CHAPTER-2
INVERSE TRIGONOMETRIC FUNCTIONS

Revision Notes

- In mathematics, the inverse trigonometric functions are the inverse function of trigonometric functions. Specifically, they are inverse of the sine, cosine, tangent, cotangent, secant and cosecant functions, and are used to obtain an angle from any of the angle's trigonometric ratios.
- **Domains and Ranges of Inverse Trigonometric Functions :**

Inverse Trigonometric Functions *i.e.*, $f(x)$	Domain/Value of x	Range/Value of $f(x)$
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\text{cosec}^{-1}x$	$\mathbb{R} - (-1, 1)$ $(-\infty, -1] \cup [1, \infty)$	$\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\sec^{-1}x$	$\mathbb{R} - (-1, 1)$ $(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$\tan^{-1}x$	$\mathbb{R}$ $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	$\mathbb{R}$ $(-\infty, \infty)$	$(0, \pi)$

Note

- The symbol $\sin^{-1}x$ is used to denote the smallest angle whether positive or negative, such that the sine of this angle will give us x. Similarly, $\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$ are defined.
- You should note that $\sin^{-1}x$ can be written as arc sin x. Similarly, other inverse trigonometric functions can also be written as arc cos x, arc tan x, arc sec x etc.
- Also, note that $\sin^{-1}x$ (and similarly other inverse trigonometric functions) is entirely different from $(\sin x)^{-1}$. In fact, $\sin^{-1}x$ is measure of an angle in Radians whose sine is x whereas $(\sin x)^{-1}$ is $\frac{1}{\sin x}$ (which is obvious as per the laws of exponents).
- Keep in mind that these inverse trigonometric relations are true only in their domains, *i.e.*, they are valid only for some values of 'x' for which inverse trigonometric functions are well defined.

- **Principal Value :**
  Numerically smallest angle is known as the principal value.
  For finding the principal value, following algorithm can be followed.
  Step 1 : First, draw a trigonometric circle and mark the quadrant in which the angle may be lie.
  Step 2 : Select anti-clockwise direction for 1st and 2nd quadrant in which the angle may be lie.
  Step 3 : Find the angles in the first rotation.
  Step 4 : Select the numerically least (magnitude wise) angle among these two values. The angle thus found will be the principal value.
  Step 5 : In case, two angles one with positive sign and the other with the negative sign qualify for the numerically least angle then, it is the convention to select the angle with positive sign as principal value.

Note

A function $f : A \to B$ is said to be invertible if f is bijective (*i.e.*, one-one and onto). The inverse of the function f is denoted by $f : B \to A$ such that $f^{-1}(y) = x$ if $f(x) = y$, $\forall\ x \in A$, $y \in B$. As trigonometric functions are many-one so, their inverse doesn't exist. But they become one-one onto by restricting their domains. Therefore, all the restrictions required so that the inverse of the concerned trigonometric functions do exist. If these restrictions are removed, the terms will represent **Inverse Trigonometric Relations** and not the functions. Note that the inverse trigonometric functions are also called as **Inverse Circular Functions**.

- **Elementary Properties of Inverse Trigonometric Functions :**

Property I

(a) $\sin^{-1}(x) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right), x \in [-1, 1]$

(b) $\operatorname{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(c) $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right), x \in [-1, 1]$

(d) $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right), x \in (-\infty, -1] \cup [1, \infty)$

(e) $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right), x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right), x < 0 \end{cases}$

(f) $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), x < 0 \end{cases}$

Property II

(a) $\sin^{-1}(\sin x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(b) $\cos^{-1}(\cos x) = x, 0 \le x \le \pi$

(c) $\tan^{-1}(\tan x) = x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

(d) $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0$

(e) $\sec^{-1}(\sec x) = x, 0 \le x \le \pi, x \ne \frac{\pi}{2}$

(f) $\cot^{-1}(\cot x) = x, 0 < x < \pi$

Property III

(a) $\sin^{-1}(-x) = -\sin^{-1}x, x \in [-1, 1]$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1]$

(c) $\tan^{-1}(-x) = -\tan^{-1}x, x \in \mathbb{R}$

(d) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x, |x| \ge 1$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x, |x| \ge 1$

(f) $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in \mathbb{R}$

Property IV

(a) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$

(b) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in \mathbb{R}$

(c) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}, x \in (-\infty, -1] \cup [1, \infty)$

Property V

(a) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(b) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$

(c) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

(d) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy > -1$

(e) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Property VI

(a) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(b) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3), \text{if } -\frac{1}{2} \le x \le \frac{1}{2}$

Property VII

(a) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1), \text{ if } 0 \le x \le 1$

(b) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), \text{if } \frac{1}{2} \le x \le 1$

Property VIII

(a) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{if } -1 < x < 1$

(b) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right), \text{if } \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Key Formulae

- **Trigonometric Formulae :**
  - **Relation between trigonometric ratios :**

(a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ (b) $\tan\theta = \frac{1}{\cot\theta}$ (c) $\tan\theta . \cot\theta = 1$

(d) $\cot\theta = \frac{\cos\theta}{\sin\theta}$ (e) $\mathrm{cosec}\,\theta = \frac{1}{\sin\theta}$ (f) $\sec\theta = \frac{1}{\cos\theta}$

- **Trigonometric Identities :**

(a) $\sin^2\theta + \cos^2\theta = 1$ (b) $\sec^2\theta = 1 + \tan^2\theta$ (c) $\mathrm{cosec}^2\theta = 1 + \cot^2\theta$

- **Addition/subtraction/formulae and some related results :**

(a) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ (b) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

(c) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(d) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(e) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ (f) $\cot(A \pm B) = \frac{\cot B\cot A \mp 1}{\cot B \pm \cot A}$

- **Multiple angle formulae involving 2*A* & 3*A* :**

(a) $\sin 2A = 2\sin A\cos A$ (b) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(c) $\cos 2A = \cos^2 A - \sin^2 A$ (d) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

(e) $\cos 2A = 2\cos^2 A - 1$ (f) $\cos 2A = 1 - 2\sin^2 A$

(g) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$ (h) $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

(i) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ (j) $\sin 3A = 3\sin A - 4\sin^3 A$

(k) $\cos 3A = 4\cos^3 A - 3\cos A$ (l) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

- **Transformation of sums/differences into products & vice-versa :**

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ (b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$ (d) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

(e) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ (f) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(g) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ (h) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

➢ **Domain and Range of trigonometric functions :**

S. No.	Function	Domain	Range
(i)	sine	R	[–1, 1]
(ii)	cosine	R	[–1, 1]
(iii)	tangent	$R - \left\{x : x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$	R
(iv)	cosecant	$R - \{x : x = n\pi,\ n \in Z\}$	$R - (-1, 1)$

(v)	secant	$R-\left\{x: x=(2n+1)\frac{\pi}{2}; n\in Z\right\}$	$R-(-1, 1)$
(vi)	cotangent	$R-\{x: x=n\pi, n\in Z\}$	R

➢ **Relations in different measures of Angle :**

(a) Angle in Radian Measure = (Angle in degree measure) × $\frac{\pi}{180°}$ radians

(b) Angle in Degree Measure = (Angle in radian measure) × $\frac{180°}{\pi}$

(c) θ (in radian measure) = $\frac{l}{r} = \frac{\text{arc}}{\text{radius}}$

Also following are of importance as well :

(a) 1 right angle = 90°

(b) 1° = 60′, 1′ = 60″

(c) $1° = \frac{\pi}{180°}$ = 0·01745 radians (Approx.)

(d) 1 radian = 57°17′45″ or 206265 seconds.

➢ **Relation in Degree & Radian Measures :**

Angles in Degree	0°	30°	45°	60°	90°	180°	270°	360°
Angles in Radian	0 rad	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$	(π)	$\left(\frac{3\pi}{2}\right)$	(2π)

➢ **Trigonometric Ratio of Standard Angles :**

Degree	**0°**	**30°**	**45°**	**60°**	**90°**
sin x	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot x	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
cosec x	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

➢ **Trigonometric Ratios of Allied Angles :**

Angles (→) **T – Ratios (↓)**	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	π – θ	π + θ	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	**2π – θ or – θ**	2π + θ
sin	cos θ	cos θ	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ
cos	sin θ	– sin θ	– cos θ	– cos θ	– sin θ	sin θ	cos θ	cos θ
tan	cot θ	– cot θ	– tan θ	tan θ	cot θ	– cot θ	– tan θ	tan θ
cot	tan θ	– tan θ	– cot θ	cot θ	tan θ	– tan θ	– cot θ	cot θ
sec	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ	sec θ	sec θ
cosec	sec θ	sec θ	cosec θ	– cosec θ	– sec θ	– sec θ	– cosec θ	cosec θ