2^2 + 3^2 \cdot C_3^2 + \ldots. + n^2 \cdot C_n^2 = n^2\;{}^{2n-2}C_{n-1}$ or $\sum_{r=1}^{n} r^2\,C_r^2 = n^2\;{}^{2n-2}C_{n-1}$